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A NEW PROOF OF THE EXPLICIT NOETHER-LEFSCHETZ THEOREM

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We will work over **C**. Let

$$Y = \{\text{algebraic surfaces of degree } d \text{ in } \mathbf{P}^3\}$$

$$\Sigma_d = \{S \in Y \mid S \text{ smooth and } \operatorname{Pic}(S) \text{ is not generated by the hyperplane bundle}\}.$$

In [2], we proved the *Explicit Noether-Lefschetz Theorem*:

Theorem 1. *For* $d \geqslant 3$, *every component of* Σ_d *has codimension* $\geqslant d - 3$ *in* Y.

As remarked in [1], where this result was conjectured, the surfaces containing a line give a component of Σ_d codimension exactly $d - 3$.

In this paper, we give a substantially easier proof of this result. As was shown in [2], the Explicit Noether-Lefschetz Theorem is a consequence of the following vanishing theorem for Koszul cohomology on projective space:

Theorem 2. *Let*

$$W \subseteq H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(d))$$

be a base-point free linear system. Then the Koszul complex

$$\Lambda^{p+1}W \otimes H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(k-d)) \to \Lambda^p W \otimes H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(k)) \to \Lambda^{p-1}W \otimes H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(k+d))$$

is exact at the middle term provided that

$$k \geqslant p + d + \operatorname{codim} W.$$

Proof of Theorem 2. Consider an increasing sequence of linear subspaces

$$W = W_c \subset W_{c-1} \subset \cdots \subset W_0 = H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(d))$$

chosen so that

$$\dim(W_i/W_{i-1}) = 1 \qquad i = 1, 2, \cdots, c.$$

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Define vector bundles M_i on $\mathbf{P}^r$ by the sequences

$$0 \to M_i \to W_i \otimes \mathcal{O}_{\mathbf{P}^r} \to \mathcal{O}_{\mathbf{P}^r}(d) \to 0.$$

In particular, we have commutative diagrams

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_i & \longrightarrow & W_i \otimes \mathcal{O}_{\mathbf{P}^r} & \longrightarrow & \mathcal{O}_{\mathbf{P}^r}(d) \to 0 \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & M_{i-1} & \longrightarrow & W_{i-1} \otimes \mathcal{O}_{\mathbf{P}^r} & \longrightarrow & \mathcal{O}_{\mathbf{P}^r}(d) \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_{\mathbf{P}^r} & = & \mathcal{O}_{\mathbf{P}^r} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Lemma 1. *$H^q(\mathbf{P}^r, \Lambda^p M_0(n)) = 0$ if $q \geq 1$ and $n + q \geq p$.*

Proof. From the exact sequence

$$0 \to M_0 \to H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(d)) \otimes \mathcal{O}_{\mathbf{P}^r} \to \mathcal{O}_{\mathbf{P}^r}(d) \to 0$$

we note that

$$H^q(\mathbf{P}^r, M_0(n)) = 0 \quad \text{if } q \geq 1 \text{ and } n + q \geq 1.$$

Thus (see [4]) M_0 is 1-regular and hence has a free resolution of the form

$$\cdots \to \oplus \mathcal{O}(-2) \to \oplus \mathcal{O}(-1) \to M_0 \to 0.$$

We note that in general if vector bundles $\mathcal{F}$, $\mathcal{G}$ have free resolutions

$$\cdots F_1 \to F_0 \to \mathcal{F} \to 0,$$
$$\cdots \to G_1 \to G_0 \to \mathcal{G} \to 0,$$

then we obtain a free resolution of the form

$$\cdots \to (F_2 \otimes G_0) \oplus (F_1 \otimes G_1) \oplus (F_0 \otimes G_2) \to (F_1 \otimes G_0) \oplus (F_0 \otimes G_1)$$
$$\to F_0 \otimes G_0 \to \mathcal{F} \otimes \mathcal{G} \to 0.$$

Thus inductively $M_0^{\otimes p}$ has a resolution of the form

$$\cdots \to \oplus \mathcal{O}(-p-1) \to \oplus \mathcal{O}(-p) \to M_0^{\otimes p} \to 0$$

and thus

$$M_0^{\otimes p} \text{ is } p\text{-regular}$$

and hence

$$H^q(\mathbf{P}^r, M_0^{\otimes p}(n)) = 0 \quad \text{if } q \geq 1 \text{ and } n + q \geq p.$$

Since $\Lambda^p M_0(n)$ is a direct summand of $M_0^{\otimes p}(n)$, the lemma follows.

Lemma 2. *For all* $i = 0, \cdots, c$, $H^q(\mathbf{P}^r, \Lambda^p M_i(n)) = 0$ *if* $q \geq 1$ *and* $n + q \geq p + i$.

Proof. As seen above, we have the exact sequence

$$0 \to M_i \to M_{i-1} \to \mathcal{O}_{\mathbf{P}^r} \to 0$$

and thus the exact sequence

$$0 \to \Lambda^{p+1} M_i \to \Lambda^{p+1} M_{i-1} \to \Lambda^p M_i \to 0.$$

Tensoring by $\mathcal{O}_{\mathbf{P}^r}(n)$ and taking the long exact sequence on cohomology, we have

$$\to H^q(\mathbf{P}^r, \Lambda^{p+1} M_{i-1}(n)) \to H^q(\mathbf{P}^r, \Lambda^p M_i(n)) \to H^{q+1}(\mathbf{P}^r, \Lambda^{p+1} M_i(n)) \to .$$

Assume $q \geq 1$ and $n + q \geq p + i$. By ascending induction on i, since

$$n + q \geq (p+1) + (i-1) \leftrightarrow n + q \geq p + i$$

the term on the left may be assumed to vanish. Since

$$n + (q+1) \geq (p+1) + i \leftrightarrow n + q \geq p + i$$

we may do a descending induction on p, the case $p > \operatorname{rank} M_i$ being automatic. Thus we may assume the term on the right vanishes, and this proves the lemma.

