J. DIFFERENTIAL GEOMETRY
26 (1987) 385–396

FLAT SPACETIMES

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Abstract

Those closed pseudo-Riemannian manifolds covered by Minkowski space M are classified up to finite covers. The simply transitive isometric actions on M are listed. Some spacetimes with 2 ends satisfying a causality condition are analyzed.

We call a 4-manifold X wtih a metric g of signature $(+,+,+,-)$ a spacetime. We will assume that g has zero curvature, i.e. X is flat. Then X is a special kind of affine manifold, namely an affine manifold with a parallel metric of the given signature. We also assume X is complete in the sense that geodesics on X extend for all time. This implies that the universal cover of X is Minkowski space M.

We will classify such X's under the assumption that X is compact. We prove in §1 that π has a solvable subgroup of finite index, using theory developed in [5] with W. Goldman. This result has been extended to higher dimensions by Goldman and Kamashima [6] but our proof is more geometric. For a noncompact counterexample see [8] and for further discussion see [9].

The classification also uses a theorem of Auslander's on unipotent simply transitive affine actions [1]. For subgroups of the isometry group $\mathscr{P}$ of Minkowski space, those are classified in §2. This is extended to all simply transitive actions in §3. Then in §4 we give our classification. It extends to dimension 4 that given by Auslander and Markus for 3-manifolds [2].

It is conceivable that if X is compact then g is automatically complete. A counterexample would be a very interesting spacetime: its curvature and global topology would not account for its failure to be complete. It would also be a valuable example in the theory of affine manifolds.

In §5 we discuss some two ended flat spacetimes with respect to their causal structure.

Received February 28, 1985, and, in revised form August 21, 1985. This work was partially supported by the National Science Foundation.

1. π is virtually solvable

A group with a solvable subgroup of finite index is called virtually solvable. We show

Theorem 1. *If $\pi \subset \mathscr{P}$ and π acts freely and properly discontinuously on M with compact quotient then π is virtually solvable.*

Proof. Taking an affine transformation to its linear part defines a natural homomorphism λ: $\mathscr{P} \to SO(3,1)$. We let $\Gamma = \lambda(\pi)$ and we let G be the algebraic hull of Γ. G is an algebraic subgroup of $SO(3,1)$. The identity component G_0 is of finite index in G, since G is algebraic. We will show G_0 is solvable.

As in [7] one knows that each $A \in \Gamma$ satisfies $\det(A - I) = 0$. As first noted by M. Hirsch, this reflects algebraically the fact that the nontrivial elements of π act without fixed points in M. It follows that this nontrivial polynomial equation holds on G. This shows $\dim G_0 < \dim SO(3,1)$.

Suppose G_0 is not solvable. Then it contains a semisimple connected subgroup S. As $\dim S < \dim SO(3,1)$, S is either $SO(3)$ or $SO(2,1)_0$, in properly chosen coordinates on M. In either case, S is maximal among the connected Lie subgroups of $SO(3,1)_0$, so $G_0 = S$. Thus G_0 fixes a vector v of nonzero length.

Let π_0 be the kernel of the natural map $\pi \to G/G_0$. Let $X_0 = M/\pi_0$ be the corresponding finite cover of our given spacetime X. Let $\tilde{Y}$ be the parallel vector field on M determined by v and let Y be the corresponding vector field on X_0. The 1-form ω on X_0 dual to Y is parallel and hence closed.

Perturb ω to a closed 1-form ω_1 with rational periods P, where P is the set of real numbers obtained by integrating ω_1 around closed loops in X_0. As π is finitely generated, P is discrete and $\mathbf{R}/P$ is a circle. Also $\omega_1(Y)$ never vanishes, assuming ω_1 is close enough to ω, since v has nonzero length.

Let $b \in X_0$ be a basepoint and define θ: $X_0 \to \mathbf{R}/P$ as the indefinite integral $\theta(y) = \int_b^y \omega_1$. Then θ is a fibration of X_0 over circle [11]. Let K be a connected component of a fiber of θ. Then K is a connected cross-section to the flow φ on X_0 generated by Y.

