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ON THE DIFFEOMORPHISM TYPES OF CERTAIN ALGEBRAIC SURFACES. II

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0. Introduction

In Part I of this paper [6], we studied the differential topology of certain algebraic surfaces via a new invariant introduced by S. Donaldson. The computation of this invariant was based on the description of moduli spaces of stable vector bundles over the surfaces in question (cf. [6, III, §1]). The purpose of this paper is to supply the proofs of these descriptions. These proofs are largely independent of Part I. The results in this paper are generalizations of results Donaldson [4] established for a particular surface ($S(2,3)$ in our notation).

The surfaces we shall be concerned with here are Dolgachev surfaces and their blowups. A *Dolgachev surface* may be defined to be a nonrational, simply connected elliptic surface with $b_2^+ = 1$. An equivalent definition is the following: A Dolgachev surface is a complex surface S, with a holomorphic map $\pi\colon S \to \mathbb{P}^1$, such that the general fiber of π is a smooth elliptic curve f, π has exactly two multiple fibers F_p and F_q of multiplicities p and q respectively with g.c.d.$(p,q) = 1$, $p_g(S) = 0$, and $b_1(S) = 0$. Such an S is necessarily an algebraic surface. For more details on Dolgachev surfaces we refer to Dolgachev's paper [2] and to Part I, especially the introduction and I, §3. Our main goal here is to describe the moduli space of L-stable rank-2 vector bundles with $c_1 = 0$ and $c_2 = 1$ over "generic" Dolgachev surfaces and their blowups, where L is a suitable ample line bundle in an appropriate sense (2.4). The main results are Theorem 3.9, Theorem 4.4, and Corollary 5.9.

The contents of this paper are as follows. §1 is an introductory section on vector bundles aimed at nonexperts. It contains little that is new. We review the notion of stability and describe some of the techniques involved in analyzing rank-2 bundles on a complex surface. In §2, we collect some of the technical results about Dolgachev surfaces that will be used later and so deal

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with the case $c_2 \leq 0$. This case is important for handling blown up Dolgachev surfaces.

§§3 and 4 discuss the crucial case $c_2 = 1$. The moduli space for such bundles is described set-theoretically in §3 and as a scheme in §4. Our results here are somewhat incomplete, as we are able to describe the full scheme structure only in the case when the Dolgachev surface has a multiple fiber of multiplicity 2. These results have also been obtained independently by Okonek and Van de Ven [12] by methods very similar to ours.

The results of §§2–4 are the first case of the general problem of classifying vector bundles on simply connected elliptic surfaces. This problem has been analyzed by the first author. However, the general situation (when c_2 is large compared to p_g) is very different from the results described here.

In §5, we study the general problem of comparing stable bundles on a blown up surface with stable bundles on the original surface. With a view toward future applications (and at no cost in additional length), we have tried to state and prove our results in rather full generality. In the case of interest to Part I, blown up Dolgachev surfaces and $c_2 = 1$, we show that for appropriate choice of ample line bundle, the moduli space of stable bundles on the blowup can be identified with the moduli space of bundles on the original surface via pullback. Such a result is not true for general surfaces or general choice of c_2, but the relationship between stable bundles on the blown up surface and on the original surface can be analyzed in much greater detail. However, we shall refrain from doing so in this paper. All of these results have gauge theory analogues: if M is a Riemannian 4-manifold, then one can compare the moduli space of anti-self-dual connections on M and on $M \# \overline{\mathbb{C}P}^2$ with an appropriate choice of metric. The appropriate gauge theory has been worked out by Taubes and Donaldson,[1] in unpublished results, and would also suffice for the applications to Part I of this paper.

Notation and conventions. The notation and conventions of Part I (essentially just standard conventions of algebraic geometry) remain in force. In particular, if V is a holomorphic vector bundle, then $\mathbf{V}$ denotes its sheaf of sections. In addition, we recall the following notation and facts about Dolgachev surfaces from Part I:

(0.1) We shall always denote by $S = S(p,q)$ a Dolgachev surface with multiple fibers of multiplicities p and q, g.c.d.$(p,q) = 1$. The associated elliptic fibration (which is *uniquely* determined by S) will be denoted $\pi\colon S \to \mathbb{P}^1$. We call S *generic* if all fibers of π are irreducible. We shall always assume that

[1] *Added in proof.* For Donaldson's work see §IV of *Connections, cohomology and the intersection forms of 4-manifolds*, J. Differential Geometry **24** (1986) 275–341.

the multiple fibers are smooth, and denote them by F_p and F_q respectively. A general fiber of π is denoted f. Thus $pF_p = qF_q = f$ as divisor classes.

(0.2) The order of divisibility of $[f]$ in $H^2(S;\mathbb{Z})$ is exactly pq. In other words, there is a primitive cohomology class $\kappa \in H^2(S;\mathbb{Z})$ with $pq\kappa = [f]$. We shall also use κ to denote the line bundle uniquely determined by κ.

(0.3) The line bundle $\mathscr{O}_S(F_p)|_{F_p}$ on the smooth elliptic curve F_p is a torsion line bundle in $\operatorname{Pic}(F_p)$ whose order is exactly p. Of course, a similar statement holds for F_q.

1. Generalities on vector bundles

Throughout this section Y is a smooth algebraic surface and V is a holomorphic (or equivalently algebraic) rank-2 vector bundle over Y. We collect some standard facts. General references for this material are [8] or [11].

We fix an ample line bundle L on Y. Recall [15], [11] that V is *L-stable* (resp. *L-semistable*) if for all sub-line bundles $\varphi\colon \mathbf{F} \to \mathbf{V}$ we have

$$c_1(F)\cdot c_1(L) < c_1(V)\cdot c_1(L)/2 \quad (\text{resp. } \leq c_1(V)\cdot c_1(L)/2).$$

Hence, if $c_1(V)=0$, then V is L-stable (resp. L-semistable) if and only if for all sub-line bundles $\varphi\colon \mathbf{F}\to\mathbf{V}$, we have

$$c_1(F)\cdot c_1(L) < 0 \quad (\text{resp. } \leq 0).$$

There are various notions of stability in the literature; the one above is sometimes called *Mumford-Takemoto stability.*

The usefulness of restricting attention to stable bundles, from the point of view of algebraic geometry, is that they naturally form an algebraic variety (or, more precisely, a scheme). Moreover, the local structure of this scheme can, in theory, be calculated.

Definition 1.1. If V is a holomorphic vector bundle, let *End* V be the sheaf of endomorphisms of V: $\mathit{End}\, V = \mathbf{V}\otimes\mathbf{V}^\vee$, where $V^\vee =$ dual of V; we set $\mathit{sl}(V) = \ker(\mathrm{tr}\colon \mathit{End}\, V \to \mathscr{O}_S)$ where tr is the naturally defined trace map. Hence there is an exact sequence of bundles (or locally free sheaves)

$$0 \to \mathit{sl}(V) \to \mathit{End}\, V \to \mathscr{O}_S \to 0.$$

The vector space $H^0(\mathit{End}\ V) = \mathrm{End}\ V$ is the space of globally defined endomorphisms of V. The following is a standard fact.

Lemma 1.2 ([11, p. 172]). *If V is stable with respect to some ample line bundle, every global endomorphism of V is multiplication by a scalar. Equivalently,*

$$H^0(\mathit{End} V) = \{\lambda\cdot \mathrm{Id} \mid \lambda\in\mathbb{C}\}, \quad \textit{i.e., } H^0(\mathit{sl}(V)) = 0. \quad \text{q.e.d.}$$

Given a holomorphic vector bundle V with $H^0(sl(V)) = 0$, there exists a *local universal deformation* of the vector bundle V over Y. This consists of the germ of an analytic space (Z, z) and of a vector bundle $\mathscr{V}$ over $Y \times (Z, z)$, together with an isomorphism

$$\mathscr{V} \mid Y \times \{z\} \cong V.$$

The germ (Z, z) is universal in an appropriate sense (i.e. it represents a functor). Intuitively, (Z, z) parametrizes all small deformations of the bundle V keeping the determinant bundle $\det V$ fixed. Moreover, it can be "calculated" in the following sense [5, §9, especially 9.7]:

(1.3) The Zariski tangent space of Z at z, $T_{Z,z}$, is naturally $H^1(sl(V))$. Moreover, there exists a convergent power series Φ near $0 \in H^1(sl(V))$, with values in $H^2(sl(V))$, such that, as germs of analytic spaces

$$(Z, z) \cong (\Phi^{-1}(0), 0).$$

Hence, if $H^2(sl(V)) = 0$, then Z is smooth at z of dimension $h^1(sl(V))$.

The following lemma applies to all the surfaces discussed in this paper:

Lemma 1.4. *If $H^1(\mathscr{O}_Y)$ and $H^2(\mathscr{O}_Y)$ are zero, then $H^i(End\ V) = H^i(sl(V))$, $i \geq 1$.*

Proof. Immediate from the definition of $sl(V)$ and the associated cohomology sequence.

Lemma 1.5. *If V is a rank-2 holomorphic bundle over the surface Y, then*

(a) $\chi(V) = \frac{1}{2}[c_1(V) \cdot (c_1(V) - K_Y)] - c_2(V) + 2\chi(\mathscr{O}_Y)$,

(b) $\chi(End\, V) = 4\chi(\mathscr{O}_Y) + c_1^2(V) - 4c_2(V)$,

where χ denotes the holomorphic Euler characteristic.

Proof. This is a very special case of the Hirzebruch-Riemann-Roch theorem [1], [9].

Lemma 1.6. *There is a canonical isomorphism $V^\vee \cong V \otimes (\det V)^{-1}$, where V is a rank-2 bundle and $\det V = \bigwedge^2 V$. In particular, if $\det V$ is trivial, then $V^\vee \cong V$. (In general, of course, $c_1(\det V) = c_1(V)$.)*

Proof. The pairing $V \otimes V \to \bigwedge^2 V = \det V$ is perfect. q.e.d.

(1.7) We briefly review some facts about Chern classes of coherent sheaves. Given a coherent sheaf $\mathscr{F}$ on a smooth projective variety, there is a theory of Chern classes $c_i(\mathscr{F})$, which satisfy appropriate analogues of the usual properties of Chern classes (e.g. Whitney product formula). In addition they satisfy:

(1.8) (a) If Z is a codimension-2 subscheme of the algebraic surface Y and I_Z is its ideal sheaf, then

$$c_2(I_Z) = l(Z), \quad \text{the } \textit{length} \text{ of } Z,$$

where by definition

$$l(Z) = \dim H^0(\mathscr{O}_Z) = \dim H^0(\mathscr{O}_Y/I_Z).$$

(b) For a divisor F on Y, we have $c_1(\mathscr{O}_Y(F) \otimes I_Z) = F$.

(1.9) We will need to know something about the local nature of sub-line bundles of a rank-2 vector bundle. If $\varphi\colon \mathscr{O}_Y(F) \to \mathbf{V}$ is a sub-line bundle, then for every $y \in Y$, there is an open set U containing y and trivialization of $\mathscr{O}_Y(F)$ and V over U so that locally

$$\varphi\colon \mathscr{O}_U \to \mathscr{O}_U \oplus \mathscr{O}_U$$

is given by $\varphi(1) = (f, g)$, with f and g in $\mathscr{O}_U$. We separate into two cases

Case 1. f and g are relatively prime. Then

$$(\mathscr{O}_U \oplus \mathscr{O}_U)/\operatorname{Im}\varphi \cong (f, g)\mathscr{O}_U = I_Z,$$

where I_Z is the ideal sheaf of the subscheme Z defined by $\{f = g = 0\}$. This is just local algebra: if $\mathscr{O}_{Y,y}$ is the local ring of Y at y, then $\mathscr{O}_{Y,y} \cong \mathbb{C}\{z_1, z_2\}$ is the ring of convergent power series in two variables, and is a UFD. It is then easy to check that the map

$$\psi\colon \mathscr{O}_U \oplus \mathscr{O}_U \to I_Z,$$

defined by $\psi((a, b)) = -ga + fb$, is surjective with kernel $\operatorname{Im}\varphi$.

