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THE SPLITTING THEOREM FOR SPACE-TIMES WITH STRONG ENERGY CONDITION

J.-H. ESCHENBURG

1. Introduction

Our aim is the proof of the following theorem:

Theorem. *Let (M, g) be a connected, time oriented, globally hyperbolic Lorentzian manifold which is timelike geodesically complete ("tgc") and satisfies* $\mathrm{Ric}(v, v) \geqslant 0$ *for every timelike tangent vector v, where* Ric *denotes the Ricci tensor of g. Let $\gamma\colon \mathbb{R} \to M$ be a line, i.e. a timelike geodesic which realizes the distance between any two of its points. Then (M, g) is isometric to* $(\mathbb{R} \times H, -dt^2 \otimes h)$ *where (H, h) is a complete Riemannian manifold, and the factor* $(\mathbb{R}, -dt^2)$ *is represented by γ.*

This is the Lorentzian version of the Cheeger-Gromoll Splitting Theorem for Riemannian manifolds of nonnegative Ricci curvature [5] which solves a problem raised by S. T. Yau [12]. Our result extends earlier work of Galloway [8] and Beem et al. [2], [3]. Galloway [8] has proved the theorem under the additional assumption that M admits a smooth function whose level sets are compact spacelike Cauchy hypersurfaces. Beem, Ehrlich, Markvorsen, and Galloway [2], [3] proved a Toponogov type splitting theorem [10] for Lorentzian manifolds, i.e. they assumed $g(R(w, v)v, w) \geqslant 0$ for any timelike vector v and any $w \perp v$. The weaker Ricci curvature assumption of our theorem, called the strong energy condition (cf. [9]), is of particular interest in General Relativity. However, we need to assume the tgc property which can be concluded from the curvature assumption in the case of Beem et al.

The proofs of Cheeger and Gromoll [5] and H. Wu [11] for the Riemannian case apply the theory of elliptic operators to the Laplace operator Δ on the manifold. This fails in the Lorentzian case. In [6], a new proof was given which is based on the following idea: The Busemann functions b^+ and b^- of the given line γ satisfy $b^+ + b^- \leqslant 0$ with equality along γ, by the triangle inequality. On the other hand, by $\mathrm{Ric} \geqslant 0$, we have $\Delta b^\pm \geqslant 0$ in the sense of support

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functions. Thus we may apply the Hopf-Calabi maximum principle [4] to $b^+ + b^-$ to show that $b^+ + b^- = 0$ which is the core of the proof. Unfortunately, the maximum principle also fails for the Laplacian of a Lorentzian manifold. Therefore, instead of $b^+ + b^-$ we consider $b^- \mid \{b^+ = a\}$ for a suitable value a. The level set $\{b^+ = a\}$ can be approximated by smooth spacelike hypersurfaces where we can use the Riemannian Laplacian.

Another difficulty in the Lorentzian case is that the Busemann function might be uncontinuous. In [3], continuity has been shown by using a triangle comparison theorem which is not valid under the weaker curvature assumption. Instead, we show the continuity of the Busemann function near the line γ (§3), and we get the splitting in a small neighborhood of γ (§6). An "open and closed subset" argument then finishes the proof (§7).

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2. Rays and co-rays

2.1. Let (M, g) be a connected, time oriented, globally hyperbolic, timelike geodesically complete ("tgc") Lorentzian manifold of dimension $n + 1$, $n \geqslant 1$. We agree that g is of type $(-, +, \cdots, +)$. Let d be the Lorentzian distance function (see [9]) and $\ll$ the timelike future relation on M: For $p, q \in M$ we say $p \ll q$ or $p \in I^-(q)$ or $q \in I^+(p)$ if there is a future oriented timelike curve from p to q. The Lorentzian distance is continuous and satisfies the reverse triangle inequality: If $p, q, r \in M$ with $p \ll q \ll r$, then

$$d(p,q) + d(q,r) \leqslant d(p,r). \tag{T}$$

A (*future oriented*) *ray* is, by definition, a timelike, future oriented unit speed geodesic $\gamma\colon [a, \infty) \to M$ (for some $a \in \mathbb{R}$) which is maximal, i.e. $d(\gamma(t), \gamma(s)) = s - t$ for all s, t with $a \leqslant t \leqslant s$. Everything will work as well for *past oriented* rays; we just have to reverse the time orientation of M. For a given ray γ let us define functions $b_s\colon M \to \mathbb{R}$ for any $s \geqslant a$ by

$$b_s(x) = s - d(x, \gamma(s)).$$

Then (T) implies for any $x \ll \gamma(s)$ and $r > s$

$$b_r(x) \leqslant b_s(s). \tag{1}$$

On the other hand, if $\gamma(a) \ll x \ll \gamma(s)$ for some $s > a$, then

$$b_s(x) \geqslant d(\gamma(a), x) + a. \tag{2}$$

Let $I(\gamma) = \{x \in M;\ \gamma(a) \ll x \ll \gamma(s) \text{ for some } s > a\}$. This is an open neighborhood of $\gamma((a, \infty))$. By (1) and (2), $b_s(x)$ is a monotonously decreasing and bounded function of s for any $x \in I(\gamma)$, so $b(x) := \lim_{s \to \infty} b_s(x)$ exists and defines a function b: $I(\gamma) \to \mathbb{R}$, called the *Busemann function* of γ. In general, the convergence need not be locally uniform and b is not necessarily continuous.

