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A GENERALIZATION OF BERGER'S THEOREM ON ALMOST 1/4-PINCHED MANIFOLDS. II

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1. Introduction

Let (M^n, g) be a compact, smooth Riemannian manifold and let $K(M, g)$, $d(M, g)$, $i(M, g)$, and $(\tilde{M}, \tilde{g})$ denote its sectional curvature, diameter, injectivity radius, and Riemannian universal cover, respectively. In this paper, we investigate Riemannian manifolds of positive sectional curvature. For normalization, we take $K(M, g) \geqslant 1$. Let $S^n(1)$, $\mathbf{R}P^n(1)$, $\mathbf{C}P^n$, $\mathbf{H}P^n$, $\mathrm{Ca}P^2$ denote the standard sphere of radius one, the projective spaces on real, complex numbers, and quaternions, and the Cayley plane with their standard metrics, respectively. $S^n(1)$ and $\mathbf{R}P^n(1)$ have constant sectional curvature 1, while the rest have $1 \leqslant K(\cdot) \leqslant 4$. The diameter of $S^n(1)$ is π, and the rest have diameter $\pi/2$. These Riemannian manifolds, except $\mathbf{R}P^n(1)$, are all of the compact simply connected symmetric spaces of rank 1, up to a constant factor of the metric.

If $K(M, g) \equiv 1$, then $(\tilde{M}, \tilde{g})$ is isometric to $S^n(1)$ [37, p. 69]. By the classical Sphere Theorem [1], [26], [7]: If $1 \leqslant K(M, g) < 4$, then $\tilde{M}$ is homeomorphic to S^n. This result is optimal by the examples above. In [1], M. Berger proved the rigidity theorem: If $1 \leqslant K(M, g) \leqslant 4$, then either $\tilde{M}$ is homeomorphic to S^n or $(\tilde{M}, \tilde{g})$ is isometric to a symmetric space of rank 1. Recently, M. Berger obtained that for even n, there exists a universal constant $\varepsilon(n) > 0$ depending only on n such that if $1 \leqslant K(M^n, g) \leqslant 4 + \varepsilon(n)$, then either $\tilde{M}^n$ is homeomorphic to S^n or diffeomorphic to $\mathbf{C}P^{n/2}$, $\mathbf{H}P^{n/4}$, or $\mathrm{Ca}P^2$ [2].

Some generalizations of the above were given involving the diameter of (M, g). Bonnet: If $K(M, g) \geqslant 1$, then $d(M, g) \leqslant \pi$ [7, p. 27]. The rigidity for the maximal diameter is obtained by Toponogov: If $K(M, g) \geqslant 1$ and

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$d(M, g) = \pi$, then (M, g) is isometric to $S^n(1)$ [7, p. 100]. Grove and Shiohama generalized the Sphere Theorem: If $K(M, g) \geqslant 1$ and $d(M, g) > \pi/2$, then M is homeomorphic to a sphere [21]. Gromoll and Grove showed that Berger's rigidity theorem can also be generalized [15]–[18]: If $K(M^n, g) \geqslant 1$ and $d(M, g) = \pi/2$, then either (i) M^n is homeomorphic to S^n, isometric to $\mathbf{C}P^{n/2}$, $\mathbf{H}P^{n/4}$, (ii) M^n is simply connected and has the cohomology ring structure as of $\mathrm{Ca}P^2$, or (iii) M^n is not simply connected, with $(\tilde{M}, \tilde{g})$ being isometric to $S^n(1)$ or $\mathbf{C}P^{n/2}$.

In this paper, we will prove some results which extend [15], [17] in the cohomological sense, and generalize [2]. These results were announced in [11].

The author wishes to thank U. Abresch, D. Gromoll, K. Grove, W. Meyer, and W. Ziller for helpful discussions and bringing to his attention that the limit metric is C^1. S. Peters proves that the limit metric is $C^{1,\alpha}$ in the general case in [32]. Using similar methods we will give a proof of the limit metric being C^1 in a particular sense (see §5.0) for the completeness of our paper, and obtain further properties which will be used in the proof of the main results.

