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A CONSTRUCTION OF STABLE BUNDLES ON AN ALGEBRAIC SURFACE

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1. Let X be a smooth projective algebraic surface over $\mathbf{C}$ and let H be an ample divisor on X. We recall that a bundle $\mathscr{E}$ of rank two and $c_1(\mathscr{E}) = 0$ is H-stable (in the sense of Mumford-Takemoto) if whenever $\mathscr{L}$ is a line bundle on X which admits a nonzero map to $\mathscr{E}$, then we have $(c_1(\mathscr{L}) \cdot H) < 0$. In this paper, we will consider the problem of constructing stable bundles $\mathscr{E}$ on X of rank two with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E})$ a prescribed number. From work of Donaldson [1], this question is a special case of the following: When does a principal SU(2) bundle on a four dimensional Riemannian manifold admit an irreducible self dual connection? In this guise, the problem has been studied by Taubes [4]. There has also been some work on higher dimensional manifolds by Uhlenbeck and Yau. The basic goal is to give conditions on the topology of X so that stable bundles $\mathscr{E}$ of the type considered exist with $c_2(\mathscr{E})$ a given integer. The topological invariant of interest here is $h^0(X, \mathscr{O}(K))$, the number of holomorphic two forms on X. Throughout the paper, we will use h^0 as an abbreviation for $h^0(X, \mathscr{O}(K))$. $[r]$ is the greatest integer in r.

Theorem 1.1. *If $n \geqslant 4([h^0/2] + 1)$, then there is an H-stable bundle $\mathscr{E}$ on X of rank two with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = n$.*

Theorem 1.2. *If $h^0 > 1000$ and $n > (3/2)h^0 + 6$, then there is an H-stable bundle $\mathscr{E}$ on X of rank two with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = n$.*

We note that Taubes constructs bundles of the above type for $n \geqslant (8/3)h^0 + 2$. Our methods are modeled on Taubes' methods, namely both methods are degeneration theoretic. My main motivation for this paper was to see Taubes' argument is an algebro-geometric setting. Actually, the argument we will use is somewhat different than Taubes'.

One's first idea in attacking this problem is to construct a torsion free coherent H-stable sheaf $\mathscr{F}$ on X and to prove that $\mathscr{F}$ can be deformed to a locally free sheaf. However, we have adopted a different but related approach

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which we now describe. Let C be a smooth curve which will function as a parameter space for our deformation and let $P \in C$. Let $Z_1 = X \times C$. Pick $x_1, \cdots, x_k \in X$ and blow up $x_i \times P$ in Z_1 to obtain a threefold Z. D will denote the proper transform of $X \times P$ and $D_1, \cdots, D_k$ will be the new exceptional divisors introduced by blowing up. Each D_i is isomorphic to $\mathbf{P}^2$. Let $\tilde{D} = D + \Sigma D_i$ and choose $v_i = (\alpha_i, \beta_i) \in \mathbf{C}^2 - \{(0,0)\}$. We assume that v_i span $\mathbf{C}^2$. For each i, we define a map

$$\phi_i\colon \mathcal{O}_Z^2 \to \mathcal{O}_{D_i}$$

by

$$\phi_i(a,b) = a\alpha_i + b\beta_i.$$

Let $\phi\colon \mathcal{O}_Z^2 \to \oplus_i \mathcal{O}_{D_i}$ be $\oplus_i \phi_i$. Let $\mathcal{E}' = \operatorname{Ker}\phi$. Thus (a,b) is a section of $\mathcal{E}'$ over an open V if $a\alpha_i + b\beta_i$ vanishes on each $D_i \cap V$. Note that on some neighborhood U_i of D_i, $\mathcal{E}'$ is a direct sum $(\mathcal{O} \oplus \mathcal{O}(-D_i))_{U_i}$. In particular, $\mathcal{E}'_{D_i} \cong \mathcal{O}_{D_i} \oplus \mathcal{O}_{D_i}(1)$, since the ideal sheaf $\mathcal{I}_{D_i}$ of D_i is isomorphic to $\mathcal{O}_{D_i}(1)$ when restricted to D_i.

Here is our basic strategy: Let $\mathcal{E}_2 = \mathcal{E}'_{2D}$. (Here $2D$ is the scheme defined by $\mathcal{I}_D^2$ and $\mathcal{E}'_{2D} = \mathcal{E}' \otimes_{\mathcal{O}_Z} (\mathcal{O}_Z/\mathcal{I}_D^2)$.) Thus $\mathcal{E}_2$ is a sheaf of locally free modules over $\mathcal{O}_Z/\mathcal{I}_D^2$.) We will analyze the obstructions to extending $\mathcal{E}_2$ to a sheaf of locally free modules over $3D$, then to $2D + \tilde{D}$ and then to $2D + 2\tilde{D}, 2D + 3\tilde{D}$, etc.

We first study how to extend $\mathcal{E}_2$ to a sheaf of modules $\mathcal{E}_3$ locally free on $3D$. D_j is just $\mathbf{P}^2$ and $D \cap D_j$ is a line L_j in $\mathbf{P}^2$, $3D \cap D_j$ is just the scheme $3L_j \subseteq \mathbf{P}^2$.

Definition 1.3. A sheaf $\mathcal{F}$ of locally free $\mathcal{O}_{3L}$ modules is nondegenerate if $\mathcal{F}$ satisfies the following conditions

a) $\Lambda^2 \mathcal{F} \cong \mathcal{O}_{3L}(1)$.

b) There is not a quotient $\mathcal{F} \to Q \to 0$ so that Q is an invertible sheaf of $\mathcal{O}_{3L}$ modules and $Q_L \cong \mathcal{O}_L$.

The existence of nondegenerate $\mathcal{E}_3$ is studied by deformation theory in §2. Assume that $\mathcal{E}_3$ satisfies our nondegeneracy condition on $3L_j$. We show that $(\mathcal{E}_3)_{3L_j}$ can be extended to a stable vector bundle $\mathcal{F}_j$ on $\mathbf{P}^2 = D_j$ with $c_1(\mathcal{F}_j) = 1$ and $c_2(\mathcal{F}_j) = 2$. The construction of the $\mathcal{F}_j$'s given in §6 is the following: Take lines L given by $x = 0$ and L' given by $y = 0$, where x and y are affine coordinates on $\mathbf{A}^2 \subseteq \mathbf{P}^2$. Construct a surjective map $\Phi\colon \mathcal{O}_{\mathbf{P}^2}^2 \to \mathcal{O}_{L'}(2)$ by

$$\Phi(a,b) = a + by^2,$$

and let $\mathscr{F}^{\vee}$ be the kernel of Φ. Then $c_1(\mathscr{F}) = 1$ and $c_2(\mathscr{F}) = 2$. Using the nondegeneracy condition on $\mathscr{E}_3$ we show that if $L = D \cap D_j \subseteq \mathbf{P}^2$, then we can choose the line L' so that the above construction gives a suitable extension.

By gluing $\mathscr{E}'$ and $\mathscr{F}_j$ together, we can construct a bundle $\mathscr{G}$ on $2D + \tilde{D}$. Let $\mathscr{G}_0 = \mathscr{G}_{\tilde{D}}$. Next we study the problem of extending $\mathscr{G}_0$ to a bundle on $2D + 2\tilde{D}$, and then to $2D + 3\tilde{D}$, etc. in §2. In each case, the obstruction to making such an extension is in

$$H^2(\tilde{D}, \operatorname{End}^0(\mathscr{G}_0) \otimes \mathscr{I}_{2D}). \tag{1.3.1}$$

Here $\operatorname{End}^0(\mathscr{E})$ is the sheaf of endomorphisms of $\mathscr{E}$ with trace zero. We suppose we have chosen the x_i's and v_i's so that (1.3.1) is zero. We can use Grothendieck's Quot scheme [3] in §5 to show that $\mathscr{G}_0$ can be extended to a bundle $\mathscr{E}$ on Z. (A minor technical point: We may have to base extend C.) We then can show using a standard semicontinuity argument that for generic $s \in C$, the bundle $\mathscr{E}_s$ is H-stable, $c_2(\mathscr{E}_s) = 2n$ and $c_1(\mathscr{E}_s) = 0$.

