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CONFORMAL DEFORMATION TO CONSTANT NEGATIVE SCALAR CURVATURE ON NONCOMPACT RIEMANNIAN MANIFOLDS

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A natural question in Riemannian geometry is whether any Riemannian manifold may be conformally deformed to achieve constant scalar curvature. It is customary to refer to this as the Yamabe Problem because Yamabe claimed in 1960 to have proven the result for compact manifolds [20]. Trudinger [18] found a deficiency in Yamabe's proof, but was able to correct the error when the total scalar curvature is nonpositive. Some cases of positive scalar curvature were solved by Aubin [1], and the remaining cases were finally resolved by Schoen [16].

The Yamabe Problem for complete, noncompact Riemannian manifolds was posed by Yau [22] and Kazdan [10], however we are not aware of any results in the literature. In this paper we shall study the case of achieving constant negative scalar curvature. From history we expect this to be the simplest case, but even here some interesting phenomena occur.

As in the compact case, the problem is studied by means of the semilinear elliptic equation

$$\frac{4(n-1)}{(n-2)}\Delta_g u - u^{(n+2)/(n-2)} = Su, \tag{1}$$

where Δ_g and S denote the Laplace-Beltrami operator and scalar curvature respectively for the Riemannian manifold (M, g) with $\dim M = n > 2$. A positive solution u of (1) will define a ("pointwise") conformal metric $\tilde{g} = u^{4/(n-2)}g$ with constant scalar curvature $\tilde{S} \equiv -1$.

The problem is clearly related to determining when a simply-connected Riemann surface is conformally equivalent to the disk, so it is not surprising that we encounter conditions on the negativity of the curvature (cf. [9], [14], [21]). Note, however, that in the present paper we always consider *pointwise* conformal metrics and conditions on the *scalar* curvature.

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Our first result is analogous to that in [14].

Theorem A. *If (M, g) is a complete Riemannian manifold with nonpositive scalar curvature S satisfying*

$$S(x) \leqslant -\varepsilon < 0 \tag{2}$$

for $x \in M \setminus M_0$, where M_0 is a compact set, then there is a complete conformal metric $\tilde{g}$ with scalar curvature $\tilde{S} \equiv -1$.

Suppose now that we allow S to vanish at infinity. Even if $S < 0$ on M, it may not be possible to solve (1). Indeed, Ni [13] has constructed metrics g in $\mathbf{R}^n$ which are uniformly equivalent and conformal to the Euclidean metric, and have $S < 0$ for $x \in \mathbf{R}^n$ although

$$|S(x)| = O\left(|x|^{-l}\right) \quad \text{as } |x| \to \infty,$$

where $l > 2$ (cf. also [11]). If $\tilde{g}$ were conformal to g with $\tilde{S} \equiv -1$, then $\tilde{g}$ would also be conformally Euclidean. Writing $\tilde{g} = v^{4/(n-2)}\,dx^2$ we find that

$$\frac{4(n-1)}{(n-2)}\Delta v - v^{(n+2)/(n-2)} = 0 \tag{3}$$

in $\mathbf{R}^n$; hence $v \equiv 0$ by [14].

Thus some negativity condition on (g, S) is required to achieve $\tilde{S} \equiv -1$. In view of [13] and [21], it is reasonable to consider

$$S(x) \leqslant -C_1(r(x))^{-l} \quad \text{for } x \in M \setminus M_0, \tag{4}$$

where $0 < l < 2$ and $r(x)$ is the geodesic distance to a fixed point x_0 in the interior of the compact set M_0. This is sufficient negativity, at least if we add an assumption on the Ricci curvature:

$$\operatorname{Ric}(\nu, \nu) \geqslant -C_2(r(x))^{-2\alpha} \quad \text{for } x \in M. \tag{5}$$

where $\nu = \partial/\partial r$ at x (whenever defined).

Theorem B. *If (M, g) is a complete Riemannian manifold with nonpositive scalar curvature $S(x)$ satisfying* (4) *and Ricci curvature satisfying* (5) *where $0 \leqslant \alpha < 1$ and $2\alpha \leqslant l < 1 + \alpha$, then there is a complete conformal metric $\tilde{g}$ with $\tilde{S} \equiv -1$.*

The required negativity of (g, S) may also be concentrated in a compact set: suppose that

$$\int \left(\frac{4(n-1)}{(n-2)} |\nabla \Phi|^2 + S\Phi^2 \right) dV < 0, \tag{6}$$

for some smooth $\Phi \geqslant 0$ with compact support. This condition implies that the "conformal Laplacian" $-\Delta + ((n-2)/4(n-1))S$ has negative first eigenvalue for Dirichlet conditions on some compact set. This strong restriction fails, for example, for the simply-connected hyperbolic space form $H^n(-1)$, but holds for certain quotients $H^n(-1)/\Gamma$ by a discontinuous group of isometries Γ, for instance in case $H^n(-1)/\Gamma$ has finite volume. In fact, it is not difficult to see that (6) holds for any complete Riemannian manifold with finite volume satisfying

$$\int_M S(x)\,dV < 0,$$

so this case is somewhat analogous to that of compact manifolds (cf. [18]).