2. Main results

Theorem I. *Let $n \geqslant 2$, $K \geqslant 4$, and $\varepsilon_0 > 0$ be given. There exists $\delta_0 = \delta_0(K, n, \varepsilon_0) > 0$ such that for any n-dimensional smooth Riemannian manifold (M, g) with*

(i) $1 \leqslant K(M, g) \leqslant K$,

(ii) $d(M, g) > \pi/2 - \delta_0$, *and*

(iii) $i(M, g) > \varepsilon_0$, *if n is odd*,

we have either

(a) *M is homeomorphic to a sphere, or*

(b) *$\pi_1(M, p) = 0$ and $H^*(M, Z)$ is a truncated polynomial ring with one generator in $H^\lambda(M, Z)$, where $n = k\lambda$, n is even, $k \in \mathbf{N}^+$, $k \geqslant 2$, $\lambda = 2, 4$ or 8, and if $\lambda = 8$ then $k = 2$ and $n = 16$, or*

(c) *$\pi(M, p) \neq 0$ and there exists a C^∞-Riemannian metric g' on M with $K(M, g') \geqslant 1$ and $d(M, g') = \pi/2$, that is $(\tilde{M}, \tilde{g}')$ is isometric to $S^n(1)$ or $\mathbf{C}P^{n/2}$.*

Remarks. (1) If n is even, condition (iii) is irrelevant and $\delta_0 = \delta_0(K, n)$ since $i(M, g) \geqslant \pi/(2\sqrt{K})$. By the work of Cheeger [6], condition (iii) can be replaced with a lower bound for the volume of M, for all n.

(2) (b) is not the best possible conclusion which should be "diffeomorphic to $\mathbf{C}P^{n/2}$, $\mathbf{H}P^{n/4}$, or $\mathrm{Ca}P^2$." Under a stronger hypothesis this can be obtained (Theorems IIA and B).

(3) In (c), if n is even, then there are at most two possibilities:

(i) (M, g') is isometric to $\mathbf{R}P^{2s}(1)$, $n = 2s$, $s \in \mathbf{N}^+$.

(ii) (M, g') is isometric to $\mathbf{C}P^s/I$, $n = 2s$, $s \geqslant 3$ and odd, where I is an orientation reversing involution of $\mathbf{C}P^s$; there is only one such space up to isometry.

If n is odd, then $(\tilde{M}, \tilde{g}')$ is isometric to $S^n(1)$ (see [17] and [37]). Hence the diffeomorphism types of M are completely determined.

Theorem IIA. *Let $n \geqslant 4$, $K \geqslant 4$ be given. There exists $\delta_1 = \delta_1(K, n) > 0$ such that for any n-dimensional smooth Riemannian manifold (M, g) with*

(i) $1 \leqslant K(M, g) \leqslant K$,

(ii) $H^*(M, \mathbf{Z}) \cong \mathbf{Z}[x]/x^{k+1}$, $k \geqslant 3$, $x \in H^\lambda(M, \mathbf{Z})$ *for* $\lambda = 2$ *or* 4, *and* $\pi_1(M, p) = 0$, *and*

(iii) $\exists p_1, p_2, p_3 \in M$ *such that* $d(p_i, p_j) > \pi/2 - \delta_1(K, n)$, $\forall 1 \leqslant i < j \leqslant 3$, *then we have M diffeomorphic to* $\mathbf{C}P^k$ *or* $\mathbf{H}P^k$.

Theorem IIB. *Let $n \geqslant 4$, $K \geqslant 4$ be given. There exists $\delta_1 = \delta_1(K, n) > 0$ such that for any n-dimensional smooth Riemannian manifold (M, g) with*

(i) $1 \leqslant K(M, g) \leqslant K$,

(ii) $\pi_1(M, p) = 0$ *and* $H^*(M, \mathbf{Z}) \cong \mathbf{Z}[x]/x^3$, $x \in H^\lambda(M, \mathbf{Z})$ *for* $\lambda = 2$, 4, *or* 8, *and*

(iii) $d(M, g) \geqslant \pi/2 - \delta_1$ *and* $\forall p_1 \forall p_2 \exists p_3$ *such that* $d(p_1, p_2) > \pi/2 - \delta_1$ *implies that* $d(p_i, p_3) > \pi/2 - \delta_1$ *for* $i = 1$ *and* 2,
then we have M diffeomorphic to $\mathbf{C}P^2$, $\mathbf{H}P^2$, *or* $\mathrm{Ca}P^2$.