Case 2. f and g are not relatively prime. Then, again because $\mathscr{O}_{Y,y}$ is a UFD, there exists (locally) a g.c.d. h of f and g. Let $f' = f/h$, $g' = g/h$, so that f' and g' are relatively prime. One sees easily that $(\mathscr{O}_U \oplus \mathscr{O}_U)/\operatorname{Im}\varphi$ has h-torsion. Moreover, if D is the divisor $\{h = 0\}$, then φ has an extension to a map

$$\hat{\varphi}\colon \mathscr{O}_U(D) \to \mathscr{O}_U \oplus \mathscr{O}_U,$$

defined by $\hat{\varphi}(1/h) = (f', g')$. (Recall that $(1/h)$ is a generating section for $\mathscr{O}_U(D)$.) We can now apply the analysis of Case 1 with (f', g') replacing (f, g). If Z' is the scheme defined by $\{f' = g' = 0\}$, then we have an exact sequence:

$$0 \to \mathscr{O}_U(D) \xrightarrow{\hat{\varphi}} \mathscr{O}_U \oplus \mathscr{O}_U \to I_{Z'} \to 0.$$

As a straightforward application of (1.9), we shall analyze rank-2 bundles which are L-semistable but not L-stable. The following proposition will not, however, be used in the sequel.

Proposition 1.10. *Let L be an ample divisor on the surface Y, and let V be a rank-2 bundle on Y with $c_1(V) = 0$. Suppose that V is L-semistable but not L-stable. Then exactly one of the following holds*:

(a) *There exists a unique destabilizing sub-line bundle $\mathscr{O}_Y(F) \to \mathbf{V}$, and $\mathbf{V}$ is given as an extension*

$$0 \to \mathscr{O}_Y(F) \to \mathbf{V} \to \mathscr{O}_Y(-F)I_Z \to 0,$$

where Z is a 0-dimensional local complete intersection subscheme.

(b) $\mathbf{V} \cong \mathcal{O}_Y(F) \oplus \mathcal{O}_Y(-F)$, *where* $2F \neq 0$ *in* $\operatorname{Pic} Y$, *and the subbundles* $\mathcal{O}_Y(F)$, $\mathcal{O}_Y(-F)$ *are the unique destabilizing sub-line bundles.*

(c) $\mathbf{V} \cong \mathcal{O}_Y(F) \oplus \mathcal{O}_Y(F)$, *where* $2F = 0$ *in* $\operatorname{Pic} Y$. *Every destabilizing sub-line bundle of* V *is isomorphic to* $\mathcal{O}_Y(F)$, *and the set of all such is identified with the set of lines in* $\operatorname{Hom}(\mathcal{O}_Y(F), \mathbf{V}) \cong \mathbf{C}^2$.

Proof. First we show that there is an exact sequence for $\mathbf{V}$ as in (a). Let $\varphi\colon \mathcal{O}_Y(F) \to \mathbf{V}$ be a destabilizing sub-line bundle. Since V is semistable, $L \cdot F = 0$. Suppose that φ does not have torsion-free cokernel. Then by Case 2 of (1.9) there exists an extension of φ to $\hat{\varphi}\colon \mathcal{O}_Y(D+F) \to \mathbf{V}$, where D is an effective nonzero divisor. But then $L \cdot (D+F) = L \cdot D > 0$, contradicting the L-semistability of V. Thus, φ has torsion-free cokernel, which must locally be of the form I_Z as in Case 1 of (1.9). Therefore, as $c_1(V) = 0$, $\mathbf{V}$ is given as an extension

$$0 \to \mathcal{O}_Y(F) \to \mathbf{V} \to \mathcal{O}_Y(-F)I_Z \to 0.$$

Now suppose that we are not in case (a), and let $\mathcal{O}_Y(G) \to \mathbf{V}$ be a destabilizing sub-line bundle. If the map $\mathcal{O}_Y(G) \to \mathbf{V}$ factors through $\mathcal{O}_Y(F)$, then $F - G = D$ is an effective divisor or zero. Since $L \cdot G = L \cdot F = 0$, $L \cdot D = 0$; so that $D = 0$ and $F = G$. If $\mathcal{O}_Y(G)$ is a different sub-line bundle, then the induced map $\mathcal{O}_Y(G) \to \mathcal{O}_Y(-F)I_Z$ is nonzero. Thus, there exists an effective divisor D linearly equivalent to $-F - G$ and containing Z in its support. But $L \cdot D = L \cdot (-F - G) = 0$, so that $D = \emptyset$ and $G = -F$. Consequently, $Z = \emptyset$. Clearly, the natural map $\mathcal{O}_Y(F) \oplus \mathcal{O}_Y(-F) \to \mathbf{V}$ is an isomorphism.

If $F \neq -F$ in $\operatorname{Pic} Y$, then $2F \neq 0$. In this case,

$$\operatorname{Hom}(\mathcal{O}_Y(F), \mathbf{V}) = H^0(\mathcal{O}_Y \oplus \mathcal{O}_Y(-2F)) \cong H^0(\mathcal{O}_Y) \oplus H^0(\mathcal{O}_Y(-2F)).$$

Note that $H^0(\mathcal{O}_Y(-2F)) = 0$, as any effective divisor D in $|-2F|$ has $L \cdot D = 0$ and hence cannot exist. Thus $\operatorname{Hom}(\mathcal{O}_Y(F), \mathbf{V})$ is one-dimensional, corresponding to the given map $\mathcal{O}_Y(F) \to \mathbf{V}$, and a similar argument works for $\mathcal{O}_Y(-F)$. This completes the analysis for (b); (c) is similar.

Remark 1.11. (a) In all cases, $c_2(V) = -F^2 + l(Z)$. Since $L \cdot F = 0$, where L is an ample divisor, $F^2 \leq 0$ and $F^2 = 0$ if and only if F is numerically equivalent to zero, i.e. $[F] = 0$ in $H^2(Y; \mathbf{R})$. Equivalently, $F^2 = 0$ if and only if the bundle $\mathcal{O}_Y(F) \oplus \mathcal{O}_Y(-F)$ is flat. Since $l(Z) \geq 0$ and $l(Z) = 0$ if and only if $Z = \emptyset$, we have shown that $c_2(V) \geq 0$, with equality if and only if V is given as an extension

$$0 \to \mathcal{O}_Y(F) \to \mathbf{V} \to \mathcal{O}_Y(-F) \to 0,$$

with $\mathcal{O}_Y(F)$ a flat line bundle. Thus, in this case $\mathbf{V}$ itself is an extension of two flat bundles (and is identified with the flat bundle $\mathcal{O}_Y(F) \oplus \mathcal{O}_Y(-F)$ in an appropriate compactification of the moduli space of stable bundles; see [7]). This is the analogue of Donaldson's result [3] for stable V: if V is L-stable for some ample L and $c_1(V) = 0$, then $c_2(V) \geq 0$, with equality if and only if V is flat.

(b) Fix $c_2(V) = c_2$. Then associated to each properly L-semistable V with $c_1(V) = 0$ and $c_2(V) = c_2$ is a divisor F, well defined up to sign with $L \cdot F = 0$ and $-F^2 \leq c_2$. Hence, if F is not numerically equivalent to zero, then the class of L lies in a wall W^α in the hyperbolic space associated to the real Neron-Severi group $\mathrm{NS}(Y) \otimes \mathbb{R} \subseteq H^{1,1}(Y;\mathbb{R}) \subseteq H^2(Y;\mathbb{R})$, where $\alpha = c_1(F)$ and $\alpha^2 = F^2 \geq -c_2$. These walls define a cell structure on the ample cone of $\mathrm{NS}(Y) \otimes \mathbb{R}$ which is the algebro-geometric analogue of the chamber structure studied in II.1 of Part I.

In particular, if $H_1(Y;\mathbb{Z}) = 0$ and L is sufficiently general, then the only potential destabilizing sub-line bundle we need to consider in studying L-semistable bundles V with rank 2, $c_1(V) = 0$ and $c_2(V) \leq c_2$ is $\mathcal{O}_Y$ itself, and every nontrivial such V is uniquely described as an extension

$$0 \to \mathcal{O}_Y \to \mathbf{V} \to I_Z \to 0.$$

2. Vector bundles over Dolgachev surfaces

In this section we gather some technical preliminaries on stable rank-2 bundles with $c_1 = 0$ over Dolgachev surfaces, and show that there are none with $c_2 = 0$. The case of main interest, $c_2 = 1$, will be treated in the next section.

(2.1) Throughout this section and the next two, $S = S(p,q)$ shall always denote a *generic* Dolgachev surface (0.1).

The following is a straightforward computation.

Lemma 2.2.

$$\begin{aligned} \text{(a)}\quad K_S &= -f + (p-1)F_p + (q-1)F_q \\ &= f - F_p - F_q = (pq - p - q)(1/pq)f \\ &= (pq-p-q)_\kappa. \end{aligned}$$

$$\text{(b)}\quad 2K_S = (p-2)F_p + (q-2)F_q.$$

$$\text{(c)}\quad 3K_S = (p-3)F_p + (q-3)F_q + f.$$

Lemma 2.3. (a) *If* $0 \leq \alpha \leq p-1$, $0 \leq \beta \leq q-1$, *and* $c \geq 0$, *the natural map* $H^0(cf) \to H^0(\alpha F_p + \beta F_q + cf)$ *is an isomorphism. In other words, with*

α, β and c as above, F_p is a fixed component with multiplicity α in the linear series $|\alpha F_p + \beta F_q + cf|$, and similarly for F_q.

(b) *$H^1(\alpha F_p + \beta F_q) = 0$ if $|\alpha| \leq p-1$, $|\beta| \leq q-1$, and at least one of α, β is nonnegative.*

(c) *$\dim H^1(\alpha F_p + \beta F_q) = 1$ for $-p \leq \alpha \leq -1$ and $-q \leq \beta \leq -1$.*

(d) *$\dim H^1(\alpha F_p + \beta F_q + f) = 1$ for $0 \leq \alpha \leq p-1$ and $0 \leq \beta \leq q-1$.*

Proof. (a) Clearly any divisor in the linear series $|\alpha F_p + \beta F_q + cf|$ is supported in the fibers of π. Thus the fixed part of $|\alpha F_p + \beta F_q + cf|$ is of the form $\alpha' F_p + \beta' F_q$, where $0 \leq \alpha' \leq \alpha$, $0 \leq \beta' \leq \beta$. Moreover, if D denotes the moving part of $|\alpha F_p + \beta F_q + cf|$, then $\mathscr{O}_S(D)|_{F_p} = \mathscr{O}_{F_p}$ and $\mathscr{O}_S(D)|_{F_q} = \mathscr{O}_{F_q}$. But $\mathscr{O}_S(\alpha F_p + \beta F_q + cf)|_{F_p} = \mathscr{O}_S(\alpha F_p)|_{F_p}$. If $\mathscr{O}_S(\alpha F_p)|_{F_p} = \mathscr{O}_S(\alpha' F_p)|_{F_p}$, where $0 \leq \alpha' \leq \alpha \leq p-1$, then $\alpha = \alpha'$ by (0.3) and similarly for F_q. Hence, the fixed component of $|\alpha F_p + \beta F_q + cf|$ is just $\alpha F_p + \beta F_q$.

(b) By Serre duality, $H^1(\mathscr{O}_{F_p}(\alpha F_p))$ is dual to $H^0(\mathscr{O}_{F_p}(-\alpha F_p))$. By (0.3) this is zero for $|\alpha| \leq p-1$, $\alpha \neq 0$. Similarly $H^0(\mathscr{O}_{F_q}(\beta F_q)) = 0$ if $|\beta| \leq q-1$ and $\beta \neq 0$. To prove (b), we induct on $|\alpha| + |\beta|$. If $\alpha = \beta = 0$, then (b) is equivalent to the statement that $H^1(\mathscr{O}_S) = 0$.

Next, without loss of generality, we may assume that $\alpha \geq 0$. If $\alpha \geq 1$, consider the sequence

$$0 \to \mathscr{O}_S((\alpha-1)F_p + \beta F_q) \to \mathscr{O}_S(\alpha F_p + \beta F_q) \to \mathscr{O}_{F_p}(\alpha F_p) \to 0.$$

By induction, $H^1((\alpha-1)F_p + \beta F_q) = 0$, and we have shown above that $H^1(\mathscr{O}_{F_p}(\alpha F_p)) = 0$. Hence $H^1(\alpha F_p + \beta F_q) = 0$ as well. If $\alpha = 0$, $\beta < 0$, then we consider the sequence

$$0 \to \mathscr{O}_S(\beta F_q) \to \mathscr{O}_S((\beta+1)F_q) \to \mathscr{O}_{F_q}((\beta+1)F_q) \to 0.$$

If $\beta = -1$, this sequence yields

$$\begin{array}{ccccccc} H^0(\mathscr{O}_S) & \to & H^0(\mathscr{O}_{F_q}) & \to & H^1(-F_q) & \to & H^1(\mathscr{O}_S) \\ \wr\| & & \wr\| & & & & \| \\ \mathbb{C} & \xrightarrow{\sim} & \mathbb{C} & & & & 0 \end{array}$$

so that $H^1(-F_q) = 0$. If $-q < \beta < -1$, then $H^0(\mathscr{O}_{F_q}((\beta+1)F_q)) = 0$. Thus, $H^1(\beta F_q) \subseteq H^1((\beta+1)F_q)$, and this last is zero by induction.