Theorem C. *If (M, g) is a complete Riemannian manifold satisfying* (6) *then there is a conformal metric $\tilde{g}$ with $\tilde{S} \equiv -1$. Moreover, $\tilde{g}$ is complete if* (2) *holds for $x \in M \setminus M_0$ where M_0 is a compact set, or if* (4) *and* (5) *hold with $0 \leqslant \alpha < 1$ and $2\alpha \leqslant l < 1 + \alpha$.*

Remark. Condition (6) is also exactly the obstruction to conformally deforming an asymptotically flat spacetime to achieve zero scalar curvature (cf. [6]).

Note that Theorem C allows unrestricted nonnegativity of S on portions of M, unlike Theorems A and B. The reasons for this can be seen from the proofs. In §§1 and 2 below we show that the existence of a lower solution is equivalent to the existence of a positive solution of (1). (This is related to results in [3] and [12] and the references therein.) Condition (6) is strong enough to guarantee such a lower solution regardless of how (g, S) behaves globally (cf. §3). However, the lower solutions for Theorems A and B depend globally on (g, S): the proof in §§3 and 5 shows that the condition $S \leqslant 0$ can be relaxed to $S \leqslant \eta$ for some $\eta = \eta(g) > 0$ but Example 6.1 shows that we cannot in general allow positivity of S on M_0. Nevertheless, it may be possible to construct a lower solution by other means. In fact, the behavior in M_0 may be irrelevant provided $M \setminus M_0$ is sufficiently "nice" as illustrated by Example 6.2.

Finally we should also mention that Bland and Kalka [5] have used Theorem A to show that every noncompact manifold admits a complete metric with constant negative scalar curvature.

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1. A priori bounds for solutions to nonlinear inequalities

Let $\overline{\Omega}$ be a compact C^∞-manifold with boundary $\partial\Omega$ and interior $\Omega = \overline{\Omega} \setminus \partial\Omega$. Suppose we have a C^∞-Riemannian metric g on $\overline{\Omega}$. We shall consider positive nonnegative weak solutions of the nonlinear equation

$$\Delta_g u \geqslant u^\alpha + Su \quad \text{in } \Omega, \tag{1.1}$$

where $\alpha > 1$, $S(x)$ is continuous, and $S(x) \geqslant -S_0$ for $x \in \Omega$. Let H_s^p denote the Sobolev space of functions with derivatives of order s in L^p.

Theorem 1.1. *For every compact set $X \subset \Omega$, there is a constant C_0 such that every nonnegative weak solution $u \in H_1^2(\Omega)$ of* (1.1) *satisfies*

$$\max_{x \in X} u(x) \leqslant C_0. \tag{1.2}$$

Proof. Since X is compact, we can find $R > 0$ and $y_1, \cdots, y_N \in X$ so that the balls $B_R(y_i)$ cover X and $\overline{B_{2R}}(y_i) \subset \Omega$. By (1.1) we have $\Delta_g u \geqslant -S_0 u$ so Theorem 8.17 of [8] states

$$\sup_{x \in X} u(x) = \sup_{x \in B_R(y_i)} u(x) \leqslant CR^{-n/p} \|u\|_{L^p(B_{2R}(y_i))}, \tag{1.3}$$

for some $i \in \{1, \cdots, N\}$, where $p > 1$ and C depends on n, p, the ellipticity constants for Δ_g in Ω, and S_0.

Now let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \equiv 1$ on $B_{2R}(y_i)$. Multiply both sides of (1.1) by $u\varphi^q$, where $q = 2(\alpha + 1)/(\alpha - 1) > 2$, and integrate by parts to obtain

$$\int u^{\alpha+1} \varphi^q \, dV \leqslant -\int \varphi^q |\nabla u|^2 \, dV - \int q\varphi^{q-1} u \nabla\varphi \cdot \nabla u \, dV + S_0 \int u^2 \varphi^q \, dV.$$

By Cauchy-Schwarz

$$-q\varphi^{q-1} u \nabla\varphi \cdot \nabla u \leqslant \frac{q^2 u^2 \varphi^{q-2}}{4} |\nabla\varphi|^2 + \varphi^q |\nabla u|^2,$$

so we obtain

$$\int u^{\alpha+1} \varphi^q \, dV \leqslant \frac{q^2}{4} \int u^2 \varphi^{q-2} |\nabla\varphi|^2 \, dV + S_0 \int u^2 \varphi^q \, dV,$$

and then the Hölder inequality applied to both terms on the right yields

$$\begin{aligned} \int u^{\alpha+1} \varphi^q \, dV \leqslant \frac{q^2}{4} &\left(\int u^{\alpha+1} \varphi^q \, dV \right)^{2/(\alpha+1)} \left(\int |\nabla\varphi|^q \, dV \right)^{(\alpha-1)/(\alpha+1)} \\ &+ S_0 \left(\int u^{\alpha+1} \varphi^q \, dV \right)^{2/(\alpha+1)} \left(\int \varphi^q \, dV \right)^{(\alpha-1)/(\alpha+1)}. \end{aligned}$$