Corollary I. *Let $n \geqslant 2$, $K \geqslant 1$ be given. $\exists \delta_1 = \delta_1(K, n) > 0$ such that any smooth Riemannian manifold (M, g) with $1 \leqslant K(M, g) \leqslant K$ and $i(M, g) > \pi/2 - \delta_1$ is homeomorphic to a sphere or diffeomorphic to* $\mathbf{R}P^n$, $\mathbf{C}P^k$, $\mathbf{H}P^k$, *or* $\mathrm{Ca}P^2$.

The proof of Corollary I follows from Theorems I, IIA, IIB, [10, Theorem 2], [21] and [37]. Obviously [2] is a corollary of Corollary I.

Corollary II (*see* [17]). *Let (M, g) be a C^∞-Riemannian manifold with $K(M, g) \geqslant 1$, $d(M, g) = \pi/2$, $\pi_1(M, g) = 0$, $H^*(M, \mathbf{Z}) \cong \mathbf{Z}[x]/x^3$, $x \in H^8(M, \mathbf{Z})$. $\forall p_1 \forall p_2 \exists p_3 \in M$, such that $d(p_1, p_2) = \pi/2$ implies that $d(p_1, p_3) = d(p_2, p_3) = \pi/2$, if and only if (M, g) is isometric to $\mathrm{Ca}P^2$ with its standard metric.*

Corollary II does not follow from the statement of Theorem IIB but it follows from its proof.

The main idea in proving these theorems is taking a sequence of C^∞-Riemannian metrics (M, g_m) with $K(M, g_m) \geqslant 1$ and $d(M, g_m) \nearrow \pi/2$, obtaining a limit metric which is not necessarily smooth and repeating a proof modelled on [15], [17]. The limit metric is $C^{1,\alpha}$ a priori and there are examples which are not C^2 in the general context [32]. Even though the first variation formula is still valid, the second variation and Jacobi field techniques fail. The proof of [15], [17] is for C^∞ metrics; so, although the main idea and steps of our proof are as in [15], [17], most of their proofs and even the proofs of some basic facts of Riemannian geometry have to be modified or changed completely.

In §3, we give the basic notation and definitions. The properties of the limit metric are developed in §§4–5. In §4, we give the proofs of basic results, and in §5, differentiability of the metric and local properties are investigated. §6 contains the proof of Theorem I. The nonsmoothness of the limit metric effects especially the proof of Theorem 6.10. The "differentiability of the metric in a particular sense" is used in constructing a local parallel translation to obtain the smoothness of the fibers of some fiber bundles in 6.17. We could not obtain the smoothness of these fiber bundles in 6.23, and this is the point where the arguments fail to obtain results on the diffeomorphism types in the general context. However with stronger hypotheses, results on diffeomorphism types can be obtained (Theorems IIA and B). §7 contains the proofs of them.

3. Basic notation

In this text, M^n denotes a compact smooth n-dimensional manifold with no boundary. If (M, g) is a C^∞-Riemannian manifold, $K(M, g)$ denotes its sectional curvature.

Let (M, g_s) be either a C^∞ or C^0-limit Riemannian metric. $d_s(p, q) = d(p, q; g_s)$ denotes the distance function of g_s, $d(M, g_s)$ denotes the diameter of (M, g_s). $i(p, M; g_s)$ and $i(M, g_s)$ denote the injectivity radius at a point p of M or of the manifold with respect to g_s. Given a C^1-submanifold A of (M, g_s), then TM, $U(M, g_s)$, $UN(A, g_s)$, $UT(A, g_s)$ and $U(M, g_s)\,|\,A$ denote the tangent bundle, unit sphere bundle, unit normal bundle to A, unit tangent bundle to A, and unit tangent bundle of M restricted to A, where inner product is taken by g_s. This g_s will be dropped only when it is g_0, i.e., the limit metric.