The Koszul sequence

$$(*) \qquad \cdots \to \Lambda^{p+1} W \otimes \mathcal{O}_{\mathbf{P}^r}(k-d) \to \Lambda^p W \otimes \mathcal{O}_{\mathbf{P}^r}(k) \to \Lambda^{p-1} W \otimes \mathcal{O}_{\mathbf{P}^r}(k+d) \to \cdots$$

breaks up into short exact sequences

$$0 \to \Lambda^{p-1} M_c(k+d) \to \Lambda^{p-1} W \otimes \mathcal{O}_{\mathbf{P}^r}(k+d) \to \Lambda^{p-2} M_c(k+2d) \to 0,$$

$$0 \to \Lambda^p M_c(k) \to \Lambda^p W \otimes \mathcal{O}_{\mathbf{P}^r}(k) \to \Lambda^{p-1} M_c(k+d) \to 0,$$

$$0 \to \Lambda^{p+1} M_c(k-d) \to \Lambda^{p+1} W \otimes \mathcal{O}_{\mathbf{P}^r}(k-d) \to \Lambda^q M_c(k) \to 0.$$

The cohomology at the middle term of $(*)$ is isomorphic to

$$H^1(\Lambda^{p+1} M_c(k-d)).$$

By the lemma, this is zero if

$$k - d + 1 \geq p + 1 + c$$

or equivalently

$$k \geq p + d + \operatorname{codim} W$$

which proves Theorem 2.

We will now sketch how Theorem 2 implies Theorem 1. Let

$$\tilde{\Sigma}_d = \{(S, L) \mid S \in \Sigma_d,\ L \in \operatorname{Pic}(S)\}.$$

The first prolongation bundle $P_1(L)$ on S sits in an exact sequence

$$0 \to \Omega_S^1 \otimes L \to P_1(L) \to L \to 0$$

which dualized and twisted by L looks like

$$(**) \qquad 0 \to \mathcal{O}_S \to P_1(L)^\vee \otimes L \to \Theta_S \to 0.$$

By standard identifications, the projection

$$\tilde{\Sigma}_d \xrightarrow{\pi} \Sigma_d$$

$$(S, L) \to S$$

gives rise to a commutative diagram

$$\begin{array}{ccc} T_{(S,L)}(\tilde{\Sigma}_d) & \xrightarrow{\pi_*} & T_S(\Sigma_d) \\ \wr\!\downarrow & & \wr\!\downarrow \\ H^1\big(S, P_1(L)^\vee \otimes L\big) & \xrightarrow{\alpha} & H^1(S, \Theta_S) \end{array}$$

where α fits into the long exact sequence of $(**)$:

$$H^1\big(S, P_1(L)^\vee \otimes L\big) \xrightarrow{\alpha} H^1(S, \Theta_S) \xrightarrow{\beta} H^2(S, \mathcal{O}_S)$$

and β is cup product with $c_1(L)$, the first Chern class of L. Let $Z_{(S,L)}$ be the union of all irreducible components of $\tilde{\Sigma}_d$ containing (S, L). The Zariski tangent space $T \subseteq H^1(S, \Theta_S)$ of $\pi(Z_{(S,L)})$ at S is $\ker \beta$.

Now assume $S \in \Sigma_d$. Without loss of generality, we may choose $L \in \mathrm{Pic}(S)$ so that $c_1(L) \in H^1_{\mathrm{prim}}(S, \Omega^1_S)$. We thus have

$$H^1(S, \Theta_S) \otimes H^1_{\mathrm{prim}}\big(S, \Omega^1_S\big) \xrightarrow{\text{cup product}} H^2(S, \mathcal{O}_S),$$

$$T \otimes c_1(L) \mapsto 0.$$

Equivalently,

$$H^1(S, \Theta_S) \otimes H^0(S, K_S) \xrightarrow{\text{cup product}} H^1_{\mathrm{prim}}\big(S, \Omega^1_S\big),$$

$$T \otimes H^0(S, K_S) \mapsto c_1(L)^\perp.$$

Using standard identifications, this is the multiplication map

$$\frac{H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d)\big)}{J_d} \otimes H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d-4)\big) \to \frac{H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2d-4)\big)}{J_{2d-4}},$$

where J_k denotes the Jacobi ideal of S in degree k. Let $\tilde{T}$ be the preimage of T in $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d))$. Then the multiplication map

$$\tilde{T} \otimes H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d-4)\big) \to H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2d-4)\big)$$

is not surjective. The nonsingularity of S implies that J_d is base-point free and hence $\tilde{T}$ is. By Theorem 2, this implies

$$\operatorname{codim} \tilde{T} \geqslant d - 3$$

and thus

$$\operatorname{codim} T \geqslant d - 3$$

as desired.

Remark. Since writing [2], I have learned of a paper by Jozefiak, Pragacz, and Weyman [3] in which they work out completely the case $c = 0$, $d = 2$ of the Koszul groups discussed in Theorem 2, and in fact their result is stronger in this case.

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