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η-INVARIANTS, THE ADIABATIC APPROXIMATION AND CONICAL SINGULARITIES

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PART I: THE ADIABATIC APPROXIMATION

0. Introduction

In this paper, we discuss a remarkable formula derived by Witten for the η-invariant of a mapping torus, $Y^{4k-2} \to N^{4k-1} \to S^1$ (see (1.56), Theorem 4.27 and [26, §IV]). Witten's derivation is based on the adiabatic approximation as it is often applied in quantum mechanics and is not rigorous. Here, in Part I, we treat explicitly the case of signature operators by heat equation methods, using Duhamel's principle and the techniques of [13] (see also [14]).[1] Bismut and Freed independently treat the case of Dirac operators, using the heat equation and probability theory (see [4], [5]).

Witten's formula is closely related to the work of Atiyah-Donnelly-Singer [1] on η-invariants of solvmanifolds (as others have independently observed). In Appendix 3 to Part I, we show how to obtain a quick proof of a similar result for the case of a 1-dimensional base space, by starting with Witten's formula. We will discuss the generalization of Witten's formula to higher dimensional base spaces and its application to η-invariants of higher dimensional solvmanifolds elsewhere.

In Part II, we discuss in detail the relation between the result of Part I and our previous work on analysis on spaces with conical singularities. The expression in Witten's formula (which is used to define the notion of *anomaly* in physics) arose there when we considered the variational derivative of the η-invariant for a family of spaces (X^{4k-1}, g_u) with conical singularities. The discussion of Part II shows that this expression is also equal to the contribution at a singular stratum, Σ^1, of dimension 1, when one calculates the L_2-signature

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[1] The idea of the proof is described in more detail in §1.

of a pseudomanifold, X^{4k}, with nonisolated conical singularities (the L_2-signature coincides with the signature in the sense of intersection homology).

Although we have restricted attention to signature operators, our discussion extends without essential modification to other operators of Dirac type. However, to extend the results of §5, a somewhat more general perturbation argument than that of §5 must be used (compare [5]).

Before closing this introduction, we remark that passing to the adiabatic limit is, up to rescaling, an example of *collapsing* (in general, without bounded curvature) in the sense of [12]. For a discussion of the topological significance of η-invariants and secondary geometric invariants of collapsed manifolds with bounded curvature, we refer to [10] and [28].

The remainder of the paper will be organized as follows.

We are indebted to J. Kaminker for calling Witten's work to our attention and to I. M. Singer for conversations concerning anomalies. Singer has also announced a proof of Witten's formula (see [24]).

Appendix 1. Existence of $\lim_{\delta \to 0} \eta(N, g_\delta, \xi)$

The main formula to be discussed in this paper calculates the limiting value as $\delta \to 0$ of the η-invariant, $\eta(N^{4k-1}, g_\delta, \xi)$, where $\{g_\delta\}$ is a certain family of Riemannian metrics on N^{4k-1} and ξ is a Hermitian vector bundle with connection (see (1.56), (4.27), (5.23)). In general, if the connection on ξ is not flat the limit must be taken in R/Z. The existence of the limit follows from the relation between the η-invariant and the appropriate Chern-Simons invariant given by the Atiyah-Patodi-Singer formula, together with the fact that the limit of the family of Riemannian connections associated to the family of Riemannian metrics $\{g_\delta\}$ exists. This argument applies in a context more general than that considered elsewhere in the paper (there we restrict attention to the case $m = 1$, of what follows directly below).

Let $Y^n \to N \overset{\pi}{\to} B^m$ be any Riemannian submersion. Let g_δ be the family of metrics obtained by multiplying the metric on the orthogonal complement, H, to the subspace, V, tangent to the fibers by a factor δ^{-2}, while leaving $H = V^\perp$ and the metric on V unchanged. Let $X_1 \cdots X_n$ be a local orthonormal frame field tangent to the fibers such that $[X_i, X_j] = 0$ at some $p \in N$. Let $N_1, \cdots, N_m$ be the horizontal lifts of a local frame field on B which is orthonormal for g, and such that $[\pi_*(N_i), \pi_*(N_j)] = 0$ at $\pi_*(p)$. Then $[N_i, N_j]$ is vertical. Moreover $\delta N_1, \cdots, \delta N_m$, $X_1, \cdots, X_n$ is orthonormal with respect to g_δ. Let ∇^δ denote the Riemannian connection of g_δ. By the standard formula for ∇^δ, we have

(A1.1) $$\left\langle \nabla^\delta_{X_i} X_j, X_k \right\rangle_\delta = 0,$$

(A1.2) $$\left\langle \nabla^\delta_{X_i} X_j, \delta N_k \right\rangle_\delta = \tfrac{1}{2}\left\{ \left\langle [\delta N_k, X_i], X_j \right\rangle_\delta + \left\langle [\delta N_k, X_j], X_i \right\rangle_\delta \right\},$$

(A1.3) $$\left\langle \nabla^\delta_{X_i} N_j, X_k \right\rangle_\delta = \tfrac{1}{2}\left\{ \left\langle [X_i, N_j], X_k \right\rangle_\delta + \left\langle [X_k, N_j], X_i \right\rangle_\delta \right\},$$

(A1.4) $$\left\langle \nabla^\delta_{X_i} N_j, \delta N_k \right\rangle_\delta = \tfrac{1}{2}\left\{ \left\langle [\delta N_k, N_j], X_i \right\rangle_\delta \right\},$$

(A1.5) $$\left\langle \nabla^\delta_{N_i} N_j, X_k \right\rangle_\delta = \tfrac{1}{2}\left\{ \left\langle [N_i, N_j], X_k \right\rangle_\delta \right\},$$

(A1.6) $$\left\langle \nabla^\delta_{N_i} N_j, \delta N_k \right\rangle_\delta = 0.$$

Let W^H and W^V denote the horizontal and vertical components of a vector W. Then we get

(A1.7) $$\nabla^\delta_{X_i} X_j = \delta^2 \left(\nabla^1_{X_i} X_j \right)^H = \delta^2 \nabla^1_{X_i} X_j,$$

(A1.8) $$\nabla^\delta_{X_i} N_j = \left(\nabla^1_{X_i} N_j \right)^V + \delta^2 \left(\nabla^1_{X_i} N_j \right)^H,$$

(A1.9) $$\nabla^\delta_{N_i} X_j = \left(\nabla^1_{N_i} X_j \right)^V + \delta^2 \left(\nabla_{N_i} X_j \right)^H,$$

(A1.10) $$\nabla^\delta_{N_i} N_j = \left(\nabla^1_{N_i} N_j \right)^V = \nabla^1_{N_i} N_j.$$

Thus, letting ∇^0 denote the limit connection, we have

(A1.11) $$\nabla^1_{X_i} X_j - \nabla^0_{X_i} X_j = \left(\nabla^1_{X_i} X_j \right)^H,$$

(A1.12) $$\nabla^1_{X_i} N_j - \nabla^0_{X_i} N_j = \left(\nabla^1_{X_i} N_j \right)^H,$$

(A1.13) $$\nabla^1_{N_i} X_j - \nabla^0_{N_i} X_j = \left(\nabla^1_{N_i} X_j \right)^H,$$

(A1.14) $$\nabla^1_{N_i} N_j - \nabla^0_{N_i} N_j = 0.$$

It now follows from the Atiyah-Patodi-Singer formula, that if $\eta(N, g_\delta \xi)$ denotes the η-invariant with coefficients in some Hermitian vector bundle, ξ, then $\lim_{\delta \to 0} \eta(N, g_\delta \xi)$ exists and is equal to the corresponding secondary geometric invariant of the limit connection (in general, the limit must be taken in R/Z). In fact, to see the existence and to calculate $\frac{d}{d\delta}\eta(N, g_\delta, \xi)$, we only need the local result that this derivative is equal to the derivative of the corresponding secondary geometric invariant.

In the remainder of the paper, we will write $\eta(N, g_\delta)$ for $\eta(N, g_\delta, \xi)$.

1. The computation and the outline of its proof

We begin by describing the setup. Then we indicate some ideas behind the proof. Finally, we do the explicit computation, the justification for which is given in the following sections.

Let $Y^{4k-2} \to N^{4k-1} \to S^1$ be a Riemannian submersion and let ξ be a Hermitian vector bundle with compatible connection over N. Let $\hat{\partial}/\partial u$ denote the horizontal lift of the unit vector field on S^1. We denote by ∂_u the operation of covariant differentiation in the direction of $\hat{\partial}/\partial u$.

By using the connection, the operation of exterior differentiation of forms extends to forms with values in ξ. Define $A : \oplus_p \Lambda^{2p} \otimes \xi \to \Lambda^{2p} \otimes \xi$ by

$$A = d * + (-1)^p d * \quad \text{on } \Lambda^{2p} \otimes \xi. \tag{1.1}$$

As usual for the Laplacian A^2, we have $* A^2 = A^2 *$. Let $\tilde{d}$ denote exterior differentiation along the fibers and $\tilde{*}$ the $*$-operator along the fibers. Let a $2p$-form $\theta + dr \wedge \omega$ with values in ξ be denoted

$$\theta + du \wedge \omega = \begin{pmatrix} \theta \\ \omega \end{pmatrix}. \tag{1.2}$$

Then

$$d = \begin{pmatrix} \tilde{d} & 0 \\ \partial_u & -\tilde{d} \end{pmatrix}, \tag{1.3}$$

and on j-forms,

$$* = \begin{pmatrix} 0 & \tilde{*} \\ (-1)^j \tilde{*} & 0 \end{pmatrix}, \tag{1.4}$$

so that on $2p$-forms

$$\begin{aligned} A &= \begin{pmatrix} (-1)^p \tilde{*} \partial_u & \tilde{d} \tilde{*} + (-1)^{p+1} \tilde{*} \tilde{d} \\ -\tilde{d} \tilde{*} + (-1)^{p+1} \tilde{*} \tilde{d} & \partial_u \tilde{*} \end{pmatrix} \\ &\overset{\text{def}}{=} \begin{pmatrix} \mathscr{D} & \mathscr{A} \\ \mathscr{A} & \mathscr{D} \end{pmatrix}. \end{aligned} \tag{1.5}$$

The above formula suggests defining

$$\beta = \begin{cases} (-1)^p \tilde{*} & 2p\text{-forms}, \\ \tilde{*} & (2p-1)\text{-forms}. \end{cases} \tag{1.6}$$

Then

$$A = \begin{pmatrix} \beta\partial_u & d\beta - \beta\tilde{d} \\ (-1)^{p+1}(\tilde{d}\beta + \beta\tilde{d}) & \partial_u\beta \end{pmatrix}, \tag{1.7}$$

where

$$\mathscr{D} = \begin{cases} \beta\partial_u & 2p\text{-forms}, \\ \partial_u\beta & (2p-1)\text{-forms}. \end{cases} \tag{1.8}$$

One checks that in all degrees,

$$\beta^2 = -1, \tag{1.9}$$

and that

$$\beta\mathscr{A} = -\mathscr{A}\beta. \tag{1.10}$$

We have

$$A^2 = \begin{pmatrix} \mathscr{A}^2 + \mathscr{D}^2 & \mathscr{A}\mathscr{D} + \mathscr{D}\mathscr{A} \\ \mathscr{A}\mathscr{D} + \mathscr{D}\mathscr{A} & \mathscr{A}^2 + \mathscr{D}^2 \end{pmatrix}. \tag{1.11}$$

Moreover, $\beta\mathscr{A} = -\mathscr{A}\beta$ implies

$$\beta\mathscr{A}^2 = \mathscr{A}^2\beta, \tag{1.12}$$

and in the case of a local product metric (where $[\mathscr{A}, \partial] = [\beta, \partial] = 0$) we have

$$A^2 = \begin{pmatrix} \mathscr{A}^2 - \partial_u^2 & 0 \\ 0 & \mathscr{A}^2 - \partial_u^2 \end{pmatrix}. \tag{1.13}$$

Note that $\mathscr{A}$ is a first order selfadjoint elliptic operator which interchanges the $\pm i$ eigenbundles of β. Viewed as such, $\mathscr{A}$ is clearly of index zero, since β is an almost complex structure.

The metric on N can be written as

$$g = du^2 + \tilde{g}(u). \tag{1.14}$$

Let g_δ denote the metric

$$g_\delta = \delta^{-2}du^2 + \tilde{g}(u). \tag{1.15}$$

If we put $v = u/\delta$, then in v-coordinates,

$$g = dv^2 + \tilde{g}(\delta v), \tag{1.16}$$

where, say, $-\frac{1}{2} \leqslant u \leqslant \frac{1}{2}$ and $-1/2\delta \leqslant v \leqslant 1/2\delta$. Thus, as $\delta \to 0$, g_δ converges locally to a product metric, but the length of S^1 is $\sim \delta^{-1}$.

If we put $\partial = \partial_v$, then in v-coordinates, for the metric g_δ, we have

$$\mathscr{D}_\delta = \begin{cases} \beta(\delta v)\partial & 2p\text{-forms}, \\ \partial\beta(\delta v) & (2p-1)\text{-forms}, \end{cases} \tag{1.17}$$

$$A_\delta = \begin{pmatrix} \mathscr{D}(\delta v) & \mathscr{A}(\delta v) \\ \mathscr{A}(\delta v) & \mathscr{D}(\delta v) \end{pmatrix}. \tag{1.18}$$

We can now give a more detailed indication of the contents of the rest of Part I. The η-invariant is defined by

$$\eta(N, g_\delta) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} \operatorname{tr}\left(A_\delta e^{-A_\delta^2 t}\right) dt. \tag{1.19}$$

We want to show that in the (adiabatic) limit, $\delta \to 0$, the η-invariant is given by (1.56). The salient feature of that formula is that the right-hand side involves the integration over the base of an expression which is only *global on the fibers*, while a priori, the η-invariant is *global on N*.

Now the trace in (1.19) entails an integration over both the fiber and base directions. We will show in §4 that for each fixed point on the base, the integral of the expression in (1.19) over only the fiber directions converges in the limit, $\delta \to 0$, to the integrand on the right-hand side of (1.56).

As we observed, the metric and connection converge locally (in v-coordinates) to a product, over the inverse images in N of larger and larger intervals centered at any fixed point of the base. It is clear however, that the lengths of such intervals must be $o(\delta^{-1})$ and, in fact, it will turn out to be convenient in analyzing the large time behavior of the integrand in (1.19) to consider intervals of length $\sqrt{2}\,|\log \delta|^{1/2} t^{1/2}$ (the reason will become apparent in §4; see also below).

Note also, that for the product case, $\operatorname{tr}(Ae^{-A^2 t}) \equiv 0$ (pointwise) because there is an orientation reversing isometry fixing any point of $R \times Y$. Thus, it is reasonable to expect that at least for $0 \leqslant t \leqslant T$, the integrand in (1.19) is *pointwise* $\sim \delta$. Since the volume of (N, g_δ) is $\sim \delta^{-1}$, we get a finite limit in (1.19), and this limit is calculated (formally) in this section.

However, since the integration in (1.19) is from 0 to ∞, of course we cannot restrict attention to a finite time interval $(0, T]$. Now it is clear from (1.13) that in the product situation the spectrum of A^2 is bounded below by that of $\mathscr{A}^2$. Thus, *if for all u, the kernel of $\mathscr{A}(u)$ is empty*, the operator A_δ^2 should be uniformly positive for sufficiently small δ. This is checked in §2. Under this assumption, which will be in force until §5, $\operatorname{tr}(e^{-A_\delta^2 t})$ decays pointwise exponentially in t (uniformly in δ). But some care must still be taken since

$\mathrm{Vol}(N) \sim \delta^{-1} \to \infty$. By an easy estimate (see [11, p. 141]) it follows in general that

$$\frac{1}{\Gamma(1/2)} \int_T^{\infty} t^{-1/2} \operatorname{tr}\left(Ae^{-A^2 t}\right) dt \leqslant \operatorname{tr}\left(e^{-A^2 T}\right). \tag{1.20}$$

If the smallest eigenvalue of A^2 is $\geqslant \lambda > 0$, then in our case

$$\frac{1}{\Gamma(1/2)} \int_T^{\infty} t^{-1/2} \operatorname{tr}\left(A_\delta e^{-A_\delta^2 t}\right) dt \leqslant ce^{-\lambda T}\delta^{-1}. \tag{1.21}$$

For $T = 2|\log \delta|/\lambda$, the right-hand side is $\leqslant c\delta$, and this shows the importance of keeping track of the t dependence of the error term in the pointwise convergence $\operatorname{tr}(A_\delta e^{-A_\delta^2 t}) \to \operatorname{tr}(A_0 e^{-A_0^2 t}) \equiv 0$. Naively, one would expect that for $0 \leqslant t \leqslant T$ the pointwise error term is of order δ^2 (where the significant term, alluded to above, is of order δ). We show in §4 (without the assumption that $\ker \mathscr{A}$, is empty) that for $t \geqslant t_0$, the error term is bounded by $c(t_0)\delta^2 |\log \delta|^{m_1} t^{m_2 - 1}$ (where the precise values of m_1, m_2 given in §4 are not important for our purposes and are not estimated with great care). For $1 \leqslant t \leqslant 2|\log \delta|/\lambda$, this implies that the pointwise error is bounded by $c(1)(2/\lambda)^{m_2}\delta^2 |\log \delta|^{m_1 + m_2} t^{-1}$. Thus, the error in the convergence of the global trace is bounded by $c(1)(2/\lambda)^{m_2}\delta |\log \delta|^{m_1 + m_2} t^{-1}$ on this interval. This, together with our previous bound, clearly suffices to prove the convergence for large time.