2.2. However, the triangle inequality (T) gives a *lower* bound for the increase of b. For $p, q \in I(\gamma)$ with $p \ll q$, we have

$$b(q) \geqslant b(p) + d(p,q). \tag{3}$$

Suppose we find two points $p_0 \ll q_0$ where we have *equality* in (3). Then the function f: $I(\gamma) \cap I^-(q_0) \to \mathbb{R}$ with

$$f(p) = b(q_0) - d(p, q_0)$$

is an upper support function of b at p_0, that means that f is defined and continuous near p_0, and we have $f \geqslant b$ with equality at p_0. Likewise, e: $I(\gamma) \cap I^+(p_0) \to \mathbb{R}$ with

$$e(q) = b(p_0) + d(p_0, q)$$

is a *lower* support function of b at q_0.

If β is a maximal geodesic connecting p_0 and q_0, then equality in (3) also holds for any two points on β between p_0 and q_0. A ray β: $[0, \infty) \to I(\gamma)$ is called a *co-ray of* γ if equality in (3) holds along β, i.e. for all $s > t \geqslant 0$,

$$b(\beta(s)) - b(\beta(t)) = s - t. \tag{4}$$

(In fact it is sufficient to check (4) for $t = 0$.) Note that the definition of a co-ray is slightly different from that in [2], [3]. Now for any $s > 0$, the function $b_{\beta,s}$: $I(\gamma) \cap I^-(\beta(s)) \to \mathbb{R}$,

$$b_{\beta,s}(x) = b(\beta(s)) - d(x, \beta(s)) = b(\beta(0)) + s - d(x, \beta(s)),$$

is an upper support function of b at $\beta(0)$ which is smooth near $\beta(0)$ since $\beta(0)$ is not in the cut locus of $\beta(s)$ (cf. [1, p. 105]). In particular, for the Busemann functions b_β of β which is defined on $I(\beta) \subset I(\gamma)$, we have

$$b_\beta \geqslant b - b(\beta(0)). \tag{5}$$

Moreover, at any point $\beta(t)$ for $t > 0$ there are smooth upper and lower support functions f and e, namely $f(x) = b(\beta(s)) - d(x, \beta(s))$ for any fixed $s > t$ and $e(x) = b(\beta(0)) + d(\beta(0), x)$. Hence b is differentiable at $\beta(t)$ with gradient $\nabla b(\beta(t)) = -\beta'(t)$, and, therefore, β is the only co-ray passing through $\beta(t)$.

2.3. Does there exist a co-ray β with $\beta(0) = q$ for any $q \in I(\gamma)$? For sufficiently large s consider a maximal unit speed geodesic segment β_s connecting q to $\gamma(s)$ (which exists by global hyperbolicity), say $\beta_s(0) = q$ and

$\beta_s(r) = \gamma(s)$. Then we have $b_s(q) = s - r$, and in particular, $r = r(s) \to \infty$ as $s \to \infty$. Moreover, for any $0 < u < r$ and $x \ll \beta_s(u)$,

$$u - d(x, \beta_s(u)) \geqslant r - d(x, \beta_s(r)) = r - s + b_s(x),$$

by (T). So we get

$$b_s(q) + u - d(x, \beta_s(u)) \geqslant b_s(x). \tag{6}$$

Now suppose that the subset $\{\beta_s'(0);\ s > 0\}$ of T_qM is bounded. This is true near γ as we will see below. Then there is a sequence $s(i) \to \infty$ such that the vectors $\beta_{s(i)}'(0)$ converge to a future oriented unit vector in T_qM. Hence the geodesics $\beta_{s(i)}$ converge to a geodesic β which is a ray since $r(s(i)) \to \infty$ and M is tgc. If $\beta_s'(u)$ is close enough to $\beta'(u)$, then β_s enters $I^+(\beta(u))$ for any $u \geqslant 0$. Thus β lies in $I(\gamma)$. So we get from (6) for $x = \beta(u)$, $u > 0$, that

$$b(\beta(0)) + u \geqslant b(\beta(u)).$$

Together with (3) we get equality which shows that β is a co-ray of γ.

2.4. Let β_k be a sequence of co-rays with $\beta_k(0) = p_k$, $\beta_k'(0) = v_k$. Assume $p_k \to p$ and $v_k \to v$. Then the rays β_k converge to a ray β.

Claim. If $b(p_k) \to b(p)$, then β is a co-ray. In fact, let $b_{k,s}$ be the upper support functions of b corresponding to β_k as defined in 2.2. Then $b_{k,s} \to b_{\beta,s}$. Thus $b \leqslant b_{\beta,s}$. In particular, for $0 < t < s$,

$$b(\beta(t)) \leqslant b_{\beta,s}(\beta(t)) = b(p) + t.$$

Comparing with (3) we get equality, and therefore, β is a co-ray.

3. The Busemann function near its ray

Let $\gamma\colon [a, \infty) \to M$ be a ray with $a < 0$. We want to study the functions b_s and b near $\gamma(0)$. Consider a small closed tubular neighborhood $U = U(R, T) \subset I(\gamma)$ of radius R around a segment $\gamma \,|\, [-T, T]$ with $-T > a$. Using Fermi coordinates, we may identify U with $I \times K$ where $I = [-T, T]$ and $K = K(R)$ is the closed ball of radius R around 0 in $\mathbb{R}^n$: We choose a parallel orthonormal basis $\gamma' = e_0, e_1, \cdots, e_n$ along $\gamma \,|\, [-T, T]$ and consider the map $\Phi\colon I \times K \to U$,

$$\Phi(x_0, x_1, \cdots, x_n) = \exp\left(\sum_{i=1}^{n} x_i e_i(x_0)\right).$$

If R is small enough, Φ is a diffeomorphism and $\Phi^{-1} = x = (x_0, \cdots, x_n)$ is a coordinate chart on U. Moreover, for small R the metric g is close to the flat Minkowski metric $g_0 = \|d\mathbf{x}\|^2 - dx_0^2$ where $\mathbf{x} = (x_1, \cdots, x_n)$, and the corresponding connections D and D^0 are close to each other.

