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SOME REMARKS ON VOLUME AND DIAMETER OF RIEMANNIAN MANIFOLDS

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In this note, we provide some remarks concerning a recent paper of Burger and Schroeder [4]. Their paper gives a relation between volume, diameter and the first eigenvalue of the Laplacian for compact quotients of rank 1 symmetric spaces. Here we will show how their results lead to analogous results for coverings of a fixed, but arbitrary, Riemannian manifold.

Theorem 2 of [4] states:

Theorem ([4]). *Let* $\mathbf{H} = \mathbf{H}^n_{\mathbf{R}}$ *for* $n \geqslant 4$, $\mathbf{H}^n_{\mathbf{C}}$, $\mathbf{H}^n_{\mathbf{H}}$, *or* $\mathbf{H}^2_{\mathbf{O}}$.

Then there are constants a_n, b_n depending only on n such that for M a compact quotient of $\mathbf{H}$,

$$\lambda_1(M) \leqslant \frac{a_n + b_n \log(\mathrm{vol}(M))}{\mathrm{diam}(M)}.$$

Note that for $\mathbf{H} = \mathbf{H}^n_{\mathbf{R}}$ or $\mathbf{H}^n_{\mathbf{C}}$ we may have $\lambda_1(M)$ arbitrarily small. The fact that this is not the case for $\mathbf{H} = \mathbf{H}^n_{\mathbf{H}}$, or $\mathbf{H}^2_{\mathbf{O}}$ follows from Kazhdan's Property T [5]. The fact that the isometry groups of these symmetric spaces have Property T is due to Kostant [11].

Our main result here is:

Theorem 1. *Let M be an arbitrary compact manifold, and M_i a family of finite coverings of M. If there exists $C > 0$ such that $\lambda_1(M_i) > C$, then there exist positive constants a, b, and c such that*

$$a < \frac{\log(\mathrm{vol}(M_i)) + c}{\mathrm{diam}(M_i)} < b.$$

Proof. We first observe that, according to [7], for each n, and in particular for $n = 4$, there exists a compact quotient

$$N \text{ of } \mathbf{H}^n_R$$

with a surjection $\pi_1(N) \to \mathbb{Z} * \mathbb{Z}$.

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Now suppose that $\pi_1(M)$ is generated by k elements. Then there is a finite covering N' of N and a surjection ϕ: $\pi_1(N') \to \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k \text{ times}} \to \pi_1(M)$.

Let N_i' be the coverings of N' induced from those of M_i—

$$\pi_1(N_i') = \phi^{-1}(\pi_1(M_i)).$$

Claim. There exist constants C', d' and k' such that

(a) $\lambda_1(N_i') > C'$,
(b) $\operatorname{vol}(N_i') = d' \operatorname{vol}(M_i)$,
(c) $\operatorname{diam}(N_i') > k' \operatorname{diam}(M_i)$.

Proof. (a) is just Theorem 4 of [3].

(b) follows with $d' = \operatorname{vol}(N')/\operatorname{vol}(N)$

(c) follows from the Milnor-Svarc lemma [8], which implies that $\operatorname{diam}(N_i')$ and $\operatorname{diam}(M_i)$ are both estimated up to constants by the group-theoretic diameter of $\pi_1(M)/\pi_1(M_i)$, relative to a fixed set of generators for $\pi_1(M)$.

We now apply the theorem of [4] to show that there is a constant a' with

$$a' < \frac{\log(\operatorname{vol}(N_i')) + c}{\operatorname{diam}(N_i')}.$$

It follows from the Claim that

$$\frac{\log(\operatorname{vol}(M_i)) + \log(d') + c}{\operatorname{diam}(M_i)} \geqslant \frac{\log(\operatorname{vol}(N_i')) + c}{k' \operatorname{diam}(N_i')}$$

$$\geqslant (\text{const}) \frac{\log(\operatorname{vol}(N_i')) + c}{\operatorname{diam}(N_i')} \geqslant (\text{const}) a'.$$

The inequality

$$\frac{\log(\operatorname{vol}(M_i))}{\operatorname{diam}(M_i)} \leqslant b$$

is true in complete generality, and follows immediately from the Comparison Theorem. b depends only on a lower bound for the curvature of M and the dimension of M. Combining these gives Theorem 1.

