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A REMARK ON THE SYZYGIES OF THE GENERIC CANONICAL CURVES

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Let C be a genus g nonhyperelliptic curve. Consider the canonical ring

$$R = \bigoplus_n H^0(\omega_C^n).$$

Set $V = H^0(\omega_C)$ and let S be the polynomial ring Symm(V). Then R can be regarded as a graded S-module. Let $\mathbf{C} = S/\mu$, where μ is the irrelevant ideal of S. Then $\mathbf{C}$ has a minimal graded Koszul resolution:

$$0 \to \wedge^g V \otimes S(-g) \to \cdots \to V \otimes S(-1) \to S \to \mathbf{C} \to 0.$$

$K_{p,q}(C)$ is defined to be the Koszul cohomology group $K_{p,q}(R)$ [1, §1] which is isomorphic to the homogeneous degree $p+q$ part of $\operatorname{Tor}_p^S(R, \mathbf{C})$. Observe that if

$$0 \to L_{g-2} \to \cdots L_1 \to L_0 \to R \to 0$$

is a minimal graded free resolution of R, then $L_p \otimes \mathbf{C} \simeq \operatorname{Tor}_p(R, \mathbf{C})$.

Mark Green conjectures that if C is generic, then $K_{p,2}(C) = 0$ for $p \leqslant [(g-3)/2]$, [1, 5.6]. It is elementary to show that $K_{p,j}(C) = 0$ for $j \geqslant 3$ (Proposition 2). Now one observes that $K_{1,2}(C) = 0$ is equivalent to Petri's theorem, which says that the homogeneous ideal of C in $\mathbb{P}(V)$ is generated by quadrics. In [2], Green and Lazarsfeld showed that if the Clifford index of C is less than or equal to m, then $K_{m,2}(C) \neq 0$. Green conjectures that the converse is also true [1, 5.1].

In this paper, we study the Koszul cohomologies of a generic curve by the degeneration method. We show that if $K_{p,2}(X) = 0$ for a curve of genus n, then $K_{p,2}(C) = 0$ for a generic curve of genus m, if $m \equiv n \pmod{p+1}$ and $m \geqslant n$.

With the aid of the computer program Macaulay, Bayer, and Stillman had showed that if C is generic and $g \leqslant 12$, then $K_{p,2}(C) = 0$ for $p \leqslant [(g-3)/2]$. Using their results, we prove that $K_{2,2}(C) = 0$ for $g \geqslant 7$ and $K_{3,2}(C) = 0$ for

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$g \geqslant 9$ as conjectured by Green. $K_{2,2}(C) = 0$ is equivalent to saying that if $\{q_1, \cdots, q_n\}$ is a basis for the quadrics containing C, then the relation among the quadrics are generated by the elements of the form $1_1 q_1 + \cdots + 1_n q_n = 0$ when $1_1, \cdots, 1_n$ are linear forms.

I would like to thank M. Green and R. Lazarsfeld for many helpful discussions. I would also like to thank Bayer and Stillman for their help. Throughout the paper, we shall work over the complex numbers.

Consider the exact sequence

$$0 \to M_C \to V \otimes \mathcal{O}_C \to \omega_C \to 0.$$

Set $Q_C = M_C^*$.

The first two propositions are well known to the experts. But I include them for the convenience of the readers.

Proposition 1. *Assume C is a nonhyperelliptic curve of genus g. Then*

(a) *There is an exact sequence,*

$$0 \to \omega_C^{-1} \otimes \mathcal{O}_C(D) \to M_C \to \sum_1^{g-2} \mathcal{O}_C(-p_i) \to 0$$

where $p_1, \cdots, p_{g-2}$ are general points on C and $D = p_1 + p_2 + \cdots + p_{g-2}$.

(b) *If $p < g - 1$, then $H^1(\wedge^p M_C \otimes \omega_C^2) = 0$.*

(c) *The natural map*

$$\phi_{p+1}\colon H^1(\wedge^{p+1} M_C \otimes \omega_C) \to H^1(\wedge^{p+1} V \otimes \omega_C)$$

is surjective. Hence

$$h^0(\wedge^{p+1} Q_C) = h^1(\wedge^{p+1} M_C \otimes \omega_C) \geqslant \binom{g}{p+1}.$$

(d) *$K_{p,2}(C) = 0$ $(p < g - 2)$ if and only if*

$$h^0(\wedge^{p+1} Q_C) \leqslant \binom{g}{p+1}.$$

Proof. (a) See 2.3 of [3].