Lemma 3.1. *If U is small enough, there is a constant $C > 0$ such that $|b_s - x_0| \leqslant C\|\mathbf{x}\|^2$ on U for all $s > 2T$.*

Proof. Choose R_1, T_1 as above and put $U_1 = U(R_1, T_1)$. Let $T = T_1/2$. If we choose $R < R_1$ small enough, then the functions

$$f(x) = 2T - d(x, \gamma(2T)), \qquad e(x) = -2T + d(\gamma(-2T), x)$$

are defined and smooth on $U = U(R, T)$ with $f(\gamma(t)) = e(\gamma(t)) = t$ for $-T \leqslant t \leqslant T$, and $Df_{\gamma(t)}$ and $De_{\gamma(t)}$ vanish on the normal space of γ. Hence there is a constant $C > 0$ such that

$$|f - x_0| \leqslant C\|\mathbf{x}\|^2, \qquad |e - x_0| \leqslant C\|\mathbf{x}\|^2.$$

By (T), we have $f \geqslant b_s \geqslant e$ for any $s > 2T$ (compare 2.2), so we get the result.

On U we also have the euclidean metric $g_1 = \|dx\|^2 = \|d\mathbf{x}\|^2 + dx_0^2$ which has the same connection D^0 as g_0. Let $\| \ \|_1$ denote the norm with respect to g_1.

Lemma 3.2. *If U is small enough, there is a constant $M > 0$ with the following property: If $q \in U$ and $s > 2T$ then every maximal unit speed geodesic segment α from q to $\gamma(s)$ satisfies $\|\alpha'(0)\|_1 \leqslant M$.*

Proof. Start with $U_0 = U(R_0, T_0)$ which satisfies the assumption of Lemma 3.1 and $R_0 < 2T_0$. Choose $R_1 \leqslant R_0$ so that $\mu := C \cdot R_1^2 \leqslant R_1/10$. Put $U_1 = U(R_1, T_0)$. Let $R = R_1/2$, $T = T_0/2$, and $U = U(R, T)$. For any $q \in U$ let α be a maximal unit speed geodesic segment from q to $\gamma(s)$, $s > 2T$. While α is in U_1, we have $\alpha(\tau) = (\alpha_0(\tau), \boldsymbol{\alpha}(\tau))$. Since α is timelike future oriented and g is close to g_0, we may assume that

$$2\alpha_0'(\tau) > \|\boldsymbol{\alpha}'(\tau)\|.$$

In particular, the function $t(\tau) = \alpha_0(\tau) - \alpha_0(0)$ has positive derivative. Choose $t_0 \in (4\mu, R_1/2)$ and let τ_0 be such that $t(\tau_0) = t_0$. Then $\alpha([0, \tau_0]) \subset U_1$.

Since α is a maximal geodesic through $\gamma(s)$, we have

$$\tau_0 = b_s(\alpha(\tau_0)) - b_s(q).$$

On the other hand, by 3.1,

$$|b_s(\alpha(\tau_0)) - b_s(q) - t_0| \leqslant 2\mu,$$

hence in particular

$$\tau_0 \geqslant t_0 - 2\mu > t_0/2.$$

Now by the mean value theorem, there exist $\tau_1 \in (0, \tau_0)$ such that $\alpha_0'(\tau_1) < 2$, thus $\|\boldsymbol{\alpha}'(\tau_1)\| < 4$ and $\|\alpha'(\tau_1)\|_1 < 5$.

Suppose that $\|D - D^0\|_1 \leqslant A$ on U_1. Then the function $f(\tau) = \|\alpha'(\tau)\|_1$ satisfies $|f'| \leqslant A \cdot f^2$ on $[0, \tau_1]$, whence $|(1/f)'| \leqslant A$ and

$$1/f(0) \geqslant 1/f(\tau_1) - A \cdot \tau_1 \geqslant 1/5 - A \cdot \tau_0.$$

Note that $\tau_0 < t_0 + 2\mu < T_0 + 2\mu$ is uniformly bounded. So we get $\|\alpha'(0)\|_1 < M := 1/10$ if R_1 was chosen so small that $A \cdot (T_0 + 2\mu) \leqslant 1/10$.

Lemma 3.3. *If U is small enough, the Busemann function is Lipschitz continuous on U with respect to the metric g_1. Moreover, for any $p \in U$ there is a co-ray β starting at p and therefore an upper support function $b_{p,s} := b_{\beta,s}$ of b at p.*

Proof. We will show that the functions $b_s = s - d_s$ with $d_s(x) = d(x, \gamma(s))$ are equi-Lipschitz continuous. The function d_s on $I^-(\gamma(s))$ is continuous but not smooth everywhere. However, for any $p \in I^-(\gamma(s))$, we get a smooth lower support function e of d_s at p as follows: Let α be a maximal unit speed geodesic segment from p to $\gamma(s)$ and choose q on α between p and $\gamma(s)$. Put $e(x) = d(x, q) + d(q, \gamma(s))$. Then e is smooth near p because there is no cut point of q near p and, by (T), e is a lower support function of d_s at p. Thus we only have to show that $\|De_p\|_1 \leqslant L$ for all $p \in U$ with a constant L not depending on s (see the Appendix). Now for any $v \in T_pM$,

$$|De_p(v)| = |g(\alpha'(0), v)| \leqslant G \cdot M \cdot \|v\|_1,$$

where M is as in 3.2 and G is an upper bound of $\|g\|_1$ on U. If we put $L = G \cdot M$, then b_s and hence b are Lipschitz continuous with Lipschitz constant L. The existence of co-rays is clear from 3.2 and 2.3.

