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COMPLETE SURFACES OF FINITE TOTAL CURVATURE

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Dedicated to S. S. Chern and Robert Osserman

Let M be a connected oriented two-dimensional surface immersed in $\mathbf{R}^n$ and complete with respect to the induced Riemannian metric. Let $K(x)$ and $B(x)$ denote the sectional curvature and the second fundamental form, respectively, of M at x. In this paper we give simple proofs of the following theorems.

Theorem 1. *If $\int_M |K^-|$ is finite, then M is of finite topological type, i.e. M is homeomorphic to $\overline{M}\setminus\{p_1,\cdots,p_k\}$, where $\overline{M}$ is a compact surface.*

Theorem 2. *If $\int_M |B|^2$ is finite, then $\int_M K$ is an integral multiple of 2π (or of 4π in case $n = 3$).*

Theorem 3. *If $\int |B|^2$ is finite and if M is nonpositively curved with respect to each normal direction, then M is properly immersed and the Gauss map extends continuously to all of $\overline{M}$.*

Theorem 1 (which is really intrinsic) is due to Huber. Indeed Huber [5] showed that M is actually conformally equivalent to a punctured Riemann surface. We include a proof to make this paper self-contained and because we need a corollary of the proof.

Theorems 2 and 3 were proved for minimal surfaces by Osserman in case $n = 3$ and by Chern and Osserman in general ([7], [3]; see also [6] for a nice exposition). Their elegant proofs, like so much of the theory of two-dimensional minimal surfaces, rely heavily on complex analysis. Indeed, the author was very surprised to realize that their hypothesis of minimality was not really required. (In the statement of their theorems, $\int |K|$ rather than $\int |B|^2$ is assumed to be finite; but for minimal surfaces $|B|^2 = -2K$ so the assumptions are equivalent. Of course, in general, finiteness of $\int |K|$ is a much weaker condition than finiteness of $\int |B|^2$.) (Mike Anderson [1] has proved analogs of the Chern-Osserman theorems for k-dimensional minimal submanifolds of $\mathbf{R}^n$.)

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The results of this paper allow one to give easy proofs of some new regularity results for embedded surfaces in a three-manifold that are minimal or stationary with respect to an even parametric elliptic integrand (see [10]).

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0. Preliminaries

1. M is a complete, connected, oriented surface. Except in §1, M is an immersed submanifold of $\mathbf{R}^n$.
2. $K(x)$ and $B(x)$ are the sectional curvature and the second fundamental form, respectively, of M at x.
3. If $x, y \in M$, $\operatorname{dist}(x, y)$ is the geodesic distance from x to y in M.
4. $G_2(n)$ is the Grassmannian manifold of oriented two-dimensional subspaces of $\mathbf{R}^n$. It is sometimes useful to identify $G_2(n)$ with the set of simple unit two-vectors in $\Lambda_2(\mathbf{R}^n)$.
5. $g: M \to G_2(n)$ is the Gauss map, which assigns to each $x \in M$ the oriented tangent plane to M at x.
6. $\Omega(r) = \{x \in M: \operatorname{dist}(x, x_0) < r\}$, where x_0 is some fixed point, and $L(r) = \operatorname{length}(\partial\Omega(r))$.

1. Finite connectivity

Theorem 1 (*Huber*). *If $\int_M |K^-| < \infty$, then $\int_M K^+ < \infty$, and M is homeomorphic to $\overline{M} \setminus \{p_1, \cdots, p_k\}$, where $\overline{M}$ is a compact 2-manifold.*

Proof. Fix $x_0 \in M$, and let

$$\Omega_r = \Omega(r) = \{x \in M : d(x, x_0) < r\},$$

where $d(x, y)$ is the geodesic distance from x to y. P. Hartman [4] has shown that $\partial\Omega_r$ is, for almost all r, a piecewise smooth, embedded closed curve. By the Gauss-Bonnet theorem,

$$\begin{aligned} &\int_{\Omega_r} (\nabla_T T) \cdot n + \sum (\text{exterior angles of } \Omega_r) \\ (1) \qquad &= 2\pi\chi(r) - \int_{\Omega_r} K\,dA = 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega_r} K\,dA. \end{aligned}$$

where $\chi(r)$, $h(r)$, and $c(r)$ are the Euler characteristic, number of handles, and the number of boundary components, respectively, of Ω_r.