We are thus left with the problem of finding conditions on the x_i and v_i and n so that nondegenerate extensions $\mathscr{E}_3$ exist and so that $\mathscr{G}_0$ can be lifted back to larger and larger infinitesimal neighborhoods of $\tilde{D}$. Let us consider the problem of showing that (1.3.1) is zero. Let $\mathscr{E} = \mathscr{G}_0 \otimes \mathscr{O}_D$. We wish to first establish conditons under which

$$H^2(D, \operatorname{End}^0(\mathscr{E}) \otimes \mathscr{O}(-2D)) = 0. \tag{1.3.2}$$

Let $E \subseteq D$ be the divisor ΣE_i, where $E_i = D \cap D_i$. The E_i are exceptional curves of the first kind on D. By Serre duality we need to show that

$$V = H^0(D, \operatorname{End}^0(\mathscr{E})(K_X - E))$$

is zero. Now $\mathscr{E}$ is a subsheaf of $\mathscr{O}_D^2$, and it is isomorphic to $\mathscr{O}_D^2$ away from the E_i's. It follows easily from Hartog's theorem that any $s \in V$ can be represented by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are holomorphic two forms on X. Further, the condition $s \in V$ implies linear relations between the values of these two forms and their derivatives at x_i. For instance, if $v_i = (1,0)$, then d must vanish at x_i and b must vanish twice at x_i, i.e., $b \in H^0(X, \mathscr{O}(K) \otimes m_{x_i}^2)$. At each x_i, the condition $s \in V$ should impose four conditions, one for the vanishing of d and three for the vanishing of b and its two partials. (Locally, we can think of b as a function.) However, these $4k$ conditions may not be independent conditions. To see the problem, let W be a subspace of $H^0(X, \mathscr{O}(K))$ and let W_x be the subspace consisting of points $b \in W$ so that b and its two partial derivatives vanish at x. Assuming $\dim W \geqslant 4$, we can easily see $d_x = \operatorname{codim}_W W_x \geqslant 2$. However if (z, w) are local coordinates at x, all the sections in W could be locally functions of z, in which case, $d_x = 2$ for x generic. The weak estimate $d_x \geqslant 2$ is all that is needed to establish Theorem 1.1. This situation can actually occur for elliptic surfaces. Specifically, if C is a curve of genus g and E is an elliptic curve, then $d_x = 2$ for $X = C \times E$ and $W = H^0(K_X)$.

To establish Theorem 1.2, we note that if $d_x = 2$ for x generic, then the linear system defined by W must map X to a curve $C \subseteq \mathbf{P}(W)$. (Of course, there may be base points.) If the dimension of W is large, we can find a hyperplane H_1 on $\mathbf{P}(W)$ which has high order contact with C at some generic point. The inverse image of H_1 in X is contained in an effective canonical divisor E which has a component of high multiplicity. §4 gives a construction of stable bundles whenever there are many canonical curves C on the surface which contain components of high order. This construction enables us to establish the existence of stable bundles with small c_2 if $d_x = 2$ for x generic if we begin with a large $h^0(K_X)$. Our construction also shows that for each $\varepsilon > 0$, then if $d \gg 0$, there are stable bundles $\mathscr{E}$ on hypersurfaces X of degree d in $\mathbf{P}^3$ with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) \leqslant \varepsilon h^0(K_X)$. This stands in contrast to a result in [1] that for a *generic* Riemannian metric on X, the existence of a self dual connection on a principal SU(2) bundle $P \to M$ requires $c_2(P) \geqslant 3/8(b - + 1 - \dim H^1_{DR})$. Evidently, the Kähler class on a hypersurface is not generic in the above sense. (If Q is the intersection matrix on H_2, $b_- = 1/2$ (rank signature Q).) §7 contains the proof of Theorems 1.1 and 1.2.

2. Let Z be a smooth threefold, D a divisor with components $D_0, \cdots, D_n$ which are smooth. We assume D_i intersect transversally and that there are no triple intersections. Let $\mathscr{E}$ be a locally free sheaf of rank two on $\Sigma n_i D_i$, i.e., $\mathscr{E}$ is a sheaf of locally free $\mathscr{O}_Z/(\Sigma n_i D_i)$ modules. We assume there is a line bundle $\mathscr{L}$ on Z so that the restriction of $\mathscr{L}$ to $\Sigma n_i D_i$ is $\Lambda^2 \mathscr{E}$. Choose a k and let

$$m_i = \begin{cases} n_i + 1 & \text{for } i \leqslant k, \\ n_i & \text{for } i > k. \end{cases}$$

We suppose $n_i > 0$ if $i \leqslant k$. We wish to study conditions under which $\mathscr{E}$ can be extended to a sheaf of locally free modules over $\Sigma m_i D_i$. Let $D' = \Sigma_{i=0}^k D_i$.

Proposition 2.1. *Suppose*

$$H^2\big(D', \operatorname{End}^0(\mathscr{E}) \otimes \mathscr{O}_{D'}(-\Sigma n_i D_i)\big) = 0,$$

where $\operatorname{End}^0(\mathscr{E})$ *is the sheaf of endomorphisms of trace zero. Then* $\mathscr{E}$ *can be extended to a bundle* $\mathscr{E}'$ *on* $(\Sigma n_i D_i + D')$ *so that* $\mathscr{L}$ *restricts to* $\det \mathscr{E}'$.

Proof. The proof uses standard ideas on deformation theory which we review. Find affine opens $U_\alpha \subseteq Z$ which cover D so that on each U_α, we can find a free bundle of rank two $\mathscr{E}_\alpha$ on $(\Sigma n_i D_i + D') \cap U_\alpha$ which restricts to $\mathscr{E}$

on $(\Sigma n_i D_i) \cap U_\alpha$. Let $\phi_{\alpha\beta}$ be isomorphisms of $\mathscr{E}_\beta$ with $\mathscr{E}_\alpha$ over $U_\alpha \cap U_\beta$ which extend the identity map on $\mathscr{E}$ when restricted to $U_\alpha \cap U_\beta \cap (\Sigma n_i D_i)$. Let

$$\psi_{\alpha\beta\gamma} = \mathrm{Id} - \phi_{\alpha\gamma} \circ \phi_{\gamma\beta} \circ \phi_{\beta\alpha}.$$

Now $\psi_{\alpha\beta\gamma}$ is an endomorphism of $\mathscr{E}_\alpha$ over $U_\alpha \cap U_\beta \cap U_\gamma = U_{\alpha\beta\gamma}$. Actually $\psi_{\alpha\beta\gamma}$ is a map of $\mathscr{E}_\alpha$ to $\mathscr{E}_\alpha \cdot \mathcal{O}(-\Sigma n_i D_i) = \mathscr{E}_{D'} \otimes \mathcal{O}_Z(-\Sigma n_i D_i)$ on $U_{\alpha\beta\gamma}$. So we can regard $\psi_{\alpha\beta\gamma}$ as a section of $\mathrm{End}(\mathscr{E})(-\Sigma n_i D_i) \otimes \mathcal{O}_{D'}$. We claim $\{\psi_{\alpha\beta\gamma}\} = \psi$ is a cocycle and so defines an element

$$\bar{\psi} \in H^2\big(D', \mathrm{End}(\mathscr{E})(-\Sigma n_i D_i)\big).$$

It suffices to check $d\psi = 0$ locally. Let U be an open so that $\mathscr{E}_\alpha$, $\mathscr{E}_\beta$ and $\mathscr{E}_\gamma$ are all restrictions of a bundle $\mathscr{F}$ on $\Sigma m_i D_i \cap U$. Then we can write $\phi_{\alpha\beta} = \mathrm{Id} + \tilde{\phi}_{\alpha\beta}$, where $\tilde{\phi}_{\alpha\beta}$ are sections of $\mathscr{F}_D \otimes \mathcal{O}(\mathscr{E}(-n_i D_i))$ over U. One checks that $d\tilde{\phi} = \psi$, and hence $d\psi = 0$.