Since Y is parallel and X_0 is flat, the flow φ has a transverse affine structure that induces an affine structure on K. Lifting φ to the universal cover M one obtains the one parameter group $\tilde{\varphi}$ of translations of M with velocity v. So $\tilde{K}$ is naturally identified with the orbit space of this flow $M/\mathbf{R}v$, cf. [4]. This orbit space is just an affine 3-space so K is complete in its induced affine structure.

A complete compact affine 3 manifold has solvable fundamental group by [5]. Thus $\pi_1 K$ is solvable. K is the fiber of a fibration of X_0 over the circle so the homotopy exact sequence of this fibration shows that π_0 is an extension of $\pi_1 S^1 = \mathbf{Z}$ by $\pi_1 K$. Hence π_0 is solvable. q.e.d.

It follows that the linear holonomy of X_0, i.e. the subgroup $\Gamma_0 = \lambda(\pi_0) \subset SO(3,1)$, is also solvable.

Now since Γ_0 has finite index in Γ, the algebraic hull $G(\Gamma_0) \subset G$ has finite index. Thus $G_0 \subset G(\Gamma_0)$, so G_0 is solvable too.

2. Unipotent groups

We will consider a subgroup $U \subset \mathscr{P}$ that is unipotent, i.e. if $Ax + v$ is an affine transformation in U then the only eigenvalue of A is 1. We suppose also that the action of U on M is simply transitive, i.e. that given m_1, $m_2 \in M$ there is exactly one $u \in U$ with $um_1 = m_2$.

Clearly such a U can be used to construct flat spacetimes $X = M/\pi$ for any discrete subgroup $\pi \subset U$. By a different procedure, these U's arise from any compact flat spacetime. We need to classify them. We write $(A \mid v)$ for the infinitesimal affine motion with linear part A and translational part v.

Theorem 2. *Let U be a unipotent subgroup of $\mathscr{P}$ that acts simply transitively on M. Then $U = \exp(L)$ where L is a nilpotent Lie algebra of infinitesimal isometries of Minkowski space M.*

In suitable linear coordinates $(w, x, y, z) = v$ on M, g is given by $g(v, v) = 2wz + x^2 + y^2$ and $L = L_{\beta,\varepsilon}$ is all pairs $(A \mid v)$ where

$$A = \begin{pmatrix} 0 & -\beta y & -\varepsilon z & 0 \\ 0 & 0 & 0 & \beta y \\ 0 & 0 & 0 & \varepsilon z \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

for fixed $\beta \geqslant 0$, $\varepsilon \geqslant 0$. The parameters β, ε are uniquely determined by U except for the rescaling

$$(\beta, \varepsilon) \to (\lambda\beta, \lambda^2\varepsilon), \qquad \lambda > 0.$$

Proof. Since U is simply transitive, it is simply connected. One can identify the affine motions of 4-space with the linear motions of 5-space that preserve an affine hyperplane. Thus we may regard U as a linear unipotent group and take logarithms of the elements in U to generate L. L is a Lie algebra of nilpotent matrices that represent infinitesimal isometries of Minkowski space. Since U is locally transitive, the correspondence $(A \mid v) \to v$ from L to $\mathbf{R}^4$ is surjective. As U is four dimensional, it is a bijective correspondence and we can write $A = A(v)$. We proceed to reduce A to the indicated normal form.

We first show

Lemma 1. $\lambda(U)$ *fixes a lightlike vector* v_0.

Proof. By Engels' theorem, some vector v_0 is fixed by $\lambda(U)$ (i.e., annihilated by the nilpotent Lie algebra $\lambda(L)$).

If v_0 is timelike then $\lambda(U)$ preserves $v_0^{\perp}$ so $\lambda(U) \subset SO(3)$. Thus U preserves a Euclidean inner product on $\mathbf{R}^4$. As a unipotent orthogonal matrix is trivial, this implies $\lambda(U) = 1$.

If v_0 is spacelike then $v_0^{\perp}$ is Minkowski 3-space and we have $\lambda(U) \subset SO(2,1)$. If $\lambda(U)$ does not fix a lightlike vector in $v_0^{\perp}$, one can again find a fixed spacelike vector v_1 in $v_0^{\perp}$. In $\{v_0, v_1\}^{\perp} = SO(1,1)$ one sees that the unipotent group $\lambda(U)$ is trivial. q.e.d.