Lemma 3.4. *Let (p_k) be a sequence in U with $p_k \to p := \gamma(0)$. Let β_k be a co-ray starting at p_k with initial vector v_k. Then $v_k \to \gamma'(0)$.*

Proof. Since $\|v_k\|_1$ is bounded, we may assume $v_k \to v$ for some $v \in T_pM$. Thus, by 2.4, the co-rays β_k converge to a co-ray β starting at p. But by 2.3, since $a < 0$, γ is the only co-ray passing through p. Thus $\beta = \gamma\,|\,[0, \infty)$ which finishes the proof.

4. Lines

A complete geodesic $\gamma\colon \mathbb{R} \to M$ is called a *line* if $\gamma\,|\,[a, \infty)$ is a ray for any $a \in \mathbb{R}$. In other words, we have $d(\gamma(t), \gamma(s)) = s - t$ for any $t < s$. Let $\gamma^-\colon \mathbb{R} \to M$, $\gamma^-(t) = \gamma(-t)$. Then $\gamma^-\,|\,[a, \infty)$ is a past oriented ray for any a. Besides $b_s^+ := b_s$, we have the analogue functions for γ^-,

$$b_s^-(x) = s - d(\gamma(-s), x),$$

and the two Busemann functions $b^\pm = \lim_{s\to\infty} b_s^\pm$, defined on

$$I(\gamma) = \{x \in M;\ \gamma(t) \ll x \ll \gamma(s) \text{ for some } s, t \in \mathbb{R}\}.$$

The triangle inequality (T) implies that $-b_r^- \leqslant b_s^+$ with equality on $\gamma\,|\,[-r, s]$ for any $r, s > 0$, and in particular we get $b^+ + b^- \geqslant 0$ with equality along γ.

Lemma 4.1. *Let $q \in I(\gamma)$ be such that $(b^+ + b^-)(q) = 0$. Assume further that there are co-rays β^+ of γ and β^- of γ^- starting at q. Then β^+ and β^- fit together to a line β whose Busemann functions $b_\beta^\pm$ satisfy $b_\beta^+ + b_\beta^- \geqslant b^+ + b^-$ on $I(\beta) \subset I(\gamma)$ with equality along β.*

Proof. Let $b^+(q) = -b^-(q) =: a$. We have $b^-(\beta^-(t)) = -a + t$ and by (3) (§2) $b^+(\beta^-(t)) \leqslant a - t$ for all $t \geqslant 0$. Hence $(b^+ + b^-)(\beta^-(t)) \leqslant 0$ and so we have equality and in particular, $b^+(\beta^-(t)) = a - t$. Let us define the broken geodesic $\beta\colon \mathbb{R} \to M$ by $\beta(t) = \beta^+(t)$ for $t \geqslant 0$ and $\beta(t) = \beta^-(-t)$ for $t \leqslant 0$. Then $b^+(\beta(t)) = a + t$ and, similarly, $b^-(\beta(t)) = -(a + t)$ for all $t \in \mathbb{R}$, and it follows from (3) (§2) that β is an unbroken maximal geodesic, hence a line. By (5) (§2) we have $b_\beta^\pm \geqslant \pm a + b^\pm$ which finishes the proof.

Lemma 4.2. *Let $\gamma\colon \mathbb{R} \to M$ be a line. Then there is a neighborhood W of $\gamma(\mathbb{R})$ such that the Busemann functions b^+ and b^- are continuous on W, and for any $q \in W$ there exist co-rays of γ and γ^- starting at q.*

Proof. This follows from Lemma 3.3.

5. The Ricci curvature condition

Let (M^{n+1}, g) be a time oriented Lorentzian manifold, $M' \subset M$ an open subset, and $f\colon M' \to \mathbb{R}$ a smooth function with $g(\nabla f, \nabla f) = -1$. Then the gradient lines of f are timelike unit speed geodesics and the level sets of f are spacelike hypersurfaces, and f can be viewed as a signed distance function of each of its level hypersurfaces, up to a constant. Moreover, if γ is a gradient line and $U(t) = D\nabla f\,|_{\gamma(t)}$ is the Hessian tensor along γ, then this tensor field along γ satisfies the Riccati equation

$$(1) \qquad U' + U^2 + R = 0$$

with $R(t)v := R(v, \gamma'(t))\gamma'(t)$. Taking the trace and putting $u(t) = \operatorname{trace} U(t) = \Delta f(\gamma(t))$, we get by the Schwarz inequality (note that $U((\gamma')^\perp) \subset (\gamma')^\perp$ and $U(\gamma') = 0$)

$$(2) \qquad u' + u^2/n + \operatorname{Ric}(\gamma', \gamma') \leqslant 0$$

with equality if and only if $U\,|(\gamma')^\perp$ is a multiple of the identity (cf. [6], [7]).

From now on suppose $\operatorname{Ric}(v, v) \geqslant 0$ for any timelike tangent vector v. Then, by (2), the function $\phi = 1/u$ satisfies $\phi' \geqslant 1/n$. Let $f = d_q := d(\cdot, q)$, defined on $I^-(q)$ outside the cut locus of q, for some point $q \in M$. Then f is as described above and any gradient line γ is future oriented with future end point q. Let $q = \gamma(s)$. Then $u(t) \to \infty$ as $t \to s$ and $t < s$, whence $\phi(s) = 0$ and so $-\phi(t) \geqslant (s - t)/n$. So we get

$$\Delta d_q \geqslant -n/d_q.$$

Therefore, for any ray γ, the functions $b_s = s - d_{\gamma(s)}$ satisfy

$$\Delta b_s \leqslant n/d_{\gamma(s)}.$$

In particular, we get from 2.2:

Lemma 5.1. *Let γ be a ray and β a co-ray with $\beta(0) = q$. Then for any $\varepsilon > 0$ there exists $r > 0$ such that $\Delta b_{\beta,s} \leqslant \varepsilon$ near q for any $s \geqslant r$.*

