J. DIFFERENTIAL GEOMETRY
27 (1988) 161–178

HARMONIC SEQUENCES AND HARMONIC MAPS OF SURFACES INTO COMPLEX GRASSMANN MANIFOLDS

JON G. WOLFSON

Introduction

Let $G(k, n)$ be the Grassmann manifold of all k-dimensional subspaces $\mathbf{C}^k$ in complex space $\mathbf{C}^n$ or, what is the same, all the $(k-1)$-dimensional projective spaces $\mathbf{C}P^{k-1}$ in projective space $\mathbf{CP}^{n-1}$. $G(k, n)$ has a canonical Kähler metric. We will study the harmonic maps of a Riemann surface M into $G(k, n)$. In particular we will describe all the harmonic maps of the two-sphere S^2 into $G(k, n)$ in terms of holomorphic data and all the harmonic maps of the torus T^2 into $G(k, n)$ in terms of holomorphic data and degree zero harmonic maps. This work completes (and extends) the program for studying harmonic maps of S^2 into $G(k, n)$, first stated by the author and S. S. Chern in [4] and partially completed in [5]. The harmonic maps of $S^2 \to G(1, n) = \mathbf{C}P^{n-1}$ were first determined by Din and Zakrzewski ([6], also see [7] and [11]). The harmonic maps $S^2 \to G(2, 4)$ were determined by Ramanathan [9] and the harmonic maps $S^2 \to G(2, n)$ were determined by the author and Chern [5]. Using techniques completely different from those of the papers cited above Uhlenbeck studied the harmonic maps of S^2 into the unitary group $U(n)$ [10]. In the course of the study she gave a description of the harmonic maps of S^2 into $G(k, n)$ by embedding $G(k, n)$ totally geodesically in $U(n)$. The description given in this paper is quite different from Uhlenbeck's and works intrinsically with $G(k, n)$.

The fundamental object of study in this paper is the transforms of a harmonic map of a surface M into $G(k, n)$. To define the ∂-transform (or $\bar{\partial}$-transform) consider a map $f: M \to G(k, n)$, when M is an oriented Riemannian surface. We write the Riemannian metric of M as $ds_M^2 = \varphi\bar{\varphi}$, where φ is a complex-valued one-form, defined up to a factor of absolute value 1. This form

Received June 30, 1986 and, in revised form December 8, 1986. Work done under partial support of NSF Grant DMS 84-05186.

φ defines a complex structure on M. For $x \in M$ the space $f(x)$ has an orthogonal space $f(x)^{\perp}$ of dimension $n - k$. We denote by $[f(x)]$ and $[f(x)^{\perp}]$ their corresponding projective spaces, of dimensions $k - 1$ and $n - k - 1$, respectively. For a vector $Z(x) \in f(x)$ the orthogonal projection of ∂Z in $f(x)^{\perp}$ is multiple of φ, and hence, by cancelling out φ, defines a point of $f(x)^{\perp}$. This defines a projective collineation $\partial : [f(x)] \to [f(x)^{\perp}]$, to be called a *fundamental collineation*. The mapping defined by sending $x \in M$ to the image of $[f(x)]$ under ∂ is called the *∂-transform*. Similarly, we define the *$\bar{\partial}$-transform*.

If the map $f: M \to G(k, n)$ is harmonic then its ∂-transform and $\bar{\partial}$-transform are also harmonic. Note that a fundamental collineation ∂ (resp. $\bar{\partial}$) may degenerate or may be zero. If it is zero then the map is antiholomorphic (resp. holomorphic). If it degenerates then the ∂-transform (resp. $\bar{\partial}$-transform) is a harmonic map $M \to G(l, n)$ where $l < k$.

By successive applications of the ∂-transform (or $\bar{\partial}$-transform) we can construct a sequence of harmonic maps

$$[f(x)] \xrightarrow{\partial} \partial[f(x)] \xrightarrow{\partial} \partial^2[f(x)] \xrightarrow{\partial} \cdots$$

called a *harmonic sequence*. If any of the fundamental collineations of the sequence degenerates then the sequence associates to f a harmonic map $g: M \to G(l, n)$, $l < k$. In §4 we will show that when M has genus zero the harmonic map f can be recovered from g by iterating a construction called *returning*. Each returning is essentially a choice of a holomorphic subbundle of a holomorphic bundle over M. In §5 we describe a construction different then returning, called *extending*, which effects the reconstruction of f from g for a surface M of any genus. Each extending, like each returning, is a choice of a holomorphic subbundle.

In §3 we will derive an inequality relating the energy of f to the degree of f, the genus of M and the singularities of the fundamental collineations of the harmonic sequence generated by f. When the genus of M is zero or when the genus of M is one and the degree of f is nonzero this inequality implies that one of the fundamental collineations must be degenerate.

Combining the results of §§3 and 4 and using induction we can prove.

Theorem 1. *Let $f: S^2 \to G(k, n)$ be a harmonic map. Then f can be constructed from holomorphic or antiholomorphic curves $S^2 \to G(l, n)$, where $1 \leqslant l \leqslant k$, using the ∂ and $\bar{\partial}$ transforms and returnings.*

Combining the results of §§3 and 5 and using induction we have

Theorem 2. *Let $f: M \to G(k, n)$ be a harmonic map, where M is a surface of genus one. Then f can be constructed using the ∂ and $\bar{\partial}$ transforms and extendings from either*:

(1) *A holomorphic or antiholomorphic curve $M^2 \to G(l, n)$, $1 \leqslant l \leqslant k$, or*

(2) *A degree zero harmonic map $M^2 \to G(l, n)$, $1 \leqslant l \leqslant k$.*

In fact the statement of Theorem 2 can be made even stronger; see Theorem 5.2. Theorem 2, with (2) deleted, holds when M is a surface of genus zero; see Theorem 5.1.

The inequality in §3 should with more careful analysis yield much interesting information about harmonic maps and harmonic sequences in $G(k, n)$.

It is a pleasure to thank R. Bryant and D. Burghelea for some interesting conversations. This paper was written while the author was visiting the Max-Planck Institute für Mathematik, Bonn, West Germany; I am indebted to the Institute for its support.