In order to carry out the analysis which has been described, we could compare the heat kernels $E_\delta(t)$, $E_0(t)$ of A_δ^2, A_0^2 by applying Duhamel's principle. But to avoid boundary terms and to simplify the job of obtaining the estimates, it is convenient to have kernels which can be regarded as living on either N or $R \times Y$. So we will replace E_δ, E_0 by semilocal parametrices $P_{\delta,b}$, $P_{0,b}$ which are supported on a neighborhood of radius b about the diagonal. These are constructed in §3, using finite propagation speed for the fundamental solution, $\cos\sqrt{A^2}\,\xi$, of the wave equation and are estimated there. The choice $b = \sqrt{2}\,|\log \delta|^{1/2} t^{1/2}$ mentioned above turns out to give an estimate on the error term G_b (where $E_b = P_b + G_b$) which enables us to establish the bound $C(t_0)\delta^2 |\log \delta|^{m_1 + m_2} t^{-1}$, just discussed.

In performing the computation of this section we will simply pretend that $E_\delta(t)$ lives on $R \times Y$ and evaluate the expression (obtained from Duhamel's principle) for differentiating the trace of a 1-parameter family of heat kernels on the same manifold. As just explained, the justification for this procedure will come in §4.

For heat kernels on a fixed manifold we expect

$$\text{(1.22)}\qquad \begin{aligned}\operatorname{tr}(A_\delta E_\delta(t))\,|_{0\times Y} &= \operatorname{tr}(A_0E_0(t))\,|_{0\times Y}\\ &\quad+\delta\frac{\partial}{\partial\delta}\operatorname{tr}(A_0E_0(t))\,|_{0\times Y}+O(\delta^2).\end{aligned}$$

Let prime denote differentiation with respect to δ at $\delta = 0$ and let $\#$ denote convolution. Then by Duhamel's principle we should have

$$\text{(1.23)}\qquad \begin{aligned}\operatorname{tr}(A_0E_0(t))' &= \operatorname{tr}(A_0'E_0(t))\\ &\quad+\operatorname{tr}\Big(A_0E_0(s)\#_0\big(A_0^2\big)'E_0(t-s)\Big).\end{aligned}$$

As explained above $\operatorname{tr}(A_0E_0(t)) \equiv 0$ and we will see momentarily that $\operatorname{tr}(A_0'E(t))\,|_{0\times Y} \equiv 0$ as well. So the main task is to evaluate (the integral over $0 \times Y$ of) the second term of (1.23) which we then substitute in (1.19).

We have

$$\text{(1.24)}\qquad E_0(t) = \begin{pmatrix}\mathscr{F}\mathscr{E} & 0\\ 0 & \mathscr{F}\mathscr{E}\end{pmatrix},$$

where $\mathscr{E}(t)$ is the heat kernel of $\mathscr{A}^2$ on $0 \times Y$ and

$$\text{(1.25)}\qquad \mathscr{F}(t) = \frac{e^{-(v_1-v_2)^2/4t}}{(4\pi t)^{1/2}}$$

is the heat kernel of R.

In the computation that follows we will write $A_0 = A$, $E_0 = E$, etc. Let dot denote differentiation with respect to $u = \delta \cdot v$. Then at $v = 0$,

$$\text{(1.26)}\qquad A' = \begin{pmatrix}0 & 0\\ 0 & \dot\beta\end{pmatrix}.$$

Since

$$\text{(1.27)}\qquad \beta^2 = -1,$$

$$\text{(1.28)}\qquad \dot\beta\beta = -\beta\dot\beta,$$

$$\text{(1.29)}\qquad \beta\mathscr{E} = \mathscr{E}\beta,$$

by standard linear algebra, we have the pointwise relation

$$\text{(1.30)}\qquad \operatorname{tr}(\dot\beta\mathscr{E}(t)) = 0.$$

Thus,

$$\text{(1.31)}\qquad \operatorname{tr}\big(A'\mathscr{E}(t)\big) \equiv 0.$$

If we take the integrated trace over $(0 \times Y)$ and use $AE = EA$, $\operatorname{tr}(TS) = \operatorname{tr}(ST)$, we can rewrite the second term on the right-hand side of (1.23) as

$$\text{(1.32)}\qquad -\int_0^t \operatorname{tr}\left\{\left[\int_{-\infty}^{\infty}\mathscr{F}(s)A(A^2)'\mathscr{F}(t-s)\,dv\right]\mathscr{E}(t)\right\}ds.$$

The diagonal terms of $A(A^2)'$ are

$$\mathscr{D}(\mathscr{A}^2+\mathscr{D}^2)'+\mathscr{A}(\mathscr{A}\mathscr{D}+\mathscr{D}\mathscr{A})'. \tag{1.33}$$

Consider first the term

$$\mathscr{D}(\mathscr{D}^2)'=\begin{cases}\beta\partial\left[(\beta\partial)^2\right]', & \text{even degrees,}\\ \partial\beta\left[(\partial\beta)^2\right]', & \text{odd degrees.}\end{cases} \tag{1.34}$$

In even degrees we get

$$\begin{aligned}\beta\partial(v\dot\beta\partial\beta\partial+\beta\partial v\dot\beta\partial)&=\beta\dot\beta\beta\partial v\partial^2+\beta^2\dot\beta\partial^2 v\partial\\&=\dot\beta(\partial v\partial^2-\partial^2 v\partial)\end{aligned} \tag{1.35}$$

and similarly in odd degrees, we get

$$=\dot\beta(\partial^2 v\partial-\partial^3 v). \tag{1.36}$$

Using

$$\partial v=1+v\partial, \tag{1.37}$$

we find that in both cases

$$\mathscr{D}(\mathscr{D}^2)'=-\dot\beta\partial^2. \tag{1.38}$$

By the linear algebra argument above, this term can be dropped.

In even degrees,

$$\begin{aligned}\mathscr{A}(\mathscr{A}\mathscr{D}+\mathscr{D}\mathscr{A})'&=\mathscr{A}(\mathscr{A}\beta\partial+\partial\beta\mathscr{A})'\\&=\mathscr{A}(v\dot{\mathscr{A}}\beta\partial+\mathscr{A}v\dot\beta\partial+\partial v\dot\beta\mathscr{A}+\partial\beta v\dot{\mathscr{A}})\\&=\mathscr{A}v\partial(\dot{\mathscr{A}}\beta+\mathscr{A}\dot\beta+\dot\beta\mathscr{A}+\beta\dot{\mathscr{A}})+\mathscr{A}(\dot\beta\mathscr{A}+\beta\dot{\mathscr{A}}).\end{aligned} \tag{1.39}$$

Since

$$(\dot{\mathscr{A}}\beta+\mathscr{A}\dot\beta+\dot\beta\mathscr{A}+\beta\dot{\mathscr{A}})=(\mathscr{A}\beta+\beta\mathscr{A})^{\cdot}=0, \tag{1.40}$$

the last line in (1.39) reduces to

$$\mathscr{A}\dot\beta\mathscr{A}-\beta\mathscr{A}\dot{\mathscr{A}}. \tag{1.41}$$

Now

$$\mathscr{A}\dot\beta=-\dot{\mathscr{A}}\beta-\dot\beta\mathscr{A}-\beta\dot{\mathscr{A}}. \tag{1.42}$$

Thus,

$$\begin{aligned}\operatorname{tr}(\mathscr{A}\dot\beta\mathscr{A}\mathscr{E})-\operatorname{tr}(\beta\mathscr{A}\dot{\mathscr{A}}\mathscr{E})&=-\operatorname{tr}(\dot{\mathscr{A}}\beta\mathscr{A}\mathscr{E})-\operatorname{tr}(\dot\beta\mathscr{A}^2\mathscr{E})\\&\quad-\operatorname{tr}(\beta\dot{\mathscr{A}}\mathscr{A}\mathscr{E})-\operatorname{tr}(\beta\mathscr{A}\dot{\mathscr{A}}\mathscr{E}).\end{aligned} \tag{1.43}$$

Using

$$-\operatorname{tr}(\dot{\mathscr{A}}\beta\mathscr{A}\mathscr{E}) = \operatorname{tr}(\dot{\mathscr{A}}\mathscr{A}\beta\mathscr{E}) = \operatorname{tr}(\dot{\mathscr{A}}\mathscr{A}\mathscr{E}\beta) = \operatorname{tr}(\beta\dot{\mathscr{A}}\mathscr{A}\mathscr{E}) \tag{1.44}$$

and

$$-\operatorname{tr}(\dot{\beta}\mathscr{A}^2\mathscr{E}) = 0 \tag{1.45}$$

(by linear algebra as above), the right-hand side of (1.43) reduces to

$$-\operatorname{tr}(\beta\mathscr{A}\dot{\mathscr{A}}). \tag{1.46}$$

In odd degrees, we find directly that

$$(\mathscr{A}\mathscr{D} + \mathscr{D}\mathscr{A})' = -\beta\mathscr{A}\dot{\mathscr{A}} + \mathscr{A}^2\dot{\beta}. \tag{1.47}$$

Now,

$$\operatorname{tr}(\mathscr{A}^2\dot{\beta}\mathscr{E}(t)) = \operatorname{tr}(\mathscr{E}(t/2)\mathscr{A}^2\dot{\beta}\mathscr{E}(t/e)) = \operatorname{tr}(\dot{\beta}\mathscr{E}(t)\mathscr{A}^2) \tag{1.48}$$

which vanishes as above. So in all degrees, we get a contribution

$$-\operatorname{tr}(\beta\mathscr{A}\dot{\mathscr{A}}\mathscr{E}). \tag{1.49}$$

For the remaining term, we have

$$\mathscr{D}(\mathscr{A}^2)' = \mathscr{D}(v\dot{\mathscr{A}}\mathscr{A} + \mathscr{A}v\dot{\mathscr{A}}) = \beta(\mathscr{A}^2)^{\cdot}\,\partial v. \tag{1.50}$$

Since $\beta\mathscr{A}^2 = \mathscr{A}^2\beta$, $\beta^2 = -1$, we have the *global* relation

$$\begin{aligned} 0 &= \operatorname{tr}(\beta\mathscr{E}(t))^{\cdot} = \operatorname{tr}(\dot{\beta}\mathscr{E}(t)) - t\operatorname{tr}(\beta(\mathscr{A}^2)^{\cdot}\mathscr{E}(t)) \\ &= -t\operatorname{tr}(\beta(\mathscr{A}^2)^{\cdot}\mathscr{E}(t)). \end{aligned} \tag{1.51}$$

Thus, the only contribution is from the term

$$-\operatorname{tr}(\beta\mathscr{A}\dot{\mathscr{A}}\mathscr{E}(t)) = \operatorname{tr}(\beta\dot{\mathscr{A}}\mathscr{A}\mathscr{E}(t)). \tag{1.52}$$

The coefficient of this term is

$$\int_{-\infty}^{\infty} \mathscr{F}(s)\mathscr{F}(t-s)\,ds\,|_{v_1=v_2=0} = \mathscr{F}(t)\,|_{v_1=v_2=0} = \frac{1}{(4\pi t)^{1/2}}. \tag{1.53}$$

Finally,

$$\begin{aligned} \lim_{\delta\to 0} \operatorname{tr}(A_\delta E_\delta(t)) &= \int_0^t \frac{1}{(4\pi t)^{1/2}} \operatorname{tr}(\beta\dot{\mathscr{A}}\mathscr{A}\mathscr{E}(t))\,ds \\ &= \frac{t^{1/2}}{2\pi^{1/2}} \cdot \operatorname{tr}(\beta\dot{\mathscr{A}}\mathscr{A}\mathscr{E}(t)). \end{aligned} \tag{1.54}$$

If we now grant that

$$\lim_{\delta\to 0} \eta(N, g_\delta) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2}\left[\int_{S^1} \operatorname{tr}(AE(t))'\right] dt, \tag{1.55}$$

we obtain

$$\lim_{\delta\to 0}\eta(N,g_\delta) = \frac{1}{\pi^{1/2}}\int_{S^1} t^{-1/2}\,\mathrm{tr}\left(\frac{t^{1/2}}{2\pi^{1/2}}\beta\dot{\mathscr{A}}\mathscr{A}\mathscr{E}(t)\right)dt$$

$$(1.56)\qquad = \frac{1}{2\pi}\int_{S^1}\int_0^\infty \mathrm{tr}(\beta\dot{\mathscr{A}}\mathscr{A}\mathscr{E}(t))\,dt$$

$$= \frac{1}{\pi}\int_{S^1}\lim_{\varepsilon\to 0}\mathrm{tr}\left(\frac{\beta}{2}\mathscr{A}^{-1}\dot{\mathscr{A}}\mathscr{E}(\varepsilon)\right),$$

which is the formula of [26], in case $\mathscr{A}$ is always invertible.

Appendix 2. Explanation of the basic formula

At the formal level, the basic formula (1.56) is a consequence of

(i) a (renormalized) version of the Atiyah-Patodi-Singer formula for the difference of η-invariants of the operator $i\nabla_{\partial/\partial u}$ acting on sections of a pair of infinite dimensional Hermitian vector bundles over S^1;

(ii) a basic point in Quillen's extension of the Weil homomorphism to superconnections (see [20], [23]).

Let $Y^{4k-2}\to N^{4k-1}\overset{\pi}{\to} S^1$ be our Riemannian submersion. For each $u\in S^1$, the Hilbert space of differential forms on $\pi^{-1}(u)$ (with coefficients in ξ) can be viewed as the fiber of an infinite dimensional Hermitian vector bundle with connection over S^1. Call this bundle Λ. The connection is induced by the metric on N and connection on ξ in the obvious way. If s is a section of Λ, we denote its covariant derivative in the direction of $\partial/\partial u$ by ∂s or $\dot{s}$. Note that if $\alpha = *^{-1}*$ one easily checks that

$$(\text{A2.1})\qquad \nabla_{\partial/\partial u}s \overset{\text{def}}{=} \partial s + \frac{\alpha}{2}(s)$$

satisfies

$$(\text{A2.2})\qquad \frac{\partial}{\partial u}\langle s_1,s_2\rangle = \langle\nabla_{\partial/\partial u}s_1,s_2\rangle + \langle s_1,\nabla_{\partial/\partial u}s_2\rangle,$$

$$(\text{A2.3})\qquad \beta\alpha = -\alpha\beta = \dot{\beta},$$

$$(\text{A2.4})\qquad \nabla\beta = \beta\nabla.$$

The bundle Λ splits as a direct sum,

$$(\text{A2.5})\qquad \Lambda = \Lambda_+\oplus\Lambda_-,$$

where $\Lambda_\pm$ are the $\pm i$-eigenbundles of β. The splitting in (A2.5) allows us to project the connection ∂ onto $\Lambda_\pm$. The connection ∇ restricts to a unitary connection on Λ_+ and Λ_-, i.e., the orthogonal projections $\pi_\pm$ on $\Lambda_\pm$ commute with ∇. Call these connections $\partial^\pm$, $\nabla^\pm$.

If, as usual, we put

$$\phi(v) + dv \wedge \omega(v) = \begin{pmatrix} \phi(v) \\ \omega(v) \end{pmatrix}, \tag{A2.6}$$

then

$$A_\delta = \begin{pmatrix} \beta(\delta v)\partial_v & \mathscr{A}(\delta v) \\ \mathscr{A}(\delta v) & \partial_v \beta(\delta v) \end{pmatrix}. \tag{A2.7}$$

Thus, for the isomorphism in (A2.6), we have

$$A_\delta(u) = \begin{pmatrix} \delta\beta(u)\partial_u & \mathscr{A}(u) \\ \mathscr{A}(u) & \delta\partial_u \beta(u) \end{pmatrix}. \tag{A2.8}$$

Set

$$C_\delta(u) = \begin{pmatrix} -\delta\beta\alpha/2 & \mathscr{A} \\ \mathscr{A} & \delta\beta\alpha/2 \end{pmatrix}. \tag{A2.9}$$

Then in terms of splitting, $\Lambda = \Lambda_+ \oplus \Lambda_-$,

$$A_\delta(u) = \begin{pmatrix} \delta i \nabla_{\partial/\partial u} & C_\delta(u) \\ C_\delta(u) & -\delta i \nabla_{\partial/\partial u} \end{pmatrix}. \tag{A2.10}$$

The operators considered so far and the relations they satisfy make equally good sense if $\dim \Lambda < \infty$. Moreover, it is clear that the derivation of (1.56) still applies in that case. On the other hand, if $\dim \Lambda < \infty$, we can compute $\eta(A_\delta)$ (for any δ) from the Atiyah-Patodi-Singer formula. We now check that the answers obtained by these two methods are consistent.