Lemma 5.2. *Let γ: $\mathbb{R} \to M$ be a line. Then for any $t \in \mathbb{R}$ we have $DD(b_s^{\pm})|_{\gamma(t)} \to 0$ as $s \to \infty$, and $R(\ ,\gamma'(t))\gamma'(t) = 0$ for any $t \in \mathbb{R}$.*

Proof. Let $U_s(t) = D\nabla(b_s^+)|_{\gamma(t)}$ for $t < s$. For fixed $v \in T_{\gamma(t)}M$, the function $s \to DD(b_s^+)(v, v) = g(U_s(t)v, v)$ is monotonously decreasing and bounded from below by $-DD(b_r^-)(v, v)$ for arbitrary $r > |t|$ (see (1), (7) in §2). Thus $U(t) := \lim_{s\to\infty} U_s(t)$ exists for any $t \in \mathbb{R}$ and solves the Riccati equation (1). Thus $u := \operatorname{trace} U \equiv 0$ since otherwise $\phi = 1/u$ would satisfy $\phi' \geqslant 1/n$ and therefore ϕ would have a zero which is impossible. Now $U \equiv 0$ follows from the equality discussion of (2). So we get the result for b_s^+ and similarly for b_s^-. In particular we get from $U = 0$ that $R(\ ,\gamma')\gamma' = 0$.

6. Local splitting

Let γ: $\mathbb{R} \to M$ be a line and W a neighborhood of $\gamma(\mathbb{R})$ as in Lemma 4.2.

Proposition 6.1. *There is a neighborhood $W_0 \subset W$ of $\gamma(\mathbb{R})$ such that $b^+ + b^- \equiv 0$ on W_0.*

Proof. Assume the contrary. Then the open set

$$P = \{b^+ + b^- > 0\} \cap W$$

is nonempty (recall that $b^+ + b^- \geqslant 0$) and has a boundary point p on $\gamma(\mathbb{R})$. We may shift the parameter so that $p = \gamma(0)$.

We may assume without restriction of generality that P contains an open coordinate ball B with $p \in \partial B$. Namely, otherwise let $V \subset W$ be a coordinate neighborhood around p, and choose $q \in P \cap V$ near p. Since $P \cap V$ is open, there exists an open coordinate ball $B' \subset P \cap V$ centered at q. By enlarging the radius of B' we will meet $\partial P \cap V$ at last. Let B'' be the smallest of these balls such that $\partial B''$ intersects ∂P, say at p_1. Then we still have $B'' \subset P$, but at $p_1 \in \partial B''$ we have $(b^+ + b^-)(p_1) = 0$. Thus by 4.1 there is a line γ_1 passing through p_1 whose Busemann functions $b_1^{\pm}$ satisfy $b_1^+ + b_1^- \geqslant b^+ + b^-$. Let W_1 be a neighborhood of $\gamma_1(\mathbb{R})$ as in 4.2. So the set $P_1 := \{b_1^+ + b_1^- > 0\} \cap W \cap W_1$ contains $P \cap W_1$ and hence $B'' \cap W_1$. Since W_1 is a neighborhood of p_1, there is a smaller coordinate ball $B \subset B'' \cap W_1$ with $p_1 \in \partial B$. Now we replace γ and P with γ_1 and P_1 and our assumption is satisfied.

Let $U = U(R, T) \subset W$ be a closed tubular neighborhood of radius R around $\gamma \,|\, [-T, T]$ as in Lemma 3.3, where R and T are chosen small enough. For suitable constants $\alpha, \mu > 0$ let

$$A = \left\{ x \in U;\ \|\mathbf{x}\|^2 > \alpha \cdot x_0^2 \right\} \subset U,$$
$$V = \left\{ v \in TU;\ \|\mathbf{v}\|^2 < \mu \cdot v_0^2 \right\} \subset TU.$$

In the subsequent Lemma 6.2, we will construct a smooth function h on U with the following properties:

(1) $$h > 0 \quad \text{on } A \setminus B,$$
(2) $$h(p) = 0,$$
(3) $$\Delta h < 0,$$
(4) $$DDh(v, v) < 0 \quad \text{for every } v \in V.$$

If R is small enough, we get from Lemma 3.1:

$$\{b^{\pm} = 0\} \cap U \subset A,$$
$$\{b^{+} \leqslant 0\} \cap U \subset \{x_0 \leqslant 0\} \cup A,$$
$$b^{-} > 0 \quad \text{on } \{x_0 \leqslant 0\} \setminus A.$$

For any $r \in (0, R]$ let $U_r = U(r, T)$. If we choose r sufficiently small, then, by Lemma 3.4, every coray β^- of γ^- with $\beta^-(0) \in U_r$ is very close to γ^-. Thus for a support function $b^-_{q,s}$ with $q \in U_r$ corresponding to a coray β^-, we get that $v^- = -\nabla(b^-_{q,s})(q)$ is close to the coordinate vector $(\partial/\partial x_0)\,|_q$. Moreover, for any smooth vector field W, we have $DD(b^-_{q,s})(W_q, W_q) \leqslant C$ with a constant C independent of q and s (recall that $DD(b^-_{q,s})(q)$ is monotonously decreasing with s; compare the proof of Lemma 5.2).