§1. Geometry of $G(k, n)$

We equip $\mathbf{C}^n$ with the standard Hermitian inner product, so that, for Z, $W \in \mathbf{C}^n$,

(1.1) $$Z = (z_1, \cdots, z_n),\ W = (w_1, \cdots, w_n),$$

we have

(1.2) $$(Z, W) = \sum z_A \bar{w}_A = \sum z_A w_{\bar{A}}.$$

Throughout this paper we will agree on the following ranges of indices

(1.3)
$$1 \leqslant A, B, C, \cdots, \leqslant n, \quad 1 \leqslant \alpha, \beta, \gamma, \cdots \leqslant k, \quad k+1 \leqslant i, j, h, \cdots \leqslant n.$$

We shall use the summation convention, and the convention

(1.4) $$\bar{z}_A = z_{\bar{A}},\ \bar{t}_{\bar{A}B} = t_{A\bar{B}}, \text{ etc.}$$

A frame consists of an ordered set of n linearly independent vectors Z_A, so that

(1.5) $$Z_1 \wedge \cdots \wedge Z_n \neq 0.$$

It is called *unitary*, if

(1.6) $$(Z_A, Z_B) = \delta_{A\bar{B}}.$$

The space of unitary frames can be identified with the unitary group $U(n)$. Writing

(1.7) $$dZ_A = \omega_{A\bar{B}} Z_B,$$

the $\omega_{A\bar{B}}$ are the Maurer-Cartan forms of $U(n)$. They are skew-Hermitian, i.e., we have

$$\omega_{A\bar{B}} + \omega_{\bar{B}A} = 0. \tag{1.8}$$

Taking the exterior derivative of (1.7), we get the Maurer-Cartan equations of $U(n)$:

$$d\omega_{A\bar{B}} = \omega_{A\bar{C}} \wedge \omega_{C\bar{B}}. \tag{1.9}$$

An element $\mathbf{C}^k$ of $G(k,n)$ can be defined by the multivector $Z_1 \wedge \cdots \wedge Z_k \neq 0$, defined up to a factor. The vectors Z_α and their orthogonal vectors Z_i are defined up to a transformation of $U(k)$ and $U(n-k)$, respectively, so that $G(k,n)$ has a G-structure, with $G = U(k) \times U(n-k)$. In particular, the form

$$ds^2 = \omega_{\alpha\bar{i}}\omega_{\bar{\alpha}i} \tag{1.10}$$

is a positive Hermitian form on $G(k,n)$, and defines an Hermitian metric. Its Kähler form is

$$\Omega = \frac{\sqrt{-1}}{2\pi}\omega_{\alpha\bar{i}} \wedge \omega_{\bar{\alpha}i}. \tag{1.11}$$

By using (1.9) it can be immediately verified that Ω is closed, so that the metric ds^2 is Kählerian.

§2. Harmonic maps of surfaces

Let M be an oriented Riemannian surface and let $f: M \to G(k,n)$ be a smooth map. Denote the Riemannian metric on M by $ds_M^2 = \varphi \cdot \bar{\varphi}$, where φ is a complex valued one-form (φ is defined up to a complex factor of absolute value one). Choose a field of unitary frames Z_A (as in §1) so that Z_α span $f(x)$, $x \in M$. Then

$$f^*\omega_{\alpha\bar{i}} = a_{\alpha\bar{i}}\varphi + b_{\alpha\bar{i}}\bar{\varphi}. \tag{2.1}$$

The *energy* of the map f is by definition,

$$E(f) = \int_M \operatorname{tr}(f^*ds^2)\, d\,\mathrm{vol}$$

where ds^2 is the metric on $G(k,n)$ and the trace is taken with respect to the metric on M. By (2.1) and (1.10) this becomes

$$E(f) = \int_M \sum_{a,i}\left(|a_{\alpha\bar{i}}|^2 + |b_{\alpha\bar{i}}|^2\right)\frac{\sqrt{-1}}{2}\varphi \wedge \bar{\varphi}. \tag{2.2}$$

A map which is a critical point of the energy functional is called *harmonic*.

The pullback of the Kähler form Ω by the map f defines an integral cohomology class $[f^*\Omega] \in H^2(M; \mathbf{Z})$. Evaluating this class on the fundamental homology class of M yields an integer $[f^*\Omega]([M])$ called the *degree* of f. The degree of f can be computed from (1.11) and (2.1) as follows:

$$\begin{aligned} \deg f &= \int_M f^*\Omega = \frac{\sqrt{-1}}{2\pi}\int_M \sum_{\alpha,i} (a_{\alpha i}\varphi + b_{\alpha i}\bar\varphi) \wedge (a_{\bar\alpha \bar i}\bar\varphi + b_{\bar\alpha \bar i}\varphi) \\ &= \int_M \sum_{\alpha,i} \left(|a_{\alpha i}|^2 - |b_{\alpha i}|^2\right)\frac{\sqrt{-1}}{2\pi}\varphi \wedge \bar\varphi. \end{aligned} \tag{2.3}$$

A map $f: M \to G(k,n)$ induces over M a vector bundle V with fibers $f(x)$, $x \in M$. In terms of our frames Z_A a local section Z of V can be written

$$Z = \xi^\alpha Z_\alpha \tag{2.4}$$

where the ξ^α are complex-valued functions on M. The bundle V is a subbundle of the trivial rank n bundle $M \times \mathbf{C}^n$. As such it inherits (by restriction) a hermitian connection ∇. We denote the $(0,1)$ part of the connection by $\nabla^{(0,1)}$. The $(0,2)$ part of the curvature of ∇ vanishes because M is a Riemann surface. Thus

$$(\nabla^{(0,1)})^2 = 0. \tag{2.5}$$

But (2.5) is the integrability condition for a holomorphic structure on V which satisfies

$$\nabla^{(0,1)} = \bar\partial.$$

Thus by the Newlander–Nirenberg theorem V has the structure of a holomorphic bundle. Clearly the same argument shows that any subbundle of the trivial bundle $M \times \mathbf{C}^n$ is equipped with a natural holomorphic structure. In the sequel when such bundles arise we will consider them equipped with this holomorphic structure. In particular the bundle W, whose fiber at $x \in M$ is the $(n-k)$ plane $f(x)^\perp$ orthogonal to $f(x)$ in $\mathbf{C}^n$, has a holomorphic structure. A local section Z of W can be written using our frame Z_A as $Z = \eta^i Z_i$ where the η^i are complex valued functions on M.