Let $L(r)$ denote the length of $\partial\Omega_r$. Note that the left side of (1) is the first variation formula for $L'(r)$ [2]; so

$$L'(r) = 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega_r} K. \tag{2}$$

Since M is complete and noncompact, $L(r) > 0$ for all $r > 0$; so

$$\begin{aligned} 0 &\leqslant \limsup L'(r) \\ &\leqslant 2\pi(2 - 2\liminf h(r) - \liminf c(r)) - \int_M |K|^+ + \int_M |K^-|. \end{aligned} \tag{3}$$

Thus the negative quantities on the right-hand side are all finite. But $h(r)$ is a nondecreasing, integer valued function of r, so this implies

$$h(r) = \text{some constant } h \quad \text{for } r > R.$$

Also, $c(r)$ is integer valued, so we can find a sequence $r_i \to \infty$ with

$$c(r_i) = c = \liminf c(r) < \infty.$$

Let A_i be the union of $\Omega(r_i)$ with those connected components of $M \sim \Omega(r_i)$ which happen to be compact. (There may not be any, in which case $A_i = \Omega(r_i)$.) Let $h(A_i)$ and $c(A_i)$ denote the number of handles and boundary components, respectively, of A_i. Then

$$h = h(\Omega(r_i)) \leqslant h(A_i) \leqslant h\big(\Omega(r_{i+j})\big) = h$$

provided j is large enough that $A_i \subset \Omega(r_{i+j})$: so

$$h(A_i) \equiv h \tag{4}$$

and clearly $c(A_i) \leqslant c(\Omega(r_i))$. By passing to a subsequence we may assume

$$c(A_i) \equiv c' \quad (\leqslant c). \tag{5}$$

By (4) and (5), the A_i are homeomorphic, with A_{i+1} obtained from A_i by attaching annuli. The result follows immediately.

Corollary 1. $L(r) \leqslant (2\pi + \int_M |K^-|)r$.

Corollary 2 (*Cohn-Vossen*). $0 \leqslant \limsup L'(r) \leqslant 2\pi\chi(M) - \int_M K$.

Remark 1. K. Shiohama [8] has independently shown by similar arguments the sharp form of Corollary 2, namely

$$\lim(L(r)/r) = 2\pi\chi(M) - \int_M K.$$

Remark 2. To deduce (3) from (2) (which only holds for almost all r) rigorously, we must know that $L(r)$ does not increase by a positive amount on a set of r's of measure 0. This fact is also contained in Hartman's paper.

2. Quantization of total scalar curvature

Lemma. *If $\int_M |B|^2 < \infty$, then there exist a sequence $r_i \to \infty$ such that $\int_{\partial\Omega(r_i)} |B| \to 0$.*

Proof.

$$\int_{\partial\Omega(r)} |B| \leqslant L(r)^{1/2}\left(\int_{\partial\Omega(r)} |B|^2\right)^{1/2}$$
$$\leqslant (Ar)^{1/2}\left(\int_{\partial\Omega(r)} |B|^2\right)^{1/2}$$

for some $A < \infty$ (by Corollary 1 to Proposition 1). Thus

$$r^{-1}\left(\int_{\partial\Omega(r)} |B|\right)^2 \leqslant A\int_{\partial\Omega(r)} |B|^2,$$

$$\int_{r=0}^{\infty} r^{-1}\left(\int_{\partial\Omega(r)} |B|\right)^2 dr \leqslant A\int_{r=0}^{\infty}\int_{\partial\Omega(r)} |B|^2\, dr = A\int_M |B|^2 < \infty.$$

But this proves the lemma since $\int_{r=a}^{\infty} r^{-1}\varepsilon\, dr = \infty$.