(c) This is proved by a very similar induction.

(d) By Serre duality, $H^1(\alpha F_p + \beta F_q + f)$ is dual to $H^1(K_S - \alpha F_p - \beta F_q - f) = H^1(-(\alpha+1)F_p - (\beta+1)F_q)$. Thus (d) follows from (c).

Lemma 2.4. *There does not exist a rank-2 bundle V on S with $c_1(V) = 0$ and $c_2(V) \leq 0$ which is stable with respect to an ample divisor L.*

Proof. Let L be an ample divisor on S and let V be L-stable, with $c_1(V) = 0$. Then, by Donaldson's results [3], $c_2(V) \geq 0$ and $c_2(V) = 0$ only if V is flat. Since S is simply connected, V would have to be trivial, i.e. $\mathbf{V} \cong \mathscr{O}_S \oplus \mathscr{O}_S$, but the trivial bundle is clearly not stable.

Remark. An easy argument using (1.10) shows that the only L-semistable bundle V as above is the trivial bundle.

Next we begin the analysis of stable bundles V with $c_2 = 1$.

Definition 2.5. A line bundle L on S is *suitable* if there exists an ample line bundle L_0 on S and an integer $n \geq (L_0 \cdot K_S)/(pq - p - q)$ such that $L = L_0 + nK_S$.

Proposition 2.6. *Let L be a suitable line bundle on S, and let V be an L-stable rank-2 bundle on S with $c_1(V) = 0$ and $c_2(V) = 1$. Then there is an exact sequence*

$$0 \to \mathscr{O}_S(D - K_S) \to \mathbf{V} \to \mathscr{O}_S(K_S - D)m_x \to 0,$$

where m_x is the maximal ideal sheaf of some point $x \in S$,

$$D = aF_p + bF_q, \qquad a, b \geq 0, \quad (a+1)q + (b+1)p < pq.$$

Proof. By the Riemann-Roch formula (1.5)(a), $\chi(V) = -1 + 2\chi(\mathscr{O}_S) = 1$, so that $h^0(V) + h^2(V) \geq 1$. If $h^0(V) \geq 1$, then V has a section, and hence there is a nonzero map $\mathscr{O}_S \to \mathbf{V}$. As $c_1(\mathscr{O}_S) = 0$, V is then not L-stable for any L. Thus $h^0(V) = 0$, $h^2(V) \geq 1$. By Serre duality, this is equivalent to $h^0(V \otimes K_S) \geq 1$. By (1.6), $V^\vee \cong V$. Hence there is a nonzero map $\mathscr{O}_S(-K_S) \to \mathbf{V}$. By (1.9), there is an effective divisor D such that this map extends to a map $\mathscr{O}_S(D - K_S) \to \mathbf{V}$ with a torsion-free cokernel. Since $c_1(V) = 0$, there is an exact sequence

$$0 \to \mathscr{O}_S(D - K_S) \to \mathbf{V} \to \mathscr{O}_S(K_S - D)I_Z \to 0,$$

where Z is a codimension-2 subscheme and I_Z its ideal sheaf.

By the stability of V, we have $L \cdot (D - K_S) < 0$. Next, we use

Lemma 2.7. *If L is a suitable line bundle, and if D is an effective divisor such that $L \cdot (D - K_S) \leq 0$, then $D = aF_p + bF_q$ with $a, b \geq 0$ and $(a+1)q + (b+1)p < pq$. Moreover, $L \cdot (D - K_S) < 0$, i.e., the inequality is automatically strict.*

Proof. We have

$$0 \geq L \cdot (D - K_S) = (L_0 + nK_S) \cdot (D - K_S) = L_0 \cdot D + n(K_S \cdot D) - (K_S \cdot L_0).$$

Suppose that $K_S \cdot D > 0$. Since $K_S = (pq - p - q)\kappa$ and $\kappa \cdot D$ is an integer, $(K_S \cdot D) \geq pq - p - q$. But then

$$\begin{aligned} &L_0 \cdot D + n(K_S \cdot D) - (K_S \cdot L_0) \\ &\qquad \geq L_0 \cdot D + [(L_0 \cdot K_S)/(pq - p - q)] \cdot (pq - p - q) - (L_0 \cdot K_S) \\ &\qquad = L_0 \cdot D + (L_0 \cdot K_S) - (L_0 \cdot K_S) = L_0 \cdot D > 0. \end{aligned}$$

The last inequality contradicts the first.

Hence $K_S \cdot D \leq 0$, and therefore $f \cdot D \leq 0$. Since f moves in a base point free pencil, D is supported in the fibers of π. As S is generic, all fibers of π are irreducible. Hence D must be a positive combination of (reductions of) fibers of π. Hence $D = aF_p + bF_q + cf$ for some nonnegative integers a, b, c. Then

$$\begin{aligned} D - K_S &= (aq + bp + cpq - (pq - p - q))\kappa \\ &= ((a+1)q + (b+1)p - pq + cpq)\kappa. \end{aligned}$$

Since $L \cdot \kappa > 0$, it follows that $c = 0$ and $(a+1)q + (b+1)p \leq pq$. Suppose equality holds. Then $p | (a+1)$, as g.c.d.$(p,q) = 1$. Hence $(a+1)q \geq pq$, so that $(b+1)p \leq 0$. This is absurd. Thus, $(a+1)q + (b+1)p < pq$, and

$$L \cdot (D - K_S) = ((a+1)q + (b+1)p - pq) \cdot (L \cdot \kappa) < 0. \quad \text{q.e.d.}$$

Returning to the proof of (2.6), we see that it suffices to show that $I_Z = m_x$ for some $x \in S$. But

$$1 = c_2(V) = -(D - K_S)^2 + l(Z).$$

Since D and K_S are rational multiples of f, $(D - K_S)^2 = 0$. Thus, $l(Z) = 1$. Hence Z is a reduced point x and $I_Z = m_x$.

3. The case $c_2 = 1$

Let $S = S(p, q)$ be a generic Dolgachev surface, and let L be a suitable line bundle over S.

Condition 3.1. (a) $D = aF_p + bF_q$ where $a, b \geq 0$ and $(a+1)q + (b+1)p < pq$.

We shall be concerned with exact sequences of the form:

$$(3.1)(\mathrm{b}) \qquad 0 \to \mathcal{O}_S(D - K_S) \to \mathbf{V} \to \mathcal{O}_S(K_S - D)m_x \to 0$$

where D satisfies (3.1)(a), $x \in S$, m_x is the maximal ideal sheaf of x, and $\mathbf{V}$ is locally free.

By (2.6), if V is a rank-2 L-stable bundle with $c_1(V) = 0$ and $c_2(V) = 1$, then there exists an extension of the form (3.1)(b) whose middle term is $\mathbf{V}$. We study a converse problem: when, for fixed D and x, there is an exact sequence as in (3.1)(b).

Lemma 3.2. *Fix D as in* (3.1)(a) *and $x \in S$. There exists an extension $\mathbf{V}$ as in* (3.1)(b) *if and only if x is in the base locus of $\mathcal{O}_S(3K_S - 2D)$.*

Proof. The extensions of $\mathcal{O}_S(K_S - D)m_x$ by $\mathcal{O}_S(D - K_S)$ are classified by a global Ext^1 [8, p. 725]. For our purposes, it suffices to know that there is an exact sequence:

$$\begin{aligned}0 &\to H^1(\mathcal{H}om(\mathcal{O}_S(K_S-D)m_x,\mathcal{O}_S(D-K_S)))\\ &\to \operatorname{Ext}^1(\mathcal{O}_S(K_S-D)m_x,\mathcal{O}_S(D-K_S))\\ &\to H^0(\mathcal{E}xt^1(\mathcal{O}_S(K_S-D)m_x,\mathcal{O}_S(D-K_S))).\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathcal{H}om(\mathcal{O}_S(K_S-D)m_x,\mathcal{O}_S(D-K_S)) &\cong \mathcal{H}om(\mathcal{O}_S(K_S-D),\mathcal{O}_S(D-K_S))\\ &\cong \mathcal{O}_S(2D-2K_S),\end{aligned}\tag{3.3}$$

where the first isomorphism is induced by the inclusion $m_x \subseteq \mathcal{O}_S$ [8, p. 690].

Also, $\mathcal{E}xt^1(\mathcal{O}_S(K_S-D)m_x,\mathcal{O}_S(D-K_S)) \cong \mathbb{C}_x$, a skyscraper sheaf with stalk $\mathbb{C}$ supported at x [8, p. 690]. Hence

$$H^0(\mathcal{E}xt^1(\mathcal{O}_S(K_S-D)m_x,\mathcal{O}_S(D-K_S))) = \mathbb{C}.$$

A class $\alpha \in \operatorname{Ext}^1$ defines a locally free extension if and only if the projection of α to $H^0(\mathcal{E}xt^1) = \mathbb{C}$ is nonzero [8, pp. 723–724].

The map

$$\begin{aligned}&\operatorname{Ext}^1(\mathcal{O}_S(K_S-D)m_x,\mathcal{O}_S(D-K_S))\\ &\qquad\to H^0(\mathcal{E}xt^1(\mathcal{O}_S(K_S-D)m_x,\mathcal{O}_S(D-K_S)))\end{aligned}$$

is dual (via Serre duality) to the map

$$\begin{aligned}H^0(\mathbb{C}_x) \xrightarrow{\partial} &H^1(\mathcal{O}_S(K_S-D)m_x \otimes \mathcal{O}_S(D-K_S)^\vee \otimes K_S)\\ &= H^1(\mathcal{O}_S(3K_S-2D)m_x),\end{aligned}$$

where ∂ is the coboundary map in the cohomology long exact sequence associated to

$$0 \to \mathcal{O}_S(3K_S-2D)m_x \to \mathcal{O}_S(3K_S-2D) \to \mathbb{C}_x \to 0$$

[8, p. 729]. By duality, then, there is an extension as in (3.1)(b) if and only if ∂ is injective, or, equivalently, if and only if the map $H^0(\mathcal{O}_S(3K_S-2D)) \to H^0(\mathbb{C}_x) = \mathbb{C}$ is zero, i.e. if and only if every section of $\mathcal{O}_S(3K_S-2D)$ vanishes at x, if and only if x is in the base locus of $\mathcal{O}_S(3K_S-2D)$. q.e.d.

We proceed to calculate this base locus.

Lemma 3.4. *Let D be as in* (3.1)(a). *Then the base locus B of $3K_S-2D$ satisfies*:

(a) $B = F_p \coprod F_q$ *if* $a \neq (p-3)/2$ *and* $b \neq (q-3)/2$;
(b) $B = F_p$ *if* $a \neq (p-3)/2$ *and* $b = (q-3)/2$;
(c) $B = F_q$ *if* $a = (p-3)/2$ *and* $b \neq (q-3)/2$
(d) $B = \varnothing$ *if* $a = (p-3)/2$ *and* $b = (q-3)/2$.

Proof. By (2.2), $3K_S - 2D = (p-3-2a)F_p + (q-3-2b)F_q + f$. Thus, (3.4) is immediate if $a \le (p-3)/2$ and $b \le (q-3)/2$, by (2.3)(a). If $a > (p-3)/2$ and $b \le (q-3)/2$, we may write

$$(p-3-2a)F_p + (q-3-2b)F_q + f = (2p-3-2a)F_p + (q-3-2b)F_q.$$

Since $a \le p-2$ by (3.1)(a), this case follows from (2.3)(a). A similar argument handles the case $a < (p-3)/2$, $b > (q-3)/2$. Finally, since a and b are integers, g.c.d.$(p,q) = 1$, and $(a+1)q+(b+1)p < pq$, we cannot have both $a > (p-3)/2$ and $b > (q-3)/2$. q.e.d.