We choose linear coordinates w, x, y, z on $\mathbf{R}^4$ so that v_0 is the unit vector in the w direction and so that g has the form $2wz + x^2 + y^2$. We also write $\mathbf{R}^4 = \mathbf{R} + \mathbf{R}^2 + \mathbf{R}$, $v = v_1 + v_2 + v_3$ so that v_1, v_3 are 1-vectors and v_2 a 2-vector. Then if J denotes the linear map that switches v_1 and v_3 and fixes v_2 the infinitesimal isometry A for g is a solution to $A^tJ + JA = 0$. As A is nilpotent, it must have the block form

$$A(v) = \begin{pmatrix} 0 & -\gamma^t & 0 \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

This defines in our setting a linear map $\gamma(v)$, $\gamma\colon \mathbf{R}^4 \to \mathbf{R}^2$ that we may express as $\gamma(v) = \gamma_1(v_1) + \gamma_2(v_2) + \gamma_3(v_3)$.

Lemma 2. $\gamma_1 = 0$, $\gamma_2^2 = 0$.

Proof. For any v, v' the commutator of $(A(v) \mid v)$ and $(A(v') \mid v')$ is of the form $(0 \mid v'')$. Since L is a Lie algebra, we must have $\gamma(v'') = 0$.

Explicitly $v_1'' = -\gamma(v)^t v_2' + \gamma(v')^t v_2$, $v_2'' = \gamma(v)v_3' - \gamma(v')v_3$, and $v_3'' = 0$.

Let $v_3 = v_3' = v_2' = 0$. Then $0 = \gamma(v'') = \gamma_1(\gamma_1(v_1') \cdot v_2)$. Letting v_1', v_2 vary we see $\gamma_1 = 0$.

Now take $v_3 = 0$ only. Then $0 = \gamma(v'') = \gamma_2(\gamma_2(v_2)v_3')$, where we have used $\gamma_1 = 0$. Letting v_2 and v_3' vary shows $\gamma_2^2 = 0$.

Conversely if $\gamma_1 = \gamma_2^2 = 0$ then indeed $\gamma(v'') = 0$. q.e.d.

So U is determined by $\gamma_2\colon \mathbf{R}^2 \to \mathbf{R}^2$ and $\gamma_3\colon \mathbf{R} \to \mathbf{R}^2$ with $\gamma_2^2 = 0$. We regard γ_3 as a 2-vector. One can make the following changes in γ_2, γ_3 by g-isometries that preserve the subspace $\mathbf{R}v_0$:

1) $\gamma_2' = \lambda\gamma_2$, $\gamma_3' = \lambda^2\gamma_3$, $\lambda \in \mathbf{R}$, $\lambda \neq 0$,

2) $\gamma_3' = B\gamma_3$, $\gamma_2' = B\gamma_2 B^{-1}$, $B \in 0(2)$,

3) $\gamma_2' = \gamma_2$, $\gamma_3' = \gamma_3 - \gamma_2(u)$, $u \in \mathbf{R}^2$.

Here 1) represents a rescaling of the w and z axes, 2) represents a rigid motion of the $x - y$ plane and 3) a change of coordinates that fixes $v_0^\perp$ (the $w - x - y$ space) and changes the z-axis.

Using 2) we can put the matrix γ_2 into the form $\binom{0\ \beta}{0\ 0}$, so $\gamma_2\binom{x}{y} = \binom{\beta y}{0}$.

Suppose $\beta \neq 0$. Then 3) can be used to put γ_3 into the form $\binom{0}{\varepsilon}$. If $\beta = 0$ then a) can be used to the same effect. So we reach the normal form for γ and A. It is not hard to arrange β, $\varepsilon \geqslant 0$ by using $\lambda = -1$ in 1) and $B = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ in 2) if necessary.

We now show that (β, ε) and (β', ε') give the same U only if $\beta' = \lambda\beta$, $\varepsilon' = \lambda^2\varepsilon$, for some $\lambda > 0$. The case $\beta = \varepsilon = 0$ is uniquely characterized by the property $\lambda(U) = 1$.