Using Young's Inequality, we can absorb the terms involving u on the right into the term on the left to obtain

$$\int u^{\alpha+1} \varphi^q \, dV \leqslant C \left(\int |\nabla\varphi|^q \, dV + \int \varphi^q \, dV \right). \tag{1.4}$$

Finally, note that $p = \alpha + 1$ implies

$$\|u\|_{L^p(B_{2R}(y_i))}^p \leqslant \int u^{\alpha+1} \varphi^q \, dV,$$

so we may combine (1.3) and (1.4) to obtain (1.2).

2. Condition for the existence of a positive solution

In this section we reduce the problem of finding a positive solution of (1) to constructing a nontrivial (weak) lower solution. Let $c_n = 4(n-1)/(n-2)$.

Proposition 2.1. *There is a positive C^∞ solution u of* (1) *if and only if there is a nonnegative continuous function $u_- \in (H_1^2)_{\text{loc}}$ satisfying $u_- \not\equiv 0$ and*

$$c_n \Delta_g u_- - u_-^{(n+2)/(n-2)} \geqslant S u_-, \tag{2.1}$$

weakly on H_1^2. Moreover, $u \geqslant u_-$ on M.

Proof. Since "only if" is trivial, suppose we have the desired lower solution u_-. If $\Omega \subset M$ is bounded, then $S(x)$ is bounded below on Ω and we may construct an upper solution $u_+ = \text{const} \geqslant \max\{u_-(x) : x \in \Omega\}$:

$$0 = c_n \Delta_g u_+ \leqslant u_+^{(n+2)/(n-2)} + S u_+ \quad \text{in } \Omega.$$

Since $u_\pm$ are bounded, and since the maximum principle applies to continuous functions in $H_1^2(\Omega)$ (cf. [17]), the monotone iteration scheme (cf. [15]) will produce a solution $u \in H_2^p(\Omega)$ of (1) satisfying $u_- \leqslant u \leqslant u_+$. Since u is bounded on Ω we can use standard elliptic theory to show $u \in C^2(\Omega)$. We may also use the Hopf maximum principle to show u is positive on Ω, assuming $u_- \not\equiv 0$ on Ω: since u satisfies $\Delta_g u - Au \leqslant 0$ in Ω for A sufficiently large, u cannot achieve a nonpositive minimum unless $u \equiv 0$, which is prevented by $u \geqslant u_- \not\equiv 0$. Thus $u^{(n+2)/(n-2)} \in C^2(\Omega)$ and we may proceed inductively with elliptic regularity to conclude $u \in C^\infty(\Omega)$.

To construct a solution on all of M, suppose $M = \bigcup\{\Omega_k\colon\ k = 1, 2, \cdots\}$, where Ω_k is bounded, $\overline{\Omega}_k \subset \Omega_{k+1}$, and $u_- \not\equiv 0$ in Ω_1. Since S is bounded below on each Ω_k we can use the above argument to construct positive solutions u_k of (1) in Ω_k with $u_k \geqslant u_-$. Let us consider the sequence $\{u_k\}_{k>3}$ on $X = \overline{\Omega}_3$. By Theorem 1.1 we have $u_k(x) \leqslant C_0$ for $x \in X$ and $k \geqslant 4$. Using interior elliptic estimates (and C as a generic constant) we find

$$\|u_k\|_{H_2^p(\Omega_2)} \leqslant C \|u_k\|_{L^p(\Omega_3)} \leqslant C.$$

Taking $p > n$ we have $\|u_k\|_{C^1(\Omega_2)} \leqslant C$ by the Sobolev embedding theorem, so interior elliptic estimates yield $\|u_k\|_{C^{2+\alpha}(\Omega_1)} \leqslant C$. The compactness of $C^{2+\alpha}(\Omega_1) \to C^2(\Omega_1)$ now yields a subsequence of $\{u_k\}$, denoted $\{u_{k1}\}$, such that u_{k1} converges to a solution of (1) on Ω_1. We repeat this procedure with $\{u_{k1}\}$ on $X = \overline{\Omega}_4$ to obtain a subsequence $\{u_{k2}\}$ which converges to a solution of (1) on Ω_2. Inductively we obtain $\{u_{kj}\}$ and finally define

$$u(x) = \lim_{k \to \infty} u_{kk}(x),$$

which is C^2 and satisfies (1) on M.

To verify that u is positive on M we can again use the Hopf maximum principle in any Ω_k. This in turn implies u is C^∞ on M and we are done.