Lemmas 3.3 and 3.4 give a characterization, in terms of D and x, of when an extension as in (3.1)(b) exists. We turn now to the question of uniqueness. It will be convenient to introduce the following technical condition:

Assumption 3.5. D is a divisor satisfying (3.1)(a), x is contained in the base locus of $3K_S - 2D$, and $H^0(\mathcal{O}_S(2K_S - 2D)m_x) = 0$.

Lemma 3.6. (a) *If D satisfies* (3.1)(a) *and $x \in F_p$ (resp. $x \in F_q$), then D and x satisfy* (3.5) *if and only if either $a > (p-3)/2$ or both $b > (q-2)/2$ and $a < (p-3)/2$ (resp. either $b > (q-3)/2$ or both $a > (p-2)/2$ and $b < (q-3)/2$).*

Now suppose that D and x satisfy (3.5).

(b) *There is an extension* $\mathbf{V}$ *as in* (3.1)(b) *and the corresponding extension class is unique* mod $\mathbf{C}^*$, *i.e. the extension is essentially unique up to isomorphism.*

(c) *The map $\mathcal{O}_S(D - K_S) \to \mathbf{V}$ is unique* mod *scalars.*

(d) *If* $\mathbf{V}$ *is also written as an extension*

$$0 \to \mathcal{O}_S(D - K_S) \to \mathbf{V} \to \mathcal{O}_S(K_S - D)m_{x'} \to 0,$$

then $x' = x$.

Proof. (a) Assume (as we may by symmetry) that $x \in F_p$. By (2.2),

$$2K_S - 2D = (p-2-2a)F_p + (q-2-2b)F_q.$$

Thus, $H^0(\mathcal{O}_S(2K_S - 2D)m_x) = 0$ if and only if either $p-2-2a \le 0$ or $q-2-2b < 0$, i.e. if and only if either $a > (p-3)/2$ or $b > (q-2)/2$. From (3.1)(a), if $b > (q-2)/2$, then $a < (p-3)/2$, and (a) follows, in view of (3.4)(a) and (b).

(b) The existence of $\mathbf{V}$ is an immediate consequence of (3.2). To prove uniqueness, by (3.3) and the proof of (3.2), it suffices to show that $H^1(2D - 2K_S) = 0$. Since D and x satisfy (3.5), and by (a), we may write

$$2D - 2K_S = (2a+2-p)F_p + (2b+2-q)F_q,$$

where either $2a+2-p \ge 0$ or $(2b+2-q) \ge 0$, and $|2a+2-p| \le p-1$, $|2b+2-q| \le q-1$. By (2.3)(b), $H^1(2D - 2K_S) = 0$.

(c) We compute $\mathrm{Hom}(\mathcal{O}_S(D-K_S), \mathbf{V}) = H^0(\mathbf{V}\otimes\mathcal{O}_S(K_S-D))$. From the exact sequence

$$0 \to \mathcal{O}_S \to \mathbf{V}\otimes\mathcal{O}_S(K_S-D) \to \mathcal{O}_S(2K_S-2D)m_x \to 0,$$

Assumption 3.5 implies that $H^0(\mathbf{V}\otimes\mathcal{O}_S(K_S-D)) \cong \mathbb{C}$, and hence (c).

(d) This follows immediately from (c) and the fact that x is the unique point of S at which $\mathbf{V}/\mathcal{O}_S(D-K_S)$ is not locally free. q.e.d.

Next, we show that we can modify the divisor D to arrange (3.5).

Lemma 3.7. *Let D satisfy* (3.1)(a) *and* $\mathbf{V}$ *be given by* (3.1)(b). *Then there exists a divisor D', an $x' \in S$, and an exact sequence*

$$0 \to \mathcal{O}_S(D'-K_S) \to \mathbf{V} \to \mathcal{O}_S(K_S-D')m_{x'} \to 0,$$

such that D' and x' satisfy (3.5).

Proof. By symmetry, we may assume that $x \in F_p$. Thus, by (3.4), $a \neq (p-3)/2$. If D and x do not satisfy (3.5), then by (3.6)(a) we have $a < (p-3)/2$ and $b \leq (q-2)/2$. Set $\overline{D}_1 = D + F_p$. Tensoring (3.1)(b) with $\mathcal{O}_S(K_S - \overline{D}_1)$, we obtain

$$0 \to \mathcal{O}_S(-F_p) \to \mathbf{V}\otimes\mathcal{O}_S(K_S-\overline{D}_1) \to \mathcal{O}_S(2K_S-2D-F_p)m_x \to 0.$$

Since $H^0(-F_p) = H^1(-F_p) = 0$, by (2.3), we have

$$H^0(\mathbf{V}\otimes\mathcal{O}_S(K_S-\overline{D}_1)) = H^0(\mathcal{O}_S(2K_S-2D-F_p)m_x).$$

But $2K_S - 2D - F_p = rF_p + sF_q$, $r \geq 1$, $s \geq 0$, by (2.2)(b). Clearly, then,

$$H^0(\mathcal{O}_S(2K_S-2D-F_p)m_x) \neq 0.$$

Thus, there is a nonzero map $\mathcal{O}_S(\overline{D}_1 - K_S) \to \mathbf{V}$. Enlarge $\overline{D}_1$ to D_1 to insure a torsion-free cokernel. By (2.7), D_1 must be of the form $a_1F_p + b_1F_q$, with $a_1 > a$ and $b_1 \geq b$ and we have an exact sequence

$$0 \to \mathcal{O}_S(D_1-K_S) \to \mathbf{V} \to \mathcal{O}_S(K_S-D_1)m_{x_1} \to 0.$$

If D_1 does not satisfy $H^0(\mathcal{O}_S(2K_S-2D_1)m_{x_1}) = 0$, this process may be repeated. Since $a+b$ strictly increases at each stage and a and b are bounded, after a finite number of repetitions we produce D' and x' satisfying (3.5). q.e.d.

We turn now to the issue of stability.

Lemma 3.8. *Let D and x satisfy* (3.5), *and let* $\mathbf{V}$ *be given by the extension* (3.1)(b). *Suppose that $x \in F_p$ (resp. $x \in F_q$). Then* $\mathbf{V}$ *is not L-stable if and only if $(a+2)q + (b+1)p \leq pq$ (resp. $(a+1)q + (b+2)p \leq pq$).*

Proof. We begin by establishing the following.

Claim. If $\mathbf{V}$ is given by (3.1)(b), then $\mathbf{V}$ is not L-stable if and only if there exists integers $c, d \geq 0$, such that, if $C = cF_p + dF_q$ and $F = K_S - D - C$, we have

(a) $x \in$ support of C;

(b) $(a+c+1)q+(b+d+1)p<pq$;

(c) the natural nonzero map $\mathcal{O}_S(F) \to \mathcal{O}_S(K_S-D)m_x$, which exists by (a), can be lifted to a map $\mathcal{O}_S(F) \to \mathbf{V}$.

Proof of the Claim. First suppose that c, d satisfying (a)–(c) exist. As

$$L \cdot F = L \cdot (K_S - D - C) = (pq-(a+c+1)q-(b+d+1)p)(L\cdot\kappa) \geq 0,$$

V is clearly not L-stable. Conversely, suppose that $\mathbf{V}$ is not L-stable, and let $\mathcal{O}_S(F)$ be a sub-line bundle of $\mathbf{V}$ such that $L \cdot F \geq 0$.

If the composite map $\mathcal{O}_S(F) \to \mathcal{O}_S(K_S-D)m_x$ were zero, there would be an induced nonzero map $\mathcal{O}_S(F) \to \mathcal{O}_S(D-K_s)$. But then F would be of the form $D-K_S-C$ for some effective C, so that $L\cdot F \leq L\cdot(D-K_S)<0$, contradicting the choice of F. Thus, the map $\mathcal{O}_S(F) \to \mathcal{O}_S(K_S-D)m_x$ is nonzero, so that $F=K_S-D-C$, where C is an effective divisor and x is in the support of C. Applying (2.7) to the divisor $D+C$, we see that C is of the form cF_p+dF_q where c and d satisfy (b). Finally, (c) is automatic by construction.

Proof of 3.8. Suppose by symmetry that $x \in F_p$. Clearly, there exist $c,d \geq 0$ satisfying (a) and (b) of the claim, if and only if $(a+2)q+(b+1)p \leq pq$. It therefore suffices to prove that, if D and x satisfy (3.5) and c,d satisfy (a) and (b) then (c) holds. The obstruction to lifting the map $\mathcal{O}_S(F)=\mathcal{O}_S(K_S-D-C) \to \mathcal{O}_S(K_S-D)m_x$ to a map $\mathcal{O}_S(F)\to\mathbf{V}$ lies in $H^1(2D-2K_S+C)$. Write

$$2D-2K_S+C=(2a+2+c-p)F_p+(2b+2+d-q)F_q=rF_p+sF_q,$$

say. The inequalities (3.6)(a) and (b) of the claim easily yield $|r|\leq p-1$, $|s|\leq q-1$, and either r or s is >0. By (2.3)(b), $H^1(2D-2K_S+C)=0$. q.e.d.

We may summarize our results as follows,

Theorem 3.9. *Let D and x satisfy* (3.5), *where $D=aF_p+bF_q$ and $x\in F_p$.*

(a) *There is an extension* $\mathbf{V}$ *as in* (3.1)(b) *which is L-stable if and only if*

(i) *either $a>(p-3)/2$ or both $b>(q-2)/2$ and $a<(p-3)/2$, and*

(ii) $(a+2)q+(b+1)p>pq$.

(b) *With* $\mathbf{V}$ *as above, there exists an exact sequence*

$$0\to\mathcal{O}_S(D'-K_S)\to\mathbf{V}\to\mathcal{O}_S(K_S-D')m_{x'}\to 0$$

with $D'=a'F_p+b'F_{q'}$ such that D' and x' also satisfy (3.5), *if and only if $(a',b')=(a,b)$ or $a\leq p-3$ and $(a',b')=(p-3-a,q-2-b)$. In this case $x'\in F_p$ as well.*

Analogous statements hold if $x\in F_q$.

Proof. Part (a) is immediate from (3.6)(a) and (3.8). To prove (b) we need the following:

Claim 3.10. *Let F be a divisor such that*

$$\mathrm{Hom}(\mathscr{O}_S(F-K_S),\mathbf{V})\neq 0 \quad \textit{and} \quad H^0(\mathscr{O}_S(2K_S-D-F)m_x)=0.$$

Then $D-F$ is effective.

Proof of (3.10). We have an exact sequence

$$0\to H^0(D-F)\to H^0(\mathbf{V}\otimes\mathscr{O}_S(K_S-F))\to H^0(\mathscr{O}_S(2K_S-D-F)m_x).$$

By hypothesis the second term is nonzero and the third is 0. Thus, $H^0(D-F)\neq 0$, i.e., $D-F$ is effective.

Now we prove the "if" direction in part (b). Set $a'=p-3-a$, $b'=q-2-b$, and $D'=a'F_p+b'F_q$. We have $\mathrm{Hom}(\mathscr{O}_S(D'-K_S),\mathbf{V})=H^0(\mathbf{V}\otimes\mathscr{O}_S(K_S-D'))$ and an exact sequence

$$\begin{aligned}0&\to H^0(D-D')\to H^0(\mathbf{V}\otimes\mathscr{O}_S(K_S-D'))\\&\to H^0(\mathscr{O}_S(2K_S-D-D')m_x)\to H^1(D-D').\end{aligned}$$

One easily verifies that D', a', and b' satisfy (3.1)(a), (3.6)(a), and (3.9)(a). Moreover, if $D-D'=rF_p+sF_q$, one verifies that one of r,s is ≥ 0 and the other is <0, and that $|r|\leq p-1$, $|s|\leq q-1$. It follows that $H^1(D-D')=0$, and that $D-D'$ is not effective. Thus, a nonzero section of $H^0(\mathscr{O}_S(2K_S-D-D')m_x)=H^0(\mathscr{O}_S(F_p)m_x)$ lifts to a nonzero homomorphism $\varphi\colon\mathscr{O}_S(D'-K_S)\to\mathbf{V}$.