Let $T^{(1,0)}$ (resp. $T^{(0,1)}$) be the cotangent bundle on M of type $(1,0)$ (resp. type $(0,1)$). Its sections can be written as $f\varphi$ (resp. $f\bar\varphi$) where f is a function on M. Define the mappings

$$\partial : \mathrm{V} \to \mathrm{W} \otimes T^{(1,0)} \qquad \bar\partial : \mathrm{V} \to \mathrm{W} \otimes T^{(0,1)} \tag{2.6}$$

by

$$\partial(\xi^\alpha Z_\alpha) = \xi^\alpha a_{\alpha i} Z_i \otimes \varphi \qquad \bar\partial(\xi^\alpha Z_\alpha) = \xi^\alpha b_{\alpha i} Z_i \otimes \bar\varphi.$$

In [5] the following theorem is proved.

Theorem 2.1. *Let $f: M \to G(k, n)$ be a smooth map, then the following conditions are equivalent*:

(i) *f is harmonic*,

(ii) *∂ is a holomorphic bundle map*,

(iii) *$\bar{\partial}$ is an antiholomorphic bundle map.*

Suppose now that f is a harmonic map. By Theorem 2.1 the image of ∂ is itself a holomorphic bundle which we denote $V_1 \otimes T^{(1,0)}$. Although ∂ is not a well-defined bundle map from V to V_1 it induces a well-defined projective bundle map from the projectivization [V] of V to the projectivization $[V_1]$ of V_1. This map and its analogue for $\bar{\partial}$ are called the *fundamental collineations* of f. By abuse of notation we will denote these projective bundle maps by ∂ and $\bar{\partial}$. $[V_1]$ is a projective subbundle of $M \times \mathbf{P}^{n-1}(\mathbf{C})$. Denote its projective rank by $k_1 - 1$. We define the ∂ transform of f

$$\partial f: M \to G(k_1, n) \tag{2.7}$$

by $\partial f(x) = [V_1]_x (= \partial[f(x)])$, $x \in M$. The $\bar{\partial}$ transform, $\bar{\partial} f$, is defined similarly.

Theorem 2.2. *Let $f: M \to G(k, n)$ be a harmonic map. Then*

(i) *The ∂ and $\bar{\partial}$ transforms, ∂f and $\bar{\partial} f$, of f are harmonic maps.*

(ii) *If $k_1 = k$, the map $\bar{\partial}(\partial f)$ is f itself. In fact if Z_i spans $f(x)$, Z_σ, $\sigma = k + 1, \cdots, 2k$, spans $\partial f(x)$ and A is the matrix representation of ∂ with respect to these frames then $-{}^t\bar{A}$ is the matrix representation of $\bar{\partial}$ with respect to the same frames. In this sense ∂ and $\bar{\partial}$ are "inverse" transforms.*

Proof. See [5].

Repeating the constructions of Theorem 2.2 we get two sequences of harmonic maps

$$f_0(=f) \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \to \cdots$$

$$f_0 \xrightarrow{\bar{\partial}} f_{-1} \xrightarrow{\bar{\partial}} f_{-2} \to \cdots$$

whose image spaces are connected by fundamental collineations. Such sequences are called *harmonic sequences*. When the k_p's are equal we can combine the sequences into one:

$$\cdots f_{-2} \underset{\bar{\partial}}{\overset{\partial}{\rightleftarrows}} f_{-1} \underset{\bar{\partial}}{\overset{\partial}{\rightleftarrows}} f_0 \underset{\bar{\partial}}{\overset{\partial}{\rightleftarrows}} f_1 \cdots .$$

By construction two consecutive spaces $[f_p(x)]$ and $[f_{p+1}(x)]$, $x \in M$, of a harmonic sequence are orthogonal. Moreover, denoting by V_p the bundle over M induced by f_p, for each p there is a holomorphic bundle map

$$V_p \xrightarrow{\partial_p} V_{p+1} \otimes T^{(1,0)} \tag{2.8}$$

and an antiholomorphic bundle map

$$(2.9) \qquad V_p \xrightarrow{\bar{\partial}_p} V_{p-1} \otimes T^{(0,1)}.$$

Example. Let $f: M \to G(1, n+1) = \mathbf{C}P^n$ be a holomorphic map. Classically there is associated to f a unitary framing $\{Z_0, \cdots, Z_n\}$ of $\mathbf{C}^n$ such that $Z_0 \cdots Z_k$ span the kth osculating space of f. This framing is called the *Frenet frame* of the curve. Analytically each element of the Frenet frame satisfies

$$(2.10) \qquad dZ_p = -\bar{a}_{p-1}\bar{\varphi}Z_{p-1} + \omega_{p\bar{p}}Z_p + a_p\varphi Z_{p+1}.$$

Moreover each Z_p defines a line bundle over M, or, what is the same, a map $M \to \mathbf{C}P^n$. These line bundles (or maps) form a harmonic sequence. The ∂ fundamental collineations are given by the scalars a_p, the $\bar{\partial}$ fundamental collineations by the scalars $\bar{a}_{p-1}$. This sequence has length at most $n+1$ and ends in an antiholomorphic curve $M \to \mathbf{C}P^n$, the polar curve of f. For more details see [11].

In the remainder of this paper we will adopt the convention that capital Roman letters (eg. L, V, W, etc.) will denote rank l complex subbundles of the trivial bundle $M \times \mathbf{C}^n$ and their associated maps $M \to G(l, n)$. We will freely identify these two corresponding objects.

§3. Harmonic sequences

In this section we discuss some of the geometry of harmonic sequences over a Riemann surface and, in particular, over the two-sphere and the torus. We begin with the simplest case, the harmonic sequences of maps $M \to G(1, n) = \mathbf{C}P^{n-1}$. Let

$$(3.1) \qquad L_0 \xrightarrow{\partial_0} L_1 \xrightarrow{\partial_1} L_2 \to \cdots \xrightarrow{\partial_{s-1}} L_s \xrightarrow{\partial_s} \cdots$$

be a harmonic sequence where each L_p is a map $M \to G(1, n)$ or, what is the same, a rank one vector bundle (a line bundle) over M. We have seen that the map ∂_p is a holomorphic bundle map:

$$(3.2) \qquad L_p \xrightarrow{\partial_p} L_{p+1} \otimes T^{(1,0)}$$

where $T^{(1,0)}$ is the holomorphic cotangent bundle of M. ∂_p has only isolated zeroes. The number of zeroes of ∂_p, counted according to multiplicity, is called the ramification index of ∂_p and will be denoted $r(\partial_p)$. The following formula is well known [8]

$$(3.3a) \qquad c_1\left(L_{p+1} \otimes T^{(1,0)}\right) = c_1\left(L_p\right) + r\left(\partial_p\right)$$

or

$$c_1(\mathrm{L}_{p+1}) = c_1(\mathrm{L}_p) + r(\partial_p) - (2g-2) \tag{3.3b}$$

where c_1 is the Chern number of the line bundle and g is the genus of M.