We remark here that one could also prove Theorem 1 by use of a graph theoretic isoperimetric inequality due to Alon and Milman [1], see also Gromov-Milman [12]. It is worth remarking that the ideas that go into the proof of [1] (which is completely elementary) are similar in many points to the ideas behind the proof of [4].

We may extend the ideas in our proof of Theorem 1 to show:

Theorem 2. *For each n, there exists a compact hyperbolic n-manifold N and coverings N_i of N such that*

(i) $\lambda_1(N_i) \to 0$ *as* $i \to \infty$.

(ii) *There exists* $C > 0$ *such that* $\log(\mathrm{vol}(N_i))/\mathrm{diam}(N_i) > C$.

Proof. Let us begin with an arbitrary manifold M with a family of coverings M_i such that $\lambda_1(M_i)$ is bounded away from 0 and $\mathrm{diam}(M_i) \to \infty$. For instance, we could choose M with $\pi_1(M) = SL(2, \mathbb{Z})$, and M_i the congruence coverings of M.

Now let $M' = M \times S^1$, and for each k let M'_k be the covering of M' whose fundamental group is $\pi_1(M_i) \oplus ([\log(\mathrm{diam}(M_i))] \times \mathbb{Z}) \subset \pi_1(M) \oplus \mathbb{Z}$.

To see that

$$\lambda_1(M'_k) \to 0 \quad \text{as } k \to \infty,$$

we compute the isoperimetric constant $h(M'_k)$. But dividing M'_k into two pieces along the fibers of antipodal points of the $[\log(\mathrm{diam}(M_i))]$-fold cover of S^1, shows that

$$h(M'_k) \leqslant \frac{2\,\mathrm{vol}(M_k)}{1/2[\log(\mathrm{diam}(M_k))]\mathrm{vol}(M_k)} \to 0 \quad \text{as } k \to \infty.$$

The fact that $\lambda_1(M'_k) \to 0$ as $k \to \infty$ then follows from Theorem 1 of [3], or can be seen directly. But $\mathrm{vol}(M'_k) = \mathrm{vol}(M_k) \times [\log(\mathrm{diam}(M_k)\,]$ and $\mathrm{diam}(M'k) \leqslant (\mathrm{const})(\mathrm{diam}(M_k) + [\log(\mathrm{diam}(M_k)\,]$ as can be seen again from the Milnor-Svarc lemma.

Hence,

$$\frac{\log(\mathrm{vol}(M'_k))}{\mathrm{diam}(M'_k)} \geqslant \frac{\log[\log(\mathrm{diam}(M_k))] + \log\mathrm{vol}(M_k)}{2\,\mathrm{const}(\mathrm{diam}(M_k))} \geqslant \mathrm{const},$$

since

$$\frac{\log(\mathrm{vol}(M_k))}{\mathrm{diam}(M_k)} \geqslant \text{const by Theorem 1}.$$

We now repeat the argument of Theorem 1 to find a hyperbolic manifold N with a surjective map $\pi_1(N) \to \pi_1(M \times S^1)$, whose coverings have the same properties.

As an example of this circle of ideas, we show:

Theorem 3. *For each $n \geqslant 2$, let us choose generators for $SL(n, \mathbb{Z})$. Then there is a constant C_n depending only on n and the choice of generators, such that* $\operatorname{diam}(SL(n, \mathbb{Z}/p)) < C_n \log p$.

Proof. We first observe that if $\pi_1(M) = SL(n, \mathbb{Z})$, the congruence coverings M^p of M satisfy $\lambda_1(M^p) > C$ for some $C > 0$.

When $n \geqslant 3$, this follows from the fact that $SL(n, \mathbf{R})$ has Property T and [3]. When $n = 2$ this follows from [3] and Selberg's Theorem [9] that $\lambda_1(\mathbf{H}^2/\Gamma_p) \geqslant 3/16$, where Γ_p is the pth congruence subgroup. The fact that [3] applies despite the noncompactness of $\mathbf{H}^2/SL(2, \mathbb{Z})$ is discussed in [2].