The classical version of the Atiyah-Patodi-Singer formula applies explicitly to the operator $\delta\beta\nabla_{\partial/\partial u}$ obtained by subtracting

$$\begin{pmatrix} 0 & C_\delta \\ C_\delta & 0 \end{pmatrix}$$

from A_δ (see however [20, §2] and Remark A2.26 below). We have (by A2.10)

$$\eta(\delta\beta\nabla_{\partial/\partial u}) \equiv 2\{\hat{c}_1(\nabla^+) - \hat{c}_1(\nabla^-)\}[S^1] \pmod{Z} \tag{A2.11}$$

(see [2]). Here $c_1(\nabla^{\pm})[S^1]$ is the Chern-Simons invariant associated to the first Chern class and the connection $\nabla^{\pm}$. It is just the phase $\pm\phi$ where $e^{\pm 2\pi\phi i}$ is the determinant of the holonomy of $\nabla^{\pm}$.

In view of Quillen's theory of superconnections, we expect that (A2.11) should actually continue to hold if $\delta\beta\nabla_{\partial/\partial u}$ is replaced by A_δ on the left-hand side. As a consequence, it follows that for $\dim \Lambda < \infty$, the right-hand side of (1.56) is just an explicit way of rewriting the right-hand side of (A2.11). To see this, pull back the connection ∇^- to a connection $\mathscr{A}^*(\nabla^-)$ on Λ^+ by means of the isomorphism $\mathscr{A}$. Then

$$\nabla^+_{\partial/\partial u} - \mathscr{A}^*(\nabla^-_{\partial/\partial u}) = \nabla^+_{\partial/\partial u} - \mathscr{A}^{-1}\frac{(1+i\beta)}{2}\{(\nabla_{\partial/\partial u}\mathscr{A}) + \mathscr{A}\nabla_{\partial/\partial u}\}$$

$$= -\mathscr{A}^{-1}\frac{(1+i\beta)}{2}\nabla_{\partial/\partial u}\mathscr{A}. \tag{A2.12}$$

The difference of Chern-Simons invariants for two connections on the same bundle is locally computable and, by the standard formula, we have

$$2\{\hat{c}_1(\nabla^+) - \hat{c}_1(\nabla^-)\}[S^1] = 2\{\hat{c}_1(\nabla^+) - \hat{c}_1(\mathscr{A}^*(\nabla^-))\}[S^1]$$

$$= \frac{1}{\pi i}\int_{S^1}\operatorname{tr}\left(-\mathscr{A}^{-1}\frac{(1+i\beta)}{2}\nabla_{\partial/\partial u}\mathscr{A}\right)du. \tag{A2.13}$$

Using

$$\operatorname{tr}\left(\mathscr{A}^{-1}\nabla_{\partial/\partial u}\mathscr{A}\right) = \frac{\partial}{\partial u}\log\det\mathscr{A} \tag{A2.14}$$

and (A2.3), it follows that the expression in (A2.13) reduces to that in (1.56) (where in (1.56) we can set $\varepsilon = 0$, since $\dim\Lambda < \infty$).

A2.15. Remark. Similarly,

$$2\{\hat{c}_1(\partial^+) - \hat{c}_1(\partial^-)\}[S^1] = 1/\pi\int_{S^1}\operatorname{tr}(\beta\mathscr{A}^{-1}\dot{\mathscr{A}})\,du. \tag{A2.16}$$

That (A2.11) continues to hold if $\delta\beta\nabla_{\partial/\partial u}$ is replaced by A_δ on the left-hand side (in case $\dim\Lambda < \infty$) follows trivially from Theorem A2.18 below, which is stated for base spaces of arbitrary dimension.

Let B^m be a compact oriented Riemannian manifold, ξ a Hermitian vector bundle with Hermitian connection, and $\mathscr{L}$ an operator of Dirac type with coefficients in ξ. Thus, $\mathscr{L}$ is functorially associated to a metric and a choice of orientation. Moreover, changing the orientation replaces $\mathscr{L}$ by $-\mathscr{L}$.

Now let ξ_0, ξ_1 be as above and let $\mathscr{L}_0, \mathscr{L}_1$ be the corresponding operators. Let $Q\colon \xi_0 \to \xi_1$ and consider the selfadjoint operator (with coefficients in $\xi_0 \oplus \xi_1$) given by

$$P_\varepsilon = \begin{pmatrix} \mathscr{L}_0 & -\varepsilon Q^* \\ \varepsilon Q & -\mathscr{L}_1 \end{pmatrix} \overset{\text{def}}{=} L + \varepsilon Q. \tag{A2.17}$$

A2.18. Theorem.

$$\eta(P_1) = \eta(P_0). \tag{A2.19}$$

The Atiyah-Patodi-Singer formula expresses $\eta(P_0)$ as a difference of Chern-Simons invariants. As above, if $Q(x)$ is invertible for all $x \in B^m$, this difference is given by a local formula on B^m.

Proof of Theorem A2.18. Since $\eta(\delta P_1) = \eta(P_1)$, it suffices to consider $\eta(\delta P_1)$. In fact, putting $\delta t = v$ shows that for all $\varepsilon \geqslant 0$,

$$\text{(A2.20)} \qquad \lim_{\delta \to 0} \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} \operatorname{tr}\left(\delta P_\varepsilon e^{-(\delta P_\varepsilon)^2 t}\right) dt = \eta(P_\varepsilon).$$

As in the usual formula involving the variation of the η-invariant, integrating the derivative of the expression in (A2.20) gives

$$\text{(A2.20)} \qquad \begin{aligned} &\frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} \operatorname{tr}\left(\delta P_1 e^{-(\delta P_1)^2 t}\right) dt \\ &= \frac{-2}{\Gamma(1/2)} \int_0^1 \operatorname{tr}\left(\delta Q e^{-(\delta P_u)}\right) du + \int_1^\infty t^{-1/2} \operatorname{tr}\left(\delta P_0 e^{-(\delta P_0) t}\right) dt. \end{aligned}$$

Suppose we can show that in the limit as $\delta \to 0$, the first term on the right-hand side of (A2.20) vanishes. Then (A2.19) follows from (A2.20).

To obtain the above vanishing, we bring in the adiabatic limit. Write $P_u(g)$ to indicate the dependence of P_u on the metric g. One checks that $\delta P_u(g)$ is conjugate to $P_{\delta u}(\delta^{-2} g)$. For example, if $D = \pm d * \pm * d$ is a signature operator and $S_\delta(\phi) = \delta^{-i}\phi$ for $\phi \in \Lambda^i(B^m)$, then

$$\text{(A2.22)} \qquad S_\delta^{-1} P_{\delta u}(\delta^{-2} g) S_\delta = \delta P_u(g).$$

Clearly, it now suffices to show that for, say, $0 \leqslant u \leqslant 1$, the expression $|\operatorname{tr}(Q e^{-P_u^2(\delta^{-2} g)})|$ stays uniformly bounded as $\delta \to 0$. By an analysis of §1, we can find an expansion for this expression of the form

$$\text{(A2.23)} \qquad \delta^{-m}\left(a_0(u) + a_1(u)\delta + \cdots + a_m(u)\delta^m\right) + O(\delta).$$

Here, the factor δ^{-m} comes from the volume blow up of $(B^m, \delta^{-2} g)$. The expression for a_k at $x \in B^m$ is a sum of terms involving i derivatives of the metric in normal coordinates and j covariant derivatives of Q, where $i + j = k$. Clearly, $|a_m(u)|$ is bounded independent of u for $0 \leqslant u \leqslant 1$. We claim that $a_0 = \cdots = a_{m-1} = 0$. By (A2.23), this implies the boundedness of $|\operatorname{tr}(Q e^{-P_u^2(\delta^{-2} g)})|$.

The argument showing $a_0 = \cdots = a_{m-1} = 0$ is completely analogous one used to establish the corresponding point in the heat equation proof of the index theorem; compare [19]. Suppose that B^m *splits isometrically* as $S^1 \times W^{m-1}$ and that Q *is parallel in the* S^1 *direction*.

Let σ: $\xi_0 \oplus \xi_1 \to \xi_0 \oplus \xi_1$ be defined by $\sigma | \xi_j = (-1)^j$; $j = 0, 1$. Let $x = (0, w) \in S^1 \times W^{m-1}$ and let I_x: $S^1 \times W^{m-1} \to S^1 \times W^{m-1}$ be the involution defined by $I_x(v, w) = (-v, w)$. Then $\sigma I_x = I_x \sigma$ satisfies $(\sigma I_x)^2 = 1$ and

$$\text{(A2.24)} \qquad (\sigma I_x)^*(P_1) = -P_1,$$

$$\text{(A2.25)} \qquad (\sigma I_x)^*(Q) = -Q.$$

From these relations it follows that the pointwise trace, $\operatorname{tr}(Qe^{-P_1})$, vanishes identically in the above case. Hence, the entire expansion of (A2.23) vanishes in this case as well.

Finally, consider the expression for the $a_1(u)$ general case (the argument for $a_k(u)$ where $0 \leqslant k \leqslant m-1$ is entirely similar). Relative to a fixed normal coordinate system, this expression can be written as the sum of m terms, each involving the derivative with respect to a single coordinate x_j, $j = 1, \cdots, m$. But each such term is just the same as that which would have been obtained from a metric and operator Q, whose 1-jets in the x_j direction coincide with those of the given metric and operator and which are *independent of* the remaining coordinates $x_1 \cdots \hat{x}_j \cdots x_{m-1}$. The vanishing now follows from the argument of the previous paragraph.

A2.26. Remark. It follows that the analog of Theorem 4.27 for higher dimensional base spaces can be viewed as a renormalized version of the Atiyah-Patodi-Singer formula for the operator $\mathscr{L}$, together with Theorem A2.18. Alternatively, the finite dimensional case can be understood in terms of the Atiyah-Patodi-Singer formula, the local index theorem for Dirac operators coupled to superconnections proved in [20, §2] and the transgression formula for the Chern character form of a superconnection proved in [23]. The case of higher dimensional base spaces will be discussed elsewhere.

A2.27. Remark. The existence of $\lim_{\delta \to 0} \eta(N, g_\delta, \xi)$ (for $N \to B^m$ and m arbitrary) also follows by an argument similar to the one just given.

A2.28. Remark. Bismut and Freed [4], [5] emphasize superconnections and interpret (1.56) in terms of the *determinant line bundle* (see also [3], [17], [23]). The relation between this interpretation and (A2.11) follows from the formula

$$\hat{c}_1(E^n) = \hat{c}_1(\Lambda^n(E^n), \nabla) \tag{A2.29}$$

for n-dimensional complex vector bundles with connection.

2. The smallest eigenvalue of A_δ^2

In order to justify the computation of the previous section in the case $\mathscr{A}$ is invertible for all u, we will need the fact that for δ sufficiently small, the spectrum of the operator A_δ^2 stays uniformly bounded away from 0. Since

$$A_\delta^2 = \begin{pmatrix} \mathscr{A}^2 + \mathscr{D}^2 & \mathscr{A}\mathscr{D} + \mathscr{D}\mathscr{A} \\ \mathscr{A}\mathscr{D} + \mathscr{D}\mathscr{A} & \mathscr{A}^2 + \mathscr{D}^2 \end{pmatrix}, \tag{2.1}$$

our assertion would be clear were it not for the presence of the off diagonal terms. However, these terms drop out in the limit, $\delta \to 0$, and we will show that for δ sufficiently small, they are dominated by the $\mathscr{A}^2$ terms on the diagonal.

In what follows we write $\beta = \beta(\delta v)$, $\mathscr{A} = \mathscr{A}(\delta v)$. A dot continues to denote differentiation with respect to $u = \delta \cdot v$, and $\partial = \partial_v$. Recall that

$$\mathscr{A}\beta + \beta\mathscr{A} = 0. \tag{2.2}$$

Then on $2p$-forms, we have

$$\begin{aligned} \mathscr{A}\mathscr{D} + \mathscr{D}\mathscr{A} &= \mathscr{A}\beta\partial + \partial\beta\mathscr{A} = \mathscr{A}\beta\partial + \delta\dot{\beta}\mathscr{A} + \beta\partial\mathscr{A} \\ &= \mathscr{A}\beta\partial + \delta\dot{\beta}\mathscr{A} + \beta\delta\dot{\mathscr{A}} + \beta\mathscr{A}\partial = \delta(\mathscr{A}\dot{\beta} + \beta\dot{\mathscr{A}}). \end{aligned} \tag{2.3}$$

Similarly, on $(2p-1)$-forms,

$$\begin{aligned} \mathscr{A}\mathscr{D} + \mathscr{D}\mathscr{A} &= \mathscr{A}\partial\beta + \beta\partial\mathscr{A} \\ &= \mathscr{A}\delta\dot{\beta} + \mathscr{A}\beta\partial + \beta\delta\dot{\mathscr{A}} - \beta\mathscr{A}\partial = \delta(\mathscr{A}\dot{\beta} + \beta\dot{\mathscr{A}}). \end{aligned} \tag{2.4}$$

Let $(\phi, \psi) \in (\Lambda^{\mathrm{ev}}(Y), \Lambda^{\mathrm{odd}}(Y))$. Then

$$\begin{aligned} \langle A_\delta^2(\phi,\psi),(\phi,\psi)\rangle &= \langle(\mathscr{A}^2 + \mathscr{D}^2)\phi,\phi\rangle + \langle(\mathscr{A}\mathscr{D} + \mathscr{D}\mathscr{A})\psi,\phi\rangle \\ &\quad + \langle(\mathscr{A}\mathscr{D} + \mathscr{D}\mathscr{A})\phi,\psi\rangle + \langle(\mathscr{A}^2 + \mathscr{D}^2)\psi,\psi\rangle, \end{aligned} \tag{2.5}$$

where

$$\int_N \langle \mathscr{D}^2\phi,\phi\rangle + \langle \mathscr{D}^2\psi,\psi\rangle \geqslant 0. \tag{2.6}$$

Substitute (2.3), (2.4) into (2.5). For the terms involving $\mathscr{A}\dot{\beta}$, we have by the Schwarz inequality on the fiber,

$$\begin{aligned} &|\langle \delta\dot{\beta}\mathscr{A}\phi,\psi\rangle + \langle \delta\dot{\beta}\mathscr{A}\psi,\phi\rangle| \\ &\qquad \leqslant \frac{\delta}{2}\left\{\|\mathscr{A}\phi\|^2 + \|\mathscr{A}\psi\|^2 + \|\dot{\beta}\phi\|^2 + \|\dot{\beta}\psi\|\right\}. \end{aligned} \tag{2.7}$$

If λ_0 is the smallest eigenvalue of $\mathscr{A}^2$, then

$$|\langle \delta\dot{\beta}\mathscr{A}\phi,\psi\rangle + \langle \delta\dot{\beta}\mathscr{A}\psi,\phi\rangle| \leqslant \tfrac{1}{8}\left(\|\mathscr{A}\phi\|^2 + \|\mathscr{A}\psi\|^2\right), \tag{2.8}$$

provided $\delta < \frac{1}{8}$ is so small that

$$\|\dot{\beta}\| \leqslant \lambda_0/8\delta. \tag{2.9}$$

Now, consider the terms involving $\beta\dot{\mathscr{A}}$. In even and odd degrees respectively, we have

$$\begin{aligned} \beta\dot{\mathscr{A}} &= \delta\left\{(-1)^{p+1}\beta(\dot{d}\beta + \beta\dot{d}) + (-1)^{p+1}\beta(\tilde{d}\beta + \dot{\beta}\tilde{d})\right\}, \\ \beta\dot{\mathscr{A}} &= \delta\{\beta(\dot{d}\beta - \beta\dot{d}) + \beta(\tilde{d}\beta - \dot{\beta}\tilde{d})\}. \end{aligned} \tag{2.10}$$

Here $\dot{\tilde{d}}$ is an operator of order zero which acts pointwise. As above, the terms involving $\dot{\tilde{d}}$ will be bounded by $\frac{1}{8}(\|\mathscr{A}\phi\|^2 + \|\mathscr{A}\psi\|^2)$ for δ sufficiently small.

Put

$$\mathscr{A} = e + f, \tag{2.11}$$

where $e = \pm\tilde{d}\beta$ and $f = \pm\beta\tilde{d}$. then

$$\beta e = -f\beta. \tag{2.12}$$

Also put

$$\alpha = *^{-1}\dot{*}. \tag{2.13}$$

Then the piece of $\dot{\mathscr{A}}$ involving $\dot{\beta}$ can be written as

$$-e\alpha + \alpha f. \tag{2.14}$$

Note that

$$\beta\alpha = \dot{\beta} = -\alpha\beta. \tag{2.15}$$

By using (2.11)–(2.15) we see easily that the remaining piece of $\beta\dot{\mathscr{A}}$ satisfies

$$\begin{aligned} \delta|\langle \beta(-e\alpha + \alpha f)\psi, \phi\rangle + \langle \beta(-e\alpha + \alpha f)\phi, \psi\rangle| &\\ = 2\delta|\langle f\psi, \dot{\beta}\phi\rangle + \langle f\phi, \dot{\beta}\psi\rangle| &\\ \leqslant \delta\{\|f\phi\|^2 + \|f\psi\|^2 + \|\dot{\beta}\phi\|^2 + \|\dot{\beta}\psi\|^2\} &\\ \leqslant \delta(\|f\phi\|^2 + \|f\psi\|^2) + \tfrac{1}{8}\{\|\mathscr{A}\phi\| + \|\mathscr{A}\psi\|^2\} & \end{aligned} \tag{2.16}$$

(where the last line follows as in (2.7)).