Theorem 1. *If $\int_M |B|^2 < \infty$, then $\int_M K = 2\pi m$ for some integer m. If $M \subset \mathbf{R}^3$, then m is even.*

Remark. (For readers familiar with geometric measure theory.) Finiteness of $\int |B|^2$ (or even of $\int |K|$ in case $n = 3$) immediately implies that the Gauss image of M is a rectifiable current in $G_2(n)$ of finite mass: in the proof below we show that it is actually a cycle, which is somewhat surprising since g need not extend continuously to $\overline{M}$ (Example 2).

Proof. By the lemma, we can find $r_i \to \infty$ so that

$$\int_{\partial\Omega(r_i)} |B| = \varepsilon_i, \qquad \varepsilon_i \to 0.$$

Note that B may be regarded as the derivative of the Gauss map. Thus the image of $\partial\Omega(r_i)$ under the Gauss map is a set of closed curves of total length

$$\operatorname{length}(g\,|\,\partial\Omega(r_i)) \leqslant \varepsilon_i.$$

Let $\mathscr{D}_i$ be a collection of disks in $G_2(n)$ having these curves as their boundaries. Note we can choose these disks so that

$$\operatorname{Area}(\mathscr{D}_i) = \mathscr{O}(\varepsilon_i^2).$$

Consider first the case $M \subset \mathbf{R}^3$. Let ω denote the volume form on the unit sphere S^2, so that

$$\int_{\Omega(r_i)} K\, dA = \int_{\Omega(r_i)} g^{\#}(\omega) = \int_{g_{\#}(\Omega(r_i))} \omega = \int_{g_{\#}(\Omega(r_i)) - \mathscr{D}_i} \omega + \int_{\mathscr{D}_i} \omega,$$

where $g_{\#}(\Omega(r_i))$ denotes the current image of $\Omega(r_i)$. Since $g_{\#}(\Omega(r_i))$ and $\mathcal{D}_i$ have the same boundary, $g_{\#}(\Omega(r_i)) - \mathcal{D}_i$ is an integral cycle. Since $H_2(S^2, \mathbf{Z})$ is a cyclic group generated by $\int_{S^2} \cdot$, we have, for some integer m_i,

$$\int_{g_{\#}(\Omega(r_i)) - \mathcal{D}_i} \omega = m_i \int_{S^2} \omega = 4\pi m_i.$$

Thus

$$\left| \int_{\Omega(r_i)} K\, dA - 4\pi m_i \right| = \left| \int_{\mathcal{D}} \omega \right| = \mathcal{O}\left(\varepsilon_i^2\right).$$

Let $i \to \infty$ and we are done in the case $M \subset \mathbf{R}^3$.

In the case $M \subset \mathbf{R}^n$, recall [6, p. 109] that $G_2(n)$ embeds naturally as the quadric $\{Z_1^2 + Z_2^2 + \cdots + Z_n^2 = 0\}$ in $\mathbf{CP}^{n-1}$ (where $Z_1, \cdots, Z_n$ are homogeneous coordinates). The argument proceeds as before, except that ω is now the Kähler form on $\mathbf{CP}^{n-1}$. Again $K\, dA = g^{\#}(\omega)$ (see the Appendix for a proof). Also, $H_2(\mathbf{CP}^{n-1}, \mathbf{Z})$ is a cyclic group that is generated by any of the $\mathbf{CP}^2$'s that sit naturally (i.e., as complex projective lines) inside $\mathbf{CP}^{n-1}$. For instance, $H_2(\mathbf{CP}^{n-1}, \mathbf{Z})$ is generated by

$$\Sigma = \{Z_2 = iZ_1;\ Z_4 = iZ_3;\ Z_5 = \cdots = Z_{n-1} = 0\}.$$

Thus (as before)

$$\int_M K\, dA = \int_{g_{\#}(M)} \omega = m \int_{\Sigma} \omega \tag{1}$$

for some integer m. One can check that if $M_0 = \{(z, z^2) \in \mathbf{C}^2 = \mathbf{R}^4 : z \in \mathbf{C}\}$, then $g_{\#}(M_0) = \Sigma$. Thus

$$\int_{M_0} K\, dA = \int_{\Sigma} \omega.$$

Combining this with (1) we obtain

$$\int_M K\, dA = m \int_{M_0} K\, dA.$$

But one can calculate that $\int_{M_0} K\, dA = -2\pi$.