Lemma 3. *If* $(\beta, \varepsilon) \neq (0,0)$ *then the w-axis W consists of all the lightlike vectors fixed by* $\gamma(U)$.

Proof. Suppose $v = (w_1, x_1, y_1, z_1)$ is lightlike and fixed, i.e. $2w_1z_1 + x_1^2 + y_1^2 = 0$, $-\beta y x_1 + \varepsilon z y_1 = 0$, $\beta y z_1 = 0$, and $\varepsilon z z_1 = 0$ for all y, $z \in \mathbf{R}$. Whether $\beta \neq 0$ or $\varepsilon \neq 0$, one has $z_1 = 0$. This gives $x_1^2 + y_1^2 = 0$ so $x_1 = y_1 = 0$. $\therefore v = (w_1, 0, 0, 0)$. q.e.d.

Thus W and its orthogonal the w, x, y, 3-space $W^\perp$ are determined by U when $\gamma(U) \neq 1$. It is now easy to check that only the coordinate changes 1), 2) and 3) above are relevant for comparing (β, ε) and (β', ε'). The special case $\beta = 0$ corresponds to $\gamma \mid W^\perp = 0$. The special case $\varepsilon = 0$ corresponds to the other possibility for rank(γ: $\mathbf{R}^4 \to \mathbf{R}^2) < 2$. It only remains to analyze the cases $\beta > 0$, $\varepsilon > 0$: but one can check that choosing $\varepsilon = 1$ forces the value of β. q.e.d.

If $(\beta, \varepsilon) \neq (0,0)$, we call the corresponding unipotent group U_ρ where $\rho = \beta^2/\varepsilon \in [0, \infty]$. Thus the only U's are the translation group T and the U_ρ's.

3. Simply transitive actions

We will consider a Lie group $H \subset \mathscr{P}$ that acts simply transitively on M. Then H must be solvable [A, M]. Auslander proved that there is an associated unipotent group U that also acts simply transitively on M, namely the unipotent radical of the algebraic hull G of $H[A]$. We will use this fact, together with the results of the previous section, to find all H's.

Suppose U is not the translation group T, so U corresponds to a nonzero pair (β, ε). If $\beta = 0$ there is a two dimensional space of parallel vector fields corresponding to the $w - x$ plane. If $\beta \neq 0$ then the $w - x$ plane is determined as the kernel of γ. Thus the flag $W \subset (w - x \text{ plane}) \subset W^\perp$ is in both cases determined by U and the normalizer N of U preserves this flag. In particular elements of N have only real eigenvalues.

Since U is the unipotent radical of G, $U \lhd G$. Thus $G \subset N$. Let L_G be the Lie algebra of G. Then elements of L_G have the form $(A \mid v)$ where

$$A = \begin{pmatrix} \lambda & -\delta^* & 0 \\ 0 & 0 & \delta \\ 0 & 0 & -\lambda \end{pmatrix}$$

relative to the splitting $\mathbf{R}^4 = \mathbf{R} \oplus \mathbf{R}^2 \oplus \mathbf{R}$. The middle block vanishes because G preserves the flag discussed above.

We will show $\lambda = 0$. Subtracting off an element of $L_{\beta,\varepsilon}$ we may suppose $v = 0$. Rescaling, we may suppose $\lambda = 1$. By choosing the z axis differently we may suppose $\delta = 0$. Computing the commutator of $(A \mid 0)$ with the typical element $(A(v) \mid v) \in L_{\beta,\varepsilon}$ gives (in this new basis which is like that used in Lemma 2)

$$\left[\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 0 & -\gamma^t & 0 & v_1 \\ 0 & 0 & \gamma & v_2 \\ 0 & 0 & 0 & v_3 \end{array}\right)\right] = \left(\begin{array}{ccc|c} 0 & -\gamma^t & 0 & v_1 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & -v_3 \end{array}\right).$$

Since $(A \mid 0)$ must normalize $L_{\beta,\varepsilon}$, we must have $\gamma(v_1, 0, -v_3) = \gamma(v_1, v_2, v_3)$. This gives $\gamma_2 = \gamma_3 = 0$, so $\beta = \varepsilon = 0$ contrary to assumption. So $\lambda = 0$.