Now,

$$\begin{aligned} \langle \mathscr{A}^2\phi, \phi\rangle + \langle \mathscr{A}^2\psi, \psi\rangle &= \|e\phi\|^2 + \|e\psi\|^2 + \|f\phi\|^2 + \|f\psi\|^2 \\ &\quad + \langle (ef + fe)\phi, \phi\rangle + \langle (ef + fe)\psi, \psi\rangle, \end{aligned} \tag{2.17}$$

where the operator $(ef + fe)$ is a curvature term (and hence acts pointwise). Let

$$\|(ef + fe)\| = K. \tag{2.18}$$

By what has been established above, if

$$\delta(\|f\phi\|^2 + \|f\psi\|^2) \leqslant \frac{\lambda_0}{8}(\|\phi\|^2 + \|\psi\|^2), \tag{2.19}$$

we have

$$\|\langle A^2(\phi, \psi), (\phi, \psi)\rangle\| \geqslant \frac{\lambda_0}{2}(\|\phi\|^2 + \|\psi\|^2). \tag{2.20}$$

So we can assume

$$\delta(\|f\phi\|^2 + \|f\psi\|^2) \geqslant \frac{\lambda_0}{8}(\|\phi\|^2 + \|\psi\|^2). \tag{2.21}$$

Suppose that in addition to the requirements above, δ is so small that

$$(2.22)\qquad (1-\delta)\frac{\lambda_0}{8\delta} - K - \frac{3}{8}\lambda_0 \geqslant \frac{\lambda_0}{2}.$$

Then we have by (2.21), (2.22),

$$\begin{aligned}
\left\|\left\langle A_\delta^2(\phi,\psi),(\phi,\psi)\right\rangle\right\| &\geqslant \left\langle \mathscr{A}^2\phi,\phi\right\rangle + \left\langle \mathscr{A}^2\psi,\psi\right\rangle - \left|\left\langle (\mathscr{A}\mathscr{D}+\mathscr{D}\mathscr{A})\psi,\phi\right\rangle\right| \\
&\quad - \left|\left\langle (\mathscr{A}\mathscr{D}+\mathscr{D}\mathscr{A})\phi,\psi\right\rangle\right| \\
(2.23)\qquad &\geqslant \|f\phi\|^2 + \|f\psi\|^2 - \tfrac{3}{8}\lambda_0 - \left|\left\langle (ef+fe)\phi,\phi\right\rangle\right| \\
&\quad - \left|\left\langle (ef+fe)\psi,\psi\right\rangle\right| - \delta\left(\|f\phi\|^2 + \|f\psi\|^2\right) \\
&\geqslant \frac{\lambda_0}{2}\left(\|\phi\|^2 + \|\psi\|^2\right).
\end{aligned}$$

Thus,

2.24. Lemma. *If for all u, the smallest eigenvalue of $\mathscr{A}^2$ is $\geqslant \lambda_0$, then for δ sufficiently small, the smallest eigenvalue of A_δ^2 is $\geqslant \lambda_0/2$.*

3. Decomposition and estimation of heat kernels

In this section we show how to decompose the heat kernel of the operator A^2 into a compactly supported piece and a piece which will turn out to be negligible in the limit, $\delta \to 0$. This will follow from the estimates on the pieces which are also given here. Our treatment follows closely that of [13] and [14]. It is based on the fact that the constant in the elliptic estimate for A_δ^2 is uniformly bounded (since the metrics and connections have bounded geometry, independent of δ) and the easily verified fact that the fundamental solution of the wave equation, $\cos\sqrt{A^2}\,\xi$, has *unit propagation speed.*

By the spectral theorem, we can write

$$(3.1)\qquad e^{-A^2t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\xi^2/4t}}{(4\pi t)^{1/2}}\cos\sqrt{A^2}\,\xi\, d\xi,$$

where $\cos\sqrt{A^2}\,\xi$ is the fundamental solution of the *wave equation* which satisfies

$$(3.2)\qquad \cos\sqrt{A^2}\,\xi(z_1,z_2)\,|_{\xi=0} = \delta_{z_2},$$

$$(3.3)\qquad \frac{\partial}{\partial\xi}\cos\sqrt{A^2}\,\xi(z_1,z_2)\,|_{\xi=0} = 0.$$

The fact that $\cos\sqrt{A^2}\,\xi$ has *unit propagation speed* means

$$\operatorname{supp}\cos\sqrt{A^2}\,\xi(z_1, z_2) \subset \left\{(z_1, z_2) \mid \overline{z_1, z_2} \leqslant |\xi|\right\}. \tag{3.4}$$

Here $\overline{z_1, z_2}$ denotes the distance from z_1 to z_2.

Write

$$1 = f_b(\xi) + (1 - f_b(\xi)), \tag{3.5}$$

where $f_b(\xi)$ is an even function such that $|f| \leqslant 1$,

$$\begin{aligned} &\operatorname{supp} f_b(\xi) \subset \{\xi \mid |\xi| \leqslant b + 1\},\\ &\operatorname{supp} 1 - f_b(\xi) \subset \{\xi \mid |\xi| \geqslant b\}, \end{aligned} \tag{3.6}$$

and

$$\left|\frac{d^i}{d\xi^i} f_b(\xi)\right| \leqslant C_i. \tag{3.7}$$

By (3.2), we have

$$\begin{aligned} e^{-A^2 t} &= \int_{-\infty}^{\infty} \frac{e^{-\xi^2/4t}}{(4\pi t)^{1/2}} \left(f_b \cos\sqrt{A^2}\,\xi + (1 - f_b)\cos\sqrt{A^2}\,\xi\right) d\xi \\ &\overset{\text{def}}{=} \left(e^{-A^2 t}\right)_b + \left(e^{-A^2 t}\right)_{b,\infty}. \end{aligned} \tag{3.8}$$

The operator norms of the operators in (3.8) satisfy

$$\left\|e^{-A^2 t}\right\| \leqslant 1, \tag{3.9}$$

$$\left\|\left(e^{-A^2 t}\right)_b\right\| \leqslant 1, \tag{3.10}$$

$$\left\|\left(e^{-A^2 t}\right)_{b,\infty}\right\| \leqslant c \cdot e^{-b^2/4t} \frac{t^{1/2}}{b}. \tag{3.11}$$

Write

$$E(t) = P_b(t) + G_b(t), \tag{3.12}$$

where $P_b(t)$, $G_b(t)$ denote the kernels of the operators $(e^{-A^2 t})_b$, $(e^{-A^2 t})_{b,\infty}$. These are smooth and the unit propagation speed of $\cos\sqrt{A^2}\,\xi$ implies

$$\operatorname{supp} P_b(t) \subset \left\{(z_1, z_2) \mid \overline{z_1, z_2} \leqslant b + 1\right\}. \tag{3.13}$$

Let $K(z_1, z_2)$ be a smooth kernel. We let $\| \ \|_\infty$ denote the pointwise norm and put

$$\|K\|_{\infty, i_1, i_2} = \|\nabla^{2i_1 2i_2} K\|_\infty. \tag{3.14}$$

where ∇^{2i_1}, ∇^{2i_2} are applied to the variables z_1, z_2 respectively. We can estimate this norm for the kernels above by the technique of [13]. We recall that this technique makes use of the unit propagation speed of $\cos\sqrt{A^2}\,\xi$, the fact that the operator norm $\cos\sqrt{A^2}\,\xi$ is 1, the elliptic estimate for A^2, and integration by parts with respect to ξ; (see [13, pp. 19–20 and 26–28]) for details. This easily gives the following estimates for, say $b \geqslant 2$ (and $n = \dim N$):

$$\|P_b(t)\|_{\infty, i_1, i_2} \tag{3.15}$$
$$= c(i_1, i_2)\begin{cases} (1 + t^{-n(n/2+2i_1+2i_2+2)}), & \overline{z_1, z_2} \leqslant 1, \\ e^{-\overline{z_1, z_2}^2/t}, & 1 \leqslant \overline{z_1, z_2} \leqslant b+1, \\ 0, & \overline{z_1, z_2} \geqslant b+1, \end{cases}$$

$$\|G_b(t)\|_{\infty, i_1, i_2} \leqslant c(i_1, i_2)\begin{cases} e^{-b^2/t}, & \overline{z_1, z_2} \leqslant b+1, \\ e^{-\overline{z_1, z_2}^2/4t}, & \overline{z_1, z_2} \geqslant b+1. \end{cases} \tag{3.16}$$

We put

$$Q_b(t) = \left(\frac{\partial}{\partial t} + A^2\right)P_b(t) = -\left(\frac{\partial}{\partial t} + A^2\right)G_b(t). \tag{3.17}$$

Then the operator norm of $Q_b(t)$ satisfies

$$\|Q_b(t)\| \leqslant ce^{-b^2/t} \tag{3.18}$$

and the pointwise norm satisfies

$$\|Q_b(t)\|_{\infty, i_1, i_2} \leqslant c(i_1, i_2)\begin{cases} e^{-b^2/t}, & \overline{z_1, z_2} \leqslant b+1, \\ 0, & \overline{z_1, z_2} \geqslant b+1. \end{cases} \tag{3.19}$$

4. The adiabatic limit

Let A_δ^2, A_0^2 be as in §§1 and 2 and let $E_\delta(t)$, $E_0(t)$ be the corresponding heat kernels. Rather than comparing these directly, we will compare the kernels $P_{b,\delta}(t)$, $P_{b,0}(t)$ corresponding to the decomposition in §3, where $b \to \infty$, with a suitable dependence on δ, t. In view of (3.16) this will suffice to compare $E_\delta(t)$, $E_0(t)$.

We will view the metrics g_δ, g_0 (and corresponding connections) as being defined on $|v| < 1/2\delta$. There they have uniformly bounded geometry for $0 \leqslant \delta \leqslant 1$, and are uniformly quasi-isometric. In particular norms and constants in the elliptic estimate for A_δ^2 can be chosen uniformly for $0 \leqslant \delta \leqslant 1$. We will assume that the geometry has been normalized so that the estimates of §3 hold.

By Taylor's theorem we can write

$$A_\delta^2 = A_0^2 + \delta\left(A_0^2\right)' + D(\delta^2, v^2), \tag{4.1}$$

where $(A_0^2)'$ is a second order operator which, when expressed in terms of the connection, has coefficients which grow linearly in v. Similarly, the remainder $D(\delta^2, v^2)$ has coefficients which grow at most like $\delta^2(1 + v^2)$.

For $|v_1|$, $|v_2|$, b, small compared to $1/\delta$, we can write Duhamel's principle as

$$\begin{aligned} P_{b,\delta}(v_1, v_2, t) &- P_{b,0}(v_1, v_2, t) * {}^{-1} *_0 \\ &= -P_{b,0}(v_1, v, t-s) \#_0 \left(A_\delta^2 - A_0^2\right) P_{b,\delta}(v, v_2 1, s) \\ &\quad + P_{b,0}(v_1, v, t-s) \#_0 Q_{b,\delta}(v, v_2, s) \\ &\quad - Q_{b,0}(v_1, v, t-s) \#_0 P_{b,\delta}(v, v_2, s). \end{aligned} \tag{4.2}$$

If we substitute the resulting expression for $P_{b,\delta}$ in the first term on the right-hand side we get, in particular,

$$\begin{aligned} &P_{b,\delta}(0,0,t) - P_{b,0}(0,0,t) \\ &\text{(I)} \quad = -P_{b,0}(0, v, t-s) \#_0 \left(A_\delta^2 - A_0^2\right) P_{b,0}(v, 0, t-s) \\ &\text{(II)} \quad + P_{b,0}(0, v, t-s) \#_0 \left(A_\delta^2 - A_0^2\right) P_{b,0}(v, w, s-u) \\ &\qquad\qquad \#_0 \left(A_\delta^2 - A_0^2\right) P_{b,0}(w, 0, u) \\ &\text{(III)} \quad - P_{b,0}(0, v, t-s) \#_0 P_{b,0}(v, w, s-u) \#_0 Q_{b,\delta}(w, 0, u) \\ &\qquad + P_{b,0}(0, v, t-s) \#_0 Q_{b,0}(v, w, s-u) \#_0 P_{b,\delta}(w, 0, u) \\ &\text{(IV)} \quad + P_{b,0}(0, v, t-s) \#_0 Q_{b,\delta}(v, 0, s) \\ &\qquad - Q_{b,0}(0, v, t-s) \#_0 P_{b,\delta}(v, 0, s). \end{aligned} \tag{4.3}$$

Note that on the supports of the kernels in (4.3), we have

$$|v|,\ |w| \leqslant b. \tag{4.4}$$

For such values, the coefficients $(A_\delta^2 - A_0^2)$ are bounded by

$$c\delta \cdot b \tag{4.5}$$

while those of $A_\delta^2 - A_0^2 - \delta(A_0^2)'$ are bounded by

$$c\delta^2 \cdot b^2. \tag{4.6}$$

We want to show that (II), (III), (IV) contribute a negligible error in the limit, $\delta \to 0$, and that the same holds for the piece of (I) coming from the remainder term in the Taylor expansion of A_δ.

We begin with (IV). Let $\| \ \|_{2,(\infty,j)}$ denote the L_2-norm with respect to either (v_1, y_1), or (v, y) and the pointwise norm with respect to (v_2, y_2). In estimating the first term, the more dangerous time parameter values are $(t - s) \leqslant t/2$, since in that case, the pointwise norm of $P_{0,b}(t-s)$ is not bounded. So we regard $Q_{b,\delta}(s)$ as living on $R \times Y$ and apply the elliptic estimate for A_0^2, together with (3.10) and (3.19) to obtain

$$
\begin{aligned}
&\| P_{b,0}(t-s) \#_0 Q_{b,\delta}(s) \|_{\infty, 2i_1, 2i_2} \\
(4.7) \qquad &\quad \leqslant \| A_0^{[n/2]+1-2i_1} P_{b,0}(t-s) \#_0 Q_{b,\delta}(s) \|_{2,(\infty,2i_2)} \\
&\quad \leqslant \| P_{\delta,0}(t-s) \#_0 A_0^{[n/2]+1+2i_1} Q_{b,\delta}(s) \|_{2,(\infty,i_2)} \\
&\quad \leqslant c(i_1, i_2, n) e^{-b^2/t} b^{1/2} t.
\end{aligned}
$$

By regarding $Q_{b,0}(t-s)$ as living on N and applying the elliptic estimate for A_δ^2 and (3.18), we obtain the same estimate for the second term (here we replace $\#_0$ by $\#_\delta$ by inserting the harmless factor $*_0^{-1} *_\delta$).

In estimating the threefold convolutions in (II) and (III) we proceed by estimating the integrand for each fixed triple of time parameters, $t-s$, $s-u$, u. We start with a pair of kernels (corresponding, say, to $s-u$, u) for which at least one of the values (say $s-u$) is $\geqslant t/3$. We let the kernel corresponding to the other value (in this case u) play the role of $P_{b,0}(t-s)$ in the case considered explicitly above. We apply the elliptic estimate for $A_{\underline{\delta}}^2$, where $\underline{\delta}$ is the subscript (δ or 0) of the kernel with time parameter (in the present example) u. We regard the kernel with time parameter $s-u$ as living on N or $R \times Y$ corresponding to the possibilities $\underline{\delta} = \delta$, $\underline{\delta} = 0$, respectively, and use (3.15), (3.19) to estimate this kernel. Then we repeat the argument with the kernel corresponding to $t-s$ playing the role of $P_{b,0}(t-s)$ in the case considered above.

The analysis just described leads to the following bound for (III):

$$(4.8) \qquad \| \mathrm{III} \|_{\infty, 2i_1, 2i_2} \leqslant c(i_1, i_2, n) e^{-b^2/t} b t^2.$$

Similarly, for (II), we have

$$(4.9) \qquad \| \mathrm{II} \|_{\infty, 2i_1, 2i_2} \leqslant c(i_1, i_2, n) \delta^2 b^2 (1 + t^{(-n/2+2i_1+2i_2+2)}) b t^2.$$

Finally, for the remainder term, R_{I}, in (I) we have

$$(4.10) \qquad \| R_{\mathrm{I}} \|_{\infty, 2i_1, 2i_2} \leqslant c(i_1, i_2, n) \delta^2 b^2 (1 + t^{(-n/2+2i_1+2i_2+2)}) b^{1/2} \cdot t.$$

Thus,

$$
\begin{aligned}
&A_0 P_{b,\delta}(0,0,t) - A_0 P_{b,0}(0,0,t) \\
(4.11) \qquad &\quad = -\delta P_{b,0}(0,v,t-s) \#_0 A_0 (A_0^2)' P_{b,0}(v,0,s) + \text{error},
\end{aligned}
$$

where the error is bounded by (4.6)–(4.9). If we neglect the error, the right-hand side can be written as

$$
\begin{aligned}
&-\delta E_0(0,v,t-s)\#_0 A_0(A_0^2)' E_0(v,0,s)\\
&\quad+\delta G_{b,0}(0,v,t-s)\#_0 A_0(A_0^2)' P_{b,0}(v,0,s)\\
&\quad+\delta P_{b,0}(0,v,t-s)\#_0 A_0(A_0^2)' G_{b,0}(v,0,t)\\
&\quad+\delta G_{b,0}(0,v,t-s)\#_0 A_0(A_0^2)' G_{b,0}(v,0,s).
\end{aligned}
\tag{4.12}
$$

The sum of the last three terms in this expression is bounded by

$$\delta b t^2 e^{-b^2/t} b^{1/2} t \tag{4.13}$$

as follows from (3.16).