We next claim that $\bar{\psi} = 0$. Indeed, let us look first at

$$\mathrm{Tr}\,\bar{\psi} \in H^2\big(D', \mathcal{O}_{D'}(-\Sigma n_i D_i)\big).$$

$\mathrm{Tr}\,\bar{\psi}$ is just the obstruction to extending $\det \mathscr{E}$ to a line bundle on $\Sigma m_i D_i$. But we are given that such an extension is possible, so the obstruction is zero. More precisely, we can assume that we have ξ_α: $\det \mathscr{E}_\alpha \tilde{\to} \mathscr{L}$ on U_α so that ξ_α is the identity on $\Sigma n_i D_i$:

$$\xi_\alpha \circ \det \phi_{\alpha\beta} \circ \xi_\beta^{-1} = \mathrm{Id} + \lambda'_{\alpha\beta}.$$

Thus

$$\det \phi_{\alpha\beta} = k_{\alpha\beta} + \lambda_{\alpha\beta},$$

where $k_{\alpha\beta} = \xi_\alpha^{-1} \circ \xi_\beta$ is a coboundary and $\lambda_{\alpha\beta}$ is zero on $\Sigma n_i D_i$.

$$\mathrm{Tr}\,\psi_{\alpha\beta\gamma} = 2 - \mathrm{Tr}\big(\phi_{\alpha\gamma}\phi_{\gamma\beta}\phi_{\beta\alpha}\big).$$

But a local computation shows that

$$\mathrm{Tr}\big(\phi_{\alpha\gamma}\phi_{\gamma\beta}\phi_{\beta\alpha}\big) = 1 + \det\phi_{\alpha\gamma}\det\phi_{\gamma\beta}\det\phi_{\beta\alpha} = 2 + \big(\lambda_{\alpha\gamma} + \lambda_{\gamma\beta} + \lambda_{\beta\alpha}\big).$$

So

$$\mathrm{Tr}\,\psi = d\lambda.$$

So since the kernel of

$$\mathrm{Tr}\colon\ H^2\big(D', (\mathrm{End}\,\mathscr{E}_{D'})(-\Sigma n_i D_i)\big) \to H^2\big(D', \mathcal{O}_{D'}(-\Sigma n_i D_i)\big)$$

is $H^2(D', \mathrm{End}^0(\mathscr{E}_{D'})(-\Sigma n_i D_i)) = 0$, we see that

$$\psi_{\alpha\beta\gamma} = d\big(\zeta_{\alpha\beta}\big)$$

where

$$\zeta_{\alpha\beta}: \mathscr{E}_\beta \to \mathscr{E}_\alpha \cdot \mathscr{O}(\Sigma - n_i D_i).$$

Let

$$\phi'_{\alpha\beta} = \phi_{\alpha\beta} + \zeta_{\alpha\beta}.$$

The $\phi'_{\alpha\beta}$ satisfies the cocycle condition and provides a lifting of $\mathscr{E}$ to $\Sigma m_i D_i$.

Now $\mathscr{M} = \det \mathscr{E} \otimes \mathscr{L}^{-1}$ is a line bundle which is trivial on $\Sigma n_i D_i$. Thus we can choose a local trivialization and present $\mathscr{M}$ as an element of $\{\eta_{\alpha\beta}\}$ of $H^1(\mathscr{O}^*)$, where $\eta_{\alpha\beta}$ reduces to 1 on $\Sigma n_i D_i$. Let $\mathscr{M}'$ be given by

$$\eta'_{\alpha\beta} = \tfrac{1}{2}(1 + \eta_{\alpha\beta}).$$

Then $(\mathscr{M}')^{\otimes 2}$ is isomorphic to $\mathscr{M}$, and so $\det(\mathscr{E} \otimes \mathscr{M}') \cong \mathscr{L}$.

We next consider the following situation: $n_0 = 2$ and all the other n_i's are zero and $m_0 = 3$ with all the other m_i's zero. Thus we have a bundle $\mathscr{E}_2$ on $2D_0$ and we wish to study the extensions of $\mathscr{E}_2$ to $3D_0$. We assume that such extension $\mathscr{E}'_3$ exists. Let $\mathscr{E}_3$ be any other extension of $\mathscr{E}_2$ to $3D_0$. Then on a suitable open cover $\{U_\alpha\}$ of $3D_0$ we choose isomorphism ϕ_α: $\mathscr{E}_3 \to \mathscr{E}'_3$ defined over U_α extending the identity on $U_\alpha \cap 2D_0$. The one cocycle $\psi = \{\psi_{\alpha\beta}\}$

$$\psi_{\alpha\beta} = \mathrm{Id} - \phi_\beta^{-1}\phi_\alpha \in H^1(D_0, \mathrm{End}(\mathscr{E})(-2D_0))$$

classifies such extensions, where $\mathscr{E} = \mathscr{E}_2 \otimes \mathscr{O}_{D_0}$.

Suppose we have a quotient Q'_3 of $\mathscr{E}'_3$ over $3D_0 \cap D_j$ for some $j > 0$. (If D_0 is locally defined by $x = 0$ and D_j is defined by $y = 0$, $3D_0 \cap D_j$ is defined by the equations $x^3 = y = 0$ as a scheme. Thus Q'_3 is an invertible module over $\mathscr{O}_Z/(x^3, y)$.) Let Q_2 be the induced quotient of $\mathscr{E}_2$. Our question is: Given $\mathscr{E}_3$ (or equivalently ψ), when does Q_2 lift to an invertible quotient of Q_3 of $\mathscr{E}_3$ over $3D_0 \cap D_j$? Let Q be the induced quotient of $\mathscr{F} = \mathscr{E}_2 \otimes \mathscr{O}_{D_0 \cap D_j}$ and let L be the kernel:

$$0 \to L \to \mathscr{F} \to Q \to 0. \tag{2.2}$$

There is a natural map from

$$\Phi: \mathrm{End}\,\mathscr{E}(-2D_0) \to \mathrm{Hom}(L, Q)(-2D_0)$$

since an endomorphism of $\mathscr{E}$ gives an endomorphism of $\mathscr{F}$ and hence a map from L to Q.

Lemma (2.3). *If Q_2 lifts to an invertible quotient Q_3 of $\mathscr{E}_3$ over $3D_0 \cap D_j$, then $\Phi(\psi_{\alpha\beta}) = 0$ in $H^1(D_0 \cap D_j, \mathrm{Hom}(L, Q)(-2D_0))$.*

Proof. If Q_2 lifts to Q_3, we can take the ϕ_α to map Q_3 to Q'_3. Then $\Phi(\psi_{\alpha\beta}) = 0$.

Lemma (2.4). *If Q_2 always lifts for any choice of $\mathscr{E}_3$ and the exact sequence* (2.2) *splits, then the kernel of the natural map*

$$H^2\big(D_0, \operatorname{End}(\mathscr{E})(-2D_0 - D_j)\big) \to H^2\big(D_0, \operatorname{End}(\mathscr{E})(-2D_0)\big)$$

has dimension $\geqslant h^1(L^\vee \otimes Q(-2D_0))$.

Proof. This follows from the long exact sequence associated to

$$0 \to \operatorname{End}(\mathscr{E})(-2D_0 - D_j) \to \operatorname{End}(\mathscr{E})(-2D_0)$$
$$\to (\operatorname{End}\mathscr{E})(-2D_0) \otimes \mathcal{O}_{D_0 \cap D_j} \to 0.$$

Corollary 2.5. *Suppose that for each j, $(\mathscr{E}_3')_{D_0 \cap D_j} = Q_j \oplus L_j$ and that Q_j lifts to an invertible quotient of $(\mathscr{E}_3')_{3D_0 \cap D_j}$. Suppose further that*

$$h^2\big(D_0, \operatorname{End}^0(\mathscr{E})(-2D_0)\big) = 0$$

and

$$h^2\big(D_0, \operatorname{End}^0(\mathscr{E})(-2D_0 - D_j)\big) < h^1\big(D_0 \cap D_j, Q_j \otimes L_j^\vee(-2D_0)\big).$$

Then we can find an extension $\mathscr{E}_3$ of $\mathscr{E}_2$ to $3D_0$ so that the quotient Q_j does not lift to an invertible quotient of $(\mathscr{E}_3)_{3D_0 \cap D_j}$ for any j and $\det \mathscr{E}_3' \cong \det \mathscr{E}_3$.