For any metric space (X, d), $p \in X$, $A \subseteq X$, $r \in [0, \infty)$, we define $B(p, r, X, d) = \{x \in X \mid d(x, p) < r\}$ and $N(A, r, X, d) = \{x \in X \mid d(x, A) < r\}$, with $\overline{B}(p, r, X, d)$ and $\overline{N}(A, r, X, d)$ their closures, respectively.

Unless otherwise stated, a normal minimal geodesic γ from p to q with respect to g satisfies $0 \leqslant d(p, \gamma(t); g) = t \leqslant d(p, q; g)$. In this case we say that γ is a $\mathrm{mg}(p, q; g)$. If γ is the only such geodesic, then it is the $\mathrm{umg}(p, q; g)$. The set of all $\mathrm{mg}(p, q; g)$ is $\mathrm{MG}(p, q; g)$. If γ is any C^1 curve,

$l(\gamma, g)$ denotes its length with respect to g. For any $X \subseteq (M, g)$, $p \in M$, γ is a mg$(p, X; g)$ means that γ is a mg(p, q) for some $q \in \overline{X}$ with $l(\gamma, g) = d(p, X; g)$.

Any letter of dependence may be dropped if there is no ambiguity.

4. Limit metric and its properties

In this section we refer to [20, particularly Chapters 3, 5, and 8] for all notation, definitions, and background.

Let $S(n, \Lambda, \varepsilon_0, D)$ be the collection of all compact smooth n-dimensional Riemannian manifolds (M, g) with $|K(M, g)| \leqslant \Lambda^2$, $d(M, g) \leqslant D$ and $i(M, g) \geqslant \varepsilon_0$ for given fixed D, ε_0, $\Lambda > 0$. Define V_Λ to be the class of Riemannian manifolds (M, g) of dimension n, where M is of class $C^{1,1}$ (that is there is a notion of differentiable functions with Lipschitz differential) with continuous metric tensor, and the distance functions d_x: $M \to [0, D]$ defined by $d_x(y) = d(x, y; g)$ are of class $C^{1,1}$ (locally and excluding x) with their derivatives Λ-Lipschitz $\forall x \in M$.

If we combine some of the results in [20, 5.3, 8.23, 25, 28 on pp. 65, 123, 125, 129] (for other proofs also see Peters [32]), we obtain

4.1.0. Theorem (*Gromov* [20], *also see* [25], [32]). *$S(n, \Lambda, \varepsilon_0, D) \subseteq V_{\Lambda'}$ for some Λ' depending on Λ. The convergence of metric structures on $S(n, \Lambda, \varepsilon_0, D)$ in the senses of Hausdorff and Lipschitz coincide. The space of pointed Riemannian manifolds $(M, g; p)$, where $(M, g) \in V_\Lambda$, $d(M, g) \leqslant D$, $i(M, g) \geqslant \varepsilon_0$, is compact with respect to Hausdorff and Lipschitz metrics. Hence, given a sequence of pointed C^∞-Riemannian manifolds (M_m^n, g_m, p_m) in $S(n, \Lambda, \varepsilon_0, D)$, there exists a convergent subsequence with Hausdorff limit $(M_0, g_0, p_0) \in V_{\Lambda'}$, and for sufficiently large m, M_m is homeomorphic to M_0.*

4.1.1. As observed in the proof of 8.28 of [20]:

(a) $(M_m, g_m, p_m) \to (M_0, g_0, p_0)$ in the sense of Lipschitz and Hausdorff means that (M_m, g_m, p_m) converges to (M_0, g_0, p_0) as metric spaces in the sense of Hausdorff, and for $r < \varepsilon_0$, $B = B(0, r) \subseteq \mathbf{R}^n$ is furnished with a Riemannian metric g_m and a distance function d_m given by the identification with $B(p_m, r; M_m)$ via normal coordinates. B is also furnished with a limit distance function d_0 satisfying the fact that d_m/d_0 converges to 1 uniformly on $(B \times B) - \operatorname{diag}(B)$.