Now rewrite the left-hand side of (4.11) as

$$
\begin{aligned}
&A_\delta E_\delta(t) + (A_0 - A_\delta)(P_{b,\delta}(t) - P_{b,0}(t))\\
&\quad+(A_\delta - A_0)E_0(t) - (A_\delta - A_0)G_{b,0}(t)\\
&\quad-A_\delta G_{b,0}(t) + A_0 G_{b,0}(t) - A_0 E_0(t).
\end{aligned}
\tag{4.14}
$$

After taking the trace, the last term in (4.14) vanishes. Since

$$(A_\delta - A_0) = \delta\beta \mid \Lambda^{2p-1} \tag{4.15}$$

at $v = 0$ (see (1.7)), after taking the trace, the third term also vanishes. The fourth, fifth, and sixth terms are bounded by (3.16). The second term is bounded by our analysis of (4.3).

Putting all this together, we find that for $t \geqslant t_0$, at $v = 0$,

$$
\begin{aligned}
&\left|\operatorname{tr}(A_\delta E_\delta(t)) + \delta \operatorname{tr}\left(E_0(0,v,t-s)\#_0 A_0(A_0^2)' E_0(v,0,s)\right)\right|\\
&\quad\leqslant c(t_0)\delta^2 t + c e^{-b^2/t} + c\delta b^{3/2} t^3 e^{-b^2/t}\\
&\quad\quad+c(t_0)(\delta^2 b^3 t^2 + \delta^2 b^{5/2} t) + c(bt^2 + b^{1/2} t)e^{-b^2/t}.
\end{aligned}
\tag{4.16}
$$

Now let $t \geqslant t_0$ and choose

$$b^2 = 2|\log\delta| \cdot t. \tag{4.17}$$

Then the right-hand side of (4.16) is bounded by

$$c(t_0)\delta^2 |\log\delta|^{3/2} \cdot t^4 \tag{4.18}$$

(where we can take $c(t_0) = t_0^{-2k+6}$).

As explained in §1, we can now use the lower bound for the spectrum of A_δ^2 established in §2 to conclude that for δ sufficiently small and $T = 4|\log\delta|/\lambda_0$,

$$(4.19)\quad \begin{aligned} &\frac{1}{\Gamma(1/2)}\int_{t_0}^{\infty} t^{-1/2}\operatorname{tr}(A_\delta E_\delta(t))\,dt \\ &= \frac{1}{\Gamma(1/2)}\int_{T}^{\infty} t^{-1/2}\operatorname{tr}(A_\delta E_\delta(t))\,dt + \frac{1}{\Gamma(1/2)}\int_{t_0}^{T} t^{-1/2}\operatorname{tr}(A_\delta E_\delta(t))\,dt \\ &= \frac{1}{\Gamma(1/2)}\int_{t_0}^{T} t^{-1/2}\operatorname{tr}(A_\delta E_\delta(t))\,dt + O(\delta) \\ &= -\int_{S^1}\frac{1}{\pi}\int_{t_0}^{T}\operatorname{tr}\left(\frac{\beta}{2}\dot{\mathscr{A}}\mathscr{A}e^{-t\mathscr{A}^2}\right)dt\,du + O(\delta) + c(t_0)\left(\frac{4}{\lambda_0}\right)^5|\log\delta|^4\delta. \end{aligned}$$

It remains to discuss the small time behavior. Put $P_{2,\delta} = P_\delta$, $P_{2,0} = P_0$, $Q_{2,\delta} = Q_\delta$, $Q_{2,0} = Q_0$. Then

$$(4.20)\qquad A_\delta E_\delta(t) = A_\delta P_\delta(t) + E_\delta(t-s)\#_\delta A_\delta Q_\delta(s),$$

$$(4.21)\qquad A_0 E_0(t) = A_0 P_0(t) + E_0(t-s)\#_0 A_0 Q_0(s).$$

Thus, at $v_1 = v_2 = 0$,

$$(4.22)\quad \begin{aligned} A_\delta E_\delta(t) = A_\delta P_\delta(t) &+ A_0 E_0(t) - A_0 P_0(t) \\ &+ (E_\delta(t-s) - E_0(t-s))\#_\delta A_\delta Q_\delta(s) \\ &+ E_0(t-s)\#_\delta(A_\delta - A_0)Q_\delta(s) \\ &+ E_0(t-s)\#_\delta A_0(Q_\delta(s) - Q_0(s)) \\ &+ E_0(t-s)\#_0(*_\delta^{-1} *_0 A_0 - A_0)Q_0(s). \end{aligned}$$

Our previous analysis easily implies that the sum of the last four terms on the right-hand side is bounded by, say,

$$(4.23)\qquad c\delta t^{-(4k-1)+6}e^{-1/t}.$$

The trace of the second and third terms vanishes identically. In general,

$$(4.24)\qquad \operatorname{tr}(A_\delta P_\delta(t)) \sim c_\delta,$$

(see [19, Theorem 2.6.1]) but since $c_0 = 0$ it follows that

$$(4.25)\qquad \operatorname{tr}(A_\delta P_\delta(t)) = O(\delta).$$

Thus, we have globally,

$$(4.26)\qquad \frac{1}{\Gamma(1/2)}\int_0^{t_0} t^{-1/2}\operatorname{tr}(A_\delta E_\delta(t)) = O(t_0^{1/2}).$$

By combining this with (4.19), we obtain, for the case in which the kernel of $\mathscr{A}$ is empty for all u,

4.27. Theorem. *If the kernel of $\mathscr{A}$ is empty for all u, then*

$$\lim_{\delta\to 0} \eta(N, g_\delta) = \frac{1}{\pi}\int_{S^1} \lim_{\varepsilon\to 0} \operatorname{tr}\left(\frac{\beta}{2}\mathscr{A}^{-1}\dot{\mathscr{A}}e^{-\varepsilon\mathscr{A}^2}\right). \tag{4.28}$$

5. The general case

In this section, we indicate how our previous discussion can be modified in the case $\mathscr{A}(u)$ is not invertible for all u. The point will be to deform $\mathscr{A}(u)$ to a family $\mathscr{A}_1(u)$, which is always invertible.

To explain the idea, we begin by assuming that $\dim\ker\mathscr{A}(u)$ is constant and hence that $\ker\mathscr{A}(u)$ is a subbundle of the bundle, Λ, of sections. Choose a smoothly varying orthogonal splitting,

$$\ker\mathscr{A}(u) = V_+(u) \oplus V_-(u), \tag{5.1}$$

such that

$$\beta(V_\pm(u)) = V_\mp(u). \tag{5.2}$$

Thus, $V_\pm(u)$ are *Lagrangian* subspaces for the symplectic form

$$\langle \beta h_1, h_2\rangle \tag{5.3}$$

on $\ker\mathscr{A}(u)$.

Now define $\mathscr{A}_1(u)$ by

$$\mathscr{A}_1(u) = \begin{cases} \mathscr{A}(u) & \text{on } (\ker\mathscr{A}(u))^\perp, \\ \pm 1 & \text{on } V_\pm(u). \end{cases} \tag{5.4}$$

5.5. Remark. The construction of $\mathscr{A}_1(u)$ in this case (and below) should be compared with the "ideal boundary conditions" which are introduced in [8, pp. 113–114].

Note that $\mathscr{A}_1(u)$ is no longer a differential operator. However, $\mathscr{A}_1(u)$ is a real selfadjoint operator which *agrees with $\mathscr{A}(u)$ on a closed subspace of finite codimension* and satisfies

$$\mathscr{A}_1(u)\beta = -\beta\mathscr{A}_1(u). \tag{5.6}$$

If $\ker\mathscr{A}(u)$ is not of constant dimension, we can construct a family satisfying the above conditions as follows. First, we choose $u_j \in S^1$, $\varepsilon_j > 0$ $(j = 0, \cdots, N-1)$ such that on each interval $[u_j, u_{j+1}]$ (where j is taken mod N) the dimension of the direct sum, $S_j(u)$, of the eigenspaces of $\mathscr{A}^2(u)$ with eigenvalue $< \varepsilon_j$ is independent of u. So defined, the subspaces $S_j(u)$ extend naturally to slightly larger intervals $(u_j - \delta, u_{j+1} + \delta) = I_j$.

Let $\pi_j(u)$ denote orthogonal projection on $S_j(u)$ over I_j. Let ϕ_j be a partition of unity subordinate to the covering $\{I_j\}$. Define a family of operators $\hat{\mathscr{A}}(u)$, by

$$\hat{\mathscr{A}}(u) = \mathscr{A}(u) - \sum_j \phi_j(u)\mathscr{A}(u)\pi_j(u). \tag{5.7}$$

Now choose Lagrangian decompositions

$$S_j(u) = V_{+,j}(u) \oplus V_{-,j}(u) \tag{5.8}$$

which are compatible on the overlaps and such that if $S_{\pm,j}(u)$ denote the subspaces of $S_j(u)$ corresponding to eigenvalues $\pm\lambda$ $(\lambda > 0)$ of $\mathscr{A}(u)$, then for $|u - u_j| < \delta$,

$$S_{\pm,j}(u) \subset V_{\pm,j}(u). \tag{5.9}$$

Let $\pi_{\pm,j}(u)$ denote orthogonal projection on $V_{\pm j}(u)$. Now define $\mathscr{A}_1(u)$ by

$$\mathscr{A}_1(u) = \hat{\mathscr{A}}(u) + \sum_j \phi_j(u)\big(\pi_{+,j}(u) - \pi_{-,j}(u)\big). \tag{5.10}$$

Then $\mathscr{A}_1(u)$ satisfies the same conditions as the previously defined $\mathscr{A}_1(u)$ (and is invertible for all u). Finally, put

$$\mathscr{A}_\varepsilon(u) = (1 - \varepsilon)\mathscr{A}(u) + \varepsilon\mathscr{A}_1(u). \tag{5.11}$$

Let $A_{\delta,\varepsilon}$ be defined as in (1.18), but with $\mathscr{A}$ replaced by $\mathscr{A}_\varepsilon$. Let $k_\varepsilon(\delta v)$ denote $A_{\delta,\varepsilon} - A_\delta$. Then $A_{\delta,\varepsilon}$ has the heat kernel

$$E_{\delta,\varepsilon}(t) = E_\delta(t) + \sum (-1)^i \overbrace{E_\delta \#_\delta k_\varepsilon E_\delta \#_\delta \cdots k_\varepsilon E_\delta}^{i+1}. \tag{5.12}$$

In generalizing our previous discussion, the following points must be taken into account.

The operator $\cos\sqrt{A^2_{\delta,\varepsilon}}\,\xi$ does not have unit propagation speed. However, it follows by an obvious modification of the proof of finite propagation speed given in [18, p. 180], that $\cos\sqrt{A^2_{\delta,\varepsilon}}\,\xi$ does satisfy

$$\sup \cos\sqrt{A^2_{\delta,\varepsilon}}\,\xi(v_1, y_1, v_2, y_2) \subset \big\{(v_1, y_1, v_2, y_2)\,\big|\,|v_1 - v_2| \leqslant |\xi|\big\}. \tag{5.13}$$

Second of all, let G_δ be a parametrix for A_δ satisfying

$$G_\delta A_\delta = I + Q_\delta, \tag{5.14}$$

where A_δ is a smooth kernel. Then

$$P_{\delta,\varepsilon} = \big(I - G_\delta k_\varepsilon + \cdots + (-1)^N (G_\delta k_\varepsilon)^N\big) G_\delta \tag{5.15}$$

is a parametrix for $A_{\delta,\varepsilon}$ satisfying

$$\begin{aligned} P_{\delta,\varepsilon} A_{\delta,\varepsilon} &= I + (-1)^N (G_\delta k_\varepsilon)^{N+1} \\ &\quad + \big(I - G_\delta k_\varepsilon + \cdots + (-1)^N (G_\delta k_\varepsilon)^N\big) Q_\delta. \end{aligned} \tag{5.16}$$

With these remarks, it is clear that (subject to obvious modifications) the discussion of §3 extends to the operators $A_{\delta,\varepsilon}$. Moreover, the discussion of §2 extends easily to $A_{\delta,1}(u)$. Thus, as in §4, we find that

$$\lim_{\delta\to 0}\eta({}_{\delta,1}(u)) = \lim_{t_0\to 0}\int_{S^1}\frac{1}{\pi}\operatorname{tr}\left(\frac{\beta}{2}\mathscr{A}_1^{-1}\dot{\mathscr{A}}_1 e^{-\mathscr{A}_1^2 t_0}\right). \tag{5.17}$$

Now recall that

$$\begin{aligned}&\frac{\partial}{\partial\varepsilon}\frac{1}{\Gamma(1/2)}\int_{t_0}^{\infty} t^{-1/2}\operatorname{tr}\left(A_{\delta,\varepsilon}e^{-A_{\delta,\varepsilon}^2 t}\right)dt\\ &\qquad = \frac{-2}{\Gamma(1/2)}t_0^{1/2}\operatorname{tr}\left(\frac{\partial A_{\delta,\varepsilon}}{\partial\varepsilon}e^{-A_{\delta,\varepsilon}^2 t_0}\right).\end{aligned} \tag{5.18}$$

Since

$$\frac{\partial}{\partial\varepsilon}A_{\delta,\varepsilon} = \begin{pmatrix} 0 & \frac{\partial}{\partial\varepsilon}\mathscr{A}_\varepsilon \\ \frac{\partial}{\partial\varepsilon}\mathscr{A}_\varepsilon, & 0\end{pmatrix}, \tag{5.19}$$

where $\frac{\partial}{\partial\varepsilon}\mathscr{A}_\varepsilon$ has finite rank, a computation like that of §1 shows that

$$\lim_{\delta\to 0}\operatorname{tr}\left(\frac{\partial}{\partial\varepsilon}A_{\delta,\varepsilon}e^{-A_{\delta,\varepsilon}^2 t_0}\right) = O\left(t_0^{1/2}\right). \tag{5.20}$$

Thus,

$$\lim_{t_0\to 0}\lim_{\delta\to 0}\frac{\partial}{\partial\varepsilon}\int_{t_0}^{\infty} t^{-1/2}\operatorname{tr}\left(A_{\delta,\varepsilon}e^{-A_{\delta,\varepsilon}^2 t}\right)dt = 0. \tag{5.21}$$

Thus, from (5.17) we conclude

5.22. Theorem.

$$\lim_{\delta\to 0}\eta(N, g_\delta) = \lim_{t_0\to 0}\int_{S^1}\frac{1}{\pi}\operatorname{tr}\left(\frac{\beta}{2}\mathscr{A}_1^{-1}\dot{\mathscr{A}}_1 e^{-\mathscr{A}_1^2 t_0}\right) \mod Z. \tag{5.23}$$

5.24. Remark. It follows after the fact that the right-hand side of (5.23) is independent of the choices made in constructing $\mathscr{A}_1$. This is to be expected in view of the interpretation put forward in Appendix 2, that the right-hand side can be thought of as a (renormalized) difference of Chern-Simons invariants. In fact, the independence of choices can be checked directly.

If $\underline{\mathscr{A}}_1$ is another such perturbation, then

$$\underline{\mathscr{A}}_1(u) = \mathscr{A}_1(u)K(u), \tag{5.25}$$

where

$$K\beta = \beta K \tag{5.26}$$

and $K(u)$ is the identity on a closed subspace of finite codimension, which (locally) can be chosen to vary smoothly with u. In particular, if K is regarded as *complex linear* relative to the complex structure defined by the almost complex structure β, then K has a well-defined *complex* determinant, whose phase, $\rho(u): S^1 \to R/2\pi Z$, is smooth. One checks easily that

$$\int_{S^1} \frac{1}{\pi} \lim_{t\to 0} \left(\frac{\beta}{2} \mathscr{A}_1^{-1} \dot{\mathscr{A}}_1 e^{-t\mathscr{A}_1^2(u)} \right) du$$

(5.27)
$$= \int_{S^1} \frac{1}{\pi} \lim_{t\to 0} \left(\frac{\beta}{2} \mathscr{A}_1^{-1} \dot{\mathscr{A}}_1 e^{-t\mathscr{A}_1^2(u)} \right) du + \frac{1}{\pi} \int_{S^1} \dot{\rho}(u)\, du,$$

which is the desired invariance.

Appendix 3. Solvmanifolds

In this appendix, we show how the 3-dimensional case of a result similar to that of [1] is an easy consequence of Theorem 5.22.[2]

Let $S \in \mathrm{SL}(2, Z)$ have two real eigenvalues. Thus,

(A3.1)
$$S = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

with

(A3.2)
$$|a + d| > 2.$$

The eigenvalues are

(A3.3)
$$\lambda_{\pm} = \left(\frac{a+d}{2} \right) \pm \sqrt{\left(\frac{a+d}{2} \right)^2 - 1}\,.$$

Let Σ^3 be the solvmanifold

(A3.4)
$$\Sigma^3 = R \times T^2/\Gamma,$$

where Γ is the subgroup of $R \times \mathrm{SL}(2, Z)$ generated by $(1, S)$. Thus, Σ^3 is a flat bundle,

(A3.5)
$$\begin{array}{ccc} T^2 & \to & \Sigma^3 \\ & & \downarrow \\ & & S^1 \end{array}$$

with holonomy S.