It follows that L_G is nilpotent, so G and H are unipotent. But any unipotent connected Lie group is Zariski closed, so $H = G$. Taking this unipotent radical we see $H = U = U_\rho$. We have

Theorem 3. *If $H \subset \mathscr{P}$ acts simply transitively on Minkowski space M then either*

1) *H is one of the unipotent groups U_ρ, $\rho \in [0, \infty]$, or*

2) *the unipotent radical U of the algebraic hull G of H is precisely the group T of translations of M.*

We now restrict to case 2) with $H \neq T$, i.e. we suppose H is not unipotent. On the one hand $\lambda(H) = H/H \cap T \subset G/T = G/U$. The quotient G_0/U is abelian (indeed it is isomorphic to a linear group of diagonal matrices). Thus $\lambda(H)$ is abelian. A nontrivial connected abelian subgroup of $SO(3,1)$ has dimension $d \leqslant 2$. As the translation subgroup $H \cap T$ is normalized by H, $\lambda(H)$ preserves the corresponding subspace of M. We find

Theorem 4. *If $H \subset \mathscr{P}$ acts simply transitively on M then either H is unipotent (and so described by Theorem 2) or $\lambda(H)$ is a nonunipotent 1-parameter subgroup of $\lambda(\mathscr{P}) = SO(3,1)$ and $H \cap T = \ker(\lambda \mid H)$ is a 3-*

dimensional invariant subspace of $\lambda(H)$. In the latter case there are linear coordinates w, x, y, z on M for which $g = 2wz + x^2 + y^2$ and the Lie algebra L_H consists of all pairs $(A \mid v)$ where

a) $$v = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \quad A(v) = \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y \end{pmatrix},$$

or

b) $$v = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \quad A(v) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & -z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Suppose $H \cap T$ has dimension 3 and choose $A_0 \neq 0$ in the Lie algebra of $\lambda(H)$. A_0 is not nilpotent.

As in the proof of Theorem 1, $\det(\Lambda - I) = 0$ for all $\Lambda \in \lambda(H)$. Thus $\det A_0 = 0$, that is A_0 is singular.

As A_0 is an infinitesimal isometry, its eigenvalues occur in pairs $\pm\mu$. We must have exactly two eigenvalues equal to 0. Rescaling A_0, we may suppose the other two are $\pm i$ or ± 1.

If they are $\pm i$, the eigenspace they span is irreducible. The invariant subspace $H \cap T$ has codimension one and so must contain this eigenspace. This gives case b), where w, x, y span $H \cap T$ and the $\pm i$ eigenspace is spanned by x, y.

If the eigenvalues are ± 1 and $H \cap T$ contains both eigenspaces, one gets case a). Otherwise, switching to $-A_0$ if necessary, we may suppose $H \cap T$ contains the $+1$ and 0 eigenspaces. This leads to a nontransitive affine group, so it does not contribute to our list.

If $H \cap T$ has dimension 2, then the Lie algebra of $\lambda(H)$ consists of singular commuting elements. This easily implies that $\lambda(H)$ is unipotent. q.e.d.

We summarize our results in the following table, listing (up to conjugacy in $\mathscr{P}$) all the simply transitive isometric actions on Minkowski space with respect to coordinates in which $g = 2wz + x^2 + y^2$. The only redundancy is that $(\beta, \varepsilon) \neq (0,0)$ and $(\beta', \varepsilon') \neq (0,0)$ determine the same U_ρ if $\rho = \beta^2/\varepsilon = (\beta')^2/\varepsilon'$. We show a typical element of each group where $r, s, t, u \in \mathbf{R}$. $(A \mid v)$ denotes the motion $p \to Ap + v$, $p \in \mathbf{R}^4$.