(b) Set $E = \sum_1^{g-2} \mathcal{O}_C(-p_i)$. Consider the sequence

$$0 \to \wedge^{p-1} E \otimes \omega_C \otimes \mathcal{O}_C(D) \to \wedge^p M_C \otimes \omega_C^2 \to \wedge^p E \otimes \omega_C^2 \to 0.$$

One sees that $H^1(\wedge^p M_C \otimes \omega_C^2) = 0$ for $p < g - 1$.

(c) Consider

$$0 \to \wedge^{p+1} M_C \otimes \omega_C \to \wedge^{p+1} V \otimes \omega_C \to \wedge^p M_C \otimes \omega_C^2 \to 0.$$

Observe that $\operatorname{cok} \phi_{p+1} = H^1(\wedge^p M_C \otimes \omega_C^2)$. So ϕ_{p+1} is surjective for $p < g - 1$. The second assertion follows from the first part by Serre's duality.

(d) Consider

$$\psi_p\colon H^0(\wedge^{p+1} V \otimes \omega_C) \to H^0(\wedge^p M_C \otimes \omega_C^2), \qquad \operatorname{cok} \psi_p \simeq K_{p,2}(C).$$

Now (d) follows from (c).

Corollary 2. *Assume C is a nonhyperelliptic curve of genus g. Then*

(a) $K_{p,3}(C) = 0$ *if* $p \neq g-2$.

(b) $K_{p,q}(C) = 0$ *if* $q \geqslant 4$.

Proof. Since the homological dimension of R is $g-2$, then $K_{p,q}(C) = 0$ for $p > g-2$. Now assume $g - 2 > p \geqslant 0$. Consider

$$H^0(\wedge^{p+1} V \otimes \omega_C^2) \xrightarrow{\alpha} H^0(\wedge^p M_C \otimes \omega_C^3) \to H^1(\wedge^{p+1} M_C \otimes \omega_C^2).$$

$K_{p,3}(C) \simeq \operatorname{cok}\alpha = 0$ by Proposition 1. Similarly $K_{p,q}(C) = 0$ for $q \geqslant 4$.

Proposition 3. *Assume C is nonhyperelliptic of genus g. Consider the minimal resolution of R,*

$$(3.1) \qquad 0 \to L_{g-2} \xrightarrow{d_{g-2}} L_{g-3} \to \cdots \to L_1 \xrightarrow{d_1} L_0 \to R \to 0.$$

Denote by $\tilde{L}_i$ the corresponding locally free sheaf on $\mathbf{P}^{g-1}$.

(a) $0 \to L_0^* \otimes S(-g-1) \xrightarrow{d_1^*} L_1^* \otimes S(-g-1) \to \cdots \to L_{g-2}^* \otimes S(-g-1)$ *is again a minimal resolution of R.*

(b) *One can recover the curve C from a boundary map d_i.*

(c) *If $0 < p < g-2$, then $\tilde{L}_p \simeq E_p \oplus F_p$ where $E_p \simeq \bigoplus \mathcal{O}_{\mathbf{P}^{g-1}}(-p-1)$ and $F_p \simeq \bigoplus \mathcal{O}_{\mathbf{P}^{g-1}}(-p-2)$. Furthermore,* $\operatorname{rank}(E_p) = \dim K_{p,1}(C)$ *and* $\operatorname{rank}(F_p) = \dim K_{p,2}(C)$.

(d) *If $K_{p,2}(C) = 0$ for an integer p ($p < g-2$), then $K_{j,2}(C) = 0$ for $j \leqslant p$.*

Proof. (a) Observe that

$$\mathcal{E}xt^j(\mathcal{O}_C, \mathcal{O}_{\mathbf{P}^{g-1}}(-g)) = \begin{cases} \omega_C = \mathcal{O}_C(1), & \text{if } j = g-2, \\ 0, & \text{otherwise.} \end{cases}$$

So

$$0 \to \tilde{L}_0^* \xrightarrow{d_1^*} \tilde{L}_1^* \to \cdots \to \tilde{L}_{g-3}^* \xrightarrow{d_{g-2}^*} \tilde{L}_{g-2}^* \xrightarrow{d_{g-1}^*} \mathcal{O}_C(g+1) \to 0$$

is an exact complex of sheaves. Set $N_j = \ker d_j^*$ $(2 \leqslant j \leqslant g-1)$. Then

$$H^1(N_{g-1}(i)) \simeq H^2(N_{g-2}(i)) \simeq \cdots \simeq H^{g-2}(\tilde{L}_0^*(i)) = 0.$$

Similarly, one shows that $H^1(N_j(i)) = 0$ for $2 \leqslant j \leqslant g-1$. Thus $(3.1)^* \otimes S(-g-1)$ is a minimal resolution of R.