(b) M_0 is an n-dimensional $C^{1,1}$ manifold since $\inf\{i(M_m, g_m) \mid m \in \mathbf{N}^+\} \geqslant \varepsilon_0$.

(c) $d_x^0 : B_0 - \{x\} \to \mathbf{R}^+$ is of class $C^{1,1}$, where $B_0 = B(x, r, M_0, d_0)$, $r < \varepsilon_0$, and $d_x^0(y) = d_0(x, y)$. In fact, $d_x^m : B_m - \{x\} \to \mathbf{R}^+$ converges uniformly to

d_x^0 in the C^1 sense on the common identification to $B - \{x\}$ as in (a). Hence $d_m/d_0 \to 1$ in the C^1 sense.

(d) The metric tensor g_0 of M_0 is continuous and $g_m \to g_0$ uniformly on some coordinate charts (see §5.0).

(e) $\forall v_1, v_2 \in TM_p - \{0\}$, $\arccos(g_m(v_1, v_2)/(\|v_1\|_m \cdot \|v_2\|_m)) = \measuredangle_m(v_1, v_2)$. Once the convergent subsequence $g_m \to g_0$ is taken, then $\lim_{m\to\infty} g_m(v_1, v_2) = g_0(v_1, v_2)$, and hence

$$\lim_{m\to\infty} \measuredangle_m(v_1, v_2) = \measuredangle_0(v_1, v_2).$$

4.1.2. Since the curvature is bounded uniformly, second derivatives of the distance functions are uniformly bounded, and the differentials dg_m are uniformly bounded in some coordinates. The distance functions and g_m are bounded on B. Using the Arzela-Ascoli Theorem, convergent subsequences are extracted [20]. (See §§5.0, 5.1.)

4.1.3. We are concerned with compact manifolds, hence by taking a finer subsequence, and omitting base points, we can work with $(M_m, g_m) \to (M_0, g_0)$.

4.1.4 (see [32]). Let M be in the same diffeomorphisms class $[M]$. In this case $(M, g_m) \to (M, g_0)$ in the sense of (4.1.0) means that on a fixed C^∞ manifold M, there is a sequence of C^∞-Riemannian metrics g_m which are converging to a C^0-Riemannian metric g_0 uniformly with the stated properties above. *Throughout this paper, the notation $g_m \to g_0$ means that the convergence is in this sense.*

4.2.1. Given p_1, p_2 in (M, g_0), there exists a curve γ from p_1 to p_2 such that the length of γ with respect to g_0 is equal to $d_0(p_1, p_2)$ (see [20], Chapter I). γ can be parametrized such that $d_0(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$. Such a curve is called a normal minimal geodesic of g_0. A curve is called geodesic if it is minimal locally.

4.2.2. Lemma. *Since M_0 is compact with no boundary, any geodesic is C^1.*

4.2.3. *Proof* (see 5.9, 5.10). Let $\varepsilon \ll \varepsilon_0$ and $\gamma: (-\varepsilon, \varepsilon) \to M_0$ be a normal geodesic. Since $d_p^m \to d_p^0$ in the C^1 sense locally, $\nabla d_p^m \to \nabla_p^0$ uniformly on $B(p, \varepsilon) - \{p\}$ $\forall p \in M$. d_p^m satisfies the first variation formula locally. It follows that γ has to be tangent to ∇d_p^0, otherwise $d_0(p, \gamma(t)) = t - t_0$, where $\gamma(t_0) = p$, $\varepsilon > t > t_0 > -\varepsilon$, would not increase linearly with constant derivative 1.

4.3. Definitions. (1) In the simply connected space form of constant sectional curvature κ, define $\rho(\alpha; a, b; \kappa)$ to be the distance between the two points which are end points of two minimal geodesics of lengths a and b starting from the same point and with an angle of α between their initial tangents, where $0 \leqslant \alpha \leqslant \pi$, $a, b \geqslant 0$, and if $\kappa > 0$ then $a, b \leqslant \pi/\sqrt{\kappa}$.