3. Proofs of Theorems A and C

To prove Theorem A we first reduce it to the case where (2) holds for all $x \in M$. Namely, let $\varphi \in C_0^\infty(M)$ satisfy $c_n\Delta_g\varphi = \delta > 0$ on M_0, but otherwise arbitrary. Choose a constant φ_0 large so that (i) $\varphi_0 + \varphi(x) > 0$ for $x \in M$, and (ii) $S(x)(\varphi_0 + \varphi(x)) - c_n\Delta_g\varphi(x) \leqslant -\delta$ for $x \in M$. (Note that (ii) uses $S \leqslant 0$ on M, but also holds if $S \leqslant \eta$, $\eta = \eta(g) > 0$ sufficiently small with $-\delta/2$ on the right-hand side of the inequality.) Now let $g_1 = (\varphi_0 + \varphi)^{4/(n-2)}g$ which has scalar curvature

$$S_1 = (\varphi_0 + \varphi)^{-(n+2)/(n-2)}\left(S\varphi_0 + S\varphi - c_n\Delta_g\varphi\right) \leqslant -\varepsilon_1,$$

on M, where $\varepsilon_1 > 0$ is a suitable constant.

But if (2) holds on M then we can choose a lower solution u_- to be a small positive constant. By Proposition 2.1, there is a positive solution u of (1). Moreover, $u \geqslant u_-$ and the completeness of g imply the completeness of $\tilde{g} = u^{4/(n-2)}g$ thus proving Theorem A.

As noted in the introduction, the hypothesis (6) in Theorem C implies that there is a positive solution Ψ of

$$-c_n\Delta\Psi + S\Psi = \lambda_1\Psi \quad \text{in } \Omega, \qquad \Psi = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain and $\lambda_1 < 0$. If we choose $\mu > 0$ so that $(\mu\Psi)^{4/(n-2)} \leqslant -\lambda_1$, then $u_- = \mu\Psi$ satisfies (2.1) in Ω and $u_- = 0$ on $\partial\Omega$. We may now extend by zero to $M \setminus \Omega$ to make u_- a weak solution of (2.1) on M. Applying Proposition 2.1 we obtain a positive solution u of (1) and hence a conformal metric $\tilde{g}$ (not necessarily complete) with $\tilde{S} \equiv -1$. (Notice that we have not used any information about (g, S) outside of Ω.)

We prove that the metric $\tilde{g}$ is complete. Assume that (2) holds in $M \setminus M_0$ where we may assume $M_0 \subset \Omega$. For $\delta > 0$ small enough, $M_1 = \{x \in \Omega: \mu\Psi(x) \geqslant \delta\} \supset M_0$ and $u_- = \delta$ is a solution of (2.1) in $M \setminus M_0$. Thus defining $u_- = \mu\Psi$ in M_1 and $u_- = \delta$ in $M \setminus M_0$ yields a weak solution of (2.1) on M. Applying Proposition 2.1 yields a solution u of (1) satisfying $u(x) \geqslant \delta$ for $x \in M$ and hence a *complete* metric $\tilde{g}$ with $\tilde{S} = -1$.

Finally, suppose that (4) and (5) hold, where we may assume $M_0 = \{x \in M: r(x) \leqslant R_0\}$. Let $w(x) = Cr^{(2-n)/2}$ for $r > R_0$ and a constant C. A calculation shows

$$\begin{aligned} &c_n\Delta_g w - w^{(n+2)/(n-2)} - Sw \\ &\qquad = \left[n(n-1) - 2(n-1)r\Delta_g r - C^{1/(n-2)} - Sr^2\right]Cr^{(-n-2)/2} \geqslant 0 \end{aligned}$$

using Proposition 4.1 (below), R_0 large, and C small. If we extend w to $r < R_0$ as a positive, smooth function, and let $g_1 = w^{4/n-2}g$, then g_1 is complete by Lemma 5.2 (below) with scalar curvature $S_1 \leqslant -1$ in $M \setminus M_0$

since

$$c_n \Delta_g w - w^{(n+2)/(n-2)} - Sw \geqslant 0 = c_n \Delta_g w + S_1 w^{(n+1)/(n-2)} - Sw.$$

Moreover, g_1 satisfies (6) since that condition is conformally invariant. Hence we may apply the preceding analysis to g_1 to obtain a complete metric $\tilde{g}$ which is conformal to g_1 (and hence to g) with $\tilde{S} \equiv -1$.

4. Ricci curvature and the Laplacian of the distance function

In this section we derive upper bounds on the Laplacian of the distance function r assuming lower bounds on the Ricci curvature. (The special case $\alpha = 0$ is well known in the literature by other methods, cf. [19] and the references therein.) We shall use this material in the next section to construct a lower solution which only depends on r.