3. The Gauss map at infinity

To insure that the Gauss map extends to all of $\overline{M}$, $\int_M |B|^2 < \infty$ is not enough (see Example 2): we need an extra condition on the second fundamental form. We say that $M \subset \mathbf{R}^n$ is nonpositively curved with respect to each normal direction at x if

(1) $\det(n \cdot B(\ ,\ )) \leqslant 0$ at x for all normals n to M at x.

This condition is equivalent to the following convex hull property:

(1)′ If $x \in M \cap \partial(L \times B)$, where L is an $(n-3)$-dimensional subspace of $\mathbf{R}^n$ such that $\mathrm{Tan}_M x \cap L = (0)$ and B is a closed three-dimensional ball in $L^{\perp}$, then $M \cap (L \times B)$ does not contain any neighborhood of x.

From (1)′, note that if M is nonpositively curved with respect to each normal at x, and if Π: $\mathbf{R}^n \to \mathbf{R}^3$ is an orthogonal projection such that $\Pi(M)$ is immersed, then $\Pi(M)$ is nonpositively curved with respect to each normal at $\Pi(x)$.

Theorem 3. *Suppose $\int_M |B|^2 < \infty$ so that, in particular, $M \approx \overline{M}\backslash\{p_1,\cdots,p_k\}$ as in Theorem* 1. *Let U be a (punctured) neighborhood of p_1 in M. If*

(1) *$M \subset \mathbf{R}^3$ and K does not change sign in U or*

(2) *$M \subset \mathbf{R}^n$ and U is nonpositively curved with respect to each normal direction, then*

(3) *the Gauss map extends continuously to p_1 (i.e., $\lim_{x\to p_1} g(x)$ exists) and*

(4) *M is properly immersed near p_1 (i.e., $\lim_{x\to p_1}|x| = \infty$).*

Proof. For simplicity assume M has only one end: $M \approx \overline{M}\backslash\{p\}$. (We can always modify M outside of U so that this is the case.) By the lemma in §2, we can find a sequence of open sets

$$\Omega_1 \subset \Omega_2 \subset \cdots; \quad \cup\,\Omega_i = M \quad \left(\overline{\Omega}_i \text{ compact}\right)$$

with $\int_{\partial\Omega_i}|B| \to 0$. As in the proof of Theorem 1, let A_i be the union of Ω_i with the compact components of $M \sim \Omega_i$. Then:

$$A_1 \subset A_2 \subset \cdots; \quad \cup A_i = M \quad \left(\overline{A_i} \text{ is compact}\right),$$

$$\partial A_i \text{ has exactly one component},$$

$$\int_{\partial A_i} |B| \to 0 \quad (\text{since } \partial A_i \subset \partial\Omega_i).$$

But

$$\mathrm{length}(g\,|\,\partial A_i) \leqslant \int_{\partial A_i} |B| \to 0;$$

so, by passing to a subsequence, we may assume that

(2) $g(\partial A_i) \subset$ an ε_i neighborhood of some $P \in G_2(n)$, $\quad \varepsilon_i \to 0$.