Let $v_{\pm}$ be the eigenvectors of S,

(A3.6)
$$v_{\pm} = \begin{pmatrix} \lambda_{\pm} - d \\ b \end{pmatrix}.$$

[2] The higher dimensional cases, which will follow from a generalization of Theorem 5.22 to higher dimensional base spaces, will be discussed elsewhere.

Put

(A3.7) $$g = du^2 + \tilde{g}(u),$$

where $\tilde{g}(u)$ is determined by the condition that

(A3.8) $$\{v_{+,u}, v_{-,u}\} \stackrel{\text{def}}{=} \{\lambda_+^{-u} v_+, \lambda_-^{-u} v_-\}$$

is orthonormal. Then g descends to Σ^3.

We now calculate the 1-form in (1.56). We now calculate the 1-form in (1.56) which, for the present case of trivial coefficients, is given by the simpler formula of (6.1) below. Let $e^*_{1,u}$, $e^*_{2,u}$ be the dual basis with respect to $g(u)$ of the standard lattice basis e_1, e_2 (with respect to which $v_\pm$ is given by (A3.6)). Then

(A3.9) $$\begin{aligned} e^*_{1,u} &= \lambda_+^{-u}(\lambda_+ - d)v_{+,u} + \lambda_-^{-u}(\lambda_- - d)v_{-,u}, \\ e^*_{2,u} &= \lambda_+^{-u} b v_{+,u} + \lambda_-^{-u} b v_{-,u}. \end{aligned}$$

The eigenfunctions of the Laplacian on functions for $(T^2, \tilde{g}(u))$ have an orthonormal basis

(A3.10) $$\left\{ e^{2\pi i(m_1 x_1 + m_2 x_2)} \left[\mathrm{Vol}(T^2)\right]^{-1/2} \right\},$$

where the coordinates correspond to the basis e_1, e_2. The associated eigenvalue is

(A3.11) $$\begin{aligned} &4\pi^2\left[\lambda_+^{-2u}(m_1(\lambda_+ - d) + m_2 b)^2 + \lambda_-^{-2u}(m_1(\lambda_- - d) + m_2 b)^2\right] \\ &\qquad \stackrel{\text{def}}{=} 4\pi^2\left[\lambda_+^{-2u}\mu_+^2 + \lambda_-^{-2u}\mu_-^2\right], \end{aligned}$$

which is $4\pi^2$ times the length squared with respect to $\tilde{g}(u)$ of the corresponding element, $m_1 e^*_{1,u} + m_2 e^*_{2,u}$, of the dual lattice. Then

(A3.12) $$2\pi e^{2\pi i(m_1 x_1 + m_2 x_2)} \tilde{*}(m_1 dx_1 + m_2 dx_2)\left[\mathrm{Vol}(T^2)\right]^{-1/2}$$

is a coexact eigen 1-form with eigenvalue given by (A3.11).

Note that the inner product and action of the $*$-operator on 1-forms of T^2 is obtained by pulling back the corresponding actions on vectors via the isomorphism defined by

(A3.13) $$dx_1 \to e^*_{1,u}, \qquad dx_2 \to e^*_{2,u}.$$

We have

(A3.14) $$\tilde{*}_u v_{+,u} = v_{-,u}, \qquad \tilde{*}_u v_{-,u} = -v_{+,u}.$$

Thus,

$$\tilde{*}_u v_+ = \lambda_-^{-2u} v_-, \qquad \tilde{*}_u v_- = -\lambda_+^{-2u} v_+, \tag{A3.15}$$

which implies

$$\dot{\tilde{*}}_u v_+ = -2\ln\lambda_-(\lambda_-)^{-2u} v_-, \qquad \dot{\tilde{*}}_u v_- = 2\ln\lambda_+(\lambda_+)^{-2u} v_+, \tag{A3.16}$$

and so,

$$\dot{\tilde{*}}_u v_{+,u} = -2\ln\lambda_- v_{-,u}, \qquad \dot{\tilde{*}}_u v_{-,u} = 2\ln\lambda_+ v_{+,u}. \tag{A3.17}$$

It follows that

$$\tilde{*}_u \dot{\tilde{*}}_u v_{+,u} = 2\ln\lambda_- v_{+,u}, \qquad \tilde{*}_u \dot{\tilde{*}}_u v_{-,u} = 2\ln\lambda_+ v_{-,u}. \tag{A3.18}$$

This gives

$$\int_{T^2} \tilde{*}_u \dot{\tilde{*}}_u (m_1 dx_1 + m_2 dx_2)\left[\mathrm{Vol}(T^2)\right]^{-1/2} \wedge (m_1 dx_1 + m_2 dx_2) \times \left[\mathrm{Vol}(T^2)\right]^{-1/2} = 2(\ln\lambda_- - \ln\lambda_+)\mu_+\mu_-. \tag{A3.19}$$

Since the norm squared of $2\pi(m_1 dx_1 + m_2 dx_2)[\mathrm{Vol}(T^2)]^{-1/2}$ is equal to the corresponding eigenvalue, we have (see (6.1))

$$\lim_{\varepsilon\to 0} \frac{1}{\pi}\operatorname{tr}\left(\dot{\tilde{*}} P_{ce} e^{-\tilde{\Delta}\varepsilon}\right) = \frac{1}{\pi}\operatorname{tr}\left(\dot{\tilde{*}} P_{ce}\tilde{\Delta}^{-s}\right)\Big|_{s=0} \tag{A3.20}$$

$$= \frac{2}{\pi}\sum_{\substack{(m_1,m_2)\\ \neq(0,0)}} (4\pi^2)\mu_+\mu_- \ln\frac{\lambda_+}{\lambda_-}\left[4\pi^2\left(\lambda_+^{-2u}\mu_+^2 + \lambda_-^{-2u}\mu_-^2\right)\right]^{-(s+1)}\Bigg|_{s=0}. \tag{A3.21}$$

To obtain $\lim_{\delta\to 0}\eta(\Sigma^3, g_\delta)$, the form in (A3.21) must be integrated over S^1 and the correction term coming from $H^1(T^2, R)$ must be calculated. In fact, the latter term is zero since the decomposition

$$H^{2k-1}(T^2, R) = V_+ \oplus V_- \tag{A3.22}$$

can be chosen to consist of eigenspaces (where in the present case, $k = 1$). Then we have

$$\operatorname{tr}(\dot{\pi}_+\pi_+) + \operatorname{tr}(\dot{\pi}_-\pi_-) = 0. \tag{A3.23}$$

In computing the integral of the form in (A3.21) over S^1, we group together the contributions coming from those pairs (m_1, m_2) lying on a Γ-orbit in $Z \times Z = \Lambda$. Then we get

$$\lim_{\delta\to 0}\eta\left(\Sigma^3, g_\delta\right) = 8\pi\int_{-\infty}^{\infty}\sum_{(\Lambda-0)/\Gamma} \mu_+\mu_- \ln\frac{\lambda_+}{\lambda_-}\left[4\pi^2\left(\lambda_+^{-2u}\mu_+^2 + \lambda_-^{-2u}\mu_-^2\right)\right]^{-(s+1)}\Bigg|_{s=0}. \tag{A3.24}$$

Put

(A3.25) $$\mu_+\mu_- = N(\mu),$$

where $N(u)$ is the norm of the algebraic number $\mu_\pm$. Also put $\lambda_\pm = \lambda^{\pm 1}$, and

(A3.26) $$z = (2\ln\lambda)\cdot u - \ln|\mu_+/\mu_-|.$$

For $\operatorname{Re} s$ large, we can interchange the summation and integration in the expression on the right-hand side of (A3.24) to get

(A3.27) $$8\pi \sum N(\mu)|N(\mu)|^{-(s+1)}\int_{-\infty}^{\infty}(8\pi^2\cosh z)^{-(s+1)}\,dz.$$

Since the L-series,

(A3.28) $$L(s) = \sum N(\mu)|N(\mu)|^{-(s+1)},$$

is known to be regular at $s = 0$ (see [20], p. 230) it follows that the analytic continuation of (A3.21) to $s = 0$ is given by

(A3.29) $$\frac{1}{\pi}\int_{-\infty}^{\infty}\cosh^{-1}z\,dz\cdot L(0) = L(0).$$

Thus, we obtain

(A3.30) $$\lim_{\delta\to 0}\eta(\Sigma^3, g_\delta) = L(0),$$

which is the analog of the result proved in [1]. There, the η-invariant of a different first order operator was considered.

A3.31. Remark. According to [11], if the family of metrics, $\tilde{g}(u)$, in (A3.9) is replaced by any other family of *flat* metrics such that g descends to Σ^3, then the limiting value of the η-invariant in (A.30) is unchanged.

PART II: CONICAL SINGULARITIES

6. Introduction

In this section we discuss the relation of our previous work on conical singularities to formula (4.28), which we rederive and interpret in that context. For simplicity we restrict attention to the case of trivial coefficients. In this case (since A^2 preserves degree) the assumption that $\mathscr{A}(u)$ is invertible for all u can be replaced by the assumption $H^{2k-1}(Y^{4k-2}, R) = 0$.

There is a connection between the more general formula (5.23) and the concept of "ideal boundary conditions" (see [8, pp. 113–114]). This will be discussed further elsewhere.

To a large extent, it is possible simply to reinterpret the discussion of Part I in the context of conical singularities, rather than rederiving it. In particular, we can do this for (5.23). But it seems to us that this reveals neither the full extent of the connection, nor its naturality.

We also point out that the derivation given below is not the full story, since it is based on the formula for the variational derivative of the η-invariant for spaces with isolated conical singularities, rather than on a direct study of the asymptotics of the heat kernel at a singular stratum of dimension one. Still, it is worth comparing and contrasting the roles played by the operation of passing to the adiabatic limit in Parts I and II.

Finally, we mention that our original derivation of (4.28) in the context of conical singularities exploited yet another connection between the subjects. Namely, if one starts with a mapping torus, $Y^{4k-2} \to N^{4k-1} \to S^1$, conical singularities can be introduced into the set-up by pinching the cross-section as in Figure 6.1. The "singular continuity method" (which is a systematization of the discussion of [7]) is used to show that the η-invariant does not jump when we pass from nonsingular space to the singular space in Figure 6.1. Then the formula for the variational derivative of the η-invariant is employed in deforming back to the product situation as in Figure 6.2. We also intend to discuss this further elsewhere.

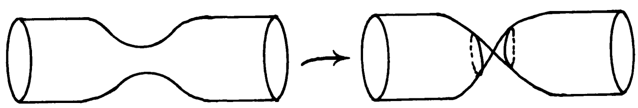

FIGURE 6.1

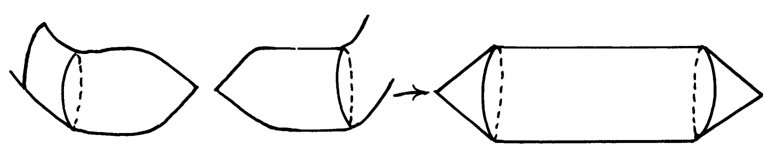

FIGURE 6.2

For the case of trivial coefficients, A^2 preserves degree and we have

$$(6.1) \qquad \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{tr}\left(\frac{\beta}{2} \mathscr{A}^{-1} \dot{\mathscr{A}} e^{-\varepsilon \mathscr{A}^2} \right) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{tr}\left(\dot{*}_{2k-1} P_{ce} e^{-\Delta_{2k-1}\varepsilon} \right),$$

where $P_{ce} e^{-\Delta_{2k-1}\varepsilon}$ denotes the coexact part of the heat kernel on $(2k-1)$-forms. The 1-form in (0.1) arose in our work on conical singularities (see [6]–[9]; especially [9, pp. 612 and 652–654]). It was emphasized there that the Atiyah-Patodi-Singer formula can be viewed as the natural geometric signature formula obtained by applying the heat equation method to calculate the L_2-signature of a manifold $X^{4k} = M^{4k} \cup C_{0,1}(N^{4k-1})$ with an isolated conical singularity. Here M^{4k} is a manifold with boundary, $\partial M^{4k} = N^{4k-1}$, and X^{4k} is obtained from M^{4k} by attaching a cone, $C_{0,1}(N^{4k-1})$, to ∂M^{4k} (see Figure

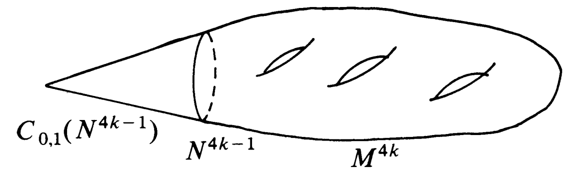

FIGURE 6.3

6.3). The expression $C_{0,A}(N)$ denotes the subset of the full cone, $C(N)$, consisting of those points, (r, x), with $r \leqslant A$ (see §1).

The heat equation method gives

$$\text{(6.2)} \qquad \operatorname{sig}_{L_2}(X) = \lim_{t\to 0} \operatorname{tr}\left(*_{2k} e^{-\Delta_{2k}t}\right) = \int_{X^4} P_L(\Omega) + \eta(N^{4k-1}).$$

The integral, $\int_{X^{4k}} P_L(\Omega)$, is the contribution to the asymptotic expansion coming from the nonsingular part of X. The η-invariant is the contribution coming from the singularity, or more precisely, from $C_{0,\varepsilon}(N)$ (for any fixed ε). Since $P_L(\Omega)\,|\,C_{0,1}(N) \equiv 0$ and $\operatorname{sig}_{L_2}(X^{4k}) = \operatorname{sig}(M^{4k})$, it follows that (6.2) is equivalent to the usual Atiyah-Patodi-Singer formula.

6.3. Remark. The L_2-cohomology of X^{4k} is isomorphic to the *middle intersection cohomology*, $IH^*(X)$ (see [8]).

The papers [6]–[9] only dealt with *nonisolated* singularities in the piecewise constant curvature case. There it is easy to see that the nonisolated singularities of positive dimension do not contribute to the asymptotic expansion for $\operatorname{tr}(*_{2k} e^{-\Delta_{2k}t})$ (although they do contribute to the expansion for $\operatorname{tr}(e^{-\Delta t})$ in general). The 1-form in (6.1) arises when we consider singular strata of dimension one for more general spaces with conical singularities. Its analogs of higher degree will enter when we consider singular strata of higher dimension.

We did observe in [9] that the 1-form in (6.1) arises in a context which is closely related to that of nonisolated singularities. We pointed out in [9] (see p. 614) that the η-invariant can be defined for spaces, X^{4k-1}, with isolated conical singularities. Let (X^{4k-1}, g_u) be a 1-parameter family of such spaces. Then one can see that the formula

$$\text{(6.4)} \qquad \dot\eta = \lim_{t\to 0} -2t^{1/2} \operatorname{tr}\left(*de^{-\Delta_{2k-1}t}\right)$$

holds as in the smooth case. However, in analogy with (6.2), the right-hand side of (6.4) splits into the usual local contribution from the interior and a contribution which comes from the singularities and, hence, depends only on the variation restricted to the cross-sections, $(Y^{4k-2}, \tilde g_u)$. This contribution is given by the expression in (6.1).

The derivation of [9, pp. 612–614] is a bit muddled because the value $\alpha = 0$ was carelessly substituted for the correct value $\alpha = 1/2$ (see §8 below where this is corrected). It is apparent that the family (X^{4k-1}, g_u) is related to the study of nonisolated conical singularities in dimension $4k$, since the *graph* of the family $C_{0,1}(Y^{4k-2}, \tilde{g}_u) \subset (X^{4k-1}, g_u)$ has such a singularity, with metric

$$du^2 + dr^2 + r^2\tilde{g}_u. \tag{6.5}$$

This connection is pursued in §9 below.

Nonisolated singularities. Let $Y^{4k-2} \to N^{4k-1} \xrightarrow{\pi} S^1$ be a Riemannian submersion. Form a space, X^{4k}, with singularities, as follows. Start with $[0,1] \times N^{4k-1}$. Attach the cone on N to the boundary component $1 \times N$ and the mapping cone of $\pi: N^{4k-1} \to S^1$ to the boundary component $0 \times N$ (that is, we cone off each fiber, (u, Y)) (see Figure 6.4). The resulting space has 2 singular strata, $\Sigma^0 = p$ and $\Sigma^1 = S^1$.