Next, we claim that φ has torsion-free cokernel. If φ vanishes along a nonzero divisor D_0, then there is an induced map $\hat{\varphi}\colon\mathscr{O}_S(D'+D_0-K_S)\to\mathbf{V}$. Hence, by (2.7), $D_0=a_0F_p+b_0F_q$, where $a_0,b_0\geq 0$, as $\mathbf{V}$ is stable. Since $D-D'$ is not effective, $D-D'-D_0$ is also not effective. It follows from (3.10) that $H^0(\mathscr{O}_S(2K_S-D-D'-D_0)m_x)\neq 0$. But $2K_S-D-D'-D_0=(1-a_0)F_p-b_0F_q$ so that a_0, b_0 are zero. This contradicts the fact that $D_0\neq 0$. The contradiction establishes that φ has torsion-free cokernel.

Now suppose that x' is defined by the exact sequence

$$0\to\mathscr{O}_S(D'-K_S)\to\mathbf{V}\to\mathscr{O}_S(K_S-D')m_{x'}\to 0.$$

By the symmetry between D and D', it follows that, if $D\neq D'$,

$$H^0(\mathscr{O}_S(2K_S-D-D')m_{x'})=H^0(\mathscr{O}_S(F_p)m_{x'})\neq 0.$$

Thus, $x'\in F_p$.

Finally, we must prove the "only if" direction of part (b). Suppose that there is an exact sequence

$$0\to\mathscr{O}_S(D''-K_S)\xrightarrow{\psi}\mathbf{V}\to\mathscr{O}_S(K_S-D'')m_{x''}\to 0,$$

where D'' and x'' also satisfies (3.5). We have an exact sequence

$$0 \to H^0(D - D'') \to H^0(\mathbf{V} \otimes \mathscr{O}_S(K_S - D'')) \to H^0(\mathscr{O}_S(K_S - D - D'')m_x).$$

If ψ is in the image of $H^0(D - D'')$, then $D - D''$ is effective. It follows easily that ψ factors through the natural inclusion $\mathscr{O}_S(D'' - K_S) \subseteq \mathscr{O}_S(D - K_S)$. As ψ has torsion-free cokernel, $D'' = D$ in this case.

Thus, we may suppose that the image of ψ in $H^0(\mathscr{O}_S(2K_S - D - D'')m_x)$ is nonzero. Setting $D'' = a''F_p + b''F_q$, we have

$$2K_S - D - D'' = (p - 2 - a - a'')F - p + (q - 2 - b - b'')F_q.$$

As $H^0(\mathscr{O}_S(2K_S - D - D'')m_x) \neq 0$, $a'' \leq p - 3 - a = a'$ and $b'' \leq q - 2 - b = b'$. Therefore, $D' - D''$ is effective. So there is an inclusion $\mathscr{O}_S(D'' - K_S) \subseteq \mathscr{O}_S(D' - K_S)$. If $D - D''$ is not effective, $H^0(D - D'') = 0$, and $\dim H^0(\mathbf{V} \otimes \mathscr{O}_S(K_S - D'')) = \dim H^0(\mathscr{O}_S(2K_S - D - D'')m_x) = 1$. Since the inclusion $\mathscr{O}_S(D'' - K_S) \subset \mathscr{O}_S(D' - K_S)$ already gives a nonzero element of $H^0(\mathbf{V} \otimes \mathscr{O}_S(K_S - D''))$, ψ factors through this inclusion. As ψ has torsion-free cokernel, $D'' = D'$ in this case. In the remaining case, $D = D''$ and $D' - D''$ are effective. It follows that $a'' \leq \min(a, a') \leq (p - 3)/2$ and $b'' \leq \min(b, b') \leq (q-2)/2$. By (3.6)(a), D'' and x'' do not satisfy (3.5). Thus, if D'' and x'' satisfy (3.5), either $D'' = D$ or $D'' = D'$, proving (3.9).

Corollary 3.11. (a) *Every L-stable rank-2 vector bundle V over $S(2,q) = S$ with $c_1(V) = 0$, $c_2(V) = 1$ is an extension as in* (3.1)(b), *of the form*

$$0 \to \mathscr{O}_S(bF_q - K_S) \to \mathbf{V} \to \mathscr{O}_S(K_S - bF_q)m_x \to 0,$$

where $x \in F_2$ and $0 \leq b \leq (q-3)/2$; moreover $\mathbf{V}$ is uniquely determined by b and x.

(b) *Every L-stable rank-2 bundle $\mathbf{V}$ over $S(3,q) = S$ with $c_1 = 0$, $c_2 = 1$, is an extension of the form* (3.1)(b), *with $D = aF_3 + bF_q$, and either*

(i) $(a, b) = (1, b)$, $0 \leq b \leq [q/3] - 1$ *and* $x \in F_3$, *or*

(ii) $(a, b) = (1, [q/3] - 1)$ *or* $(a, b) = (0, [2q/3] - 1)$, *and* $x \in F_q$.

In case (i), $\mathbf{V}$ *is uniquely determined by b and x. In case* (ii), *either choice of (a, b) leads to the same set of bundles, and, having chosen (a, b), $\mathbf{V}$ is uniquely determined by x.*

Remark 3.12. We can be more precise about the connection between the bundles corresponding to (a, b) and (a', b'), in the notation of (3.9)(b). If $a \leq p-3$, then $H^0(\mathbf{V} \otimes \mathscr{O}_S(K_S))$ has dimension 2. Identifying sections mod $\mathbb{C}^*$ gives a family of maps $\mathscr{O}_S(-K_S) \to \mathbf{V}$ parametrized by $\mathbb{P}^1$. Two points of this $\mathbb{P}^1$ correspond to maps vanishing along D and D', respectively. The remaining $\mathbb{C}^*$ consists of maps vanishing along a divisor $D'' = a''F_p + b''F_q$, where $a'' = \min(a, a')$ and $b'' = \min(b, b')$. There is thus a 1-parameter family (indexed by $\mathbb{C}^*$) of ways to describe $\mathbf{V}$ as an extension

$$(*) \qquad 0 \to \mathscr{O}_S(D'' - K_S) \to \mathbf{V} \to \mathscr{O}_S(K_S - D'')m_{x''} \to 0,$$

where in fact x'' is independent of the particular extension. To account for these, note that by (2.3)(c), $H^1(2D'' - 2K_S) = \mathbb{C}$. By the discussion of (3.3), the set of all locally free extensions $(*)$ as above, mod scalars, is a principal homogeneous space over $H^1(2D'' - 2K_S)$. A calculation with Yoneda pairing shows that exactly one such extension is unstable, and so does not yield $\mathbf{V}$. The remaining $\mathbb{C}^*$ precisely compensates for the nonuniqueness of the map $\mathscr{O}_S(D'' - K_S) \to \mathbf{V}$.

4. Construction of the moduli space

In this section we construct the moduli space of L-stable rank-2 bundles V over $S = S(p, q)$ with $c_1(V) = 0$ and $c_2(V) = 1$ and discuss its local structure. We keep the notation of the preceding sections.

(4.1) Construction of the moduli space. Let (a, b) be a pair of integers satisfying the hypotheses of (3.9)a). We shall construct a universal bundle $\mathscr{V}_{(a,b),p}$ over $S \times F_p$.

Notation. $\eta = \mathscr{O}_S(F_p)|_{F_p}$. Recall by (0.3) that η is a torsion line bundle of order exactly p;

p_i is the projection of $S \times F_p$ to the ith factor ($i = 1$ or 2);

$j \colon F_p \to S$ and $i = (j, \mathrm{Id}) \colon F_p \to S \times F_p$ are the inclusions;

$\Delta = i(F_P) \subset S \times F_p$ is the image of F_p; and

I_Δ is the ideal sheaf of Δ in $S \times F_p$.

Theorem 4.2. *Given $D = aF_p + bF_q$, where a and b satisfy the hypotheses of* (3.9)(a), *there is a unique bundle $\mathscr{V}_{(a,b),p}$ over $S \times F_p$ which is an extension of the form*

$$0 \to p_1^*\mathscr{O}_S(D - K_S) \otimes p_2^*\eta^{\otimes(p-2a-3)} \to \mathscr{V}_{(a,b),p} \to p_1^*\mathscr{O}_S(K_S - D)I_\Delta \to 0.$$

Proof. We use the analogue of the Ext sequence of (3.2) in this situation. The set of such extensions is classified by

$$\mathrm{Ext}^1(p_1^*\mathscr{O}_S(K_S - D)I_\Delta, p_1^*\mathscr{O}_S(D - K_S) \otimes p_2^*\eta^{\otimes(p-2a-3)}).$$

There is an exact sequence

$$\begin{array}{c} 0 \\ \downarrow \\ H^1(\mathit{Hom}(p_1^*\mathscr{O}_S(K_S - D)I_\Delta, p_1^*\mathscr{O}_S(D - K_S) \otimes p_2^*\eta^{\otimes(p-2a-3)})) \\ \downarrow \\ \mathrm{Ext}^1(p_1^*\mathscr{O}_S(K_S - D)I_\Delta, p_1^*\mathscr{O}_S(D - K_S) \otimes p_2^*\eta^{\otimes(p-2a-3)}) \\ \downarrow \alpha \\ H^0(\mathit{Ext}^1(p_1^*\mathscr{O}_S(K_S - D)I_\Delta, p_1^*\mathscr{O}_S(D - K_S) \otimes p_2^*\eta^{\otimes(p-2a-3)})), \end{array}$$

which, for convenience of notation, we denote $0 \to A_1 \to A_2 \to A_3$. By a straightforward local calculation, $\mathit{Ext}^1(I_\Delta, \mathscr{O}_{S\times F_p}) = i_* \det N$, where N is the normal bundle of the map i. As T_{F_p}, the tangent bundle of F_p, is trivial, we have

$$\begin{aligned} \det N &= \det(i^*T_{S\times F_p}) \otimes T_{F_p}^{-1} = \det(j^*T_S \oplus T_{F_p}) \\ &= \det(j^*T_S) = \text{normal bundle of } F_p \text{ in } S = \eta. \end{aligned}$$

Thus

$$\begin{aligned} &\mathit{Ext}^1(p_1^*\mathscr{O}_S(K_S - D)I_\Delta, p_1^*\mathscr{O}_S(D - K_S) \otimes p_2^*\eta^{\otimes(p-2a-3)}) \\ &= i_*(\det N \otimes i^*p_1^*\mathscr{O}_S(2D - 2K_S) \otimes i^*p_2^*\eta^{\otimes(p-2a-3)}) \\ &= i_*(\eta \otimes \eta^{\otimes(2a-p+2)} \otimes \eta^{\otimes(p-2a-3)}) = i_*\mathscr{O}_{F_p}. \end{aligned}$$

Thus, there is an everywhere nonvanishing section of Ext^1 (i.e., $A_3 = \mathbb{C}$), which in fact was the reason for twisting by $p_2^*\eta^{\otimes(p-2a-3)}$.

By Serre duality, α is dual to the map

$$\begin{array}{ll} \partial\colon H^1(\mathscr{O}_{F_p}) \to & H^2(p_1^*\mathscr{O}_S(3K_S - 2D) \otimes p_2^*\eta^{\otimes-(p-2a-3)} \otimes I_\Delta) \\ \quad\ \| \quad\ \nearrow & \\ H^1(\mathscr{O}_\Delta) & \end{array}$$

which is the coboundary map arising from the short exact sequence

$$\begin{aligned} 0 &\to p_1^*\mathscr{O}_S(3K_S - 2D) \otimes p_2^*\eta^{\otimes-(p-2a-3)} \otimes I_\Delta \\ &\to p_1^*\mathscr{O}_S(3K_S - 2D) \otimes p_2^*\eta^{\otimes-(p-2a-3)} \to \mathscr{O}_\Delta \to 0. \end{aligned}$$

By the Künneth formula, $H^1(p_1^*\mathscr{O}_S(3K_S - 2D) \otimes p_2^*\eta^{\otimes-(p-2a-3)}) = 0$, since, as η is of order p and degree 0, $H^i(F_p, \eta^{\otimes-(p-2a-3)}) = 0$ for all i. Hence ∂ is injective and α surjective. Thus, there exists an extension $\mathscr{V}_{(a,b),p}$, which is moreover locally free since the local extension class is everywhere nonvanishing. Finally, we check the uniqueness of $\mathscr{V}_{(a,b),p}$.

By the Künneth formula and local algebra,

$$A_1 = H^1(p_1^*\mathscr{O}_S(2D - 2K_S) \otimes p_2^*\eta^{\otimes(p-2a-3)}) = 0.$$

Therefore, by the exact sequence for Ext, A_2 is one-dimensional, and the corresponding nontrivial extension $\mathscr{V}_{(a,b),p}$ is unique. This concludes the proof of (4.2). q.e.d.