Proof. We have to show there is $\alpha \in H^1(D_0, \operatorname{End}(\mathscr{E})(-2D_0))$ which has nonzero image in $H^1(D_0 \cap D_j, (L_j^\vee \otimes Q_j)(-2D_0))$ where $(\mathscr{E}_3)_{D_0 \cap D_j} = Q_j \oplus L_j$. Lemma 2.4 shows that such an α_j exists for each j. Some linear combination of the α_j works as α, since the field is infinite.

Remark. We will be interested in applying the results of this section in the case Z is the variety constructed in §1. D_i is the divisor D_i of the introduction for $i \geqslant 1$ and D_0 is D, the blow up of X. The $\mathscr{E}_3'$ will be $\mathscr{E}_{3D}'$ of §1 and Q_j is $\mathcal{O}_{E_j}$. Thus $Q_j \otimes L_j^\vee(-2D_0)$ has degree -3 on E_j. So

$$h^1\big(Q_j \otimes L_j^\vee(-2D_0)\big) = 2.$$

3. Let X be the algebraic surface of §1 and let $P_1, \cdots, P_k$ be points of X in general position. Let D be blow up of X at $P_1, \cdots, P_k$. $E_1, \cdots, E_k$ will denote the exceptional divisors. Let $E = \Sigma E_i$. At each point P_i, choose

$$v_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \in \mathbf{C}^2 - \{(0,0)\}.$$

We produce a new vector bundle $\mathscr{E}$ on D by the following construction: For each E_i, consider the map

$$\phi_i(f, g) = \alpha_i \bar{f} + \beta_i \bar{g}$$

from $\mathcal{O}_D^2$ to $\mathcal{O}_{E_i}$, where $\bar{f}$ is the restriction of a local section f of $\mathcal{O}_D$ to $\mathcal{O}_{E_i}$. Let $\phi = \oplus_i \phi_i$, so

$$\phi \colon \mathcal{O}_D^2 \to \bigoplus_i \mathcal{O}_{E_i}.$$

Thus $\mathscr{E}$ is the subsheaf of $\mathcal{O}_D^2$ whose local sections consist of pairs of functions (f, g) with $\alpha_i f + \beta_i g$ vanishing on E_i. We seek conditions on the P_i and v_i so that

$$h^2(D, \operatorname{End}^0(\mathscr{E})(2E)) = 0 \tag{3.1.1}$$

and

$$h^2(D, \operatorname{End}^0(\mathscr{E})(2E - E_i)) \leqslant 1 \tag{3.1.2}$$

for all i. Let K_D be the canonical divisor on D. We have

$$K_D = K_X + E$$

where K_X denotes the pull back of the canonical bundle of X. It suffices to show that

$$V = H^0(D, \operatorname{End}^0(\mathscr{E})(K_X - E)) = 0$$

and that for

$$W_i = H^0(D, \operatorname{End}^0(\mathscr{E})(K_X - E + E_i))$$

we have $\dim W_i \leqslant 1$.

First, notice that

$$H^0(D - E, \operatorname{End}^0(\mathscr{E})(K_X)) = H^0(X - (\cup x_i), \mathcal{O}(K)^3) = H^0(X, \mathcal{O}(K)^3).$$

Thus any sections of V or W_i can be represented as a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are in $H^0(D, \mathcal{O}(K_X))$ and $\operatorname{Tr} A = 0$.

We analyze the conditions on a, b, c, d for s to be in V. Suppose $\beta_i = 1$. We claim that $s_1 = a - \alpha_i b$ and $s_2 = c - \alpha_i d$ vanish at least once on E_i, and that $s_3 = b\alpha_i^2 + (d - a)\alpha_i - c$ vanishes twice on E_i. Note that $(1, -\alpha_i)$ is a section of $\mathscr{E}$ near E_i, since $\phi_i(1, -\alpha_i) = (0, 0)$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -\alpha_i \end{pmatrix}$$

must be a section of $\mathscr{E}(-E_i + K_X)$. In particular, it is a section of $\mathcal{O}_D^2(-E_i + K_X)$ in a neighborhood of E_i. Thus s_1 and s_2 have the required properties. Further, $(a - \alpha_i b, c - \alpha_i d)$ must be in the kernel of the natural map of $\mathcal{O}_D^2(K_X - E_i)$ to $\mathcal{O}_{E_i}(K_X - E_i)$, i.e., s_3 must vanish on E_i as a section of $\mathcal{O}_D(K_X - E_i)$, i.e., it vanishes twice on E_i as a section of $\mathcal{O}_D(K_X)$. If $\beta_i = 0$, the corresponding conditions are that d vanishes at least once on E_i and b vanishes at least twice on E_i.

Proposition 3.2. *Let* $n = [h^0/2] + 1$ *and* $k = 2n$. *Let* $v_i = (1,0)$ *for* $i = 1, \cdots, n$ *and* $v_i = (0,1)$ *for* $i = n+1, \cdots, k$. *If the* P_i *are chosen generically, then* (3.1.1) *and* (3.1.2) *are satisfied.*

Proof. Let $V_i = H^0(D, \mathcal{O}(K(-2E_1 \cdots -2E_i)))$. We claim that as long as $\dim V_i \geqslant 2$, the codimension of V_{i+1} in V_i must be at least two. Indeed, let s_1 and s_2 be two independent sections of V_i. Then $f = s_1/s_2$ is a nonconstant meromorphic function, so we can choose P_{i+1} so that $s_2(P_{i+1}) \neq 0$ and $(df)_{P_{i+1}} \neq 0$. Then

$$s' = s_1 - \frac{s_1(P_{i+1})}{s_2(P_{i+1})} s_2$$

vanishes exactly once on E_{i+1}, so no nontrivial linear combinations of s_2 and s' are in V_{i+1}. Thus our claim is established. In particular, $V_n = 0$.

Let

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and suppose $s \in H^0(D, \mathrm{End}^0(\mathcal{E})(K - E))$. Since $V_n = 0$, we have $b = c = 0$. Since $k \geqslant h^0$, and the P_i are generic, $a - d$ is zero since $a - d$ vanishes at the P_i. We have $a + d = 0$, since the matrix is traceless. So $s = 0$.

Suppose $s, t \in H^0(D, \mathrm{End}^0(\mathcal{E})(K - E + E_k))$ are linearly independent. Let

$$t = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

Since $c, c_1 \in V_{n-1}$ are linearly dependent, we can assume that $c_1 = 0$ by replacing t by a linear combination of s and t. As before $b_1 = 0$ and then $a_1 = d_1 = 0$. So (3.1.2) is satisfied.

Proposition 3.3. *Suppose* $V \subseteq H^0(X, K_X)$ *has dimension* $\geqslant 21$. *Then either*

i) *for generic* $x \in X$, *the natural map from* V *to* $H^0(X, \mathcal{O}(K)/m_x^2 \cdot \mathcal{O}(K))$ *is onto, or*

ii) *for a generic point* $x \in X$ *there is a curve* D *so that* $20D + E = K$ *where* E *is effective.*

Proof. Let $\mathcal{F} \subseteq \mathcal{O}(K_X)$ be the subsheaf generated by the sections in V and let $z_1, \cdots, z_r$ be the points at which $\mathcal{F}$ is not invertible and let $X' = X - \{x_1, \cdots, x_r\}$. The linear system V then defines a map Φ of X' to $\mathbf{P}(V)$. If $\overline{\Phi(X')}$ is a surface, then (i) holds. Otherwise, $\overline{\Phi(X')}$ is a curve $\subseteq \mathbf{P}(V)$ not contained in a hyperplane. If $x \in \Phi(X')$ is a generic point, we can find a hyperplane H which has contact 20 or more with $\overline{\Phi(X')}$ at x. Let $D = \Phi^{-1}(H)$. Then (ii) is valid.