Now consider first the case $M \subset \mathbf{R}^3$, so $P \in S^2$ ($\approx G_2(n)$). Since $|K| \leqslant |B|^2$, we have

$$\left| \int_{A_{i+1} \sim A_i} K \right| \leqslant \int_{A_{i+1} \sim A_i} |B|^2 < \frac{1}{2} \tag{3}$$

for sufficiently large i. On the other hand, since K is the Jacobian determinent of the Gauss map,

$$\int_{A_{i+1} \sim A_i} K = \int_{v \in S^2} \deg(v), \tag{4}$$

where $\deg(v)$ is the degree of $g \mid A_{i+1} \sim A_i$ at v:

$$\begin{aligned} \deg(v) = {} & \#\{x \in A_{i+1} \sim A_i : g(x) = v \text{ and } K > 0\} \\ & - \#\{x \in A_{i+1} \sim A_i : g(x) = v \text{ and } K < 0\}. \end{aligned}$$

Since K does not change sign,

$$|\deg(v)| = \#\{x \in A_{i+1} \sim A_i : g(x) = v \text{ and } K \neq 0\}.$$

Thus by equation (3) and (4)

$$\int_{v \in S^2} |\deg(v)| < \frac{1}{2}. \tag{5}$$

Since $\deg(v)$ is constant (modulo sets of measure 0) on connected components of $S^2 \sim g(\partial(A_{i+1} \sim A_i))$ and since

$$g(\partial(A_{i+1} \sim A_i)) \subset \mathbf{B}_{\varepsilon_i}(P),$$

it follows that $\deg(v)$ is constant on $S^2 \sim \mathbf{B}_{\varepsilon_i}(P)$. But then by (5), this constant must be 0. Thus

$$\{x \in A_{i+1} \sim A_i : g(x) = v \text{ and } K \neq 0\} = \varnothing \quad \text{if } v \notin \mathbf{B}_{\varepsilon_i}(P),$$

i.e.,

$$g\{x \in A_{i+1} \sim A_i : K(x) \neq 0\} \subset \mathbf{B}_{\varepsilon_i}(P). \tag{6}$$

But if N is any compact surface with boundary immersed in $\mathbf{R}^3$, then

$$g\{x \in N : K(x) \neq 0\} \cup g(\partial N)$$

is dense in $g(N)$. (One shows with the Codazzi equations that $\operatorname{int}\{x \in N : K(x) = 0\}$ consists of straight line segments along which g is constant: see [9, V2] or any other treatment of developable surfaces.) Thus (6) implies

$$g(A_{i+1} \sim A_i i) \subset \mathbf{B}_{\varepsilon_i}(P).$$

Since $\varepsilon_i \to 0$, $\lim_{x \to p} g(x) = P$.

To prove conclusion (4) of the theorem, suppose P is the e_1, e_2 plane and let $\Pi: \mathbf{R}^3 \to P$ be orthogonal projection. Let U be a small enough neighborhood of P that

$$|g(x) - P| < \tfrac{1}{2} \quad \text{if } x \in U \tag{7}$$

(so the tangent plane at M at each $x \in U$ is sloped $\leqslant 30°$ with respect to the horizontal). Let Γ be the Riemannian metric on $\overline{U}$ induced by the orthogonal projection $\Pi: \overline{U} \to P$. By (7), Γ is a nondegenerate metric and is pointwise comparable to the original metric on $U \subset M$ (i.e., the metric induced from $\mathbf{R}^3$). In particular, $\overline{U}$ is also complete with respect to the metric Γ. Now let $x \in U$ and let $\gamma \subset \overline{U}$ be a curve (a Γ-geodesic) which realizes the Γ-distance from x to ∂U. Then $\Pi(\gamma)$ must be a straight line segment joining $\Pi(x)$ to $\Pi(\partial U)$. By (7),

$$\begin{aligned} \operatorname{dist}_M(x, \partial U) &\leqslant (\sin 30°)^{-1}(\text{length of } \Pi(\gamma)) \\ &\leqslant (\sin 30°)^{-1}(\operatorname{dist}(\Pi(x), \Pi(\partial U)) + \operatorname{diam} \Pi(\partial U)). \end{aligned}$$

As $x \to p$, the LHS $\to \infty$, hence $|\Pi(x)| \to \infty$, which proves conclusion (4).