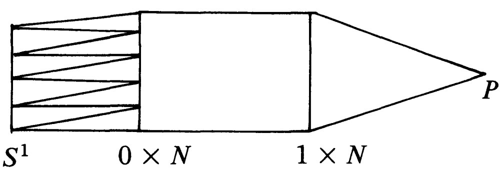

FIGURE 6.4

Suppose we use the product metric $dw^2 + g_\delta$ on $[0,1] \times N$, the metric $C_{0,\delta^{-1}}(N, \delta^2 g_\delta)$ on the cone on N, and the metric $C_{0,\delta^{-1}}(Y, \delta^2\tilde{g}_u)$ on Y. Since the cross-sections of $C_{0,\delta^{-1}}(N, \delta^2 g_\delta)$ and $C_{0,\delta^{-1}}(Y, \delta^2\tilde{g}_u)$ at distance δ^{-1} from their vertices are (N, g_δ) and $(Y, \tilde{g}_u)$, the metrics on the cross-sections match up at both ends. However, the metric is not smooth since the interval direction, w, on $[0,1] \times N$ changes to the radial direction on the cones. But in the limit, $\delta \to 0$, this lack of smoothness disappears and in fact, as we will see in §9, makes no contribution even though S^1 becomes infinitely long (the lack of smoothness at $1 \times N$ could have been neglected since the metric can be smoothed conformally).

Now imagine that we apply the heat equation method to calculate the L_2-signature of X^{4k} and pass to the limiting metric. One can see that

$$\operatorname{sig}_{L_2}(X^{4k}) = \operatorname{sig}_{IH^*}(X^{4k}) = 0. \tag{6.6}$$

Of course, the local term on $[0,1] \times N$ is 0. The (complete) contribution from the cone on N is

$$\lim_{\delta \to 0} \eta(N, g_\delta). \tag{6.7}$$

The complete contribution from the mapping cone will be seen to be

$$\int_{S^1} \frac{1}{\pi} \operatorname{tr}\left(\dot{*} P_{ce} e^{-\tilde{\Delta}_{2k-1}t}\right) \stackrel{\text{def}}{=} \eta_1. \tag{6.8}$$

Thus,

$$0 = \lim_{\delta \to 0} \eta(N, g_\delta) - \int_{S^1} \eta_1. \tag{6.9}$$

6.10. Remark. Clearly one can do the above construction starting with a fibration for which the base space, Z^{j+4i-1}, has arbitrary dimension. In this case, forms, η_{j+4i-1}, of higher degree arise.

6.11. Remark. The geometric construction above can be rephrased in purely analytic terms. The conditions that forms be in L_2 near the singularities are equivalent to imposing two different *global boundary conditions* at the boundary components $0 \times N$, $1 \times N$ of the smooth manifold with boundary, $[0,1] \times N$. The resulting *fiberwise global boundary condition* at $0 \times N$ appears to be of a type not previously considered. In this spirit more general spaces with singularities for which the singular strata have other singular strata in their closures can be replaced by manifolds with *piecewise smooth boundary* and the appropriate global boundary conditions.

6.12. Remark. The discussion of the following sections does generalize to the signature operator with coefficients in a bundle and to spinors (see [16] for a discussion of the Dirac operator on cones).

7. Functional calculus on cones and the η-invariant

We begin this section by outlining the functional calculus for the Laplacian on cones which was introduced in [7]. The fundamental idea behind this calculus is what we call the *strong form* of the method of separation of variables. First we use the Hankel inversion formula in the radial direction to obtain the (distribution) kernel, k_f, of $f(\Delta)$, in the form of an eigenfunction expansion on the cross-section with the radial variables (r_1, r_2) as parameters. We regard this expression as a parametrized family of kernels for an associated family of functions of the Laplacian, $\tilde{\Delta}$, on the cross-section and employ the functional calculus for $\tilde{\Delta}$ to "sum the series" (the words "strong form" refer to this second step). For details of this calculus see [6]–[8] and [14]–[16]. We point out that the "strong form" can be used in any problem involving separation of variables.

The metric cone on a Riemannian manifold N^m with metric $\tilde{g}$ is the completion of the Riemannian manifold, $R^+ \times N^m$, where in polar coordinates, (r, x), the metric is given by

$$dr^2 + r^2 \tilde{g}. \tag{7.1}$$

When we do analysis on the cone, $C(N, \tilde{g})$, we always work on the nonsingular part, $R^+ \times N^m$.

Let ϕ be a coexact eigen i-form, with eigenvalue μ of the Laplacian, $\tilde{\Delta}$, on i-forms of N^m.

Put

$$\alpha = (1 + 2i - m)/2, \tag{7.2}$$

$$\nu = \sqrt{\mu + \alpha^2}\,. \tag{7.3}$$

Let J_ν denote the Bessel function of order ν. Then

$$r^\alpha J_\nu(\lambda r)\phi \tag{7.4}$$

is a coexact eigenform of the Laplacian $d\delta + \delta d$ on $C(N^m, \tilde{g})$, with eigenvalue $\lambda^2 > 0$. Such an eigenform is said to be of type 1. Similarly, there are eigenforms of types, 2, 3, 4 (described most easily as being obtained by applying the operations, d, $* d$, and $*$ to the form in (7.4)). Thus, types 1 and 3 are coexact while types 2 and 4 are exact. In case there exist harmonic i-forms, h, on N ($\mu = 0$) the eigenform

$$r^\alpha J_\nu(\lambda r)h \tag{7.5}$$

is called type E and its $*$ is called type 0. These can be treated in the same way as types 1 and 4. But if N is of dimension $2l$ and $i = l$, the solutions $r^\alpha J_{-\nu}(\lambda r)h$ must also be discussed (this is excluded by our assumption $H^{2k-1}(Y^{4k-2}, R) = 0$).

Given a suitable function, f, we let k_f denote the kernel of the operator $f(\Delta)$ defined by the spectral theorem. Then k_f is the sum of four terms corresponding to the types 1–4 above. For type 1-forms, the Hankel inversion formula leads to the expression

$$(r_1 r_2)^\alpha \sum_j \int_0^\infty f(\lambda^2) J_{\nu_j}(\lambda r_1) J_{\nu_j}(\lambda r_2)\lambda\, d\lambda\, \phi_j(x_1) \otimes \phi_j(x_2). \tag{7.6}$$

The corresponding expression for type 2 is obtained by replacing $f(\lambda^2)$ by $f(\lambda^2)/\lambda^2$ in (7.6) and applying exterior differentiation to the (r_1, x_1) and (r_2, x_2) variables (see [9, p. 590]).

As illustrated in [7], [8], [14], k_f can be calculated explicitly for important functions f by employing the evaluation of certain classical integrals. For our purposes, the Weber Schaftheitlin formula [25, p. 401] is particularly significant;

$$\int_0^\infty \lambda^{1-2s} J_\nu(\lambda r) J_\nu(\lambda r)\, d\lambda = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\nu - s + 1)}{\Gamma(\nu + s)} \frac{\Gamma(s - 1/2)}{\Gamma(s)}. \tag{7.7}$$

This formula allows us to calculate explicitly the pointwise zeta function on $C(N)$ (the analytic continuation of $\operatorname{tr}(\Delta^{-s})$) and related quantities.

We now indicate the derivation of the Atiyah-Patodi-Singer formula which was given in [9, §6]. Let $X^{4k} = M^{4k} \cup C_{0,1}(N^{4k-1})$. We have

$$\operatorname{sig}_{L_2}(X) = \lim_{t\to 0} \operatorname{tr}\left(*_{2k} e^{-\Delta_{2k}t}\right). \tag{7.8}$$

By employing a suitable cutoff function, the right-hand side of (7.8) can be split into two pieces, the usual local term,

$$\lim_{t\to 0} \int_M \operatorname{tr}\left(*_{2k} e^{-\Delta_{2k}t}\right) = \int_M P_L(\Omega), \tag{7.9}$$

and a term coming from $C_{0,1}(N) \subset X$. Since the heat kernel on the cone $C(N)$ is a parametrix for the heat kernel of X near the singularity, the latter term is equal to

$$\lim_{t\to 0} \int_{C_{0,1}(N)} \operatorname{tr}(*_{2k} \mathscr{E}_{2k}(t)), \tag{7.10}$$

where $\mathscr{E}_{2k}(t)$ is the heat kernel on $C(N)$.

Consider, for the moment, an arbitrary Riemannian manifold, Y^{4k}, with metric g, volume from ω_g, and heat kernel $e^{-\Delta_{2k,g}t}$. If we write the pointwise trace, $\operatorname{tr}(*_{2k} e^{-\Delta_{2k,g}t})$, as

$$\operatorname{tr}\left(*_{2k} e^{-\Delta_{2k,g}t}\right) = F_g(t)\,\omega_g, \tag{7.11}$$

then it is easy to see that

$$F_{c^2 g}(t) = F\left(t/c^2\right)c^{-4k}. \tag{7.12}$$

Now take $Y = C(N)$. We put $F = F(r,x,t)$ and recall that $C(N)$ has 1-parameter family of dilations. Then, by using the relation

$$\omega = dr \wedge r^{4k-1}\beta, \tag{7.13}$$

where β is the volume form on N, we obtain

$$\begin{aligned} \int_{C_{0,1}(N)} \operatorname{tr}(*_{2k} \mathscr{E}_{2k}(t)) &= \int_N \int_0^1 F(r,x,t)\, dr \wedge r^{4k-1}\beta \\ &= \int_N \left(\int_0^1 F\left(1,x,t/r^2\right) r^{-1} dr \right)\beta. \end{aligned} \tag{7.14}$$

If we set $t/r^2 = u$, this becomes

$$\tfrac{1}{2}\int_N \int_t^\infty F(1,x,u)u^{-1}\, du\beta, \tag{7.15}$$

and we get

$$\lim_{t\to 0} \int_{C_{0,1}} \operatorname{tr}(*_{2k}(\mathscr{E}_{2k}(r,x,t))) = \tfrac{1}{2}\int_N \int_0^\infty t^{-1} \operatorname{tr}(*_{2k} \mathscr{E}_{2k}(1,x,t)). \tag{7.16}$$

In view of the usual Mellin transform formalism (and as explained in [9, §6]) the right-hand side of (7.16) can be identified with the analytic continuation to $s = 0$ of the pointwise trace,

$$\tfrac{1}{2}\int_N \operatorname{tr}(\Gamma(s) *_{2k}\Delta^{-s}(1,x))\,|_{s=0}. \tag{7.17}$$

The contribution to (7.17) from forms of type 1 contains only terms involving the expression $\phi_j \wedge \phi_j$. Since this $4k$-form (on $C(N)$) does not involve dr, it vanishes. Similarly the type 4 contribution vanishes and the type 2 and 3 contributions are equal.

To evaluate the type 2 contribution, choose an orthonormal basis of coexact eigen $(2k-1)$-forms, ϕ_j, on N^{4k-1} such that

$$\bar{*}\, d\phi_j = \pm\nu_j\phi_j = \mu_j\phi_j \tag{7.18}$$

(note that $\alpha = (1 + 2(2k-1) - (4k-1))/2 = 0$). As explained after (7.6), we must evaluate the analytic continuation to $s = 0$ of

$$\begin{aligned}\Gamma(s)\int_N\int_0^\infty \sum_j \lambda^{-1-2s} d\{J_{\nu_j}(\lambda r_1)\phi_j(x_1)\} \wedge d\{J_{\nu_j}(\lambda r_2)\phi_j(x_2)\}\Big|_{r_1=r_2=1} d\lambda \\ = 2\Gamma(s)\sum_j \int_0^\infty \lambda^{-2s}J_\nu'(\lambda)J_\nu(\lambda)\,d\lambda\phi_j \wedge d\phi_j.\end{aligned} \tag{7.19}$$

Integration by parts gives

$$\begin{aligned}2\Gamma(s+1)\sum\int_N\int_0^\infty \lambda^{1-2(s+1)}J_\nu(\lambda)J_\nu(\lambda)\,d\lambda\phi_j \wedge d\phi_j) \\ = \sum_j \frac{1}{\sqrt{\pi}}\frac{\Gamma(\nu_j - s)}{\Gamma(\nu_j + s + 1)}\Gamma\left(s + \frac{1}{2}\right)\phi_j \wedge d\phi_j.\end{aligned} \tag{7.20}$$

As a consequence of Stirling's formula, we have for ν large

$$\frac{\Gamma(\nu - s)}{\Gamma(\nu + s)} = \nu^{1-2s}\frac{1}{\nu^2 - s^2}\left(1 + s\sum_{l=1}^{\infty}(-1)^{(l-1)}\frac{B_l}{l}\nu^{-2l} + O(s^2)\right), \tag{7.21}$$

where B_l is the lth Bernoulli number. From this and the fact that the analytic continuation of

$$\eta(s) = \sum_j \int_N \nu^{-1-2s}\phi_j \wedge d\phi_j \tag{7.22}$$

is regular for $s > -1/2$ (see [19, Theorem 2.61] and [8, p. 610]) it follows that the analytic continuation to $s = 0$ of the right-hand side of (7.20) can be identified with $\eta(0)$. This completes the derivation of the Atiyah-Patodi-Singer formula.

8. The η-invariant and its variation for spaces with conical singularities

For N^{4k-1} smooth, the derivative of the η-invariant under change of metric is local. We have

$$\begin{aligned}
\dot{\eta} &= \lim_{\varepsilon\to 0}\frac{1}{\Gamma(1/2)}\int_\varepsilon^\infty t^{-1/2}\operatorname{tr}\left(*de^{-\Delta_{2k-1}t}\right)^{\boldsymbol{\cdot}}\,dt \\
&= \lim_{\varepsilon\to 0}\frac{1}{\Gamma(1/2)}\left\{\int_\varepsilon^\infty t^{-1/2}\operatorname{tr}(\dot{*}\,de^{-\Delta t}) - \int_\varepsilon^\infty t^{-1/2}\operatorname{tr}(*\,d\dot{\Delta}e^{-\Delta t})\right\}dt \\
&= \lim_{\varepsilon\to 0}\frac{1}{\Gamma(1/2)}\left\{\int_\varepsilon^\infty t^{-1/2}\operatorname{tr}(\dot{*}de^{-\Delta t}) - 2\int_\varepsilon^\infty t^{1/2}\operatorname{tr}(*\,de^{-\Delta t})^{\boldsymbol{\cdot}}\right\}dt \qquad (8.1)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\Gamma(1/2)}\left\{\int_\varepsilon^\infty t^{-1/2}\operatorname{tr}(\dot{*}de^{-\Delta t})\,dt\right. \\
&\qquad\left. +2\int_\varepsilon^\infty t^{1/2}\frac{d}{dt}\operatorname{tr}(\dot{*}\,de^{-\Delta t})\,dt\right\}
\end{aligned}$$

Integrating the second term by parts gives

$$\frac{d}{du}\eta = -2\lim_{t\to 0} t^{1/2}\frac{1}{\Gamma(1/2)}\operatorname{tr}(\dot{*}de^{-\Delta t}). \qquad (8.2)$$

We now consider the η-invariant for spaces $X^{4k} = C_{0,1}(Y^{4k-2}) \cup M^{4k-1}$, with isolated conical singularities. As we pointed out in [9], this is equivalent to considering the η-invariant of M^{4k-1} for a suitable *global boundary condition*.

As explained in [9, p. 612], the kernel $*\,d\mathscr{E}_{2k-1}(t)$ of $C(Y^{4k-2})$ satisfies

$$\operatorname{tr}(*\,d\mathscr{E}_{2k-1}(t)) \equiv 0. \qquad (8.3)$$

Hence, the heat kernel $E_{2k-1}(t)$ of X^{4k-1} satisfies

$$\operatorname{tr}(*\,dE_{2k-1}(t)) \sim a_0 + O(t^{1/2}). \qquad (8.4)$$

In particular, the η-invariant of X^{4k-1} exists. Let g_u be a 1-parameter family of metrics with conical singularities on X^{4k-1}, which is smooth away from the singularity (in particular the metrics, $\tilde{g}_u$, on the cross-sections vary smoothly). It follows by standard arguments that $\frac{d}{du}\eta(X^{4k-1}, g_u)$ exists and is given by

$$-2\lim_{t\to 0}\frac{t^{1/2}}{\Gamma(1/2)}\operatorname{tr}(\dot{*}dE_{2k-1}(t)), \qquad (8.5)$$

provided this limit exists. As usual the expression in (8.5) can be split into a contribution coming from, say, $C_{0,1}(Y^{4k-2})$ and the previous local contribution away from the singularity. We must investigate the contribution on $C_{0,1}(Y^{4k-2})$ for which it suffices to consider

$$-2\lim_{t\to 0}\frac{t^{1/2}}{\Gamma(1/2)}\int_{C_{0,1}(Y^{4k-2})}\operatorname{tr}(\dot{*}d\mathscr{E}_{2k-1}(t)). \tag{8.6}$$

Write the pointwise trace, $\operatorname{tr}(\dot{*}de^{-\Delta_{2k-1}t})$, on an arbitrary Riemannian manifold as

$$\frac{d}{du}\left(\operatorname{tr} *_{g+uA}de^{-\Delta t}\right)\Big|_{u=0} = G_{g,A}(t)\,\omega_g, \tag{8.7}$$

where ω_g is the volume element. Then it is easy to check that

$$G_{\rho^2 g,\rho^2 A}(t) = G_{g,A}(t/\rho^2)\rho^{-4k}, \tag{8.8}$$

$$G_{\rho^2 g,\rho^2 A}(t)\,\omega_{\rho^2 g} = \rho^{-1}G_{g,A}(t/\rho^2)\,\omega_g. \tag{8.9}$$

Now for the cones $C(Y,\tilde{g}_u)$, write $G(r,y,t)$ for $G_{g,A}(t)$ at (r,y). Then by using the 1-parameter group of dilations of $C(N,\tilde{g}_u)$, it follows that

$$\begin{aligned}\frac{-2t^{1/2}}{\Gamma(1/2)}&\int_{C_{0,1}(Y^{4k-2})}\operatorname{tr}(\dot{*}d\mathscr{E}_{2k-1}(t))\\ &= \frac{-t^{1/2}}{\Gamma(1/2)}\int_0^1\int_Y G(1,y,t/r^2)r^{-4k}(r^{4k-2}dr\wedge\beta)\\ &= \frac{1}{\Gamma(1/2)}\int_t^\infty\int_Y w^{-1/2}G(1,y,w)\beta\,dw\end{aligned} \tag{8.10}$$

(where β denotes the volume element on Y^{4k-2}). Thus, the contribution to $\dot{\eta}$ is

$$\frac{1}{\Gamma(1/2)}\int_0^\infty\int_Y w^{-1/2}G(1,y,w)\beta\,dw \tag{8.11}$$

provided the integral converges at $w = 0$. Note that this convergence is equivalent to the vanishing of the coefficient of $t^{-1/2}$ in the asymptotic expansion of the pointwise trace $(-2/\Gamma(1/2))\operatorname{tr}(\dot{*}d\mathscr{E}_{2k-1}(t))$ at $r = 1$, which, in turn, is the local contribution to $\dot{\eta}$ at $r = 1$. Thus, we are asserting that this local contribution vanishes for *all variations through conical metrics*.