Simply Transitive Motions of Minkowski Space

$$T:\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & r \\ 0 & 1 & 0 & 0 & s \\ 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 1 & u \end{array}\right).$$

U_ρ:

$$\left(\begin{array}{cccc|c} 1 & -\beta t & -\varepsilon u & -\frac{1}{2}(\beta^2 t^2 + \varepsilon^2 u^2) & r - \frac{1}{2}(\beta st + \varepsilon tu) - \frac{1}{2}(\beta^2 t^2 u + \varepsilon^2 u^3) \\ 0 & 1 & 0 & \beta t & s + \frac{1}{2}\beta tu \\ 0 & 0 & 1 & \varepsilon u & t + \frac{1}{2}\varepsilon u^2 \\ 0 & 0 & 0 & 1 & u \end{array}\right).$$

$$\mathscr{A}:\left(\begin{array}{cccc|c} e^t & 0 & 0 & 0 & r \\ 0 & 1 & 0 & 0 & s \\ 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & e^{-t} & u \end{array}\right).$$

$$\mathscr{B}:\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & r \\ 0 & \cos u & \sin u & 0 & s \\ 0 & -\sin u & \cos u & 0 & t \\ 0 & 0 & 0 & 1 & u \end{array}\right).$$

4. Discrete groups

Let π be a discrete subgroup of a simply transitive group H of motions of Minkowski space M. Then the orbit space $X = M/\pi$ is a flat spacetime. The possible discrete subgroups π depend only on the structure of H as a Lie group and not on its embedding in $\mathscr{P}$. Since H is known up to conjugacy by §3, we can easily find the possible subgroups π.

By a theorem in [5], every compact complete affine manifold $X = A/\Gamma$ with virtually solvable Γ has a *crystallographic hull* $H(\Gamma)$. This group $H(\Gamma)$ has an identity component $H_0(\Gamma)$ that acts simply transitively on A. Also $H(\Gamma)$ has only finitely many components, each of which meets Γ. These properties determine $H(\Gamma)$ uniquely in case Γ (or some subgroup of finite index in Γ) consists of matrices with only real eigenvalues.

We turn to the special case $A = M$, $\Gamma \subset \mathscr{P}$. Then A/Γ is the most general compact complete flat spacetime, since §1 assures us that Γ is virtually solvable. The identity component $H_0(\Gamma)$ must occur in our table of simply transitive motions. By Bieberbach's theorem, any discrete subgroup of B meets

T in a subgroup of finite index [12]. Thus in all cases Γ has a subgroup of finite index with only real eigenvalues. So the hull $H(\Gamma)$ is unique in our setting. If $\pi = \Gamma \cap H_0(\Gamma)$ and $H = H_0(\Gamma)$, the quotient M/π is a finite regular cover of A/Γ, canonically associated to Γ. So by classifying the discrete subgroups π, we will classify up to a natural finite cover all the compact, complete, flat spacetimes.

Now we lose little by not discussing $\mathscr{B}$ further, since it does not arise as $H_0(\Gamma)$. The case $H = T$ is obvious: π is any lattice in $\mathbf{R}^4$. Only the cases $H = U_\rho$ and $H = \mathscr{A}$ give interesting groups.

Theorem 5. *X is a flat complete compact spacetime with fundamental group Γ. There is a uniquely determined simply transitive group H of motions of the universal cover $\tilde{X}$ such that $H \cap \Gamma = \pi$ has finite index in Γ.*

The group π is a semidirect product $\mathbf{Z}^3 \times_A \mathbf{Z}$. $A \in SL(3, Z)$ has characteristic polynomial $p_A(t) = \det(t - A) = (t - 1)(t^2 - bt + 1)$, with $b \geqslant 2$ an integer.

The similarity class of A over $\mathbf{Q}$ (or $\mathbf{C}$) determines the group H up to isomorphism as a Lie group. In the finer classification as a subgroup of $\mathscr{P}$, H is conjugate to

a) $\mathscr{A}$, *if* $b > 2$,

b) U_ρ *with* $\rho > 0$, *if* $(A - I)^3 = 0, (A - I)^2 \neq 0$,

c) U_0, *if* $(A - I)^2 = 0$, $A - I \neq 0$,

d) T, *if* $A = I$.

In each case, if $\alpha = \log A$ then H is the semidirect product of $\mathbf{R}^3$ by $\mathbf{R}$ where $\mathbf{R}$ acts on $\mathbf{R}^3$ by $e^{t\alpha}$, $t \in \mathbf{R}$.

Proof. The first paragraph has been shown and all possibilities for H are known.