For the rest of the section, we will assume that there are no canonical divisors on X with components of multiplicity 20 passing through a generic x, so case i) of Proposition 3.3 always holds. In particular, by choosing the x_i's generically we can assume that

$$h^0\left(D, \mathcal{O}\left(K_X - \sum^{l} 2E_i\right)\right) = h^0 - 3l \tag{3.3.1}$$

as long as $h^0 - 3l \geqslant 18$. We define integers k_1, k_2, k_3 by

$$k_1 = \left[\frac{5}{16}h^0\right] + 1, k_2 = \left[\frac{5}{8}h^0\right] - \left[\frac{5}{16}h^0\right],$$
$$k_3 = 2h^0 - 3\left(\left[\frac{5}{8}h^0\right]\right).$$

Let $v_i = \binom{1}{0}$ for $i = 1$ to $k_1, v_i = \binom{0}{1}$ for $i = k_1 + 1$ to $k_2 + k_1$ and $v_i = \binom{\alpha_i}{1}$ for $i = k_2 + k_1 + 1$ to $k_1 + k_2 + k_3$.

Proposition 3.4. *If the x_i and α_i are generic and $h^0 \geqslant 1000$, then $h^0(D, \mathrm{End}^0(\mathcal{E})(K - E + E_j)) = 0$ for any j.*

Proof. We will treat the case $j = 1$ first. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an element of $H^0(D, \mathrm{End}^0(\mathcal{E})(-E + K + E_1)$. Then b vanishes twice on E_i for $1 < i \leqslant k_1$ and c vanishes twice on E_i for $k_1 < i \leqslant k_1 + k_2$. On the other hand, we have $\alpha_i^2 b + c$ vanishes on E_i for $k_1 + k_2 < i$. Notice that if $W \subseteq \oplus^2 H^0(X, K_X)$ is any nonzero subspace, then the condition $\alpha_i^2 b = -c$ is nontrivial for some α_i, i.e., there is a pair $(b, c) \in W$ violating the condition. Hence if $k_3 \geqslant \dim W$, the conditions $\alpha_i^2 b = -c$ at k_3 points implies $b = c = 0$. In our case

$$W = H^0\left(D, \mathcal{O}\left(K - 2\sum_{i=2}^{k_1} E_i\right)\right) \oplus H^0\left(D, \mathcal{O}\left(K - 2\sum_{i=k_1+1}^{k_2} E_i\right)\right),$$

so if

$$k_3 \geqslant h^0\left(D, \mathcal{O}\left(K - 2\sum_{i=2}^{k_1} E_i\right)\right) + h^0\left(D, \mathcal{O}\left(K - 2\sum_{i=k_1+1}^{k_2} E_i\right)\right), \tag{3.3.2}$$

then any (b, c) satisfying the conditions $\alpha_i^2 b = -c$ is zero. On the other hand,

$$h^0 - 3k_i \geqslant 18 \quad \text{for } i = 1, 2$$

since $h^0 \geqslant 1000$ and $k_i \leqslant [(5/16)h^0] + 1$. So (3.3.1) shows that (3.3.2) is valid using our definition of k_3.

If $e = a - d$, then e vanishes twice on E_i for $i > k_2 + k_1$ and once at the $k_1 + k_2 - 1$ curves E_i where $1 < i \leqslant k_1 + k_2$. Now

$$k_3 \leqslant 2h^0 - \frac{15}{8}h^0 \leqslant \frac{1}{8}h^0.$$

So $h^0 - 3k_3 \geqslant 18$. So (3.3.1) shows that

$$h^0\left(D, \mathcal{O}\left(K - \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} 2E_i\right)\right) = h^0 - 3k_3$$

and since

$$k_1 + k_2 - 1 \geqslant h^0 - 3k_3$$

by elementary algebra, we see that $e = a - d = 0$. Hence $a = d = 0$.

The cases where $j > 1$ can be treated similarly.

4. In this section we consider a construction of stable bundles which is useful if there are curves of low genus on X. We begin with a well-known lemma.

Lemma 4.1. *Let C be a reduced and irreducible curve of arithmetic genus g in X. Let $\mathcal{M}$ be a line bundle of degrees $\geqslant 3g$. Then $\mathcal{M}$ is generated by its global sections.*

Proof. Let $x \in C$. Let π: $\tilde{C} \to C$ be the normalization of C. The image of $\pi^*(m_x)$ in $\mathcal{O}_{\tilde{C}}$ is a sheaf of ideals $\mathcal{I}$. We claim $\deg \mathcal{I} \geqslant -(g+1)$. Indeed, if $\mathcal{L}$ is a line bundle of very large degree on C and $\tilde{\mathcal{L}} = \pi^*(\mathcal{L})$

$$1 + \deg(\mathcal{I} \otimes \tilde{\mathcal{L}}) \geqslant h^0(\tilde{C}, \mathcal{I} \otimes \tilde{\mathcal{L}}) \geqslant h^0(C, m_x \otimes \mathcal{L})$$
$$\geqslant h^0(C, \mathcal{L}) - 1 \geqslant \deg \mathcal{L} - g.$$

Since $\deg(\mathcal{I} \otimes \tilde{\mathcal{L}}) = \deg \mathcal{I} + \deg \mathcal{L}$, we have established our claim.

Note that $\mathcal{M}$ is generated by global sections if $h^1(m_x \otimes \mathcal{M}) = 0$ for all $x \in C$. If $\mathcal{M}$ is not generated by global sections, Serre duality shows we have a nonzero map from $m_x \otimes \mathcal{M}$ to ω_C, where ω_C is the sheaf of dualizing differentials on C. This in turn gives a nonzero map for $\mathcal{I} \otimes \tilde{\mathcal{M}}$ to $\tilde{\omega}_C$. Since $\deg \mathcal{M} \geqslant 3g$, such a map is necessarily zero.

To construct our bundle, we suppose we are given two distinct algebraically equivalent curves C and C' of arithmetic genus g. We suppose C and C' are reduced and irreducible and $C \cdot K \geqslant 0$. Select divisors F and F' on C and C' respectively so that the points of F and F' are smooth points of C and C' and the support of F and F' is disjoint from $C \cap C'$. We suppose the degrees of F and F' are $\geqslant 3g$. We first construct a surjective map

$$\Phi: \mathcal{O}_X(C) \oplus \mathcal{O}_X(C') \to \mathcal{O}_C(C + F).$$

Indeed such a map is given by a pair (s, s'), where s is a section of $\mathcal{O}_C(F)$ and s' is a section of $\mathcal{O}_C(F + C - C')$. Since both these line bundles are generated by global sections by Lemma 4.1, taking s, s' generic produces a surjective map Φ. We can similarly construct a surjective map

$$\Phi' \colon \mathcal{O}_X(C) \oplus \mathcal{O}_X(C') \to \mathcal{O}_{C'}(C' + F')$$

given by sections t of $\mathcal{O}_{C'}(C' - C + F')$ and t' of $\mathcal{O}_{C'}(F')$. At a given point P of $C \cap C'$, we can choose $s(P) = 0$ and $t'(P) = 0$. Thus

$$\Psi = \Phi \oplus \Phi' \colon \mathcal{O}_X(C) \oplus \mathcal{O}_X(C') \to \mathcal{O}_C(C + F) \oplus \mathcal{O}_{C'}(C' + F')$$

is onto at P. Since we are free to choose s, t' generically, we can assume that Ψ is surjective. Let $\mathcal{E} = \operatorname{Ker} \Psi$. We compute $c_2(\mathcal{E})$.