Continuing with the construction of the moduli space, let I_p be the set of pairs (a,b) satisfying the hypotheses of (3.9)(a), modulo the equivalence relation $(a,b) \sim (p-3-a, q-2-b)$ if $a \le p-3$. For each $(a,b) \in I_p$, we obtain $\mathscr{V}_{(a,b),p} \to S \times F_p$. Performing the same construction with the roles of p and q reversed (i.e. for I_q) we obtain

$$\overline{\mathfrak{M}} = \coprod_{I_p} F_p \amalg \coprod_{I_q} F_q \quad \text{and} \quad \overline{\mathscr{V}} \to S \times \overline{\mathfrak{M}}.$$

Proposition 4.3. *$\overline{\mathfrak{M}}$ is the normalization of $\mathfrak{M}_{\mathrm{red}}$, where $\mathfrak{M}$ is the fine moduli space classifying L-stable rank-2 bundles V over S with $c_1(V) = 0$, $c_2(V) = 1$ (see [10]).*

Proof. If we tensor the exact sequence in (4.2) by $\mathscr{O}_{p_2^{-1}(x)}$, $x \in F_p$, we obtain the exact sequence

$$\mathscr{O}_S(D-K_S) \xrightarrow{i} \mathscr{V}_{(a,b),p}|p_2^{-1}(x) \to \mathscr{O}_S(K_S - D)I_\Delta \otimes \mathscr{O}_{p_2^{-1}(x)} \to 0.$$

Since the map i on the left-hand side is injective at a generic point of S and $\mathscr{O}_S(D-K_S)$ is torsion free, i is an injection. A straightforward calculation shows that $I_\Delta \otimes \mathscr{O}_{p_2^{-1}(x)} = m_x$. Thus, restricting $\mathscr{V}_{(a,b),p}$ to $p_2^{-1}(x)$ yields the sequence (3.1)(b). By the universal property of $\mathfrak{M}$, there is an induced morphism $\mu\colon \overline{\mathfrak{M}} \to \mathfrak{M}$, which by construction is a bijection on geometric points. Thus, μ identifies $\mathfrak{M}$ with the normalization of $\mathfrak{M}_{\mathrm{red}}$.

Theorem 4.4. *Let $S = S(p,q)$ be a generic Dolgachev surface, and let L be a suitable line bundle over S. The moduli space $\mathfrak{M}$ of L-stable rank-2 vector bundles V with $c_1(V) = 0$ and $c_2(V) = 1$ is compact. Each component of $\mathfrak{M}_{\mathrm{red}}$ is identified either with F_p or F_q. If a component $(\mathfrak{M}_i)_{\mathrm{red}}$ of $\mathfrak{M}_{\mathrm{red}}$ is identified with F_p (resp. F_q), and $\mathscr{V}_i$ is the restriction of the universal bundle over $S \times \mathfrak{M}$ to $S \times (\mathfrak{M}_i)_{\mathrm{red}}$, then $\mu_i = (p_i)_* c_2(\mathscr{V}_i)$ is Poincaré dual to $[F_p]$ (resp. $[F_q]$). Lastly, if $p = 2$, then $\mathfrak{M}$ is reduced and consists of $(q-1)/2$ components each identified with F_2.*

Proof. By (4.3) the normalization $\overline{\mathfrak{M}}$ of $\mathfrak{M}_{\mathrm{red}}$ is identified with a disjoint union of copies of F_p and F_q. Let $\mathfrak{M}_1$ be an F_p-component of $\overline{\mathfrak{M}}$. Let $\mathscr{V}_i$ be as in the statement of (4.4). Since $c_1(\eta) = 0$, $c_1(\mathscr{V}_i) = 0$ as well. Hence, in the notation of (4.2),

$$c_2(\mathscr{V}_i) = -(p_1)^*(D-K_S)^2 + [\Delta].$$

Thus, $(p_1)_*(c_2(\mathscr{V}_i)) = (p_1)_*[\Delta] = [F_p]$.

Lastly, if $p = 2$, then by (4.3) $\overline{\mathfrak{M}}$ consists of $(q-1)/2$ components each identified with F_q. Thus, to complete the proof of Theorem 4.4, we need only show that $\mathfrak{M}$ is smooth if $p = 2$. This will be an immediate consequence of Lemma 4.5 and Theorem 4.6(b). q.e.d.

Next, we discuss the local structure of the fine moduli space. Let V correspond to the point $v \in \mathfrak{M}$.

Lemma 4.5. *$\mathfrak{M}$ is smooth at v if and only if $H^2(\operatorname{End} V) = 0$.*

Proof. We know that $\dim \mathfrak{M} = 1$ and that $\dim H^0(\operatorname{End} V) = 1$ by (1.2). Moreover

$$h^0(\operatorname{End} V) - h^1(\operatorname{End} V) + h^2(\operatorname{End} V) = 0,$$

by (1.5). Thus $\mathfrak{M}$ is smooth at v if and only if $h^1(\operatorname{End} V) = 1$ if and only if $h^2(\operatorname{End} V) = 0$.

Theorem 4.6. *Let D and x satisfy* (3.9)(a), *and let V be as in* (3.1)(b), *with $x \in F_p$.*

(a) *$h^2(\operatorname{End} V) \leq 2$ and $h^2(\operatorname{End} V) \leq 1$ if $a = p-2$ or if p is even and $a = p/2 - 2$.*

(b) *$H^2(\operatorname{End} V) = 0$ if and only if $p = 2$.*

Proof. Since $\operatorname{End} V = V \otimes V^\vee$ is self-dual, $h^2(\operatorname{End} V) = h^0(\operatorname{End} V \otimes K_S)$. We have the exact sequence

$$0 \to \mathcal{O}_S(D) \otimes \mathbf{V} \to \operatorname{End} V \otimes K_S \to \mathcal{O}_S(2K_S - D)m_x \otimes \mathbf{V} \to 0,$$

and therefore the corresponding sequence

$$(*) \qquad \begin{aligned} 0 \to H^0(\mathcal{O}_S(D) \otimes \mathbf{V}) &\to H^0(\operatorname{End} V \otimes K_S) \\ &\to H^0(\mathcal{O}_S(2K_S - D)m_x \otimes \mathbf{V}). \end{aligned}$$

Claim 4.7. (a) $\dim H^0(\mathcal{O}_S(D) \otimes \mathbf{V}) = 0$ if p is even and $a = p/2 - 1$, and $\dim H^0(\mathcal{O}_S(D) \otimes \mathbf{V}) = 1$ otherwise.

(b) $\dim H^0(\mathcal{O}_S(2K_S - D)m_x \otimes \mathbf{V}) = 0$ if $a = p-2$ or if p is even and $a = p/2 - 2$, and $\dim H^0(\mathcal{O}_S(2K_S - D)m_x \otimes \mathbf{V}) = 1$ otherwise.

Proof that (4.7) *implies* (4.6). The proof of (a) and the "if" direction of (b) are clear, using the exact sequence $(*)$ and (3.11)(a). For the "only if" direction of (b), assume that $H^2(\operatorname{End} V) = 0$. This assumption is independent of how V is written as an extension (3.1)(b). By assumption, D and x satisfy (3.5). If $a \neq p-2$, by (3.9)(b) possibly after replacing (a, b) by $(p-3-a, q-2-b)$, we may assume that $a < (p-3)/2 < p/2 - 1$. Using (4.7)(a) and the sequence $(*)$, we have $h^0(\operatorname{End} V \otimes K_S) \geq 1$. This shows that $a - p - 2$, and, by (4.7)(a) again, $a = p/2 - 1$. Hence $p = 2$.

Proof of (4.7)(a). We start with the exact sequence

$$0 \to \mathcal{O}_S(2D - K_S) \to \mathcal{O}_S(D) \otimes \mathbf{V} \to \mathcal{O}_S(K_S)m_x \to 0.$$

Since $H^0(\mathcal{O}_S(K_S)m_x) \subseteq H^0(\mathcal{O}_S(K_S)) = 0$,

$$H^0(\mathcal{O}_S(D) \otimes \mathbf{V}) = H^0(\mathcal{O}_S(2D - K_S)).$$

As $2D - K_S = (2a - p + 1)F_p + (2b - q + 1)F_q + f$, $H^0(2D - K_S) = 0$ if and only if $2a - p + 1 < 0$ and $2b - q + 1 < 0$. But since D and x satisfy (3.5), by (3.6)(a) either $a > (p-3)/2$ or $b > (q-2)/2$. Therefore, as there are no integers b with $(q-2)/2 < b < (q-1)/2$, if $H^0(2D - K_S) = 0$, then $(p-3)/2 < a < (p-1)/2$. Since a is an integer, p must be even and $a = p/2 - 1$. Moreover, by (3.1)(a), at least one of $2a - p + 1$ and $2b - q + 1$ is negative, so that in all cases $\dim H^0(2D - K_S) \leq 1$. This proves (a).

Proof of (4.7)(b). We have an inclusion $H^0(\mathcal{O}_S(2K_S - D)m_x \otimes \mathbf{V}) \subseteq H^0(\mathcal{O}_S(2K_S - D) \otimes \mathbf{V})$. Moreover, there is an exact sequence

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(2K_S - D) \otimes \mathbf{V} \to \mathcal{O}_S(3K_S - 2D)m_x \to 0.$$

As $H^0(K_S) = H^1(K_S) = 0$, $H^0(\mathcal{O}_S(2K_S - D) \otimes \mathbf{V}) = H^0(\mathcal{O}_S(3K_S - 2D)m_x)$. Since x is in the base locus of $3K_S - 2D$, $H^0(\mathcal{O}_S(3K_S - 2D)m_x) \neq 0$. In fact, $3K_S - 2D = (2p - 3 - 2a)F_p + (q - 3 - 2b)F_q$ if $a > (p-3)/2$ and $3K_S - 2D = (p - 3 - 2a)F_p + (2q - 3 - 2b)F_q$ if $b > (q-2)/2$. In either case, we write $3K_S - 2D = rF_p + sF_q$ with $r \geq 1$, $s \geq 0$. Hence,

$$\dim H^0(\mathcal{O}_S(3K_S - 2D)m_x) = 1.$$

and therefore $\dim H^0(\mathcal{O}_S(2K_S - D) \otimes \mathbf{V}) = 1$.

To complete the proof of (4.7)(b), we must determine when a nonzero section σ of $\mathcal{O}_S(2K_S - D) \otimes \mathbf{V}$ lies in $H^0(\mathcal{O}_S(2K_S - D)m_x \otimes \mathbf{V})$, i.e. when σ vanishes at $x \in F_p$. Let $\pi\colon H^0(\mathcal{O}_S(2K_S - D) \otimes \mathbf{V} \to H^0(\mathcal{O}_S(3K_S - 2D)m_x)$ be the natural projection. Choose local coordinates z_1, z_2 near x so that F_p is defined by $\{z_1 = 0\}$. This choice trivializes $\mathcal{O}_S(D)$ near x, and we choose trivializations of V and K_S as well. Then the exact sequence

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(2K_S - D) \otimes \mathbf{V} \to \mathcal{O}_S(3K_S - 2D)m_x \to 0$$

is given locally in some open set U containing x by

$$0 \to \mathcal{O}_U \xrightarrow{\varphi_1} \mathcal{O}_U \oplus \mathcal{O}_U \xrightarrow{\varphi_2} m_x \to 0.$$

For appropriate choices, we may further assume that $\varphi_1(1) = (z_1, z_2)$, $\varphi_2(f, g) = z_2 f - z_1 g$, and $\pi(\sigma)(z_1, z_2) = z_1^r \cdot u$, where u is a unit in $\mathcal{O}_{U,x}$. Hence $\sigma \equiv (0, -z_1^{r-1} \cdot u) \bmod (z_1, z_2)$. In particular, σ does not vanish at x if and only if $r = 1$. But $r = 2p - 3 - 2a$ if $a > (p-3)/2$ and $r = p - 3 - 2a$ if $a < (p-3)/2$. Hence, $H^0(\mathcal{O}_S(2K_S - D)m_x \otimes \mathbf{V}) = 0$ if $a = p - 2$ or if $a = p/2 - 2$, and $H^0(\mathcal{O}_S(2K_S - D)m_x \otimes \mathbf{V}) = H^0(\mathcal{O}_S(2K_S - D) \otimes \mathbf{V}) = H^0(\mathcal{O}_S(3K_S - 2D)m_x) = \mathbb{C}$ otherwise.