In each case H has an abelian subgroup H' isomorphic to $\mathbf{R}^3$. For $\mathscr{A}$ there is the subgroup $T \cap \mathscr{A}$, for U_ρ the subgroup defined by $t = 0$. This shows that H is a semidirect product by some one parameter group $e^{t\alpha}$. One sees that α has eigenvalues all 0 if $H = U_\rho$ or $H = T$ and eigenvalues 0, $\pm\lambda$, some $\lambda \in \mathbf{R}$, if $H = \mathscr{A}$.

In case $H = \mathscr{A}$, one can check that $H' = T \cap \mathscr{A}$ is the maximum connected normal nilpotent subgroup of H. By [10, 3.5] π meets H' in a lattice. Since $H' \cong \mathbf{R}^3$, $\pi \cap H' \cong \mathbf{Z}^3$. The quotient $\pi/\pi \cap H'$ is discrete in $H/H' \cong \mathbf{R}$, so it is infinite cyclic. The action A of $\mathbf{Z}$ on $\mathbf{Z}^3$ has eigenvalues $\mu, 1, \mu^{-1}$, $\mu > 1$ and so $\det(t - A)$ is of form $(t - 1)(t^2 - bt + 1)$, $b \geqslant 3$.

In the unipotent cases, the argument is similar. One only needs to show that H' can be chosen so $\pi \cap H'$ is a lattice in H'.

For $H = T$, this is trivial.

For $H = U_0$ (i.e. $\beta = 0$, $\varepsilon = 1$) the center Z of H is 2-dimensional. By [10, 2.17] $Z \cap \pi$ is a lattice in Z. The quotient $\pi/Z \cap \pi$ is a lattice in $\mathbf{R}^2$. Taking an element of π that corresponds to an indivisible vector in $\pi/Z \cap \pi$ and adjoining it to $Z \cap \pi$ gives a free abelian subgroup π' of rank 3. The quotient π/π' is infinite cyclic. One takes H' to be the Malcev completion of π'.

Now suppose $H = U_\rho$, $\rho > 0$, say $\beta = 1$. Now the center Z is 1-dimensional and the quotient H/Z is the Heisenberg group. Let G be the discrete subgroup $\pi/\pi \cap \mathbf{Z}$ of H/Z. It is a lattice and so it has the form $\langle x, y, z \,|\, x \leftrightarrow y, z, yz = x^a zy \rangle$. The central extension π of G by $\mathbf{Z}$ has presentation $\langle w, x, y, z \,|\, w \leftrightarrow x, y, z, xy = w^b yx, xz = w^c zx, yz = w^d x^a zy \rangle$ for suitable integers a, b, c, d. One finds immediately that $xy^m z^n = w^{bm+cn} y^m z^n x$. Choose m, n relatively prime with $bm + cn = 0$. Then the subgroup π' generated by w, x, and $y^m z^n$ is free abelian with quotient group $\mathbf{Z}$. Again we take H' to be the Malcev completion of π' in H. q.e.d.

The quotient groups Γ/π are not very large if $H \neq T$. For instance, recall that the groups U_ρ have a natural flag: this must be preserved by Γ. The graded action of Γ is a sum of 4 1×1 matrices and it has finite order and determinant 1. Thus $[\Gamma : \pi]$ divides 8. If $H = T, \Gamma$ is on the list of crystallographic groups in dimension 4, [3]. Thus the finite extensions are, in principle, routinely computable.

Note that we have shown

Corollary. *Every complete compact flat spacetime is finitely covered by a T^3 bundle over the circle.*

5. Causality

Let us call a motion $Ap + v$ of Minkowski space *causal* if the displacement vector $Ap - p + v$ is spacelike or zero for all points p. If $\pi \subset \mathscr{P}$ acts freely and properly discontinuously on M and each element of π is causal, we will say that π and $X = M/\pi$ are a causal group and a causal spacetime, respectively. Here no timelike or lightlike geodesic in X returns to its starting point, reflecting the physical notion of causality.