For small enough r let $S_r = (\partial U_r) \cap \{b^+ \leqslant 0\}$. Then $b^- \geqslant 0$ on S_r and $S_r \subset \{b^- > 0\} \cup \{h > 0\}$. Thus there exists $\varepsilon_r > 0$ such that $f := b^- + \varepsilon \cdot h > 0$ on S_r for any $\varepsilon \in (0, \varepsilon_r]$. Let $U_r^- = U_r \cap \{b^+ \leqslant 0\}$. Then

$$\partial U_r^- \subset S_r \cup \left(\{b^+ = 0\} \cap \operatorname{Int}(U) \right).$$

We have $f(p) = 0$ by (2), and $p \in U_r^-$. Thus f takes a minimum $m \leqslant 0$ on U_r^-, say at q. Since $f \,|\, S_r > 0$, we have

$$q \in \operatorname{Int}(U_r^-) \cup \left(\{b^+ = 0\} \cap \operatorname{Int}(U_r) \right).$$

By Lemma 3.3, there is a co-ray β^- of γ^- starting at q, and b^- has smooth upper support functions $b^-_{q,s}$ at q. Thus $f_s := b^-_{q,s} + \varepsilon \cdot h$ is a smooth upper support function of f at q, and therefore, also, f_s takes its minimum on U_r^- at q.

By (3) and Lemma 5.1 we have $\Delta f_s(q) < 0$ if s is large enough, depending on ε. Moreover, we claim that $DDf_s(v, v) < 0$ for $v = \nabla f_s(q)$ for arbitrary $s \geqslant 1$, provided that ε is small enough. In fact, we have $v = -v^- + \varepsilon \cdot w$ where $w = \nabla h(q)$ and $v^- = -\nabla(b^-_{q,s})(q) = (\beta^-)'(0)$. Since r is small, v^- and hence $-v$ are close enough to the coordinate vector $\partial/\partial x_0$ to satisfy the assumption of (4) (see Lemma 3.4). In fact, we may assume that $DDh(v, v) < -\delta$ for some $\delta > 0$ depending only on r. Further, v^- is in the kernel of the Hessian $DD(b^-_{q,s})$ at q for any s, so

$$DD(b^-_{q,s})(v, v) = \varepsilon^2 DD(b^-_{q,s})(w, w) \leqslant C \cdot \varepsilon^2.$$

Thus

$$\begin{aligned} DDf_s(v, v) &= \varepsilon^2 \cdot DD(b^-_{q,s})(w, w) + \varepsilon \cdot DDh(v, v) \\ &\leqslant \varepsilon^2 C - \varepsilon\delta < 0, \end{aligned}$$

if ε is small enough.

Let $a := b^+(q) \leqslant 0$. Put $g_s := b^+_{q,s}$ and let H_s be the level set $\{g_s = a\}$ passing through q. This is a smooth spacelike hypersurface near q. We have $g_s \geqslant b^+$ with equality at q. Therefore $H_s \subset U_r^-$ and $f_s \mid H_s$ takes a minimum at q. Hence the gradients of f_s and g_s at q are linearly dependent. Since the co-rays $\beta^\pm$ starting at q are integral curves of $-\nabla(b^\pm_{q,s})$ and β^+ (β^-) is future (past) oriented, we get $\nabla f_s(q) = -\lambda \cdot \nabla g_s(q)$ for some positive λ close to 1. On the other hand, along H_s we have $\Delta g_s = -g(\eta, \nabla g_s)$ where η denotes the mean curvature normal field on H_s (recall that g_s is the distance function of H_s, up to a constant) and

$$\Delta f_s = \Delta_s f_s - g(\eta, \nabla f_s) - DDf_s(N, N),$$

where N is the unit normal field on H_s and Δ_s is the Laplacian of the induced (Riemannian) metric on H_s. So

$$\Delta_s f_s(q) = \Delta f_s(q) + \lambda \cdot \Delta g_s(q) + DDf_s(N_q, N_q) < 0$$

for sufficiently large s, since N_q and $v = \nabla f_s(q)$ are linearly dependent. But this is a contradiction to the minimality of $f_s \mid H_s$ at q. This finishes the proof of Proposition 6.1.

Lemma 6.2. *There exists a function h on U with the properties* (1)–(4) *as above.*

Proof. As before, let $x = (x_0, \cdots, x_n)\colon U \to \mathbb{R}^{n+1}$ be the Fermi coordinate system on U, and put $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{x}^* = (x_1, \cdots, x_{n-1})$. We may assume that $B \cap U = \{\psi < 0\}$ with

$$\psi = x_0^2 + \|\mathbf{x}^*\|^2 + x_n^2 - 2S \cdot x_n$$

for some radius $S > 0$. Assume $R < S/12 \ll 1$ and $T < 2R$. Choose $\mu \in (R/2, S/12)$ and $\alpha > 1 + 4\mu$. Put

$$\phi = \|\mathbf{x}^*\|^2/(2\mu) - x_n - x_0^2.$$

Claim. If $x \in A$ and $\phi(x) \leqslant 0$, then $\psi(x) < 0$.

In fact, from

$$\text{(a)} \qquad |\mathbf{x}^*|^2 > \alpha \cdot x_0^2 - x_n^2,$$

$$\text{(b)} \qquad |\mathbf{x}^*|^2 \leqslant 2\mu\left(x_n + x_0^2\right),$$

we get $x_n^2 + 2\mu \cdot x_n > (\alpha - 2\mu)x_0^2 > (1 + 2\mu)x_0^2$. Thus either $x_n < -2\mu < -R$ (which is impossible) or

$$x_n > \mu \cdot \left\{\left[1 + x_0^2(1 + 2\mu)/\mu^2\right]^{1/2} - 1\right\}.$$

Since $x_0^2 < T^2 < 4R^2 < 16\mu^2$, we may assume

$$x_0^2(1 + 2\mu)/\mu^2 < 24,$$

and since the slope of the square root function is bigger than $1/10$ on the interval $(1, 25)$, the inequality above gives

$$x_n > x_0^2(1 + 2\mu)/(10\mu).$$

Now using (b) we have

$$\begin{aligned}\psi(x) &\leqslant x_0^2(1 + 2\mu) + 2\mu \cdot x_n + x_n^2 - 2S \cdot x_n \\ &< (12\mu + x_n - 2S) \cdot x_n < (x_n - S) \cdot x_n < 0,\end{aligned}$$

since $x_n \leqslant R < S$. This proves the claim. So we get $\phi > 0$ on $A \setminus B$.