$$\chi(\mathcal{E}) = -c_2(\mathcal{E}) + 2\chi(\mathcal{O}_X), \tag{4.1.1}$$

$$\chi\big(\mathcal{O}(C) \oplus \mathcal{O}(C')\big) = C^2 - C \cdot K + 2\chi(\mathcal{O}_X), \tag{4.1.2}$$

$$\chi\big(\mathcal{O}_C(C + F)\big) = \deg F - \tfrac{1}{2}(C^2 - C \cdot K), \tag{4.1.3}$$

$$\chi\big(\mathcal{O}_{C'}(C' + F')\big) = \deg F' - \tfrac{1}{2}(C^2 - C \cdot K), \tag{4.1.4}$$

so

$$c_2(\mathcal{E}) = \deg F + \deg F' \geqslant 6g.$$

Let $\mathcal{E}(s, s', t, t')$ be the bundle $\mathcal{E}$ we have constructed. Let us check the stability of such $\mathcal{E}(s, s', t, t')$ if s, s', t, t' are chosen generically. First, if $\mathcal{E}(s, s', t, t')$ is not H-stable for generic s, s', t, t', there is a line bundle $\mathcal{M}$ mapping to $\mathcal{O}(C) \oplus \mathcal{O}(C')$ so that $\Phi(\mathcal{M}) = 0$, $\Phi'(\mathcal{M}) = 0$ and $(c_1(\mathcal{M}) \cdot H) \geqslant 0$. By a standard semicontinuity argument (see §5) such an $\mathcal{M}$ would have to exist for all s, s', t, t'. In particular, take $s' = t = 0$. Say the map of $\mathcal{M}$ to $\mathcal{O}(C)$ is nontrivial. The map of $\mathcal{M}$ to $\mathcal{O}(C)$ would have to vanish on C. Hence $\mathcal{M}$ would map to $\mathcal{O}$. Since $(c_1(\mathcal{M}) \cdot H) \geqslant 0$, this implies that $\mathcal{M} = \mathcal{O}$. By our semicontinuity argument, we can assume that the generic $\mathcal{E}(s, s', t, t')$ is destabilized by a line bundle algebraically equivalent to zero. Since $2g - 2 = C(C + K)$ and $C \cdot K \geqslant 0$, we see that $\deg F \geqslant 3g > C^2$. Now the kernel $\mathcal{L}_1$ of the map $\Phi_{|C}$

$$\Phi_{|C} \colon \mathcal{O}_C(C) \oplus \mathcal{O}_C(C') \to \mathcal{O}_C(C + F)$$

is a line bundle on C of degree $C^2 - \deg F < 0$. Hence the map of $\mathcal{M}_C$ to $\mathcal{L}_1$ is zero since $\mathcal{M}$ has degree zero on C. So the map Ψ of $\mathcal{M}$ to $\mathcal{O}(C) \oplus \mathcal{O}(C')$ vanishes on C. Similarly Ψ vanishes on C'. So $\mathcal{M}$ maps to $\mathcal{O}(-C') \oplus \mathcal{O}(-C)$, which contradicts the $(c_1(\mathcal{M}) \cdot H) \geqslant 0$. We have established.

Proposition 4.2. *If $n \geqslant 6g$, there is a stable bundle $\mathcal{E}$ of rank two with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = n$.*

We remark that this Proposition establishes Theorem 1.2 unless X is of general type. Indeed if $h^0 > 1000$ and X is not of general type, then X must be elliptic. Thus we can apply the above theory when C and C' are elliptic.

Suppose that X is a surface of general type which has no exceptional curves of the first kind and that there are effective divisors E and E' so that $20C + E$ and $20C' + E'$ are canonical divisors.

Proposition 4.3. *Suppose $h^0 \geqslant 1000$ and $n \geqslant (3/2)h^0$. Then there is a stable bundle $\mathscr{E}$ on X with $c_1(\mathscr{E}) = 0, c_2(\mathscr{E}) = n$.*

Proof. We have Noether's formula

$$1 - h^1(\mathscr{O}) + h^2(\mathscr{O}) = \chi(\mathscr{O}_X) = \frac{1}{12}\big(K^2 + c_2(T)\big),$$

where T is the tangent bundle. We have $h^2(\mathscr{O}) = h^0(K)$, and the Miyoka-Yau inequality

$$3c_2(T) \geqslant K^2.$$

Combining these, we obtain

$$h^0(K) \geqslant \tfrac{1}{9}K^2 - 1.$$

Let us compute an estimate for the genus of C.

$$2g - 2 = C(K + C).$$

We have

$$0 \leqslant 20(C \cdot K) \leqslant K^2,$$

since $K \cdot E \geqslant 0$. Also

$$K^2 \cdot C^2 \leqslant (C \cdot K)^2 \leqslant \frac{1}{(20)^2}(K^2)^2.$$

So

$$C^2 \leqslant \frac{1}{400}K^2.$$

Thus

$$2g - 2 \leqslant \left(\frac{1}{20} + \frac{1}{400}\right)K^2,$$

$$2g - 2 \leqslant \left(\frac{1}{20} + \frac{1}{400}\right)\big(9h^0(K) + 1\big).$$

Since $h^0(K) \geqslant 1000$, then

$$6g \leqslant \tfrac{3}{2}h^0(K),$$

and the Proposition follows by Proposition 4.2.

Suppose X is a smooth hypersurface of degree d in $\mathbf{P}^3$ and that H is just a hyperplane section. Let C and C' also be hyperplane sections. Then the genus g of C is $\frac{1}{2}(d-1)(d-2)$, since C is a plane curve of degree d. On the other hand, we have

$$h^0(X, \mathcal{O}(K)) = \binom{d-1}{3} = \frac{1}{6}(d-1)(d-2)(d-3).$$

So there are stable bundles on X with $c_1(E) = 0$ and $c_2(E) = n$, as long as $n > 3(d-1)(d-2)$ and $d \geqslant 3$.

5. We retain the notation of §1. Let $\mathcal{E}$ be a bundle on $\tilde{D}$. We suppose that $\mathcal{E}_D$ is a subsheaf of $\mathcal{O}_D \oplus \mathcal{O}_D$ and that $H^0(D, \mathcal{E}_D) = 0$. We further assume that $\Lambda^2\mathcal{E}$ is isomorphic to $\mathcal{O}_{\tilde{D}}(+\Sigma n_i D_i)$ for some appropriate $n_i \in \mathbf{Z}$.

Our main object in this section is to establish:

Lemma 5.1. *Suppose that for each n, $\mathcal{E}$ can be extended to a bundle on $n\tilde{D}$. Then we can find a stable bundle $\mathcal{F}$ on X with $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = c_2(\mathcal{E})$.*

Proof. Let $\mathcal{L}$ be a very ample line bundle on Z so that $H^i(\mathcal{L} \otimes \mathcal{E}) = 0$ for $i > 0$ and $\mathcal{L} \otimes \mathcal{E}$ is generated by global sections. Let

$$P(n) = \chi(\mathcal{E} \otimes \mathcal{L}^{n+1}).$$

Let $N = h^0(\mathcal{E} \otimes \mathcal{L})$. Let $Q \to C$ be Grothendieck's Quot scheme. Thus there is a coherent sheaf $\mathcal{G}$ on $Q \times_C Z$ which is flat over Q and such that the Euler-Poincaré Polynomial of $\mathcal{G}$ over each closed point in Q is P and there is a given surjective map $\pi\colon \mathcal{O}^N \to \mathcal{G}$. Further π and $\mathcal{G}$ are universal with respect to these properties. In particular, choose a basis of $H^0(\mathcal{E} \otimes \mathcal{L})$. This choice determines a surjection $\mathcal{O}_{\tilde{D}}^N \to \mathcal{E} \otimes \mathcal{L}$. Let q be the corresponding closed point in Q.