5. Stable bundles on blowups

We will use the following notation throughout this section: $\rho\colon \tilde{Y} \to Y$ is the blowup of the smooth algebraic surface Y at x, and E is the exceptional fiber $\rho^{-1}(x)$. We identify a divisor D on Y with ρ^*D on $\tilde{Y}$. Every divisor $\tilde{D}$ on $\tilde{Y}$ is linearly equivalent to $D+aE$, where the linear equivalence class of the divisor D on Y and the integer $a \in \mathbb{Z}$ are uniquely determined by $\tilde{D}$. If L is any ample divisor on Y, set $\tilde{L} = \tilde{L}_N = NL - E$. For some $N_0 = N_0(L, x) \geq 0$ and for all $N \geq N_0$, $\tilde{L}_N$ is ample. Finally, V will always denote a rank-2 bundle on Y with $c_1(V) = 0$.

The following lemma is well known,

Lemma 5.1. (a) *If $n \geq 0$, then $\rho_*\mathcal{O}_{\tilde{Y}}(nE) = \mathcal{O}_Y$.*

(b) *If $n < 0$, say $n = -a$, then $\rho_*\mathcal{O}_{\tilde{Y}}(nE) = m_x^a$.*

Proposition 5.2. *For a rank-2 bundle $\tilde{V}$ on $\tilde{Y}$ with $c_1(\tilde{V}) = 0$, the following are equivalent:*

(a) *$\tilde{\mathbf{V}} = \rho^*\mathbf{V}$ for some bundle V on Y.*

(b) *$\rho_*\tilde{\mathbf{V}}$ is locally free.*

Proof that (a) $\Rightarrow$ (b). Suppose that $\tilde{\mathbf{V}} = \rho^*\overline{\mathbf{V}}$. We have the projection formula $\rho_*\tilde{\mathbf{V}} = \rho_*\rho^*\mathbf{V} = \mathbf{V} \otimes \rho_*\mathcal{O}_{\tilde{Y}} = \mathbf{V}$. Thus, $\rho_*\tilde{\mathbf{V}}$ is locally free.

Proof that (b) $\Rightarrow$ (a). Suppose that $\mathbf{V} = \rho_*\tilde{\mathbf{V}}$ is locally free. There is a generically injective map

$$\psi\colon \rho^*\mathbf{V} = \rho^*\rho_*\tilde{\mathbf{V}} \to \tilde{\mathbf{V}},$$

which is the identity outside of E. Moreover, since $Y - \{x\} \cong \tilde{Y} - E$, $V|(Y - \{x\}) \cong \tilde{V}|(\tilde{Y} - E)$, and $c_1(\tilde{V}) = 0$, we have $c_1(V) = 0$. Thus, $c_1(\rho^*\mathbf{V}) = 0$ as well, i.e. $\bigwedge^2 \rho^*\mathbf{V} \cong \mathcal{O}_{\tilde{Y}} \cong \bigwedge^2 \tilde{\mathbf{V}}$. The induced map $\bigwedge^2 \psi$: $\bigwedge^2 \rho^*\mathbf{V} \to \bigwedge^2 \tilde{\mathbf{V}}$ is consequently constant. Since it is nonzero, it is everywhere nonvanishing, so that $\psi\colon \rho^*\mathbf{V} \xrightarrow{\cong} \tilde{\mathbf{V}}$. q.e.d.

Let $\tilde{V}$ be a rank-2 bundle on $\tilde{Y}$ with $c_1(\tilde{V}) = 0$. Set $\mathbf{V} = (\rho_*\tilde{\mathbf{V}})^{\vee\vee}$; it is a locally free rank-2 sheaf on Y, corresponding to the vector bundle V with $c_1(V) = 0$. Define the sheaf Q by the following exact sequence:

$$0 \to \rho_*\tilde{\mathbf{V}} \to V \to Q \to 0.$$

Clearly, Q is supported at x.

Lemma 5.3. *We have*

$$c_2(\tilde{V}) - c_2(V) = l(Q) + l(R^1\rho_*\tilde{\mathbf{V}}) \geq 0.$$

Thus, if $\tilde{V} \neq \rho^*V$, then $c_2(V) < c_2(\tilde{V})$.

Proof. Clearly, $c_2(\rho_*\tilde{\mathbf{V}}) - c_2(V) = l(Q)$. To calculate $c_2(\rho_*\tilde{\mathbf{V}})$, we use the Grothendieck-Riemann-Roch formula [1]: $\mathrm{ch}(\rho_!\tilde{\mathbf{V}}) \cdot Td(Y) = \rho_*((\mathrm{ch}\,\tilde{\mathbf{V}}) \cdot Td(\tilde{Y}))$. Recalling that $\rho_!(\tilde{\mathbf{V}}) = [\rho_*\tilde{\mathbf{V}}] - [R^1\rho_*\tilde{\mathbf{V}}]$ in $K_0(\tilde{Y})$, the Grothendieck group of coherent sheaves in $\tilde{Y}$, and that ch is additive, we get

$$\mathrm{ch}(\rho_*\tilde{\mathbf{V}}) - \mathrm{ch}(R^1\rho_*\tilde{\mathbf{V}}) = \rho_*[(\mathrm{ch}\,\tilde{V}) \cdot Td(\tilde{Y})] \cdot Td(Y)^{-1}.$$

Expanding the right-hand side and using the fact that $c_1^2(\tilde{Y}) + c_2(\tilde{Y}) = 12\chi(\mathcal{O}_{\tilde{Y}}) = 12\chi(\mathcal{O}_Y) = c_1^2(Y) + c_2(Y)$, we get $2 - c_2(\rho_*\tilde{\mathbf{V}}) - c_2(R^1\rho_*\tilde{\mathbf{V}}) = 2 - c_2(\tilde{V})$. Thus $c_2(\tilde{V}) - c_2(\rho_*\tilde{\mathbf{V}}) = l(R^1\rho_*\tilde{\mathbf{V}})$, so that $c_2(\tilde{V}) - c_2(V) = l(Q) + l(R^1\rho_*\tilde{\mathbf{V}})$. In particular, $c_2(V) \leq c_2(\tilde{V})$, and, if equality holds, then $l(Q) = 0$, i.e., $\mathbf{V} = \rho_*\tilde{\mathbf{V}}$. By (5.2)(b), this is equivalent to $\tilde{\mathbf{V}} = \rho^*V$. q.e.d.

Note. It is easy to give a proof of (5.3) which does not involve the Grothendieck-Riemann-Roch theorem.

Remark 5.4. There is a more precise result. By the Grothendieck theorem on vector bundles over $\mathbb{P}^1$ [11, p. 22] and the hypothesis that $c_1(\tilde{V}) = 0$, $\tilde{\mathbf{V}}|_E \cong \mathcal{O}_E(k) \oplus \mathcal{O}_E(-k)$ for some nonnegative integer k. One can show that

$$k \leq c_2(\tilde{V}) - c_2(V) \leq k^2.$$

In particular, if $k = 0$, then $\mathbf{V} = \rho_*\tilde{\mathbf{V}}$ and hence $\tilde{V} = \rho^*V$. This is a result first proved by Schwarzenberger [14, Theorem 5, p. 613].

The following is the main result of this section.

Theorem 5.5. *Let $\tilde{V}$ be a rank-2 bundle on $\tilde{Y}$ with $c_1(\tilde{V}) = 0$. Set $\mathbf{V} = (\rho_*\tilde{\mathbf{V}})^{\vee\vee}$. Fix an ample divisor L on Y, and fix*

$$N \geq \max(N_0(L, x), (2c_2(\tilde{V}) + 2)^{1/2}).$$

(a) *If V is L-stable, then $\tilde{V}$ is $\tilde{L}_N$-stable.*

(b) *If $\tilde{V}$ is $\tilde{L}_N$-stable, then V is L-semistable.*

(c) *If $\tilde{V}$ is $\tilde{L}_n$-stable and V is not L-stable, then there exists a divisor F on Y with $L \cdot F = 0$ and a nonzero map $\mathcal{O}_Y(F) \to V$ with torsion-free cokernel such that every meromorphic section of* $\mathrm{Hom}(\rho^*\mathcal{O}_Y(F), \tilde{\mathbf{V}})$ *which is regular away from E has a pole along E of order ≥ 1.*

(d) *If $\tilde{V} = \rho^*V$, then $\tilde{V}$ is $\tilde{L}_N$-stable if and only if V is L-stable.*

(e) *If $\tilde{V}$ is $\tilde{L}_N$-stable, if $H_1(Y; \mathbb{Z}) = 0$ and if $c_2(\tilde{V}) = 1$, then V is actually L-stable.*

Proof. We divide the proof into two steps.

Step I. Suppose that V is stable. To prove (a), we must show that $\tilde{V}$ is $\tilde{L}_N$-stable. Let $\mathcal{O}_{\tilde{Y}}(\tilde{F}) \to \tilde{\mathbf{V}}$ be a sub-line bundle. We can write $\tilde{F} = \rho^*F + aE$ for a unique $a \in \mathbb{Z}$. First suppose that $a \leq 0$, say $a = -n$, $n \geq 0$. From (5.1) and the projection formula, we get a sequence of injections

$$\rho_*(\rho^*\mathcal{O}_Y(F) \otimes \mathcal{O}_{\tilde{Y}}(-nE)) = \mathcal{O}_Y(F)m_x^n \to \rho_*\tilde{\mathbf{V}} \to \mathbf{V}.$$

Taking double duals, we get a map $\mathscr{O}_Y(F) \to \mathbf{V}$. As V is L-stable, $L \cdot F < 0$. But then $\tilde{L}_N \cdot \tilde{F} = (NL - E) \cdot (F + aE) = N(L \cdot F) + a < 0$ as well. This completes the proof of Step I when $a \leq 0$.

We may therefore suppose that $a > 0$. For $\tilde{F} = \rho^* F + aE$, we are given a nonzero map $\mathscr{O}_{\tilde{Y}}(\tilde{F}) \to \tilde{\mathbf{V}}$, and we wish to show that $\tilde{L}_N \cdot \tilde{F} = N(L \cdot F) + a < 0$. Without loss of generality, we may assume that $\tilde{\mathbf{V}}/\mathscr{O}_{\tilde{Y}}(\tilde{F})$ is torsion-free. Hence we have an exact sequence

$$0 \to \mathscr{O}_{\tilde{Y}}(\tilde{F}) \to \tilde{\mathbf{V}} \to \mathscr{O}_{\tilde{Y}}(-\tilde{F}) I_W \to 0.$$

Thus, $c_2(\tilde{V}) = -(\tilde{F})^2 + l(W) = -F^2 + a^2 + l(W) \geq -F^2 + a^2$. From the natural map $\mathscr{O}_{\tilde{Y}}(\rho^* F) \to \tilde{\mathbf{V}} \otimes \mathscr{O}_{\tilde{Y}}(-aE)$, we get a nonzero map $\mathscr{O}_Y(F) \to \rho_*(\tilde{\mathbf{V}} \otimes \mathscr{O}_{\tilde{Y}}(-aE)) \subseteq \mathbf{V}$. As $\tilde{\mathbf{V}}/\mathscr{O}_{\tilde{Y}}(\tilde{F})$ is torsion-free, $\mathbf{V}/\mathscr{O}_Y(F)$ is torsion-free except possibly at x, and so (by (1.9)) everywhere. Thus, we have an exact sequence

$$0 \to \mathscr{O}_Y(F) \to \mathbf{V} \to \mathscr{O}_Y(-F) I_Z \to 0.$$

It follows that $c_2(V) = -F^2 + l(Z)$. Finally, by the stability of $\mathbf{V}$, $L \cdot F \leq -1$.

First suppose that $F^2 \leq 0$. As $a^2 + (-F^2) \leq c_2(\tilde{V})$, we get $a^2 \leq c_2(\tilde{V})$. Thus, since $N \geq (2c_2(\tilde{V}) + 2)^{1/2}$, $N \geq (2a^2 + 2)^{1/2} > a$. It follows that $N(L \cdot F) + a \leq -N + a < -a + a \leq 0$.

To handle the case where $F^2 > 0$, we use:

Lemma 5.6. *We have* $(F^2)(L^2) \leq (L \cdot F)^2$.