Proposition 4.1. *Suppose (M, g) is a complete Riemannian manifold and $r(x) = d(x, x_0)$ denotes the geodesic distance to a fixed point $x_0 \in M$. If the Ricci curvature satisfies*

$$\text{(4.1)} \qquad \operatorname{Ric}(v, v) \geqslant -C_1^2 \min\left[1, (r(x))^{-2\alpha}\right],$$

for $x \in M$, where $C_1 > 0$, $0 \leqslant \alpha < 1$, and $v = \partial/\partial r$ at x (whenever defined), then there is a constant $C_2 = C_2(C_1, n, \alpha) > 0$ such that

$$\text{(4.2)} \qquad \Delta_g r \leqslant C_2 \max(r^{-1}, r^{-\alpha})$$

holds weakly on M.

Proof. We shall use the ideas of Calabi [7]. Write $M \setminus \{x_0\}$ as the disjoint union $Y(x_0) \cup Z(x_0)$ where $Y(x_0)$ is the set of points x connected to x_0 by a unique minimal geodesic γ which has no conjugate points, and $x \in Z(x_0)$ if the length-minimizing geodesic γ is not unique or contains conjugate points.

If $p \in Y(x_0)$, then r is differentiable at p, p is a regular point on the geodesic sphere $S_r = \{x \in M \colon r(x) = r\}$, and the mean curvature H of S_r at p satisfies

$$\text{(4.3)} \qquad (n-1)H = -\Delta_g r.$$

Moreover, along the geodesic γ we have

$$\text{(4.4)} \qquad \partial H/\partial r \geqslant H^2 + \frac{1}{n-1}\operatorname{Ric}(v, v),$$

where v is the unit normal (cf. (4.9) in [7]). Let $C = C_1/\sqrt{n-1}$.

Now suppose $0 < r(p) \leqslant 1$ so by (4.1)

$$\partial H/\partial r \geqslant H^2 - C^2. \tag{4.5}$$

Notice that $H \to -\infty$ as $r \to 0$, so there are two possibilities:

(i) $H < -C$ and $\partial H/\partial r > 0$ along γ, or

(ii) $H(\gamma(s)) \geqslant -C$ for $s_0 \leqslant s \leqslant s_1$ where $\gamma(s_1) = p$.

In case (ii) we find immediately that $-H(p) \leqslant C$ or by (4.3)

$$\Delta_g r(p) \leqslant C(n-1). \tag{4.6}$$

In case (i) we may integrate (4.5) along γ to find

$$\ln\left(\frac{H(r) - C}{H(r) + C}\right) - \ln\left(\frac{H(r_0) - C}{H(r_0) + C}\right) \geqslant 2C(r - r_0),$$

where $0 < r_0 < r$. Letting $r_0 \to 0$ we find

$$\ln\left(\frac{H(r) - C}{H(r) + C}\right) \geqslant 2Cr.$$

Since $H(r) + C < 0$, we obtain

$$-H(r) \leqslant \left(\frac{C(1 + e^{2Cr})}{(e^{2Cr} - 1)}\right) \leqslant \frac{1}{r}(1 + Cr),$$

where the last inequality is easily verified; so by (4.3)

$$\Delta_g r(p) \leqslant \frac{(n-1)}{r(p)}(1 + C \cdot r(p)). \tag{4.7}$$

Next suppose $r(p) > 1$ so by (4.1)

$$\frac{\partial H}{\partial r} \geqslant H^2 - C^2 r^{-2\alpha} \quad \text{at } p.$$

Again there are two possibilities:

(i) $H < -Cr^{-\alpha}$ and $\partial H/\partial r > 0$ along γ, or

(ii) $H(\gamma(s)) \geqslant -C(r(\gamma(s))^{-\alpha}$ for $s_0 \leqslant s \leqslant s_1$ where $\gamma(s_1) = p$. Again in case (ii) we find that

$$Ar \leqslant C(n-1)r^{-\alpha}, \tag{4.8}$$

whereas for case (i) we let H_1 denote the solution of the ordinary differential equation

$$\begin{aligned} \frac{d}{dr}H_1 &= H_1^2 - C^2 r^{-2\alpha} && \text{for } r > 1, \\ H_1 &= -(1 + C) && \text{at } r = 1. \end{aligned}$$

The solution of this Riccati equation clearly exists for all $r > 1$ and satisfies $H_1'(r) > 0$ and $H_1(r) < -Cr^{-\alpha}$. The standard substitution $H_1 = -v'/v$ yields the second-order linear equation

$$v'' - C^2 r^{-2\alpha} v = 0. \tag{4.9}$$

The leading behavior of (4.9) as $r \to \infty$ is (using $\alpha < 1$)

$$v(r) \sim \exp\left[\pm Cr^{1-\alpha}/(1-\alpha) + (\alpha \ln r)/2 + b\right], \tag{4.10}$$

where b is a constant. Plugging into H_1 we find we must take $+$ in (4.10) and then

$$H_1(r) \sim -Cr^{-\alpha} \quad \text{as } r \to \infty.$$

In particular, we may choose $C_3 = C_3(C_1, n, \alpha)$ such that $H_1 \geqslant -C_3 r^{-\alpha}$ for $r \geqslant 1$. Since $\partial H/\partial r \geqslant dh_1/dr$ for $r > 1$, and $H \geqslant H_1$ for $r = 1$, we find $H \geqslant -C_3 r^{-\alpha}$ along γ; so

$$\Delta_g r(p) \leqslant (n-1) C_3 (r(p))^{-\alpha}. \tag{4.11}$$

Combining (4.6), (4.7), (4.8), and (4.11) yields (4.2).