Let t be a uniformizing parameter at $P \in C$. By shrinking C, we may assume that t vanishes only at P. We claim t does not vanish identically on Q_{red} in any neighborhood of q. Suppose not. Then for some n, t^n would vanish identically on Q near q since Q is a finite type over C. This means that we cannot lift the inclusion of mP into C to a map of mP to Q if $m > n$. But $\mathcal{E}$ can be extended to a bundle $\mathcal{E}_m$ on $m\tilde{D}$ and since $h^i(\mathcal{E} \otimes \mathcal{L}) = 0$, the sections of $\mathcal{E} \otimes \mathcal{L}$ extend to $\mathcal{E}_m \otimes \mathcal{L}$. But $mP \times_C Z = m\tilde{D}$. So the universal property of the Quot scheme gives a lifting of mP to Q. So our claim is established.

In particular, we can find a reduced curve C' in Q passing through q so that t does not vanish identically on C'. Let $Z' = Z \times_C C'$. For $s \in C'$, let Z'_s be the fiber of Z' over s. There is a coherent $\mathcal{F}$ on Z' so that $\mathcal{F}_q = \mathcal{F} \otimes \mathcal{O}_{Z'_q}$ is our original $\mathcal{E}$. (Note $Z'_q \cong \tilde{D}$.) By shrinking C', we may assume $\mathcal{F}$ is locally free and that $q \in C'$ is the only point mapping to P. Note $\det \mathcal{F}_r$ is algebraically equivalent to zero for $r \neq q$ since $\det \mathscr{F}_q$ is a sheaf of ideals. Thus $c_1(\mathscr{F}_r) = 0$. Let H be an ample line bundle on X and suppose that $\mathscr{F}_r$ is not H-stable for an infinite number of $r \in C'$. H stability is an open condition, so $\mathscr{F}_r$ must be H unstable for an uncountable number of s. Since there are only a countable number of line bundles mod algebraic equivalence, we can select a connected component A of the Picard group of X so that for an infinite number of $r \in C'$, there is an L_r in A with $h^0(L_r \otimes \mathscr{F}_r) \neq 0$ and $(c_1(L_r) \cdot H) \leqslant 0$. The set $T \subseteq A \times (C' - q)$ consisting of points (L, r) so that $h^0(L \otimes \mathscr{F}_r) \neq 0$ is closed and has infinite image in C'. There is a curve $C'' \subseteq T$ which has infinite image in C'. Let $\overline{C''}$ be the closure of C''. Then $\overline{C''}$ maps onto C'. Replacing C' by $\overline{C''}$, we see that we can assume that there is a line bundle $\mathscr{M}$ on $X \times C'$ so that $h^0(\mathscr{M}_r \otimes \mathscr{F}_r) \neq 0$ for $r \neq q$. We can pull back $\mathscr{M}$ to a line bundle again denoted by $\mathscr{M}$ on Z'. (This Z' is the fiber product of the original Z' by the base extensions we have made.) Thus $\mathscr{M}_q$ is trivial on the exceptional divisors D_i and $c_1(\mathscr{M}_D) \cdot H \leqslant 0$ on D. But semicontinuity, there is a nonzero section s of $\mathscr{M}_q \otimes \mathscr{E}$. We claim this is impossible. First, s must vanish on D. Since $\mathscr{E}_D \subseteq \mathscr{O} \oplus \mathscr{O}$, s would give a section of $(\mathscr{M}_q \oplus \mathscr{M}_q)_D$. Since $(c_1(\mathscr{M}_q) \cdot H) \leqslant 0$, $\mathscr{M}_q \mid_D \cong \mathscr{O}_D$. So $\mathscr{E}_D$ would have a section, which contradicts our assumptions. Consider s on each D_i. s vanishes on $D \cap D_i$, which is a line in $D_i = \mathbf{P}^2$. So s is a section of $\mathscr{F}_i(-1)$. But $\mathscr{F}_i$ is stable and $c_1(\mathscr{F}_i) = 1$. So s vanishes on D_i, and hence s vanishes.

Our bundle $\mathscr{F}_r$, $r \in C'$ must be H-stable for all but finitely many r. Since there are only a countable number of ample divisors mod algebraic equivalence, an infinite number of those $\mathscr{F}_r$ must be H-stable for any H.

6. In this section, we consider vector bundles on $\mathbf{P}^2$. Let L be a line in $\mathbf{P}^2$ and let $\mathscr{E}_3$ be a bundle on $3L$ so that $\mathscr{E}_2 = \mathscr{E}_3 \otimes \mathscr{O}_{2L}$ is isomorphic to $(\mathscr{O} \oplus \mathscr{O}(1))_{2L}$ and $\det \mathscr{E}_3 \cong (\mathscr{O}(1))_{3L}$. We suppose that if $\mathscr{L}$ is an invertible sheaf on $3L$ of degree -1, then $h^0(\mathscr{E}_3 \otimes \mathscr{L}) = 0$ (Such an $\mathscr{L}$ need not be $\mathscr{O}_{3L}(-1)$.)

Proposition 6.1. *There is a stable bundle $\mathscr{G}$ on $\mathbf{P}^2$ so that $\mathscr{G}_{3L} \cong \mathscr{E}_3$ and $c_2(\mathscr{G}) = 2$.*

Proof. There is an exact sequence

$$0 \to \mathscr{E}_1(-2) \to \mathscr{E}_3 \to \mathscr{E}_2 \to 0$$

where $\mathscr{E}_1 = (\mathscr{E}_3)_L$. Since $h^1(\mathscr{E}_1(-2)) = 1$, and $h^0(\mathscr{E}_2) = 4$, we see that at least 3 independent sections of $\mathscr{E}_2$ lift to $\mathscr{E}_3$. We claim there are two sections s and t of $H^0(\mathscr{E}_3)$ so that $s \wedge t$ maps to a nonzero element of $H^0(\wedge^2 \mathscr{E}_1)$. Let s_1 and s_2 be two sections of $\mathscr{E}_3$ which map to independent sections of $H^0(\mathscr{E}_1)$. (s_1 and s_2 exist, since the kernel of the map from $H^0(\mathscr{E}_2)$ to $H^0(\mathscr{E}_1)$ has dimension 1.) If $s_1 \wedge s_2 = 0$, they both must be sections of the subbundle

$\mathcal{O}_L(1) \subseteq \mathcal{E}_1$. Since s_1 and s_2 map to zero in the quotient $\mathcal{O}_L$ of $\mathcal{E}_1$, they must map to zero in the quotient $\mathcal{O}_{2L}$ of $\mathcal{E}_2$, since $H^0(\mathcal{O}_L) = H^0(\mathcal{O}_{2L})$. So $s_1 \wedge s_2$ maps to zero in $H^0(\det \mathcal{E}_2)$. But $H^0(\deg \mathcal{E}_2) = H^0(\deg \mathcal{E}_3)$, so s_1 and s_2 would be dependent in $\mathcal{E}_3$. But s_1 and s_2 generate $\mathcal{O}_L(1)$. So if $\mathcal{L}$ is the line bundle generated by s_1 and s_2, $\mathcal{L}$ would have degree 1. This contradicts our original assumption. So s_1 and s_2 generate $\mathcal{E}_3$ at a generic point.

We use s_1 and s_2 to define a map from $\mathcal{O}_{3L} \oplus \mathcal{O}_{3L}$ to $\mathcal{E}_3$. Dualizing we have a map Φ: $\mathcal{E}_3^\vee \to \mathcal{O}_{3L} \oplus \mathcal{O}_{3L}$. We can choose Φ so that the induced map of $\mathcal{E}_2^\vee$ to $\mathcal{O}_{2L} \oplus \mathcal{O}_{2L}$ maps the unique section of $\mathcal{E}_2^\vee$ to $(1,0)$. $\wedge^2\Phi$ is a map from $\mathcal{O}_{3L}(-1)$ to $\mathcal{O}_{3L}$, and so is represented by a section of $H^0(\mathcal{O}_{3L}(1)) = H^0(\mathbf{P}^2, \mathcal{O}(1))$. Thus there is a line L' so that $\wedge^2\Phi$ vanishes on L'. We can choose affine coordinates on $\mathbf{P}^2$ so that L is given by $y = 0$ and L' by $x = 0$. Locally around $(0,0)$, we can find a section $(1, g(x,y))$ of $\mathcal{O}_{3L} \oplus \mathcal{O}_{3L}$ which is in the image of Φ. Note that $g(0,y)$ can be represented as a polynomial $G(y)$ of degree $\leqslant 2$. Define a map

$$\Phi' \colon \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2} \to \mathcal{O}_{L'}(2)$$

by $\Phi'(h,l) = -G(y)h + l$, where we regard $H^0(\mathcal{O}_{L'}(2))$ as the polynomials in y of degree $\leqslant 2$. l is then a polynomial of degree zero. We claim Φ' is onto. Indeed $\Phi'(1,0) = -G(y)$. But g maps to zero in $\mathcal{O}_{2L}$, so $G(y) \equiv 0 \bmod (y^2)$. Hence G has degree 2 and Φ' is onto.