Now we put $h = 1 - e^{-\sigma \cdot \phi}$ for some sufficiently big constant σ. Then (1) and (2) are true. Since ϕ satisfies $g_0(D\phi, D\phi) > 0$ and thus $g(D\phi, D\phi) > 0$ if R is small enough, we have

$$\Delta h = e^{-\sigma\phi}\left(-\sigma^2 \cdot g(\nabla\phi, \nabla\phi) + \sigma \cdot \Delta\phi\right) < 0$$

if σ is big enough, so (3) holds. Moreover,

$$DDh(v, v) = e^{-\sigma\phi}\left(-\sigma^2 \cdot g(\nabla\phi, v)^2 + \sigma \cdot DD\phi(v, v)\right).$$

Since the Hessian of ϕ with respect to g_0, namely

$$D^0D\phi(v, v) = \|\mathbf{v}^*\|^2/\mu - 2v_0^2,$$

is negative if $\|\mathbf{v}^*\|^2 \leqslant \|\mathbf{v}\|^2 < 2\mu \cdot v_0^2$, we get $DD\phi(v, v) < 0$ if $\|\mathbf{v}^*\|^2 < \mu \cdot v_0^2$ provided that R is small enough. This shows (4).

Proposition 6.3. *There is a neighborhood W_1 of $\gamma(R)$ which splits. More precisely, there exists an n-dimensional Riemannian manifold H and an isometry $j\colon \mathbb{R} \times H \to W_2$ such that $j|\mathbb{R} \times \{p\} = \gamma$ for some $p \in H$.*

Proof. The argument is similar to the Riemannian case (cf. [6, Chapter 4], and also [2], [3]). Let W_0 be as in Proposition 6.1. By 4.1 and 5.2, at any $q \in W_0$ there are smooth upper support functions $b^{\pm}_{q,s}$ of $b^{\pm}$ with $DD(b^{\pm}_{q,s})|_q$ arbitrarily close to zero. Thus for any geodesic c: $I \to W_0$, the functions $b^{\pm} \circ c$ have upper support functions with second derivatives arbitrarily close to zero, at any $t \in I$. The same is true for $b^{\pm} \circ c - g$ where g: $I \to \mathbb{R}$ is the affine function with $b^{\pm} \circ c = g$ on ∂I. Thus $b^{\pm} \circ c \geqslant g$ by the one-dimensional maximum principle. So $b^{\pm} \circ c$ is concave, and therefore, $b^{+} = -b^{-}$ is concave and convex. Hence on W_0, the level sets of b^+ are totally geodesic and b^+ is smooth with parallel gradient field $\nabla b^+ =: V$. For any $q \in W_0$, let γ_q be the line passing through q (see 4.1). Put

$$H = \{b^+ = 0\} \cap W_0.$$

Let j: $\mathbb{R} \times H \to M$, $j(t, q) = \gamma_q(t) = \exp_q(t \cdot V_q)$. Since γ_q is a line, j is defined on all of $\mathbb{R} \times H$; moreover we have $R(\ ,\gamma_q')\gamma_q' = 0$ by 5.2. So for any $v \in T_qH$, the Jacobi field $J(t) = Dj_{(t,q)}(v)$ along γ_q with initial values $J(0) = v$, $J'(0) = D_v(\nabla b^+) = 0$ is parallel. Therefore, j is a local isometry. Since the lines γ_q are co-rays of γ, they do not intersect each other, so j is one-to-one. Thus putting $W_1 = j(\mathbb{R} \times H)$, we finish the proof.

7. Global splitting

By a *flat strip*, we mean a totally geodesic isometric immersion f of $(\mathbb{R} \times I, -dt^2 + ds^2)$ into M for some real interval I, such that $f | \mathbb{R} \times \{s\}$ is a line for any $s \in I$.

Proposition 7.1. *Let* γ *be a line and* c: $[0,1] \to M$ *any geodesic with* $c(0) = \gamma(0)$. *Then there is a flat strip containing* γ *and* c.

Proof. We may assume that c is not contained in γ. Let F be the set of all parameters $u \in [0,1]$ such that there is a flat strip containing γ and $c|[0,u]$. Obviously $0 \in F$. Let $v = \sup F$. It follows from the local splitting (Proposition 6.3) that $v > 0$. There is a flat strip f: $\mathbb{R} \times [0, a) \to M$ containing γ and $c|[0, v)$ for some $a \in (0, \infty]$. More precisely, $\gamma(t) = f(t, 0)$ and $c(u) = f(k \cdot u, m \cdot u)$ for $0 \leqslant u < v$, where $k, m \in \mathbb{R}$, and $a = m \cdot v < \infty$. Since c is not contained in γ, we have $m \neq 0$.

Claim. f can be extended to $\mathbb{R} \times [0, a]$.

In fact, put $\gamma_u(t) = f(k \cdot u + t, m \cdot u)$ for $0 \leqslant u < v$. This is a line. Let X denote the parallel vector field along c with $X(0) = \gamma'(0)$. Then $X(u) = \gamma_u'(0)$

for $0 < u < v$. Let γ_v be the geodesic with $\gamma_v'(0) = X(v)$. Then $\gamma_v(t) = \lim_{u \to v} \gamma_u(t)$, hence γ_v is a line. Now for $0 \leqslant s \leqslant a$ and $t \in \mathbb{R}$ we may put

$$f(t, s) = \gamma_{s/m}(t - k \cdot s/m);$$

this is a smooth extension of the previous map f and hence a flat strip containing γ and $c\,|\,[0, v]$.