(2) $S^n(\kappa)$ denotes the sphere of radius $\kappa^{-1/2}$ with the standard metric, where $\kappa > 0$.

(3) In a geodesic triangle in $S^2(1)$, with sides of length a, b, and c, all $\leqslant \pi$, define $\alpha(a, b, c)$ to be an angle between the sides of length b and c.

4.4. Lemma. *There exists a unique* $\exp: TM \to (M, g_0)$ *compatible with* g_0, *that is it takes rays of* TM_p *emanating from* 0 *to geodesics of* (M, g_0) *from* p, $\forall p \in M$, *and all geodesics are obtained in this fashion.*

Proof. Assume 5.9, 5.10.

4.4.1. Let $g_m \to g_0$, as in 4.1. Let $r_0 \in M$ be such that $r_0 \in B(p_0, \varepsilon_0, M, g_0)$, where p_0 is as in 4.1.1, and $r_0 \in B$. Let r_m represent r_0 in (B, g_m). For sufficiently large m_0 and small $R \ll \varepsilon_0$, $\exp_{r_m, g_m}: B' = B(0, R, TB_{r_0}, g_0) \to B = (B, g_m)$ is defined $\forall m \geqslant m_0$. $\|d(\exp_{r_m})\|_{C^0} \leqslant C(\Lambda, R)$ on B'. Hence $f_m = \exp_{r_m, g_m}$ is a bounded and equicontinuous family on B'. There exists a subsequence which we denote by f_s converging uniformly to a continuous function f_0: $B' \to B$. We define $\exp_{r_0}: B' \to (B, g_0)$ to be f_0.

4.4.2. Let $q_0 \in B - \{r_0\}$ such that $d_0(q_0, r_0) < R$. $d_s \to d_0$, so, for sufficiently large s, $q_0 \in f_s(B')$, and $\exists v_s \in B'$ with $q_0 = f_s(v_s)$ and $\gamma_s(t) = f_s(v_s t/\|v_s\|_s)$ is a mg$(r_0, q_0; g_s)$. v_s has to converge to a unique $v_0 \in B'$, since $f_s \to f_0$ uniformly. $v_0 \neq 0$ by $q_0 \neq p_0$, $d_s/d_0 \to 1$ and $i(M, g_s) \geqslant \varepsilon_0$. $\gamma_0(t) = f_0(tv_0/\|v_0\|_0)$ is the limit of the mg$(r_0, q_0; g_s)$'s $\gamma_s(t)$, hence its length is $d_0(r_0, q_0)$ between r_0 and q_0 and it is mg$(r_0, q_0; g_0)$ (see 5.9, 5.10). This shows that f_0 maps rays from 0 in B' to minimal geodesics from r_0 in (B, g_0) locally and f_0 is onto $B(r_0, R, B, g_0)$.

4.4.3. Let $\gamma(t)$ be a normal geodesic in (B, g_0) such that $\gamma(0) = r_0$. By 4.2.3, γ is C^1 and tangent to $\nabla d^0_{r_0}$ which is Lipschitz (see 4.1.1(c)). Even though $\nabla d^0_{r_0}(r_0)$ is not defined, $\gamma'(t)$ is well defined $\forall t \geqslant 0$. By the uniqueness of the solutions of first order ODE given by Lipschitz functions, and f_0 being onto locally, we have $f_0(t\gamma'(0)) = \gamma(t)$. Hence:

4.4.4. Around r_0 all geodesics from r_0 are only given by f_0. f_0 is well defined, it does not depend on the choice of the convergent subsequence of f_m. In fact once $g_m \to g_0$ is fixed, then $f_m \to f_0$; in order to be compatible with g_0, f_0 is unique and all subsequences of f_m converge to f_0.