It will emerge from the explicit computation which follows that

$$G(1,y,w)dr\wedge\beta \sim a_{-1/2}dr\wedge\beta t^{-1/2} + O(t), \tag{8.12}$$

where

$$a_{-1/2} = \frac{1}{\pi}\operatorname*{Res}_{s=0} dr\wedge[\operatorname{tr}(\Sigma\mu^{-s}\,\dot{*}\,\phi\wedge *\phi)] \tag{8.13}$$

and for large s,

$$\sum \mu^{-s}\phi \otimes \phi = \operatorname{tr}\tilde{\Delta}^{-s}_{ce,2k-1}. \tag{8.14}$$

The right-hand side of (8.13) is unchanged if the family of metrics on the cross-sections is replaced by a scaled family, $c^2\tilde{g}_u$ (since $\tilde{*}_{2k-1}$, $\dot{\tilde{*}}_{2k-1}$ on Y^{4k-2} are invariant under scaling and we evaluate at $s = 0$).

Let $G_c(r, y, t)$ denote $G(t)$ for the family $C(Y, c^2\tilde{g}_u)$. Then using (8.8) and the dilations of $C(Y, c^2\tilde{g}_u)$

$$G_c(1, y, t)\,dr \wedge \beta_c = c^{-1}G_c(1/c, y, t/c^2)c^{-(4k-2)}dr \wedge \beta_c. \tag{8.15}$$

Since, $\beta_c = c^{-(4k-2)}\beta$, it suffices to show the vanishing in the limit as $c \to 0$ of $a_{-1/2,c}$, where

$$G_c(1/c, y, t) \sim a_{-1/2,c}t^{-1/2}. \tag{8.16}$$

Note that if we put $r - 1/c = z$, the metric on $C(Y, c^2\tilde{g})$ is

$$dz^2 + (1 + cz)^2\tilde{g}, \tag{8.17}$$

and we recognize that this family is obtained from $dz^2 + (1 + z)^2\tilde{g}$ by passing to the adiabatic limit. The vanishing in the limit of $a_{-1/2,c}$ is now clear because, in the limit, the variation is compatible with an orientation reversing isometry.

The above argument is in the spirit of the scaling argument of [9, p. 612], which gave an alternative proof of the regularity of the η-function at $s = 0$ in the smooth case.

To evaluate the quantity in (8.11), we first derive a general formula for $\operatorname{tr}(*df(\Delta))$ (for suitable f). We will apply it to $f(\Delta) = \Delta^{-2s}$. The type 1 component is

$$\begin{aligned}&\sum_j \int_0^\infty f(\lambda^2)\lambda * d\left(r_1^{1/2}J_{\nu_j}(\lambda r_1)\phi_j(x_1)\right)\\ &\qquad \wedge *\left(r_2^{1/2}J_{\nu_j}(\lambda r_2)\phi_j(x_2)\right)\Big|_{r_1=r_2=1} d\lambda\\ &= \sum_j \int_0^\infty f(\lambda^2)\lambda\left\{\frac{1}{2}J_{\nu_j}^2(\lambda) + \lambda J_{\nu_j}'(\lambda)J_{\nu_j}(\lambda)\right\} d\lambda\, dr \wedge \dot{\tilde{*}}\phi \wedge \tilde{*}\phi,\end{aligned} \tag{8.18}$$

where $\tilde{*}$ is the $*$-operator of Y^{4k-2}. For suitable f, the second term can be integrated by parts, and (8.18) becomes

$$-\sum_j \int_0^\infty \left(\frac{1}{2}f(\lambda^2) + f'(\lambda^2)\lambda^2\right)\lambda J_{\nu_j}^2(\lambda)\, d\lambda\, dr \wedge \dot{\tilde{*}}\phi \wedge \tilde{*}\phi. \tag{8.19}$$

On type 3 forms, we have

$$(8.20)\qquad \sum_j \int_0^\infty \frac{f(\lambda^2)}{\lambda^2} \lambda {*}d * d\left(r_1^{1/2} J_{\nu_j}(\lambda r_1)\phi_j(x_1)\right) \wedge *^2 d\left(r_2^{1/2} J_{\nu_j}(\lambda r_2)\phi(x_2)\right)\Big|_{r_1=r_2=1} d\lambda.$$

Using ($*^2 = 1$ and)

$$(8.21)\qquad \begin{aligned} {*}d * d\left(r_1^{1/2} J_{\nu_j}(\lambda r_1)\right) &= {*}{*}\,\Delta\left(r_1^{1/2} J_{\nu_j}(\lambda r_1)\right) \\ &= {*}{*}\,\lambda^2\left(r_1^{1/2} J_{\nu_j}(\lambda r_1)\right), \end{aligned}$$

it follows easily that the contribution from type 3 forms is also given by (8.19). The contribution from types 2 and 4 of course is zero, since these forms are annihilated by ${*}d$.

As in §1, we can identify the quantity in (8.11) with the value at $s = 1/2$ of the analytic continuation of $-\operatorname{tr}({*}d\Delta^{-s})$. Note that a priori, this expression might have a pole at $s = 1/2$ (since $\dim X = 4k - 1$ is odd). However, by putting $f(\lambda^2) = \lambda^{-2s}$ in (8.19) and multiplying by 2 (to include the type 3 contribution) we get

$$(8.22)\qquad 2\left\{\sum_j \int_0^\infty (s - 1/2)\lambda^{1-2s} J_{\nu_j}^2(\lambda)\, dr \wedge \dot{*}\,\phi_j \wedge \tilde{*}\,\phi_j\right\} d\lambda.$$

By using (7.7) we get

$$(8.23)\qquad \frac{1}{\pi}\int_N \sum_j \frac{\Gamma(\nu_j - s + 1)}{\Gamma(\nu_j + s)} \frac{\Gamma(s - 1/2)(s - 1/2)}{\Gamma(s)}\, \dot{*}\,\phi_j \wedge \tilde{*}\,\phi_j.$$

Our previous discussion ((8.12)–(8.17)), now implies that the expression in (8.23) is actually regular at $s = 1/2$ and is equal to the value of the analytic continuation of

$$(8.24)\qquad \frac{1}{\pi}\int_N \sum_j \mu_j^{-s}\, \dot{*}\,\phi_j \wedge \tilde{*}\,\phi_j$$

at $s = 0$. The expression in (8.24) can now be written as

$$(8.25)\qquad \frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{tr}\left(\dot{*}\, P_{ce} e^{-\Delta_{2k-1}\varepsilon}\right).$$

9. The adiabatic limit and local terms

Let X^{4k} be the space in Figure 6.4. We want to see that the contribution to the constant term of $\mathrm{tr}(*\, e^{-\Delta_{2k-1}t})$ from the singular stratum $\Sigma^1 = S^1$ is $\int_{S^1} \eta_1$. In fact we will see that this holds locally on S^1. Implicit in this assertion is the fact that on a singular space of the type under consideration, the heat kernel has a parametrix which is *local* on the singular stratum. In the case under consideration, $H^{2k-1}(Y^{4k-2}, R) = 0$, this follows by arguing as in [8, p. 618]. (The argument uses a suitable cutoff function.) As indicated in this section the argument does not apply and the assertion that the trace localizes on $\Sigma^1 = S^1$ is false if $H^{2k-1}(Y^{4k-2}, R) \neq 0$.

Granted the localization we can go on to calculate the contribution by introducing an auxiliary space W^{4k} $(= W^{4k}(a, b))$ as follows. The space W has five pieces. The middle piece is $[a, b] \times Y^{4k-2} \times [a, b]$, where we think of $[a, b] \times Y^{4k-2} \times b$ and $[a, b] \times Y^{4k-2} \times a$ as two copies of the corresponding subset $\pi^{-1}([a, b]) \subset N^{4k-1}$ (where $\pi: N \to S^1$). Now attach to $[a, b] \times Y^{4k-2} \times a$ and $[a, b] \times Y^{4k-2} \times b$ the mapping cones of the projections onto $[a, b]$. Call the corresponding piece of W pieces I, II, III (see Figure 9.1).

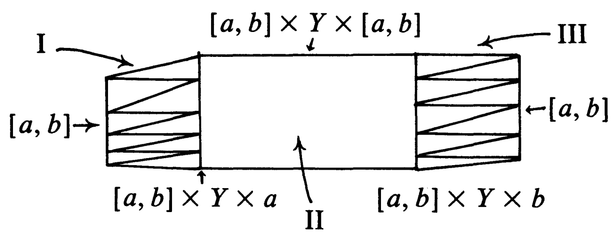

FIGURE 9.1

Observe that topologically W is $[a, b] \times \Sigma(Y)$, where $\Sigma(Y)$ denotes the *suspension* of Y. Thus, we can regard W as a subset of $\Sigma^2(Y)$, the *double suspension*, by coning off the boundary components $a \times \Sigma(Y)$ and $b \times \Sigma(Y)$, call these pieces IV, V (see Figure 9.2).

We now introduce a family of metrics, g_δ, parametrized by δ on W^{4k}. The metric will be given on each piece. The metrics on the pieces will *not* quite match smoothly on the boundaries and will not be smooth along the diagonal $u + v = a + b$ of piece II (which is dashed in Figure 9.2).

(I) The metric on piece I is

$$g_\delta = \delta^{-2} du^2 + dr^2 + r^2\big(\delta^{-2}\tilde{g}(u)\big), \tag{9.1}$$

where

$$dr^2 + r^2\big(\delta^{-2}\tilde{g}(u)\big) \tag{9.2}$$

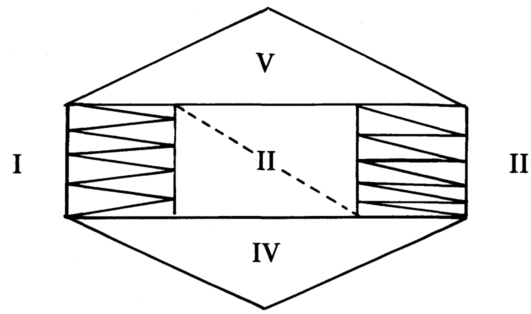

FIGURE 9.2

is the metric on the cone $C_{0,\delta^{-1}}(Y, \delta^{-2}\tilde{g})$. Thus, the metric on piece I *coincides* with the metric on the corresponding subset, $\pi^{-1}([a, b])$, of the mapping cone, which is the left most of the three pieces in Figure 6.2 (which depicts X^{4k}). This is the basic connection between the space W^{4k} and the space X^{4k}.

(II) The metric on II is

$$\delta^{-2}du^2 + h(u,w) + \delta^{-2}dw^2, \tag{9.3}$$

where

$$h(u,w) = \begin{cases} \tilde{g}(u), & a \leqslant u \leqslant (a+b) - w, \\ \tilde{g}(a+b-w), & (a+b) - w \leqslant u \leqslant b. \end{cases} \tag{9.4}$$

(III) The metric on III is

$$\delta^{-2}du^2 + dr^2 + r^2\big(\delta^{-2}g(a)\big). \tag{9.5}$$

(IV) Let ρ_a denote the metric on the boundary component $a \times \Sigma(Y)$ of I $\cup$ II $\cup$ III. The metric on IV is

$$ds^2 + s^2\big(\delta^2\rho_a\big), \tag{9.6}$$

where s is the radial parameter on $C_{0,\delta^{-1}}(a \times \Sigma(Y), \delta^2\rho_a)$.

(V) Similarly the metric on V is

$$ds^2 + s^2\big(\delta^2\rho_b\big) \tag{9.7}$$

on $C_{0,\delta-1}(b \times \Sigma(Y), \delta^2\rho_b)$.

If the metric just constructed is smoothed slightly we can apply the heat equation method to $\Sigma^2(Y)$ to compute its L_2-signature which is zero. Ignoring for a moment the lack of smoothness we get

$$0 = C_{\mathrm{I}} + C_{\mathrm{II}} + C_{\mathrm{III}} + C_{\mathrm{IV}} + C_{\mathrm{V}}. \tag{9.8}$$

(C_{I}) The contribution, C_{I}, which is the contribution we wish to identify, is identical with the contribution of the corresponding piece of X^{4k}.

(C_{II}) The contribution, C_{II}, is locally computable and will be shown to vanish.

(C_{III}) We have $C_{\mathrm{III}} = 0$, since the metric is a product in the u-direction and hence, there is a local orientation reversing isometry.

(C_{IV}) C_{IV} is given by the η-invariant of the cross-section $a \times \Sigma(Y)$,

$$C_{\mathrm{IV}} = \eta(a \times \Sigma(Y), \rho_a). \tag{9.9}$$

In fact $C_{\mathrm{IV}} = 0$ because there is a local orientation reversing isometry (see (9.5)). (Actually we will not need this.)

(C_{V}) C_{V} is given by minus the η-invariant of the cross-section $b \times \Sigma(Y)$,

$$C_{\mathrm{V}} = -\eta(b \times \Sigma(Y), \rho_b). \tag{9.10}$$

If we still disregard the lack of smoothness and accept the vanishing of C_{II}, we get (in the limit, $\delta \to 0$)

$$C_{\mathrm{I}} = \eta(b \times \Sigma(Y), \rho_b) - \eta(a \times \Sigma(Y), \rho_a). \tag{9.11}$$

Now integrate the formula derived in §2 for the variation of the η-invariant to compute the right-hand side of (9.11). The cones lying in I contribute

$$\int_{u_1}^{u_2} \eta_1. \tag{9.12}$$

The cones lying in III contribution 0 since the metrics on these cones do not change ($\eta_1 \equiv 0$ there); see (9.5).

Finally we get the usual local contribution, D_{II}, from the interior which will be shown to vanish in the limit, $\delta \to 0$. Thus, we obtain

$$\int_{u_1}^{u_2} \eta_1 = C_{\mathrm{I}}. \tag{9.13}$$

This is our basic result. It immediately yields Theorem 4.27 in our context.

To see the vanishing of C_{II}, observe that where it is smooth, the metric has local orientation reversing isometries (reflection in the u-axis for $u < a + b - w$ and reflection in the w-axis for $u > a + b - w$). Moreover, the metric also has a local orientation reversing isometry at the diagonal $u + w = a + b$, namely, reflection in the diagonal. The metric can be smoothed compatibly with these local isometries and in the limit, $\delta \to 0$, the smoothing plays no role at all (even at the ends of the diagonal) since the limit metric is actually smooth at the diagonal.

Now consider the boundary of regions I and II. The metric is not smooth because we pass from a canonical metric to a product metric as the r-coordinate changes to the w-coordinate. In the limit, $\delta \to 0$, smoothing this produces a local term which would, in general, be of the form

$$\delta \cdot a(\delta). \tag{9.14}$$

In our situation however, in fact

$$a(\delta) = O(\delta) \tag{9.15}$$

because the limiting metric has a local orientation reversing isometry, reflection in the u-axis. Thus, even though the volume of this boundary component is $O(\delta^{-1})$ we get no contribution in the limit.

At the boundary of regions II and III reflection in the u-axis actually is a local orientation reversing isometry (with which the smoothing can be made compatible).

Similiar arguments apply at the boundaries between region II and regions IV and V.

Finally consider the term D_{II}. If we ignore the nonsmoothness at $u + w = a + b$, this term vanishes because away from the diagonal the variation is either compatible with reflection in the u-axis (for $u < a + b - w$) or is constant ($u > a + b - w$).

Near the diagonal, we can smooth the variation (or, more precisely, the induced metric on its graph) in a strip who's width is 1 when measured with respect to the metric. We can do this in such a way that the first derivatives are $O(\delta)$ and higher derivatives are $O(1)$. Since the diagonal has length $O(\delta^{-1})$ this would in general produce a contribution to the formula for $\dot{\eta}$ which remains bounded independent of δ. But since the limit metric possesses an additional local orientation reversing isometry (reflection in the w-axis), the contribution vanishes in the limit. This completes the argument.

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