(b) Let $M_j = \ker d_j$. Then

$$\mathcal{E}xt^{g-2}(\mathcal{O}_C, \mathcal{O}_{\mathbf{P}^{g-1}}(-g-1)) \simeq \mathcal{O}_C(1) \simeq \mathcal{E}xt^{g-j-3}(M_j, \mathcal{O}_{\mathbf{P}^{g-1}}(-g-1)).$$

(c) By Noether's theorem and (a), we conclude that $\tilde{L}_0 \simeq \mathcal{O}_{\mathbf{P}^{g-1}}$ and $\tilde{L}_{g-2} \simeq \mathcal{O}_{\mathbf{P}^{g-1}}(-g-1)$. Since C is nondegenerate in $\mathbf{P}^{g-1}$ and $K_{1,j}(C) = 0$ for $j \geqslant 3$, $\tilde{L}_1 \simeq E_1 \oplus F_1$ where

$$E_1 \simeq \bigoplus \mathcal{O}_{\mathbf{P}^{g-1}}(-2) \quad \text{and} \quad F_1 \simeq \bigoplus \mathcal{O}_{\mathbf{P}^{g-1}}(-3).$$

Since (3.1) is a minimal resolution, $K_{p,q}(C)=0$ for $q \leqslant 0$ and $p \geqslant 1$. By Corollary 2, this implies that $\tilde{L}_p \simeq E_p \oplus F_p$ $(p < g-2)$ where $E_p \simeq \oplus \mathcal{O}_{\mathbf{P}^{g-1}}(-p-1)$ and $F_p \simeq \oplus \mathcal{O}_{\mathbf{P}^{g-1}}(-p-2)$. Furthermore, rank $E_p = \dim K_{p,1}(C)$ and rank $F_p \simeq \dim K_{p,2}(C)$.

(d) If $K_{p,2}(C)=0$, then $\tilde{L}_p \simeq E_p$. Suppose for contradiction that $K_{p-1,2}(C) \neq 0$. Then $\tilde{L}_{p-1} = E_{p-1} \oplus F_{p-1}$ where $F_{p-1} \neq 0$. We can decompose d_p as $f_p \oplus g_p$ where $f_p \in \operatorname{Hom}(E_p, E_{p-1})$ and $g_p \in \operatorname{Hom}(E_p, F_{p-1})$. Since (3.1) is a minimal resolution, $g_p = 0$. Set $B_{p-2} = \operatorname{cok} d_p$. Then $B_{p-2} \simeq F_{p-1} \oplus B'_{p-2}$. Now consider

$$\beta\colon 0 = H^0\big(\tilde{L}^*_{p-2} \otimes \mathcal{O}_{\mathbf{P}^{g-1}}(-p-1)\big) \to H^0\big(B^*_{p-2} \otimes \mathcal{O}_{\mathbf{P}^{g-1}}(-p-1)\big).$$

Observe that β is not surjective. This contradicts that (3.1)* is a minimal resolution of $R(g+1)$. Thus $K_{p-1,2}(C)=0$. It follows by induction that $K_{j,2}(C)=0$ for $j \leqslant p$.

Theorem 4. *Let X be a nonhyperelliptic genus n curve. Assume $K_{p,2}(X)=0$ for an integer p where $1 \leqslant p \leqslant n-3$. Then*:

(a) *If C is a general curve of genus $n+p+1$, then $K_{p,2}(C)=0$.*

(b) *If C is a general curve of genus m, where $m \equiv n \bmod(p+1)$ and $m \geqslant n$, then $K_{p,2}(C)=0$.*

Proof. (a) Consider a stable curve $C_0 = X \cup Y$, where $Y \simeq \mathbf{P}^1$ and $X \cap Y = q_1 + q_2 + \cdots + q_{p+2}$ are $p+2$ general points on X. Now consider a one-parameter degeneration $\pi\colon \mathscr{C} \to T$ where $\mathscr{C}$ is a surface and T is an affine curve. Assume that π is proper and flat and there is a point $t_0 \in T$ such that $\pi^{-1}(t_0) \simeq C_0$. Furthermore if $t \neq t_0$ in T, then $\pi^{-1}(t) = C_t$ is a smooth curve of genus $n+p+1$. Now consider the following line bundle on $\mathscr{C}$: $\mathscr{L} = \omega_{\mathscr{C}/T} \otimes \mathcal{O}_{\mathscr{C}}(X)$. Observe that $\mathscr{L}|_{C_t} = \omega_{C_t}$ for $t \neq t_0$, $\mathscr{L}|_X = \omega_X$, and $\mathscr{L}|_Y \simeq \mathcal{O}_{\mathbf{P}^1}(2p+2)$.