This completes the proof in the case $M \subset \mathbf{R}^3$.

Now suppose $M \subset \mathbf{R}^n$, $n > 3$, and let A_i be as above so that (2) holds. Let Π be orthogonal projection of $\mathbf{R}^n$ onto a three-dimensional subspace and let

$$M' = \Pi(M), \quad A_i' = \Pi(A_i), \quad K' = \text{sectional curvature of } M'.$$

Suppose we can choose Π so that, for sufficiently large i, $M' \sim A_i'$ is immersed and (2) and (3) still hold, i.e., so that

$$(2)' \quad g(\partial A_i') \subset \text{an } \varepsilon_i \text{ neighborhood of some } P' \in S^2, \qquad \text{where } \varepsilon_i \to 0,$$

$$(3)' \qquad \left| \int_{M' \sim A_i'} K' \right| < \frac{1}{2}.$$

Note $M' \sim A_i'$ will also be nonpositively curved at each point. Then the proof above (for $n = 3$) shows that $M' \sim A_i'$ has a well-defined Gauss map at p'. Furthermore, if we can find several different such projections Π, then this forces $M \sim A_i$ to have a well-defined Gauss map at p, thus proving conclusion (3) of the theorem; conclusion (4) then follows exactly as it did in the $n = 3$ case.

How do we know that enough such projections Π exist? First, (2)$'$ will hold provided $(\ker \Pi) \cap P = (0)$. That there exist many Π's satisfying (3)$'$ is a consequence of the following integral geometry theorem.

Proposition. *Let M be a surface in $\mathbf{R}^n$ such that the image of M under its Gauss map has area $A < \infty$. For $v \in \mathbf{R}^n$, $|v| = 1$, let*

$$\Pi_v : \mathbf{R}^n \to (v)^{\perp}, \qquad \Pi_v(u) = u - (u \cdot v)v.$$

If (1) $n \geqslant 5$, *or if* (2) $n = 4$ *and* A *is sufficiently small, then there exists a set of unit vectors* $v \in \mathbf{R}^n$ *of positive measure such that* $\Pi_v(M)$ *is immersed and the image of* $\Pi_v(M)$ *under its Gauss map has area* $\leqslant c_n A$.

Proof. Let

$$\mathscr{S} = \{(\omega, v) : \omega \in G_2(n),\ v \in \omega,\ |v| = 1\},$$
$$\rho_1 : \mathscr{S} \to G_2(n), \quad \rho_1((\omega, v)) = \omega,$$
$$\rho_2 : \mathscr{S} \to \mathbf{R}^n, \quad \rho_2((\omega, v)) = v$$

(so $\rho_1 : \mathscr{S} \to G_2(n)$ is the natural circle bundle on $G_2(n)$). Note if $v \in \partial\mathbf{B}^n$ is not tangent to M at any point, i.e., if $v \notin \rho_2(\rho_1^{-1}(g(M)))$, then $\Pi_v(M)$ is immersed. Since

$$\mathscr{H}^3(\rho_2(\rho_1^{-1}(g(M)))) \leqslant \mathscr{H}^3(\rho_1^{-1}(g(M))) \leqslant c\mathscr{H}^2(g(M)) \leqslant cA,$$
$$\mathscr{H}^3(\partial\mathbf{B}^n \sim \rho_2(\rho_1^{-1}(g(M)))) \leqslant \mathscr{H}^3(\partial\mathbf{B}^n) - cA.$$

So if $n \geqslant 5$, then for almost all v, $\Pi_v(M)$ is immersed; and if $n = 4$, then there exists a set of $v \in \partial\mathbf{B}^4$ of measure $\geqslant \frac{2}{3}\mathscr{H}^3(\partial\mathbf{B}^4)$ for which $\Pi_v(M)$ is immersed, provided $cA \leqslant \frac{1}{3}\mathscr{H}^3(\partial\mathbf{B}^4)$.