Proof. This is an easy consequence of the Hodge index theorem. q.e.d.

Returning to the proof of Step I, note that if $a < (2c_2(\tilde{V}) + 2)^{1/2}$, we must have

$$N(L \cdot F) + a \leq -N + a < -a + a \leq 0.$$

We may therefore assume that $a^2 \geq 2c_2(\tilde{V}) + 2$. Since $L \cdot F < 0$ and L^2 is a positive integer, by (5.6) we have

$$L \cdot F \leq -(F^2)^{1/2}.$$

On the other hand, $F^2 \geq a^2 - c_2(\tilde{V}) > a^2/2$, by our assumption. Thus, $L \cdot F < -(a/\sqrt{2})$, and $N \geq \sqrt{2}$ by construction, so that

$$N(L \cdot F) + a < -a + a \leq 0.$$

It follows that $\tilde{V}$ is $\tilde{L}_N$-stable, concluding the proof of Step I.

Step II. We now analyze the case where $\tilde{V}$ is $\tilde{L}_N$-stable. Let $\mathscr{O}_Y(F)$ be a sub-line bundle of V. Thus, on $\tilde{Y}$, we are given a section of $\mathrm{Hom}(\mathscr{O}_{\tilde{Y}}(\rho^* F), \tilde{\mathbf{V}})$ over $\tilde{Y} - E$. In turn, this yields a meromorphic section of $\mathrm{Hom}(\mathscr{O}_{\tilde{Y}}(\rho^* F), \tilde{\mathbf{V}})$ which is regular away from E, and hence a nonzero map $\mathscr{O}_{\tilde{Y}}(\rho^* F + aE) \to \tilde{\mathbf{V}}$ for some $a \in \mathbb{Z}$.

If $a \geq 0$, then, as $\tilde{\mathbf{V}}$, is $\tilde{L}_N$-stable,

$$\tilde{L}_N \cdot (F + aE) = N(L \cdot F) + a < 0,$$

and thus $L \cdot F < 0$. Thus we may as well assume that $a < 0$, say $a = -n$ and $n > 0$. In checking for stability, we may always take $\mathbf{V}/\mathscr{O}_Y(V)$ torsion-free. Hence $\tilde{\mathbf{V}}/\mathscr{O}_{\tilde{Y}}(F - nE)$ is torsion-free except along E. By taking n as small as possible, we may assume that $\tilde{\mathbf{V}}/\mathscr{O}_{\tilde{Y}}(F - nE)$ is torsion-free along E as well. Thus, there is an extension

$$0 \to \mathscr{O}_{\tilde{Y}}(F - nE) \to \tilde{\mathbf{V}} \to \mathscr{O}_{\tilde{Y}}(nE - F)I_W \to 0,$$

where I_W is the ideal sheaf of some 0-dimensional subscheme W of $\tilde{Y}$. Set $c = c_2(\tilde{V})$. Then $c = -F^2 + n^2 + l(W)$, i.e., $F^2 = n^2 + l(W) - c$. By (5.6) (as L^2 is a positive integer)

$$(L \cdot F)^2 \geq F^2 = n^2 + l(W) - c \geq n^2 - c.$$

From the stability of $\tilde{V}$, we also have

$$(NL - E) \cdot (F - nE) = N(L \cdot F) - n < 0.$$

Thus, $(L \cdot F) < n/N$ and $(L \cdot F)^2 \geq n^2 - c$.

Now suppose that V is not stable, and let $\mathscr{O}_Y(F)$ be a destabilizing sub-line bundle, i.e. $L \cdot F \geq 0$. If $L \cdot F \geq 1$, then $n > N$, so that $n^2 > N^2 \geq 2c + 2$. We have $n^2 - c \leq (L \cdot F)^2 < n^2/N^2$, so that

$$c > n^2(1 - 1/N^2) \geq \tfrac{1}{2}n^2, \quad \text{i.e. } n^2 < 2c,$$

a contradiction. Thus, $(L \cdot F) = 0$, so that V is semistable. In addition, we have shown that the meromorphic map $\mathscr{O}_{\tilde{Y}}(\rho^* F) \to \tilde{\mathbf{V}}$ has a pole of order precisely $n > 0$ along E. This establishes (b) and (c) of (5.5), and (d) is clear from (c) and (a) (or by a simple direct argument using (a)).

Finally, we must prove (e). Thus, suppose that $c_2(\tilde{V}) = 1$ and that V is not L-stable. The proof of (b) and (c) shows that there exists a sub-line bundle $\mathscr{O}_Y(F) \to \mathbf{V}$ with $L \cdot F = 0$ and exact sequences

$$0 \to \mathscr{O}_Y(F) \to \mathbf{V} \to \mathscr{O}_Y(-F)I_Z \to 0,$$
$$0 \to \mathscr{O}_{\tilde{Y}}(F - nE) \to \tilde{\mathbf{V}} \to \mathscr{O}_{\tilde{Y}}(-F + nE)I_W \to 0, \qquad n > 0.$$

Since V is not L-stable, $\tilde{V} \neq \rho^* V$ by (d). It follows from (5.3) that $c_2(V) \leq c_2(\tilde{V}) - 1 = 0$. Hence $-F^2 \leq 0$. By the Hodge index theorem, $F^2 \leq 0$, and so $F^2 = L \cdot F = 0$, and F is numerically equivalent to 0. As $H_1(Y; \mathbb{Z}) = 0$, F is in fact 0. Hence $\tilde{V}$ is given by an exact sequence

$$0 \to \mathscr{O}_{\tilde{Y}}(-nE) \to \tilde{\mathbf{V}} \to \mathscr{O}_{\tilde{Y}}(nE)I_W \to 0,$$

where $n \geq 1$. But $1 = c_2(\tilde{V}) = n^2 + l(W)$, so that $n = 1$ and $l(W) = 0$, i.e. $W = \emptyset$. We have a long exact cohomology sequence

$$H^0(\tilde{Y}; \tilde{\mathbf{V}}) \to H^0(\mathscr{O}_{\tilde{Y}}(E)) \to H^1(\mathscr{O}_{\tilde{Y}}(-E)).$$

From the sequence $0 \to \mathscr{O}_{\tilde{Y}}(-E) \to \mathscr{O}_{\tilde{Y}} \to \mathscr{O}_E \to 0$ and the fact that $H^1(\mathscr{O}_{\tilde{Y}}) = 0$ by our assumption on Y, $H^1(\mathscr{O}_{\tilde{Y}}(-E)) = 0$. But then a nonzero section in $H^0(\mathscr{O}_{\tilde{Y}}(E))$ lifts to a nonzero section of $\tilde{V}$, i.e., a nonzero map $\mathscr{O}_{\tilde{Y}} \to \tilde{\mathbf{V}}$. This contradicts the stability of $\tilde{V}$. q.e.d.

Remark 5.7. The exceptional cases in (5.5)(c) really do arise, and can be more precisely analyzed. In the typical case where every destabilizing sub-line bundle of a semistable V is $\mathscr{O}_Y$ (cf. Remark (1.11)(b)), the stable bundles $\tilde{V}$ on $\tilde{Y}$ such that V is only L-semistable will all be described as extensions

$$0 \to \mathscr{O}_{\tilde{Y}}(-nE) \to \tilde{\mathbf{V}} \to \mathscr{O}_{\tilde{Y}}(nE)I_Z \to 0$$

for appropriate $n > 0$ and Z.

Similarly, if $H_1(Y; \mathbb{Z}) \neq 0$, there exist $\tilde{L}_N$-stable bundles $\tilde{V}$ with $c_2(\tilde{V}) = 1$, such that V is just L-semistable. These will all be of the form

$$0 \to \mathscr{O}_{\tilde{Y}}(\rho^*F - E) \to \tilde{\mathbf{V}} \to \mathscr{O}_{\tilde{Y}}(-\rho^*F + E) \to 0,$$

where F is a nontrivial divisor on Y numerically equivalent to 0, and where the extension does not split.

Finally, we apply our results to the Dolgachev surfaces and to blowups of $\mathbb{P}^2$. First, we need a result on the local structure of the moduli spaces, which essentially says that they are locally isomorphic (with multiplicities) at points corresponding to V respectively ρ^*V.

Lemma 5.8. *Let V be a vector bundle on Y and $\tilde{V} = \rho^*V$. Let* Def_V *be the functor from Artinian local $\mathbb{C}$-algebras to sets defined by the deformation functor of V (cf.* [13]*), and let* $\mathrm{Def}_{\tilde{V}}$ *be similarly defined in terms of $\tilde{V}$. Then* Def_V *and* $\mathrm{Def}_{\tilde{V}}$ *are naturally isomorphic.*

Proof. Let R be an Artinian local $\mathbb{C}$-algebra. Given a deformation $\mathscr{V}$ of V over $\operatorname{Spec} R$, i.e. a vector bundle over $Y \times \operatorname{Spec} R$ which restricts to V over the closed fiber, we define $\tilde{\mathscr{V}} = (\rho \times \mathrm{Id})^*\mathscr{V}$. It is a deformation of $\tilde{V}$ over $\operatorname{Spec} R$. Conversely, given a deformation $\tilde{\mathscr{V}}$ of $\tilde{V}$, set $\mathscr{V} = (\rho \times \mathrm{Id})_*\tilde{\mathscr{V}}$. By (5.2) and Nakayama's lemma, $\mathscr{V}$ is locally free and gives a deformation of V over $\operatorname{Spec} R$. Finally, these constructions are easily seen to be mutual inverses.

Corollary 5.9. *Let $S = S(p,q)$ be a generic Dolgachev surface and L a suitable line bundle on S. Let $\rho\colon \tilde{S} \to S$ be the blowup of S at r distinct points. Let $E_1, \cdots, E_r$ be the exceptional curves on $\tilde{S}$. Then there exist positive integers N and m_i, $1 \leq i \leq r$, with m_i/N arbitrarily small, $1 \leq i \leq r$,* *and such that, if we set*

$$\tilde{L} = N\rho^* L - \sum_{i=1}^{r} m_i E_i$$

(a) *$\tilde{L}$ is ample on Y.*

(b) *The moduli space $\tilde{\mathfrak{M}}$ of $\tilde{L}$-stable rank-2 vector bundles on $\tilde{S}$ with $c_1 = 0$ and $c_2 = 1$ is isomorphic as a scheme to $\mathfrak{M}$, the moduli space of L-stable rank-2 vector bundles on S with $c_1 = 0$ and $c_2 = 1$. This isomorphism is induced by ρ^*.*

(c) *The universal bundle $\tilde{\mathscr{V}}$ over $\tilde{S} \times \tilde{\mathfrak{M}}$ is the pull-back via $(\rho \times \mathrm{id})$ of the universal bundle $\mathscr{V}$ over $S \times \mathfrak{M}$.*

More precisely, we may take $\tilde{L}$ to be of the form

$$N_r(\cdots(N_2(N_1 L - E_1) - E_2) - \cdots) - E_r,$$

for any integers $N_r \gg N_{r-1} \gg \cdots \gg N_1 \gg 0$.

Proof. There is a natural map $\mathfrak{M} \to \tilde{\mathfrak{M}}$, which exists by the universal property of $\tilde{\mathfrak{M}}$, and is defined on closed points by the correspondence $V \to \rho^* V$. This map is a bijection for appropriate choice of the N_i, by (5.5)(a) and (e) and (2.5). It is an isomorphism of schemes by (5.8). The statement about universal bundles is then clear by construction. q.e.d.

Corollary 5.10. *Let X be the blowup of $\mathbb{P}^2$ at n points. Then, for an appropriate ample line bundle on X, the moduli space of stable rank-2 bundles V on X with $c_1 = 0$, $c_2 \leq 1$, is empty. If in addition $-K_X$ is effective this statement is true for every ample line bundle.*

Proof. We prove the second statement first. By the Riemann-Roch theorem, as in the proof of (2.6), if $c_1(V) = 0$ and $c_2(V) \leq 1$, there is a nonzero map

$$0 \to \mathscr{O}_X(-K_X) \to \mathbf{V}.$$

As $-K_X$ is linearly equivalent to an effective divisor, $L \cdot (-K_X) > 0$ for every ample L, hence V cannot be stable. The second statement then implies the first, by (5.5)(e), since $-K_{\mathbf{P}^2} = \mathscr{O}_{\mathbf{P}^2}(3H)$ is effective.

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