Thus $\operatorname{Ker}\Phi' = \mathcal{F}$ is locally free. Note that $\mathcal{F}_{3L} \supseteq \mathcal{E}_{3L}^\vee$ since on $L' \cap 3L$, the image of any other section of $\mathcal{E}_{3L}^\vee$ is dependent on $(1,g)$. Both $\mathcal{F}_{3L}$ and $\mathcal{E}_{3L}^\vee$ have determinant $\mathcal{O}(-1)$, so they must be isomorphic, since there is a map between them which is an isomorphism at a generic point.

We claim $\mathcal{F}$ is stable. If $\mathcal{F}$ were not stable, $\mathcal{F}(k)$ would have a section which vanished only at a finite number of points for some $k \leqslant 0$. In particular, we would have a section s of $\mathcal{E}_{3L}^\vee(k)$. Such an s would give a nonzero solution of $(\mathcal{O}_L \oplus \mathcal{O}_L(-1)(k)$. Thus $k = 0$. Further s is nowhere vanishing and so defines a subbundle of degree 0 of $\mathcal{E}_{3L}^\vee$, which contradicts our original assumption. We let $\mathcal{G} = \mathcal{F}$. One checks $c_2(\mathcal{G}) = 2$.

7. We continue with the notation of §1. We will now establish Theorem 1.1 and Theorem 1.2. Let us first turn to Theorem 1.1. Suppose $k \geqslant 2([h^0/2] + 1)$. Proposition 3.2 shows that with appropriate choice of x_i and v_i, we have

$$h^2\big(D, \operatorname{End}^0(\mathcal{E}_2 \otimes \mathcal{O}_D)(-2D)\big) = 0, \tag{7.1.1}$$

$$h^2\big(D, \operatorname{End}^0(\mathcal{E}_2 \otimes \mathcal{O}_D)(-2D - E_i)\big) \leqslant 1. \tag{7.1.2}$$

The remark at the end of §2 shows that we can find an extension of $\mathcal{E}_3$ of $\mathcal{E}_2$ to $3D$ which is nondegenerate over each E_j.

Using §6 we can then construct $\mathscr{F}_j$ on D_j so that $(\mathscr{F}_j)_{3D\cap D_j}$ is isomorphic to $(\mathscr{E}_3)_{3D\cap D_j}$ and $c_1(\mathscr{F}_j) = 1$, $c_2(\mathscr{F}_j) = 2$. Consequently, we can construct $\mathscr{G}_0$ on $2D + \tilde{D}$ which restricts to $\mathscr{F}_j$ on D_j and restricts to $\mathscr{E}_3$ and $3D$. We now show that

$$(7.1.3) \qquad h^2(\tilde{D}, \mathrm{End}^0(\mathscr{G}_0)(-2D)) = 0.$$

Let ω be the dualizing sheaf of $\tilde{D}$. Then $\omega_{D_j} \cong \mathscr{O}_{D_j}(-2)$ and $\omega_D = \mathscr{O}(K_X + 2E)$. Suppose

$$s \in H^0(\tilde{D}, \mathrm{End}^0(\mathscr{G}_0)(+2D) \otimes \omega).$$

If we show $s = 0$, (7.1.3) follows by Serre duality. First, s restricts to section s_j of $\mathrm{End}^0(\mathscr{G}_0) \otimes \omega \otimes \mathscr{O}_{D_j}(2D)$. But $\omega \otimes \mathscr{O}_{D_j}(2D) \cong \mathscr{O}_{D_j}$. Since $\mathscr{F}_j$ are stable, $H^0(D_j, \mathrm{End}^0(\mathscr{F}_j)) = 0$. Thus each s_j is zero, and s is actually a section of $H^0(D, \mathrm{End}^0(\mathscr{G}_0) \otimes \omega(2D - \Sigma E_j))$ which is

$$(7.1.4) \qquad H^0(D, \mathrm{End}^0(\mathscr{G}_0) \otimes K_D(2D)).$$

By (7.1.1) and Serre duality on D, (7.1.4) is zero, so $s = 0$. By the results of §2 $\mathscr{G}_0$ can be lifted to arbitrary large infinitesimal neighborhoods of D_0. After a suitable base extension, §5 shows that $\mathscr{G}_0$ can be lifted to Z. Thus Theorem 1.1 is established as n is even. We even see that the bundle $\mathscr{E}$ constructed satisfies $h^2(X, \mathrm{End}^0(\mathscr{E})) = 0$. The theorem follows for odd n by the following:

Lemma 7.2. *Let $\mathscr{E}$ be an H-stable bundle on X with $c_1(\mathscr{E}) = 0$ and $h^2(X, \mathrm{End}^0(\mathscr{E})) = 0$. Then for any $n \geqslant c_2(\mathscr{E})$, there is an H-stable bundle $\mathscr{E}'$ with $c_2(\mathscr{E}') = n$, $c_1(\mathscr{E}') = 0$ and $h^2(X, \mathrm{End}^0(\mathscr{E}')) = 0$.*

Proof. We construct the variety Z of §1 with $k = 1$. Let $\mathscr{E} = \mathscr{E}'_D$. $\mathscr{E}_{E_1}$ is $\mathscr{O} \oplus \mathscr{O}(1)$. There is a stable bundle $\mathscr{F}_1$ on $D_1 = \mathbf{P}^2$ which is isomorphic to $\mathscr{O}_{E_1} \oplus \mathscr{O}_{E_1}(1)$ when restricted to the line E_1 and with $c_2(\mathscr{F}_1) = 1$. We can then produce a bundle $\mathscr{G}$ on $\tilde{D}$ by gluing $\mathscr{F}_1$ to $\mathscr{E}$. Suppose $s \in H^0(X, \mathrm{End}^0(\mathscr{G}) \otimes \omega)$. We claim $s = 0$. ω_{D_1} is $\mathscr{O}(-2)$, so s must vanish on D_1. Thus s is a section of $H^0(D, \mathrm{End}^0(\mathscr{G}) \otimes \mathscr{O}(K_D))$. If $s \neq 0$, we would get a nonzero section of $H^0(X, \mathrm{End}^0(\mathscr{E}) \otimes \mathscr{O}(K_X))$. Arguing as before, we can produce an H-stable $\mathscr{F}$ on X with $c_2(\mathscr{F}) = c_2(\mathscr{E}) + 1$ and $h^2(X, \mathrm{End}^0(\mathscr{F})) = 0$.

Next we establish Theorem 1.2. If $k - 1 = k_1 + k_2 + k_3$ in the notation of §3, then $h^0(D, \mathrm{End}^0(\mathscr{E})(K - E + E_i)) = 0$. Arguing as before, we can construct an H-stable $\mathscr{E}$ with

$$c_2(\mathscr{E}) = 2(k_1 + k_2 + k_3 + 1),$$

i.e.,

$$c_2(\mathscr{E}) = 4\left(h^0 - \left[\frac{5}{8}h^0\right]\right) + 2,$$

with the property that $h^2(X, \mathrm{End}^0(\mathscr{G})) = 0$. Theorem 1.2 follows as before.

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