On the other hand the Chern class of the line bundle L_p can be computed as follows. Choose a unitary framing $\{Z_1, \cdots, Z_n\}$ of $\mathbf{C}^n$ adapted so that span $\{Z_{p-1}\} = \mathrm{L}_{p-1}$, span $\{Z_p\} = \mathrm{L}_p$ and span $\{Z_{p+1}\} = \mathrm{L}_{p+1}$. (To choose such a frame requires the additional assumption that the map L_p is conformal. However, the result to follow does not depend on this assumption. When we discuss the general case we will not make this assumption.) (3.1) and harmonicity give

$$\begin{aligned} dZ_{p-1} &= \sum_{\sigma=1}^{p-2} (\ \ )\bar{\varphi} Z_\sigma + \omega_{p-1,\overline{p-1}} Z_{p-1} \\ &\quad + a_{p-1}\varphi Z_p + (\ \ )\bar{\varphi} Z_{p+1} + \sum_{\tau=p+2}^{n} (\ \ )\bar{\varphi} Z_\tau \\ dZ_p &= -\bar{a}_{p-1}\bar{\varphi} Z_{p-1} + \omega_{p\bar{p}} Z_p + a_p \varphi Z_{p+1} \\ dZ_{p+1} &= \sum_{\sigma=1}^{p-2} (\ \ )\varphi Z_\sigma + (\ \ )\varphi Z_{p-1} - \bar{a}_p \bar{\varphi} Z_p \\ &\quad + \omega_{p+1,\overline{p+1}} Z_{p+1} + \sum_{\tau=p+2}^{n} (\ \ )\varphi Z_\tau \end{aligned} \tag{3.4}$$

where $\bar{a}_{p-1}$, and a_p are functions representing the $\bar{\partial}$ and ∂ fundamental collineation of $\mathrm{L}_p \cdot \omega_{p\bar{p}}$ is the connection 1-form of the bundle L_p. The curvature of L_p can then be computed from the Maurer-Cartan equations of $U(n)$:

$$\begin{aligned} d\omega_{p\bar{p}} &= (-\bar{a}_{p-1}\bar{\varphi}) \wedge (a_{p-1}\varphi) + (a_p\varphi) \wedge (-\bar{a}_p\bar{\varphi}) \\ &= \left(|a_{p-1}|^2 - |a_p|^2\right)\varphi \wedge \bar{\varphi}. \end{aligned} \tag{3.5}$$

Thus

$$c_1(\mathrm{L}_p) = \frac{i}{2\pi}\int_M \left(|a_{p-1}|^2 - |a_p|^2\right)\varphi \wedge \bar{\varphi}. \tag{3.6}$$

Note that from (3.4) it is immediate that the only (0,1) form among the coframing of dZ_{p+1} is $\omega_{p+1,\bar{p}} = -\bar{a}_p\bar{\varphi}$. Applying the above reasoning to L_{p+1} we get

$$c_1(\mathrm{L}_{p+1}) = \frac{i}{2\pi}\int_M \left(|a_p|^2 - |a_{p+1}|^2\right)\varphi \wedge \bar{\varphi}, \tag{3.7}$$

for some function a_{p+1} representing the ∂ fundamental collineation of L_{p+1}. It follows that

$$\begin{aligned}\sum_{p=0}^{s} c_1(L_p) &= \frac{i}{2\pi}\int_M \left(|a_{-1}|^2 - |a_s|^2\right)\varphi\wedge\bar{\varphi} \\ &\leqslant \frac{i}{2\pi}\int_M \left(|a_{-1}|^2\right)\varphi\wedge\bar{\varphi}.\end{aligned} \tag{3.8}$$

By (3.3),

$$\begin{aligned}\sum_{p=0}^{s} c_1(L_p) &= \sum_{p=0}^{s}\left\{c_1(L_0) + \sum_{q=0}^{p-1} r(\partial_q) - p(2g-2)\right\} \\ &= (s+1)c_1(L_0) + \sum_{p=1}^{s}\sum_{q=0}^{p-1} r(\partial_q) - (2g-2)\frac{s(s+1)}{2}.\end{aligned} \tag{3.9}$$

Theorem 3.1. *Let* (3.1) *be a harmonic sequence for the map* $L_0: M \to G(1,n)$ *where M has genus g and the ramification index of* ∂_p *is* $r(\partial_p)$. *Then for any s*

$$\begin{aligned}(s+1)c_1(L_0) + \sum_{p=1}^{s}\sum_{q=0}^{p-1} r(\partial_q) - (g-1)s(s+1) \\ < \frac{1}{\pi}\cdot \text{energy}(L_0).\end{aligned} \tag{3.10}$$

Proof. The energy of L_0 is $\frac{i}{2}\int_M(|a_{-1}|^2 + |a_0|^2)\varphi\wedge\bar{\varphi}$. Moreover $|a_0| = 0$ if and only if L_0 is antiholomorphic, (equivalently $a_0 = 0$ if and only if $\partial_0 = 0$).

Corollary 3.2. *When* $g = 0$ *the harmonic sequence* (3.1) *must terminate.*

Suppose $g = 0$ and that L_t is the last element of the harmonic sequence (3.1). Then $L_t: M \to G(1,n)$ is an antiholomorphic map. The construction of the harmonic sequence of a holomorphic or antiholomorphic curve in $\mathbf{C}P^{n-1}$ is precisely the classical construction of the curve's Frenet frame. Hence L_0 is an element of the Frenet frame of L_t and we have proved the result of Din-Zakrzewski [6]. (For this version of this theorem see [11].)

Applying to the above considerations to the harmonic sequence

$$\xleftarrow{\bar\partial} L_{-s} \xleftarrow{\bar\partial} \cdots \xleftarrow{\bar\partial} L_{-1} \xleftarrow{\bar\partial} L_0. \tag{3.1a}$$

It follows that

$$\begin{aligned}-\frac{1}{\pi}\text{ energy }(L_0) &< -\frac{i}{2\pi}\int_M |a_0|^2\varphi\wedge\bar{\varphi} \\ &\leqslant (s+1)c_1(L_0) - \sum_{p=1}^{s}\sum_{q=0}^{p-1} r(\bar{\partial}_{-q}) \\ &\quad + (g-1)s(s+1).\end{aligned} \tag{3.11}$$

Proposition 3.3. *When $g = 1$ and* $\deg \mathrm{L}_0 < 0$ *then the harmonic sequence* (3.1) *must terminate. When $g = 1$ and* $\deg \mathrm{L}_0 > 0$ *or when $g = 0$ then the harmonic sequence* (3.1a) *must terminate.*

Proof. $\deg \mathrm{L}_0$ is the degree of the map $\mathrm{L}_0 :: M \to \mathbf{C}P^{n-1}$. As $\deg \mathrm{L}_0 = -c_1(\mathrm{L}_0)$ the first statement follows from (3.10) and the second statement follows from (3.11).