On the other hand, if $p \in Z(x_0)$ let γ be a length-minimizing geodesic between x_0 and p, and let x_ε be the point on γ with $r(x_\varepsilon) = \varepsilon$. Then $p \in Y(x_\varepsilon)$, so $r_\varepsilon(x) = d(x, x_\varepsilon)$ is differentiable near p. Let γ_ε be the unique geodesic between x_ε and p. If $0 < r(p) \leqslant 1$ then we have $\mathrm{Ric} \geqslant -C_1^2$ along γ_ε and the proof above shows

$$\Delta_g r_\varepsilon \leqslant \frac{n-1}{r_\varepsilon}(1 + Cr_\varepsilon) \leqslant \frac{n-1}{r-\varepsilon}(1 + C). \tag{4.12}$$

If $r(p) > 1$ then $r_\varepsilon(p) > 1$ provided ε is sufficiently small. Moreover, $\mathrm{Ric} \geqslant -C_1^2 (r_\varepsilon)^{-2\alpha}$ along γ_ε with the same C_1 as in (4.1) since $r_\varepsilon < r$. Repeating the above argument shows

$$\Delta_g r_\varepsilon \leqslant (n-1) C_3 r_\varepsilon^{-\alpha} \leqslant (n-1) C_3 (r - \varepsilon)^{-\alpha}, \tag{4.13}$$

with the same C_3 as above. Moreover, (4.12) and (4.13) hold in a neighborhood $V_\varepsilon(p)$, and $r - r_\varepsilon$ achieves its minimum at p; arguing as in [7] shows that we can take $\varepsilon = 0$ and obtain weak inequalities at p. Combining these as before yields the weak inequality (4.2).

5. Proof of Theorem B

Theorem B follows from Proposition 4.1 and the succeeding more general, but more technical, result.

Proposition 5.1. *Suppose (M, g) is a complete Riemannian manifold with nonpositive scalar curvature S, which is strictly negative outside a compact set, and distance $r(x)$ to a fixed x_0 satisfying: there is an $R > 0$ such that*

$$\left[2(n-1)\Delta_g r + \frac{(2n-1)}{r}\right] \leqslant -rS \tag{5.1}$$

holds weakly for all $r(x) > R$. Then there is a complete conformal metric $\tilde{g}$ with $\tilde{S} \equiv -1$.

Proof. First note that the reduction used in §3 enables us to assume $S < 0$ on M; observe that condition (5.1) continues to hold. Next define

$$u_-(r) = (r^2 + b)^{-(n-2)/4}.$$

For this to be a lower solution, a calculation shows we must find $b > 0$ so that

$$\frac{(n-1)(n+2)r^2}{(r^2+b)^2} - \frac{2n-1}{r^2+b} - \frac{2(n-1)r\Delta r}{r^2+b} \geqslant S \tag{5.2}$$

holds weakly on M. We shall ignore the first term since it is $\geqslant 0$. Now (5.1) implies there is an $R_0 \geqslant 0$ such that

$$-\frac{2n-1}{r^2} - \frac{2(n-1)\Delta r}{r} \geqslant S \quad \text{for } r > R_0,$$

so for any $b > 0$ we have

$$-\frac{2n-1}{r^2+b} - \frac{2(n-1)r\Delta r}{r^2+b} \geqslant S \quad \text{for } r > R_0.$$

This establishes (5.2) for $r > R_0$. For $r < R_0$ we have $S < -\delta$ and Δr bounded, so we can take b large to achieve (5.2) on M. Hence Proposition 2.1 implies there is a positive solution u, and we need only verify that $\tilde{g} = u^{4/n-2}g$ is complete. But note that

$$u^{2/n-2}(x) \geqslant C/r(x) \quad \text{for } r(x) > 1. \tag{5.3}$$

This is exactly what we need by the following.