Now Π_v induces a map $\tilde{\Pi}_v$ on two-planes (i.e., simple unit 2-vectors in $\Lambda_2\mathbf{R}^n$) by

$$\tilde{\Pi}_v(\omega) = \frac{(\omega \wedge v) \llcorner v}{|(\omega \wedge v) \llcorner v|}.$$

(So if ω is the simple unit 2-vector associated with a plane $p \subset \mathbf{R}^n$, then $\tilde{\Pi}_v(\omega)$ is the simple unit 2-vector associated with $\Pi(P)$.) From this formula one readily calculates

$$|D\tilde{\Pi}_v(\omega)| \leqslant |\omega \wedge v|^{-1}.$$

Thus

$$\int_{v \in \partial\mathbf{B}^n} (\text{Area of Gauss image of } \Pi_v(M))\, dv$$
$$= \int_{v \in \partial\mathbf{B}^n} (\text{Area of } \tilde{\Pi}_v(g(M)))\, dv$$
$$\leqslant \int_{v \in \partial\mathbf{B}^n} \int_{\omega \in g(M)} |D\tilde{\Pi}_v(\omega)|^2\, d\omega\, dv$$
$$\leqslant \int_{\omega \in g(M)} \int_{v \in \partial\mathbf{B}^n} |\omega \wedge v|^{-2}\, dv\, d\omega$$
$$= [\text{Area}(g(M))] \cdot \int_{v \in \partial\mathbf{B}^n} |(e_1 \wedge e_2) \wedge v|^{-2}\, dv$$
$$= [\text{Area}(g(M))] \cdot \int_{v \in \partial\mathbf{B}^n} \left(1 - (v \cdot e_1)^2 - (v \cdot e_2)^2 S\right)^{-1} dv.$$

One readily computes that this integral is finite if $n \geqslant 4$. Thus there is a set of $v \in \partial \mathbf{B}^n$ of measure $\geqslant \frac{2}{3}\mathscr{H}^n(\partial \mathbf{B}^n)$ for which

$$\text{Area of Gauss image of } \Pi_v(M) \leqslant c_n A.$$

4. Examples and open questions

1. If M is the graph of $z = \lambda(1 + x^2 + y^2)^{1/2}$, then

$$\int_M K = \text{Area of } g(M)$$

may be made arbitrarily close to 0 by choosing λ small. Note also that $\lim_{x \to \infty} g(x)$ does not exist.

2. Let M be the graph of a smooth function $z = u(x, y)$ which away from the origin is given by

$$u(x, y) = x \sin(\log(\log r)) \qquad \left(r = \sqrt{x^2 + y^2}\right).$$

Then $|Du|$ is bounded, so

$$|B|^2\, dA \leqslant C|D^2u|^2\, dx\, dy.$$

Also, $|Du|^2 = \mathcal{O}((r \ln r)^{-2})$ so

$$\int_M |B|^2 \leqslant C \int |D^2u|^2 < \infty.$$

However

$$\frac{\partial u}{\partial x}(x, 0) = \sin(\log(\log r)) + \mathcal{O}\left((\log r)^{-1}\right)$$

so the Gauss map is not well defined at ∞.

3. Let M be a smooth surface in $\mathbf{R}^4$ which, away from the origin, is the image of

$$X(r, \theta) = r(\cos\theta \cos(\ln\ln r), \sin\theta\cos(\ln\ln r),$$
$$\cos\theta \sin(\ln\ln r), \sin\theta \sin(\ln\ln r)).$$

Then

$$|\partial X/\partial r|^2 = 1 + (\ln r)^{-2}, \quad |\partial x/\partial\theta|^2 = r^2, \quad (\partial X/\partial r)\cdot(\partial X/\partial\theta) = 0.$$

From this one readily sees that M is complete and calculates that

$$K = -r^{-2}\left(1 + (\ln r)^2\right)^{-2} < 0, \qquad |B|^2 = \mathcal{O}\left((\ln r)^{-2} r^{-2}\right).$$

So $\int |B|^2 < \infty$. On the other hand, for r large the tangent plane to M at $X(r, \theta)$ is approximately

$$\left\{(z, w) \in \mathbf{C}^2 = \mathbf{R}^4 : z \sin(\ln\ln r) = w\cos(\ln\ln r)\right\}.$$

Thus $\lim_{r\to\infty} g(X(r,\theta))$ does not exist. Hence negative sectional curvature is not sufficient for Theorem 3.