Thus when $g = 1$ and $\deg \mathrm{L}_0 \neq 0$ there is a terminal element to the left or the right of the harmonic sequence

$$(3.12) \qquad \cdots \leftarrow \mathrm{L}_{-1} \overset{\bar\partial}{\leftarrow} \mathrm{L}_0 \overset{\partial}{\to} \mathrm{L}_1 \overset{\partial}{\to} \cdots .$$

Suppose, without loss of generality, that L_{-t}, $t > 0$, is the terminal element. Then $\mathrm{L}_{-t} : M \to \mathbf{C}P^{n-1}$ is a holomorphic curve and the harmonic map L_0 occurs as an element of the Frenet frame of L_{-t}. This result is due to Eells and Wood [7]. Their proof is different from the one given here.

We remark that if a harmonic sequence (3.12) terminates in one direction then it must terminate in the other direction and it contains at most n elements. This is an immediate consequence of the construction of the Frenet frame of a holomorphic or antiholomorphic curve in $\mathbf{C}P^{n-1}$.

We now turn to the general case of a harmonic sequence

$$(3.13) \qquad \mathrm{V}_0 \overset{\partial_0}{\to} \mathrm{V}_1 \overset{\partial_1}{\to} \cdots \overset{\partial_{s-1}}{\to} \mathrm{V}_s \overset{\partial_s}{\to} \cdots$$

$$(3.13\mathrm{a}) \qquad \overset{\bar\partial_{-s}}{\leftarrow} \mathrm{V}_{-s} \cdots \overset{\bar\partial_{-1}}{\leftarrow} \mathrm{V}_{-1} \overset{\bar\partial_0}{\leftarrow} \mathrm{V}_0$$

where each V_p is a map $M \to G(k, n)$ or a rank k vector bundle over M. We would like to find conditions under which one of the ∂ or $\bar\partial$ fundamental collineations degenerates, that is, has rank less than k.

We can change the sequence (3.13) into a sequence of line bundles by taking the kth exterior power of each bundle

$$(3.14) \qquad \Lambda^k \mathrm{V}_0 \overset{\det \partial_0}{\to} \Lambda^k \mathrm{V}_1 \overset{\det \partial_1}{\to} \Lambda^k \mathrm{V}_2 \to \cdots \overset{\det \partial_{s-1}}{\to} \Lambda^k \mathrm{V}_s \to \cdots .$$

In (3.14) the map $\det \partial_p$ is a holomorphic bundle map

$$(3.15) \qquad \Lambda^k \mathrm{V}_p \overset{\det \partial_p}{\to} \Lambda^k \mathrm{V}_{p+1} \otimes \left(T^{(1,0)}\right)^k .$$

Formula (3.3) can be written

$$(3.16) \qquad c_1\left(\Lambda^k \mathrm{V}_{p+1}\right) = c_1\left(\Lambda^k \mathrm{V}_p\right) + r\left(\det \partial_p\right) - k(2g - 2).$$

We remark that (3.16) is a "Plucker formula" for harmonic maps $M \to G(k, n)$.

The Chern number $c_1(\Lambda^k V_p)$ can be computed as follows: First, it is an elementary and basic fact of k-plane bundles that if the connection form of V_p is given by $(\pi_{\alpha\beta})$, $1 \leqslant \alpha, \beta \leqslant k$, then the connection form of $\Lambda^k V_p$ is given by $\operatorname{tr}(\pi_{\alpha\beta}) = \sum_{\alpha=1}^k \pi_{\alpha\alpha}$. Thus

$$c_1(V_p) = c_1(\Lambda^k V_p). \tag{3.17}$$

To compute $c_1(V_p)$ we adapt a unitary framing $\{Z_1 \cdots Z_n\}$ of $\mathbf{C}^n$ to the map V_p as in §1, that is the vectors Z_α span V_p, where $1 \leqslant \alpha, \beta \leqslant k$. Then we have

$$d\begin{pmatrix} Z_1 \\ \vdots \\ Z_k \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} \pi_p & A_p\varphi + B_p\bar{\varphi} \\ -{}^t\bar{A}_p\bar{\varphi} - {}^t\bar{B}_p\varphi & \cdots \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_k \\ \vdots \\ Z_n \end{pmatrix},$$

where π_p is a $k \times k$ skew-hermitian matrix of 1-forms and A_p and B_p are $k \times (n-k)$ matrices of functions. In fact in the notation of §2

$$\pi_p = (\omega_{\alpha\bar{\beta}}),$$
$$A_p\varphi + B_p\bar{\varphi} = (\omega_{\alpha\bar{i}}),$$
$$A_p = (a_{\alpha\bar{i}}), \qquad B_p = (b_{\alpha\bar{i}}).$$

π_p is the connection 1-form of V_p. By the Maurer-Cartan equations, the curvature of V_p is

$$d\pi_p - \pi_p \wedge \pi_p = \left(-A_p{}^t\bar{A}_p + B_p{}^t\bar{B}_p\right)\varphi \wedge \bar{\varphi}.$$

Thus

$$\begin{aligned} c_1(V_p) &= \frac{i}{2\pi} \int \operatorname{tr}(d\pi_p - \pi_p \wedge \pi_p) \\ &= \frac{i}{2\pi} \int \left[\operatorname{tr}\left(B_p{}^t\bar{B}_p\right) - \operatorname{tr}\left(A_p{}^t\bar{A}_p\right)\right] \varphi \wedge \bar{\varphi}. \end{aligned} \tag{3.18}$$