Now the local splitting 6.3 implies again that the flat strip can be extended beyond γ_v. Ths is a contradiction to the choice of v unless we have $v = 1$ which finishes the proof.

Two lines γ_1 and γ_2 are called *strongly parallel* if they bound a flat strip, i.e. there is a flat strip $f\colon \mathbb{R} \times [a_1, a_2] \to M$ with $\gamma_i = f\,|\,(\mathbb{R} \times \{a_i\})$ for $i = 1, 2$. They are called *parallel* if there exist lines $\gamma_1 = \beta_0, \beta_1, \cdots, \beta_k = \gamma_2$ such that β_{j-1} and β_j are strongly parallel for $j = 1, \cdots, k$.

Lemma 7.2. *If γ_1, γ_2 are parallel lines, then $I(\gamma_1) = I(\gamma_2)$ and the Busemann functions $b_1^{\pm}$ and $b_2^{\pm}$ of γ_1 and γ_2 agree.*

Proof. We may assume that γ_1 and γ_2 are strongly parallel and bound the flat strip $f\colon \mathbb{R} \times [0, a] \to M$. Then $\gamma_1 \subset I(\gamma_2)$ and $\gamma_2 \subset I(\gamma_1)$, hence $I(\gamma_1) = I(\gamma_2)$. Further, for $t > a$ and $0 \leqslant s \leqslant a$ we have $d(f(0, s), \gamma_i(t))^2 \geqslant t^2 - a^2$, thus $b_i^+(f(0, s)) \leqslant 0$. Likewise, $b_i^-(f(0, s)) \leqslant 0$, and therefore $b_i^{\pm}(f(0, s)) = 0$, due to $b_i^+ + b_i^- \geqslant 0$. Moreover, $\gamma_1^{\pm}$ is a co-ray of $\gamma_2^{\pm}$ and vice versa. So, by (5) in §2, we have $b_1^{\pm} \geqslant b_2^{\pm} \geqslant b_1^{\pm}$ and we get the result.

Now consider a fixed line γ. Let $P_\gamma \subset M$ be the set of points which lie on a parallel line. It follows from 7.2 and 2.2 that b^+ is differentiable at any point $q \in P_\gamma$ and that there is exactly one parallel line γ_q passing through q.

Lemma 7.3. *P_γ is a connected component of M.*

Proof. P_γ is open by the local splitting 6.3. We show that ∂P_γ is empty. Suppose that there exists a point $q \in \partial P_\gamma$. Let B_q be a geodesically convex open coordinate ball around q. Then $B_q \cap P_\gamma$ is open and nonempty. Choose $p \in B_q \cap P_\gamma$ close to q. There is a geodesically convex ball $B_p \subset B_q \cap P_\gamma$ such that ∂B_p hits $(\partial P_\gamma) \cap B_q$, say at q'. Let $c\colon [0, 1] \to B_q$ be a geodesic segment with $c(0) = p$, $c(1) = q'$. There is a line γ_1 passing through p which is parallel to γ. By 7.1, there is a line γ_2 through q' which is strongly parallel to γ_1. Thus $q' \in P_\gamma$. But since P_γ is open, it does not intersect its boundary, a contradiction.

Now we can prove the Theorem. Since M is connected, through every point $q \in M$ there is exactly one line γ_q which is parallel to the given line γ, by 7.3. Let V_q be the tangent vector of γ_q at q. By the local splitting 6.3, this defines a parallel timelike vector field V on M. Thus $V^{\perp}$ is a parallel distribution; in particular, it is integrable. Let H be the maximal integral leave through

$p = \gamma(0)$. Then the map

$$j\colon \mathbb{R} \times H \to M, \qquad j(t,q) = \gamma_q(t),$$

is the desired isometry. So in particular, $\mathbb{R} \times H$ with the product metric is globally hyperbolic which implies that H is a complete Riemannian manifold (cf. [1, p. 65]).

Appendix

Proposition. *Let U be an open convex domain in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ a continuous function. Assume that for any $p \in U$ there is a smooth lower support function f_p at p, i.e. f_p is defined in a neighborhood of p with $f_p \leqslant f$ and $f_p(p) = f(p)$, and that $\|D(f_p)_p\| \leqslant L$. Then f is Lipschitz with Lipschitz constant L, i.e. for all $x, y \in U$ we have*

$$|f(x) - f(y)| \leqslant L \cdot \|x - y\|.$$

Proof. *Case* 1: $n = 1$. Then U is an open interval I. Assume that there are $x, y \in I$ such that $|f(x) - f(y)| > L \cdot |x - y|$. We may assume $x < y$ and $f(x) < f(y)$ (otherwise replace f with $-b$). Let $l\colon \mathbb{R} \to \mathbb{R}$ be an affine function with $l(x) > f(x)$, $l(y) < f(y)$, and slope $l' = L_0 > L$. Let $p = \sup\{t \in [x, y]; l(t) > f(t)\}$. Then $l(p) = f(p) = f_p(p)$ and $f_p(t) \leqslant f(t) < l(t)$ for $t < p$. But this implies $f_p'(p) \geqslant L_0 > L$ which is a contradiction.

Case 2: $n > 1$. For given $x, y \in U$ let c be the line $c(t) = x + t(y - x)$, restricted to the interval $I = c^{-1}(U)$. Now apply Case 1 to the function $f \circ c\colon I \to \mathbb{R}$.

Added in proof. Very recently, G. J. Galloway succeeded in proving the splitting theorem without assuming the timelike geodesical completeness, but using the existence and regularity of certain maximal spacelike hypersurfaces obtained by R. Bartnik; see G. J. Galloway, *The Lorentzian splitting theorem without completeness assumption*, Preprint, University of Miami, Coral Gables, FL, 1987.

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