It follows from Theorem 1 that

$$\frac{\log(\operatorname{vol}(SL(n, \mathbb{Z}/p)))}{\operatorname{diam}(SL(n, \mathbb{Z}/p))} \geqslant a$$

for some $a \geqslant 0$. But $\log(\operatorname{vol}(SL(n, \mathbb{Z}/p))) \leqslant \log(p^{n^2}) = C_n \cdot \log(p)$ and the theorem is proved.

Corollary 4. *For p a prime number, consider the set $Vp = \{0, 1, \cdots, p-1, \infty\}$. Then there is a C independent of p such that any $a, b \in Vp$ can be joined by a sequence of at most $C \log(p)$ moves of the type $x \to x+1$, $x-1$, $x \to \bar{x}$, where $\bar{x}$ is the multiplicative inverse of* $x(\operatorname{mod} p)$, $\bar{0} = \infty$, *and* $\overline{\infty} = 0$.

Proof. This is the graph of $SL(2, \mathbb{Z})/\Gamma_p^*$, where

$$\Gamma_p^* \supset \Gamma_p \text{ is the Hecke group} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \atop (\operatorname{mod} p) \right\}.$$

We close this paper with the following example, shown to us by John Millson and based on work of R. Livne [6]:

Theorem 5. *There exists a compact quotient M of $\mathbf{H}^2_{\mathbf{C}}$, such that $\pi_1(M)$ surjects onto $\mathbb{Z} * \mathbb{Z}$.*

At present, we don't have examples of $\mathbf{H}^n_{\mathbf{C}}$, $n > 2$, whose fundamental group surjects onto $\mathbb{Z} * \mathbb{Z}$.

We remark that Theorem 5 allows us to extend those results of [4] (in particular remark (iii)) and the present paper which only applied to $\mathbf{H}^n_{\mathbf{R}}$ also to $\mathbf{H}^2_{\mathbf{C}}$.

Proof. We consider the following situation: for each N, let $X(N)$ be the compactified moduli space for elliptic curves with level N structure, and $E(N)$ the universal elliptic curve of level N. Then when $N \geqslant 3$, $X(N)$ is a smooth Riemann surface, and there is a submersion $E(N) \to X(N)$ which, away from finitely many points of $X(N)$, is a fibration whose fibers are elliptic curves.

There are N^2 sections of this fibration taking a point in $X(N)$ to one of the N^2 points of order N on the fiber.

For each integer d, let $S_d(N)_\Delta$ be a d-fold cyclic branched covering of $E(N)$ which is totally ramified along these N^2 sections, and is a covering away from these sections.

By calculating Chern numbers, and appealing to a characterization of quotients of $\mathbf{H}^2_{\mathbf{C}}$ due to Yau [10], Livne showed in [6] that $S_d(N)_\Delta$ can be realized as a compact quotient of $\mathbf{H}^2_{\mathbf{C}}$ precisely when (N, d) is one of the pairs $(7,7)$, $(8,4)$, $(9,3)$ or $(12,2)$. For these values, he also explicitly constructs a realization of $\pi_1(S_d(N)_\Delta)$ as a discrete, cocompact subgroup of $PU(2,1)$. In all of these cases, X is a Riemann surface of genus $\geqslant 2$.

We now claim:

Claim: $\pi_1(S_d(N)_\Delta)$ surjects onto $\mathbb{Z}*\mathbb{Z}$.

Proof. It suffices to show that $\pi_1(S_d(N)_\Delta)$ surjects onto $\pi_1(X(N))$.

So pick a base-point p in $S_d(N)_\Delta$, and a point p_E in $E(N)$ lying over p and not a point of order N.

If γ is any loop at p, we jiggle it slightly if necessary to guarantee that γ avoids the singular values on $X(N)$. We then use the fact that $E(N) \to X(N)$ is a fibration away from the singular values to lift γ to a curve on $E(N)$ starting at p_E. Since the fibers are connected, we may close this curve up to a loop $\tilde{\gamma}$ based at p_E which projects onto γ.

We may now jiggle $\tilde{\gamma}$ so that it avoids the N^2 sections, and so lift it to a curve on $S_d(N)_\Delta$. Again since the fibers are connected, we may close this up to a closed curve on $S_d(N)_\Delta$ whose homotopy class projects onto that of γ, showing that $\pi_1(S_d(N)_\Delta)$ surjects onto $\pi_1(X(N))$. This completes the claim, and hence the theorem.

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