Recall that the energy of the map $M \to G(k,n)$ determined by V_p is given by

$$\begin{aligned} E(V_p) &= \frac{i}{2} \int \left(\sum_{\alpha,j} |a_{\alpha j}|^2 + \sum_{\alpha,j} |b_{\alpha j}|^2 \right) \varphi \wedge \bar{\varphi} \\ &= \frac{i}{2} \int \left(\operatorname{tr}\left(A_p{}^t\bar{A}_p\right) + \operatorname{tr}\left(B_p{}^t\bar{B}_p\right)\right)\varphi \wedge \bar{\varphi}. \end{aligned} \tag{3.19}$$

We define the holomorphic or ∂ energy of V_p by

$$E(\partial_p) = \frac{i}{2} \int \operatorname{tr}\left(A_p{}^t\bar{A}_p\right). \tag{3.20}$$

Similarly the antiholomorphic or $\bar\partial$ energy of V_p is by definition

$$E(\bar\partial_p) = \frac{i}{2}\int \operatorname{tr}\left(B_p {}^t\bar{B}_p\right)\varphi \wedge \bar\varphi. \tag{3.21}$$

Thus

$$E(V_p) = E(\partial_p) + E(\bar\partial_p), \tag{3.22}$$

and

$$c_1(V_p) = \frac{1}{\pi}E(\bar\partial_p) - \frac{1}{\pi}E(\partial_p). \tag{3.23}$$

Now consider the ∂-transform of V_p, namely V_{p+1}. We have, by the above argument

$$c_1(V_{p+1}) = \frac{1}{\pi}E(\bar\partial_{p+1}) - \frac{1}{\pi}E(\partial_{p+1}), \tag{3.24}$$

where $\bar\partial_{p+1}$ and ∂_{p+1} are the $\bar\partial$ and ∂ transforms, respectively, of V_{p+1}. Recall Theorem 2.2(ii). It is an immediate consequence of this result that

$$E(\bar\partial_{p+1}) = E(\partial_p). \tag{3.25}$$

Thus

$$c_1(V_{p+1}) = \frac{1}{\pi}E(\partial_p) - \frac{1}{\pi}E(\partial_{p+1}). \tag{3.26}$$

Hence we have

$$\sum_{p=0}^{s} c_1(\Lambda^k V_p) = \frac{1}{\pi}E(\bar\partial_0) - \frac{1}{\pi}E(\partial_s) < \frac{1}{\pi}E(V_0), \tag{3.27}$$

Theorem 3.4. *If* (3.13) (*resp.* 3.13a) *is a harmonic sequence for the map* $V_0 : M \to G(k, n)$ *where M has genus g and if none of the fundamental collineations of* (3.13) (*resp.* 3.13a) *degenerates then for any s*

$$\begin{aligned}(s+1)c_1(V_0) + \sum_{p=1}^{s}\sum_{q=0}^{p-1} r(\det\partial_q) - k(g-1)s(s+1) \\ = \frac{1}{\pi}\left(E(\bar\partial_0) - E(\partial_s)\right) < \frac{1}{\pi}E(V_0)\end{aligned} \tag{3.28}$$

$$\left(\text{resp., } -\frac{1}{\pi}E(V_0) < -\frac{1}{\pi}\left(E(\partial_0) - E(\bar\partial_s)\right)\right. \tag{3.29}$$
$$\left. = (s+1)c_1(V_0) - \sum_{j=1}^{s}\sum_{q=0}^{p-1} r\left(\det\bar\partial_{-q}\right) + k(g-1)s(s+1)\right).$$

Proof. To prove (3.28) combine (3.16) and (3.27). To prove (3.29) modify the above arguments to the harmonic sequence (3.13a).

Remark. By (2.3), (3.17) and (3.18)

$$c_1(\mathrm{V}_p) = c_1(\Lambda^k \mathrm{V}_p) = -\deg(\mathrm{V}_p) \tag{3.30}$$

where $\deg(\mathrm{V}_p)$ is the degree of the map $M \to G(k, n)$ induced by V_p. Consequently the inequalities (3.28) and (3.29) relate the degree of a harmonic map $\mathrm{V}_p: M \to G(k, n)$ to its energy.

Corollary 3.5. *If $g = 0$ or if $g = 1$ and $c_1(\mathrm{V}_0) > 0$ (resp. $c_1(\mathrm{V}_0) < 0$) then the harmonic sequence* (3.13) *(resp.* (3.13a)) *must have a degenerate ∂ (resp. $\bar{\partial}$) fundamental collineation.*

Using (3.30) we have

Theorem 3.6. *Suppose that $\mathrm{V}_0: M \to G(k, n)$ is a harmonic map and that M has genus g. If $g = 0$ or if $g = 1$ and* $\deg \mathrm{V}_0 \neq 0$ *then the harmonic sequence generated by by V_0 has a degenerate fundamental collineation.*

Remark. In fact we have proved more than is stated in Theorem 3.6. If M has genus 1 and V_0 is a harmonic map $M \to G(k, n)$ then the harmonic sequence generated by V_0 must have a degenerate fundamental collineation if any of the ∂ and $\bar{\partial}$ transforms of V_0 have nonzero degree. This means that the only harmonic sequences over the torus that we cannot prove have a degenerate fundamental collineation are those such that every map in the sequence has degree zero. Note that by (3.16) every fundamental collineation of such a sequence has ramification index zero. In $\mathbf{C}P^n$ every nonsuperminimal minimal torus belongs to such a sequence. (For details see [11].) In particular, the Clifford torus in $\mathbf{C}P^2$ generates a cyclic harmonic sequence consisting of three maps all of degree zero.

Let $f: S^2 \to G(k, n)$ be a harmonic map. Denote the pth ∂ (resp., $\bar{\partial}$) transform of f by f_p, (resp. f_{-p}) so that $f_{p+1} = \partial f_p$, $p = 0, 1, 2, \cdots$ (resp., $f_{-p-1} = \bar{\partial} f_{-p}$, $p = 0, 1, 2, \cdots$). By applying Corollary 3.5 repeatedly we can associate two sequences of pairs of integers to f as follows: Let l_σ be the length of the ∂ harmonic sequence from f to the σth map of the sequence with a degenerate ∂ fundamental collineation. In other words, f_{l_1} is the first map in the ∂ harmonic sequence of f to have a degenerate ∂ fundamental collineation, f_{l_2} is the second such map and so on. Set $k_\sigma = \operatorname{rank} \partial_{l_\sigma}$, so that if $\partial_{l_\sigma} \neq 0$ the ∂ transform of f_{l_σ}, $f_{l_\sigma+1}$, is a map $S^2 \to G(k_\sigma, n)$. Clearly $l_\sigma < l_{\sigma+1}$, $k_{\sigma+1} < k_\sigma < k$ and $1 \leqslant \sigma \leqslant k$. The sequence of pairs

$$\{(l_1, k_1), (l_2, k_2), \cdots, (l_\tau, 0)\} \tag{3.31}$$

is uniquely associated to f. Note that the map $f_{l_\tau}: S^2 \to G(k_{\tau-1}, n)$ is antiholomorphic. The same consideration applied to the $\bar{\partial}$ harmonic sequence of f leads to a similar sequence again uniquely associated to f.