Lemma 5.2. *If g is complete and the positive function u satisfies* (5.3), *then $\tilde{g} = u^{4/n-2}g$ is complete.*

Proof. Let $M_1 = \{x \in M : r(x) < 1\}$ and suppose γ: $[0, b) \to M$ is a geodesic for $\tilde{g}$ with $\gamma(0) = x_0$ and which is not extendible to b; we must show the length of γ is infinite. Since (M, g) is complete, γ cannot remain in any compact subset of M. In particular, with respect to M_1 there are two possibilities: (i) γ leaves M_1 in finite time and does not return, or (ii) γ returns to M_1 infinitely many times. In both cases we are interested in the length of γ while outside of M_1, so suppose we have $0 < \alpha < \beta < b$ with γ: $[\alpha, \beta] \to M \setminus M_1$. Consider the partition $\alpha < t_1 < t_2 < \cdots < t_N < \beta$ such that there exists a geodesic ball $B(\gamma(t_j), R_j)$ with center $\gamma(t_j)$ and radius R_j, $\gamma(t_{j+1}) \in B(\gamma(t_j), R_j)$, and in $B(\gamma(t_j), R_j)$ there is a chart and coordinates in which we may write the metric g as

$$g = (dr_j)^2 + r_j^2 g_\theta, \tag{5.4}$$

with r_j being the geodesic distance from $\gamma(t_j)$. For $t_{j-1} < t < t_{j+1}$ and h small we have

$$r(\gamma(t+h)) \leqslant r(\gamma(t)) + r_j(\gamma(t+h)) - r_j(\gamma(t)),$$

which implies

$$\frac{dr(\gamma(t))}{dt} \leqslant \frac{dr_j(\gamma(t))}{dt}.$$

The length in $\tilde{g}$ of $\{\gamma(t): t_j < t < t_{j+1}\}$ is given by

$$\begin{aligned} L(\gamma;\, t_j, t_{j+1}) &= \int_{t_j}^{t_{j+1}} [\tilde{g}(\dot{\gamma}(t), \dot{\gamma}(t))]^{1/2}\, dt \\ &= \int_{t_j}^{t_{j+1}} u^{2/(n-2)}(\gamma(t))[g(\dot{\gamma}(t), \dot{\gamma}(t))]^{1/2}\, dt. \end{aligned}$$

Using (5.3) and (5.4), we obtain

$$\begin{aligned} L(\gamma;\, t_j, t_{j+1}) &\geqslant \int_{t_j}^{t_{j+1}} \frac{C_1}{r(\gamma(t))} \frac{dr_j(\gamma(t))}{dt}\, dt \\ &\geqslant C_1\left[\ln r(\gamma(t_{j+1})) - \ln r(\gamma(t_j))\right]. \end{aligned}$$

Thus

$$L(\gamma;\, \alpha, \beta) \geqslant C_1[\ln r(\gamma(\beta)) - \ln r(\gamma(\alpha))]. \tag{5.5}$$

Since $\gamma(t)$ cannot remain in any compact set we may find $b_j \to b$ such that $r(\gamma(b_j)) \to \infty$. Now suppose (i) γ leaves M_1 in finite time, i.e. for some $a \in (0, b)$ we have $\gamma(t) \in M \setminus M_1$ for $a < t < b$. Then by (5.5) we have $L(\gamma;\, a, b_j) \to \infty$ so that γ has infinite length in $\tilde{g}$. On the other hand, if (ii) γ returns to M_1 infinitely many times, then let $b_j \to b$ with $b_{j-1} < a_j < b_j$ such that $r(\gamma(b_j)) \to \infty$, $\gamma(a_j) \in \partial M_1$, and $\gamma(t) \in M \setminus M_1$ for $a_j < t < b_j$. By (5.5) we have

$$\begin{aligned} L(\gamma;\, a_j, b_j) &\geqslant C_1\left[\ln r(\gamma(b_j))\text{-}\ln r(\gamma(a_j))\right] \\ &\geqslant C_1\left[\ln r(\gamma(b_j)) - \ln d_0\right], \end{aligned}$$

where $d_0 = \max\{r(x): x \in \partial M_1\}$. Summing over j we find that γ has infinite $\tilde{g}$-length. Thus $\tilde{g}$ is complete.

6. Two examples

Both examples in this section involve "cylindrical" metrics

$$g = dr^2 + f(r)^2 h \tag{6.1}$$

on some portion of $\mathbf{R}_r \times N$ where (N, h) is a compact Riemannian manifold and $f(r)$ is a smooth positive function. We may compute the sectional curvatures of (6.1) by direct calculation; the answer depends on whether the relevant two-dimensional plane in the tangent space contains the direction $\partial/\partial r$. If so, the result is the "radial curvature" and is found to be

$$k_{\text{rad}} = -\frac{f''(r)}{f(r)}. \tag{6.2}$$

If the sectional curvature is computed for a plane perpendicular to $\partial/\partial r$ we obtain

$$k_{\text{perp}} = \frac{k_N - (f'(r))^2}{(f(r))^2}, \tag{6.3}$$

where k_N is the sectional curvature in the metric h of the associated plane in the tangent space of N. Similarly we may compute the scalar curvature of (6.1) to find

$$S = -2(n-1)\frac{f''}{f} - (n-1)(n-2)\frac{(f')^2}{f^2} + \frac{S_N}{f^2}, \tag{6.4}$$

where S_N is the scalar curvature of N.