If π is causal, one cannot have X compact. The strongest compactness property one can assume corresponds to the compactness of the "special directions" in X. This is to assume that X has two ends, i.e. that there is a compact set C (separating X into two unbounded open components but no compact C' gives more than two. Clearly this holds if $X = K \times \mathbf{R}$, K compact: the 2 ends correspond to those of $\mathbf{R}$, i.e., to $\pm\infty$, or the infinite future and infinite past.

We will further suppose that π is contained in a simply transitive group of motions H. Then we find

Theorem 6. *Let H be a simply transitive group of motions of M and let π be a causal subgroup of H such that $X = M/\pi$ has two ends. Then there is an affine fibration of X over $\mathbf{R}$ with 3-tori as fibers. In particular π is free abelian of rank 3 and X is diffeomorphic to $T^3 \times \mathbf{R}$.*

Proof. Suppose to begin that $H = \mathscr{A}$. The element $Ap + v$ of $\mathscr{A}$ is causal if $2(r + (e^t - 1)w)(u + (e^{-t} - 1)z) + s^2 + t^2 > 0$ for all $w, z \in \mathbf{R}$. If $t \neq 0$ this is absurd. So $t = 0$ on π. Thus $\pi \subset T$.

Similarly if $H = \mathscr{B}$, one can show $\pi \subset T$.

But for $\pi \subset T$, the condition that X has two ends means just that π has rank 3. Let T' be the vector space spanned by π. The projection $X \to M/T'$ is the desired affine fibration. The fibers are flat Euclidean tori, except in the case when $T' = v^{\perp}$, v lightlike, $\mathbf{R}v \cap \pi = 0$.

We now suppose $H = U_\rho$. In the notation of our table, let $(A \mid v) \in U_\rho$ correspond to the parameters r, s, t, u. Let p be the vector with coordinates w, x, y, z and write v_i for the coordinates of v, δ_i for those of $Ap - p + v$. Then

$$\delta_1 = -\beta tx - \varepsilon vy - \tfrac{1}{2}\left(\beta^2 t^2 + \varepsilon^2 u^2\right)z + v_1,$$
$$\delta_2 = \beta tz + v_2,\ \delta_3 = \varepsilon uz + v_3,\ \delta_4 = v_4 = u,$$

and causality means $2\delta_1\delta_4 + \delta_2^2 + \delta_3^2 > 0$ for all x, y, z. Considering the terms in x and y, one sees that $\beta tu = \varepsilon u^2 = 0$ at all elements of π.

Suppose there is some element of π with nonzero u. Then $\varepsilon = 0$. As $(\beta, \varepsilon) \neq (0,0)$, we have $\beta \neq 0$. Thus $tu = 0$ on π. Since $\varepsilon = 0$, the map ϕ: $U_\rho \to \mathbf{R}^2$, $\phi(A \mid v) = \binom{t}{u}$ is a group homomorphism. As $\phi(\pi)$ is a subgroup of $\mathbf{R}^2$ and lies in the coordinate axes, we must have $t = 0$ on π. Thus $2\delta_1\delta_4 + \delta_2^2 + \delta_3^2 = 2ru + s^2$ and $\pi \subset T$. So again π has rank 3. It follows that r, s, u can be chosen on π with $2ru + s^2 < 0$ since any cocompact lattice in Minkowski 3-space meets the interior of the light cone. This contradiction shows $u = 0$ on π. Causality holds provided s, t does not vanish simultaneously on nonzero elements of π.

Since $u = 0$, π preserves the z-coordinate. This gives the fibration of X over $\mathbf{R}$. One can check that $\ker u \subset U_\rho$ is an abelian subgroup, so the fiber is a 3-torus. q.e.d.

For each fixed z, the action of $\pi \subset U_\rho$ is given by

$$\left(\begin{array}{ccc|c} 1 & -t & 0 & r - \frac{1}{2}st - \frac{1}{2}t^2 z \\ 0 & 1 & 0 & s + tz \\ 0 & 0 & 1 & t \end{array}\right),$$

where we have set $\beta = 1$ for simplicity (if $\beta = 0$ then $\pi \subset T$). Thus the affine structures on the fibers vary with z. They are never Euclidean. The vector field $\partial/\partial w$ gives bounded light rays on each toral fiber, so the causality of these examples is very delicate.

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