Question. Is there a relation between these sequences and Uhlenbeck's uniton number of f as defined in [10]? If so, what is it?

It is also possible to associate a sequence similar to (3.31) to a harmonic map $f: M \to G(k, n)$ where M has genus one. We leave the details to the reader.

§4. Turning and harmonic maps of the two-sphere

In this section we study the degenerate harmonic maps, that is, the harmonic maps one of whose fundamental collineations is degenerate. For use later we order the Grassmann manifolds as follows. We say $G(l, n)$ is "smaller" than $G(k, n)$ if $l < k$.

Let V_0 be a harmonic map $M \to G(k, n)$ regarded as a rank k bundle. Suppose that the ∂ fundamental collineation is singular of rank l where $0 < l < k$. Let W_0 denote the harmonic map $M \to G(l, n)$ determined by the image of ∂. Then we have

$$\mathrm{V}_0 \xrightarrow{\partial} \mathrm{W}_0. \tag{4.1}$$

The vector bundle V_0 decomposes as the orthogonal direct sum of the rank $(k - l)$ bundle $\ker\partial$ and the rank l bundle $\mathrm{W}_{-1} = (\ker\partial)^{\perp}$. W_{-1} describes a harmonic map $M \to G(l, n)$. In fact W_{-1} is the $\bar{\partial}$-transform of W_0. Let W_{-2} denote the $\bar{\partial}$-transform of W_{-1}. Define the bundle V^1 by

$$\mathrm{V}^1 = \mathrm{span}\{\mathrm{W}_{-2}, \ker\partial\}.$$

Note that in general W_{-2} and $\ker\partial$ are not orthogonal. However we have

Lemma 4.1. *V^1 is a vector bundle (i.e. V^1 has constant rank).*

To prove the lemma we need the following proposition which will be used implicitly in §5

Proposition 4.2. (1) *The bundle* $\ker\partial$ *is a holomorphic subbundle of* V_0.

(2) *The bundle* W_{-1} *is an antiholomorphic subbundle of* V_0.

Proof. Because $\ker\partial \oplus \mathrm{W}_{-1} = \mathrm{V}_0$ the two statements in the proposition are equivalent. We will prove the first statement. Choose a unitary framing $\{Z_1, \cdots, Z_n\}$ of $\mathbf{C}^n$ adapted so that Z_σ span $\ker\partial$ and Z_r span W_{-1}, where the indices have the ranges

$$1 \leqslant \sigma, \tau \leqslant k - l, \quad k - l + 1 \leqslant r, s \leqslant k, \quad k + 1 \leqslant i, j \leqslant n.$$

Then $a_{\sigma\bar{i}} = 0$ and the matrix $(a_{r\bar{i}})$ has rank l. Since V_0 is harmonic, it follows from Theorem 2.1 in [5] that

$$\omega_{\sigma\bar{r}} a_{r\bar{i}} \equiv 0 \mod \varphi.$$

This implies that

$$\omega_{\sigma\bar{r}} \equiv 0 \mod \varphi.$$

Hence

$$dZ_\sigma \equiv 0 \mod Z_\tau, Z_i, \varphi.$$

Proof of the lemma. Let $\bar{\partial}_{V_0}$ denote the $\bar{\partial}$ fundamental collineation of V_0 and $\bar{\partial}_{V_0}(W_{-1})$ denote the image of W_{-1} under $\bar{\partial}_{V_0}$. Then

$$V^1 = \bar{\partial}_{V_0}(W_{-1}) \oplus \ker \partial.$$

Since W_{-1} is an antiholomorphic subbundle, the map $\bar{\partial}_{V_0}$ restricted to W_{-1} can be regarded as an antiholomorphic map. Thus $\bar{\partial}_{V_0}(W_{-1})$ has constant rank.

Theorem 4.3. *The bundle* V^1 *gives a harmonic map* $M \to G(k_1, n)$ *where* $k_1 \leqslant k$. *If* $k_1 = k$ *then the* ∂*-transform of* V^1 *is* W_{-1} *and*

$$V^1 \xrightarrow{\partial} W_{-1} \underset{\bar{\partial}}{\overset{\partial}{\rightleftarrows}} W_0 \tag{4.2}$$

is a harmonic sequence. If $k_1 < k$ *then the* ∂*-transform of* V^1 *lies inside* W_{-1}.

Proof. Left to the reader

The construction of (4.2) is called *turning*. This construction generalizes the construction of the same name described in [5].

Remarks. (1) If $k_1 \geqslant l$ then "generically" the ∂-transform of V^1 is W_{-1} and similarly if $k_1 \leqslant l$ the $\bar{\partial}$-transform of W_{-1} is "generically" V^1. For this reason we call a turning *regular* if

(a) The ∂-transform of V^1 is W_{-1} when $k_1 \geqslant l$.

(b) The $\bar{\partial}$-transform of W_{-1} is V^1 when $k_1 \leqslant l$.

Theorem 4.3 says that if $k_1 = k$ then the turning is regular.

(2) It is interesting (and important) to determine how to reverse the operation of turning , that is, how to recover the map V_0 from the map V^1. V^1 is a holomorphic rank k_1 bundle over M where by construction $k_1 \geqslant (k - l)$. Choose an antiholomorphic rank $(k - l)$ subbundle B of V^1. Then the bundle $B \oplus W_{-1}$ has rank k and its ∂-transform is W_0. For appropriate choice of B this bundle will be V_0. This operation is called *returning*. Note that when the turning is regular, the returning depends on V^1 and the choice of B alone (because in this case W_{-1} is determined by V^1). Whereas when the turning is not regular the returning depends on V^1, the choice of B, and W_{-1}.