Claim 4.1. $h^0(\mathscr{L}|_{C_0}) = n+p+1$ and $\mathscr{L}|_{C_0}$ is generated by its sections. Consider

$$0 \to \mathcal{O}_{\mathbf{P}^1}(p) \to \mathscr{L}|_{C_0} \to \omega_X \to 0, \tag{4.1.1}$$

$$0 \to \omega_X\left(-\sum_1^{p+2} q_i\right) \to \mathscr{L}|_{C_0} \to \mathcal{O}_{\mathbf{P}^1}(2p+2) \to 0. \tag{4.1.2}$$

By (4.1.1), $h^0(\mathscr{L}|_{C_0}) = n+p+1$, $h^1(\mathscr{L}|_{C_0}) = 1$, and $H^0(\mathscr{L}|_{C_0})$ maps onto $H^0(\omega_X)$. Since the q_i's are general points,

$$h^1\left(\omega_X\left(-\sum_1^{p+2} q_i\right)\right) = h^1(\mathscr{L}|_{C_0}) = 1.$$

Thus $H^0(\mathscr{L}|_{C_0})$ maps onto $H^0(\mathscr{O}_{\mathbf{P}^1}(2p+2))$. So $\mathscr{L}|_{C_0}$ is generated by its sections. After replacing T by a smaller open set if necessary, we may assume $\pi_*\mathscr{L} \simeq (n+p+1)\mathscr{O}_T$ and $\mu\colon \pi^*\pi_*\mathscr{L} \to \mathscr{L}$ is surjective. Set $M_{\mathscr{C}} = \ker\mu$, and $Q_{\mathscr{C}} = M_{\mathscr{C}}^*$. Observe that

$$Q_{\mathscr{C}}|_{C_t} \simeq Q_{C_t}, \qquad Q_{\mathscr{C}}|_X = Q_X \oplus (p+1)\mathscr{O}_X,$$
$$Q_{\mathscr{C}}|_Y \simeq (n-p-2)\mathscr{O}_{\mathbf{P}^1} \oplus (2p+2)\mathscr{O}_{\mathbf{P}^1}(1).$$

Claim 4.2. $h^1(\Lambda^{p+1}Q_{\mathscr{C}}|_{C_0}) \leqslant \binom{n+p+1}{p+1}$. Consider

$$0 \to \Lambda^{p+1}Q_{\mathscr{C}}|_Y \otimes \mathscr{O}_{\mathbf{P}^1}(-p-2) \to \Lambda^{p+1}Q_{\mathscr{C}}|_{C_0} \to \Lambda^{p+1}Q_{\mathscr{C}}|_X \to 0.$$

Observe that

$$h^0\big(\Lambda^{p+1}Q_{\mathscr{C}}|_Y \otimes \mathscr{O}_{\mathbf{P}^1}(-p-2)\big) = 0,$$
$$h^0\big(\Lambda^{p+1}Q_{\mathscr{C}}|_X\big) = \sum_{k=0}^{p+1}\binom{p+1}{p+1-k}h^0(\Lambda^k Q_X)$$
$$= \sum \binom{p+1}{p+1-k}\binom{n}{k} = \binom{n+p+1}{p+1}$$

by Proposition 1 and Proposition 3. Thus $h^0(\Lambda^{p+1}Q_{\mathscr{C}}|_{C_0}) \leqslant \binom{n+p+1}{p+1}$. It follows that for generic t, $h^0(\Lambda^{p+1}Q_{C_t}) \leqslant \binom{n+p+1}{p+1}$. Thus $K_{p,2}(C_t) = 0$ by Proposition 1.

(b) This follows from (a) and induction.

Theorem 5. *Let C be a general curve of genus g.*

(a) $K_{2,2}(C) = 0$ *if* $g \geqslant 7$.

(b) $K_{3,2}(C) = 0$ *if* $g \geqslant 9$.

(c) $K_{4,2}(C) = 0$ *if* $g \geqslant 11$ *and* $g \equiv 1$ *or* $2 \bmod 5$.

Proof. (a) Using the computer program Macaulay, Bayer, and Stillman had checked that $K_{p,2}(C) = 0$ for $p \leqslant [(g-3)/2]$ if $g \leqslant 12$. So $K_{2,2}(C) = 0$ for $g = 7$, 8, or 9. Then Theorem 4 will imply that $K_{2,2}(C) = 0$ if $g \geqslant 7$. Similarly one can prove (b) and (c).

References

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