4. Does $\int_M |B|^2 < \infty$ imply that M is properly immersed?

5. In Theorem 3, does $M \subset \mathbf{R}^3$, $K \geqslant 0$ in U imply that U is totally geodesic?

6. Can the case $n > 3$ of Theorem 3 be proved directly, without projecting M into a three-dimensional space?

Appendix

Proposition. (1) *Up to a scalar factor, there is a unique two-form ω on $G_2(n)$ which is invariant under the $\mathcal{O}(n)$ action on $G_2(n)$. In particular, the restriction to $G_2(n)$ of the Kähler form η on $\mathbf{C}P^{n-1}$ to $G_2(n)$ is such a form and is not zero, so any such ω is a multiple of η.*

(2) *If M is a surface in $\mathbf{R}^n$, then $K\,dA = g^{\#}(\omega)$ for such a two-form ω.*

Proof. Let P be an oriented two-plane in $\mathbf{R}^n$: we will determine ω at P. Let $e_1, e_2, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-2}$ be an orthonormal basis for $\mathbf{R}^n$ such that e_1, e_2 is an oriented basis for P. Note that the two-planes near P are in one-to-one correspondence with simple two-vectors of the form

$$\left(e_1 + \sum \alpha^i \varepsilon_i\right) \wedge \left(e_2 + \sum \beta^j \varepsilon_j\right).$$

Thus we have as an orthonormal basis for the tangent space to $G_2(n)$ at P

$$u_1 = e_i \wedge \varepsilon_i, \quad v_i = \varepsilon_i \wedge e_2 \qquad (1 \leqslant i \leqslant n-2).$$

Let $\{U^i, V^i\}$ be the dual basis of one-forms. Then

$$(*) \qquad \omega(P) = \sum_{i<j} \alpha_{ij} U^i \wedge U^j + \sum \beta_{ij} U^i \wedge V^j + \sum_{i<j} \gamma_{ij} V^i \wedge V^j.$$

Let ϕ be the isometry of $\mathbf{R}^n$ which maps ε_1 to $-\varepsilon_1$ and which leaves the other basis vectors fixed. Let

$$(\phi^{\#}\omega)(P) = \sum_{i<j} \alpha'_{ij} U^i \wedge U^j + \sum \beta'_{ij} U^i \wedge V^j + \sum_{i<j} \gamma'_{ij} V^i \wedge V^j.$$

Then

$$\alpha'_{12} = (\phi^{\#}\omega)(u_1, u_2) = \omega(\phi_{\#}(u_1), \phi_{\#}(u_2)) = \omega(-u_1, u_2) = -\alpha_{12}.$$

But $\alpha'_{12} = \alpha_{12}$ by the invariance of ω. In the same way one sees that

$$\alpha_{1j} = \beta_{1j} = \beta_{j1} = \gamma_{1j} = 0 \quad \text{unless } j = 1$$

and (repeating the argument with ε_i instead of ε_1):

$$\alpha_{ij} \equiv \gamma_{ij} \equiv 0; \quad \beta_{ij} = 0 \qquad (\text{unless } i = j)$$

so

$$\omega(P) = \sum \beta_{ii} U^i \wedge V^i.$$

Since this is invariant under any isometry that permutes the ε_i's

$$\omega(P) = \beta \sum U^i \wedge V^i.$$

Finally, since any other plane may be mapped to P by an isometry of $\mathbf{R}^n$, β does not depend on P.

As for (2), it is not hard to check that

$$K\,dA = g^{\#}\left(\sum U^i \wedge V^i\right).$$

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