It is clear that the construction of turning can be iterated to construct the sequence

$$V^s \xrightarrow{\partial} W_{-s} \underset{\bar{\partial}}{\leftarrow} W_{-s+1} \underset{\bar{\partial}}{\leftarrow} \cdots \underset{\bar{\partial}}{\leftarrow} W_0. \tag{4.3}$$

Suppose that V^s is a rank k_s bundle where $k_s < k$ and that each V^σ, $\sigma < s$, constructed before V^s is a rank k bundle. If the final turning is regular then V_0 can be constructed from V^s by a sequence of returnings. If the final turning is not regular then V_0 can be constructed from V^s and W_{-s} through a sequence of returnings. In both cases note that the harmonic map $V_0: M \to G(k, n)$ can be constructed, by returnings, from harmonic maps of M into smaller Grassmann manifolds. In the nongeneric (that is, the not regular) case more data (namely, W_{-s}) is required to reconstruct V_0.

Theorem 4.4. *Let V_0 be a harmonic map $M \to G(k, n)$. Let W_0 denote the ∂ transform of V_0 and suppose that W_0 is a bundle of* rank l, $l < k$. *If M has genus zero or if M has genus one and the map W_0 has positive degree then V_0 can be constructed by returnings from maps of M into smaller Grassmann manifolds.*

Proof. The hypothesis on M insure that the $\bar\partial$ harmonic sequence of W_0 must contain a singular $\bar\partial$ fundamental collineation. This in turn insures that some V^s has rank strictly less than k.

By combining Theorem 3.6 and Theorem 4.4 we have

Theorem 4.5. *If M has genus zero then any harmonic map $M \to G(k, n)$ can be constructed from either*

(1) *a holomorphic or antiholomorphic curve $M \to G(k, n)$ using the ∂ or $\bar\partial$ transforms, or*

(2) *one, or possibly two, harmonic maps $M \to G(k_i, n)$ $i = 1, 2$, where $k_i < k$; using the ∂ and $\bar\partial$ transforms and using returnings.*

Now by induction, we have

Corollary 4.6. *If M has genus zero then any harmonic map $M \to G(k, n)$ can be constructed from holomorphic or antiholomorphic curves $M \to G(l, n)$, $1 \leqslant l \leqslant k$, using the ∂ and $\bar\partial$ transforms and returnings.*

We remark that turning and returning can be formulated for the case of a harmonic map V_0 with degenerate $\bar\partial$ fundamental collineation. We leave this to the reader.

§5 Extending and harmonic maps of the two-sphere and the torus

We begin by describing another technique which, like returning, reconstructs a harmonic map from its degenerate ∂-transform (or $\bar\partial$-transform).

Using the same notation as in §4 we let V_0 denote a harmonic map $M \to G(k, n)$ with degenerate ∂ fundamental collineation and W_0 denote the ∂ transform of V_0, so that W_0 is a harmonic map $M \to G(l, n)$, $0 < l < k$. By a result of [5] the map $W_0^\perp$ determined by the space orthogonal to W_0 is also harmonic. $W_0^\perp$ is a holomorphic vector bundle over M. Let W_{-1} denote the $\bar\partial$ transform of W_0. W_{-1} is a rank l antiholomorphic subbundle of $W_0^\perp$. Now choose an antiholomorphic rank k subbundle V of $W_0^\perp$ satisfying the condition that W_{-1} is an antiholomorphic subbundle of V. A straightforward local computation shows that the map $M \to G(k, n)$ defined by V is harmonic. Moreover, for appropriate choice of V we have $V = V_0$. This operation is called *extending*. (The bundle V "extends" the bundle W_{-1}.)

Suppose V_0 has a degenerate $\bar\partial$ fundamental collineation and U_0 denotes its $\bar\partial$ transform. Let U_1 denote the ∂ transform of U_0. Then to "extend" U_1 we choose a rank k holomorphic subbundle V of $U_0^\perp$ satisfying the condition that U_1 is a holomorphic subbundle of V. Again V describes a harmonic map $M \to G(k, n)$ and for appropriate choice of V we have $V = V_0$.

We have

Theorem 5.1. *If M has genus zero then any harmonic map $M \to G(k, n)$ can be constructed from one holomorphic (or one antiholomorphic) curve $M \to G(l, n)$, $1 \leqslant l \leqslant k$, using the ∂ and $\bar\partial$ transforms and extendings.*

Proof. Apply Theorem 3.6 repeatedly.

We can also use extending to give the following description of the space of harmonic maps of the torus into $G(k, n)$.

Theorem 5.2. *A harmonic map of a surface M of genus one into $G(k, n)$ can be constructed using the ∂ and $\bar\partial$ transforms and extendings from either*

(1) *a holomorphic or antiholomorphic curve $M \to G(l, n)$ $1 \leqslant l \leqslant k$ or*

(2) *a degree zero harmonic map $M \to G(l, n)$, $1 \leqslant l \leqslant k$.*

In fact in case (2) *the degree zero map can be taken to be an element of a harmonic sequence consisting only of degree zero harmonic maps.*

Proof. Apply Theorem 3.6 repeatedly.

References

[1] F. Burstall & S. Salamon, *Tournaments, flags and harmonic maps*, preprint.

[2] S. S. Chern & S. I. Goldberg, *On the volume-decreasing property of a class of real harmonic mappings*, Amer. J. Math. **97** (1975) 133–147.

[3] S. S. Chern & J. Wolfson, *Minimal surfaces by moving frames*, Amer. J. Math. **105** (1983) 59–83.

[4] ______, *Harmonic maps of the two-sphere into a complex Grassmann manifold*, Proc. Nat. Acad. Sci. U.S.A. **82** (1985) 2217–2219.

[5] ______, *Harmonic maps of the two-sphere into a complex Grassmann manifold*. II, Ann. of Math. **125** (1987) 301–335.

[6] A. M. Din & W. J. Zakrzewski, *General classical solutions in the* $\mathbf{CP}^{n-1}$ *model*, Nucl. Phys. B **174** (1980) 397–406.

[7] J. Eells & J. C. Wood, *Harmonic maps from surfaces to complex projective spaces*, Advances in Math. **49** (1983) 217–263.

[8] P. Griffiths & J. Harris, *Principles of algebraic geometry*, J. Wiley and Sons, New York, 1978.

[9] J. Ramanathan, *Harmonic maps from S^2 to $G(2,4)$*, J. Differential Geometry **19** (1984) 207–219.

[10] K. Uhlenbeck, *Harmonic maps into Lie groups (Classical solutions of the chiral model)*, preprint.

[11] J. G. Wolfson, *On minimal two-spheres in Kähler manifolds of constant holomorphic sectional curvature*, Trans. Amer. Math. Soc. **290** (1985) 627–646.

TULANE UNIVERSITY