Example 6.1. Let $M = \mathbf{R}_z \times T^{n-1}$, where T^{n-1} is the standard flat torus whose metric we denote simply by $d\Theta^2$. The formula $z = \int_0^r \exp(s^2)\,ds$ defines $r(z)$ and let $f(r) = \exp(-r^2)$. Consider the conformal metric

$$g = f(r(z))^2(dz^2 + d\Theta^2) = dr^2 + f(r)^2\, d\Theta^2,$$

which is clearly complete. If we let $u = f^{(n-2)/2}$ then we may compute the scalar curvature from (1) or (6.4),

$$S = 4(n-1)(1 - nr^2).$$

Notice that $S(0) > 0$ although (2) is satisfied outside of the compact set $r^2 \leqslant 1$. However, this complete metric cannot be conformally deformed to $\tilde{g} = v^{4/(n-2)}(dz^2 + d\Theta^2)$ with $\tilde{S} \equiv -1$ since

$$c_n \Delta v - v^{(n+2)/(n-2)} = 0$$

admits no positive solution on the flat cylinder M. Thus we cannot in general allow S to be positive inside M_0 in Theorem A. Notice that $dV_g = u^{2n/n-2}\,dz\,d\Theta = \exp[(1-n)r^2]\,dr\,d\Theta$ so that g has finite volume, however the total scalar curvature is positive:

$$\int_M S\,dV_g = 4(n-1)\int_M (1-nr^2)\exp\left[(1-n)r^2\right] dr\,d\Theta$$
$$= 4(n-1)(1-n/2(n-1))\int_M \exp\left[(1-n)r^2\right] dr\,d\Theta > 0,$$

so there is no contradiction with Theorem C.

Example 6.2. Suppose (M, g) is a noncompact manifold with a cylindrical end, i.e. $M = M_0 \cup M^+$ where M_0 is a compact manifold with boundary and $M^+ = \mathbf{R}^+ \times N$ with

$$g = dr^2 + f(r)^2 h \quad \text{on } M^+,$$

as in (6.1). Let us suppose $f(r)$ satisfies

$$(6.5) \qquad f'(r) > 0,$$

$$(6.6) \qquad \lim_{r\to\infty} f(r) = \lim_{r\to\infty} f'(r)/f(r) = \lim_{r\to\infty} f''(r)/f(r) = +\infty,$$

for example $f(r) = \exp[r^2]$. Clearly the manifold (M, g) is complete and the scalar curvature S satisfies $S \leqslant -\varepsilon$ for r sufficiently large by (6.4) and (6.6). By reparametrization we may assume

$$(6.7) \qquad S \leqslant -\varepsilon \quad \text{for } r > 0.$$

Theorem A does not apply since S may be positive on M, however we shall now show that the desired conclusion holds regardless of how g behaves in M_0.

Claim. *There is a complete conformal metric $\tilde{g}$ on M^+ (hence on M) with $\tilde{S} \equiv -1$.*

Proof. We consider the function

$$(6.8) \qquad \Psi = \begin{cases} \delta & \text{for } r \geqslant r_1, \\ \delta\left(1 - \left(r_1^2 - r^2\right)^2/r_1^4\right) & \text{for } 0 < r < r_1, \\ 0 & \text{otherwise,} \end{cases}$$

where r_1, $\delta > 0$ are to be specified. Using $|\nabla r|^2 = 1$, a calculation shows that for $0 < r < r_1$

$$c_n\Delta\Psi - \Psi^{(n+2)/(n-2)} - S\Psi = \frac{4c_n\delta}{r_1^4}\left[\left(r_1^2 - r^2\right)(1 + r\Delta r) - 2r^2\right]$$
$$-\delta^{(n+2)/(n-2)}\left(1 - \left(r_1^2 - r^2\right)^2/r_1^4\right)^{n+2/n-2}$$
$$-S\delta\left(1 - \left(r_1^2 - r^2\right)^2/r_1^4\right).$$

Since we can take δ small, it suffices by (6.7) to show

$$4(r_1^2 - r^2)(1 + r\Delta r) - 8r^2 - \varepsilon'(r_1^2 - r^2)^2 + \varepsilon' r_1^4 > 0,$$

where $\varepsilon' = \varepsilon/c_n$. But

$$\Delta r = (n-1)f'(r)/f(r),$$

so, using (6.5), it suffices to verify

$$4(r_1^2 - r^2) - 8r^2 - \varepsilon'(r_1^2 - r^2)^2 + \varepsilon' r_1^4 > 0. \tag{6.9}$$

Elementary calculus shows that (6.9) holds for $0 < r < r_1$ provided r_1 is taken sufficiently large. Thus Ψ is a lower solution for $0 < r < r_1$. Since δ is very small, Ψ is also a lower solution for $r > r_1$ by (6.7), and hence Ψ is a lower solution on all of M. By Proposition 2.1 there is a conformal metric $\tilde{g}$ as desired which must be complete since $\Psi = \delta > 0$ near infinity.

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