4.4.5. Suppose given $w_1, w_2 \in B' - \{0\}$ with $\|w_1\|_0 = \|w_2\|_0$. For sufficiently large m,

$$d_m(f_m(v_1), f_m(v_2)) \geqslant \tfrac{1}{2} d_0(f_0(w_1), f_0(w_2)) := C_1.$$

By Toponogov's Theorem [7, p. 42],

$$\measuredangle_m(\dot{\gamma}^1_m(0), \dot{\gamma}^2_m(0)) \geqslant C_2(C_1, \Lambda, \|w_1\|_0) \quad \text{if } m > 0,$$

where $\gamma_m^i(t) = \exp_{r_m}(tw_i/\|w_i\|_m)$ for $m \geqslant 0$. If $C_1 > 0$, then $C_2 > 0$ and $\measuredangle_0(\dot\gamma_0^1(0), \dot\gamma_0^2(0)) \neq 0$ (see 5.10). Hence any geodesic is uniquely determined locally with its initial point and tangent vector. This is true for all r_0 (see 4.1.3) since M is compact, $\partial = \varnothing$. It follows that $\exp_0$: $TM \to (M, g_0)$ is globally defined, it is unique, and all normal geodesics of (M, g_0) are obtained from it.

4.4.6. Remark. exp: $TM \to (M, g_0)$ is continuous, but it may not be differentiable. It is differentiable only in the radial directions. Also $d(\exp_{r_m})(r_m) =$ Identity $\forall m$, this makes 4.4.5 possible. $f_m \to f_0$ in the C^0 sense, not C^1.

4.5. Toponogov lemma. *Suppose given $p, q, r \in (M, g_0)$, a* $\mathrm{mg}(p, q; g_0)$ γ, *and a* $\mathrm{mg}(p, r; g_0)$ θ, *where $(M, g_m) \to (M, g_0)$ as in* 4.1 *with $K(M, g_m) \geqslant \kappa$ $\forall m \geqslant 1$. Let d_m $\forall m \geqslant 0$ be the associated distance function to g_m. Then*

$$d_0(q, r) \leqslant \rho\big(\measuredangle_0(\gamma'(p), \theta'(p));\, d_0(p, q),\, d_0(p, r); \kappa\big).$$

4.5.1. Remarks. (1) If g_0 is C^∞, then this is the classical Toponogov Theorem [7, p. 43].

(2) A local version of this lemma is given in [2, p. 138, Lemma 3]. We do not assume that the triangle obtained by attaching a $\mathrm{mg}(q, r; g_0)$ lies in $B(p, \varepsilon_0, M, g_0)$.

4.5.2. *Proof.* Define $q_s = \gamma(d_0(p, q) - 1/s)$ and $r_s = \theta(d_0(p, r) - 1/s)$ for sufficiently large s. Assume 5.9, 5.10.

Let s be fixed. Consider γ_m and θ_m to be any $\mathrm{mg}(p, q_s; g_m)$ and any $\mathrm{mg}(p, r_s; g_m)$, respectively. $l(\gamma_m, g_m) = d_m(p, q_s)$. Since $d_m/d_0 \to 1$ in the C^1 sense when $d_m, d_0 \leqslant \varepsilon_0$, and uniformly on M, given $\delta > 0$, $\exists N = N(\delta)$ such that $\forall m \geqslant N(\delta)$

$$\left|\frac{l(\gamma_m, g_m)}{l(\gamma_m, g_0)} - 1\right| < \delta \quad \text{and} \quad \left|\frac{d_m(p, q_s)}{d_0(p, q_s)} - 1\right| < \delta.$$

Hence, $\forall m \geqslant N(\delta)$,

$$\left|\frac{l(\gamma_m, g_0)}{d_0(p, q_s)} - 1\right| < \frac{2\delta}{1 - \delta}.$$

Therefore $l(\gamma_m, g_0) \to d_0(p, q_s)$ as $m \to \infty$. Hence, we take a convergent subsequence of γ_m converging to γ_0, a $\mathrm{mg}(p, q_s; g_0)$. γ_0 has to coincide with γ between p and q_s; since

$$l(\gamma_0, g_0) + l\left(\gamma\left|\left[d_0(p, q) - \frac{1}{s},\, d_0(p, q)\right], g_0\right.\right) = d_0(p, q),$$

4.2.2 implies that $\gamma_0'(q_s) = \gamma'(q_s)$ and 4.4.5. This shows that γ_0 does not depend on the choice of the convergent subsequence of γ_m. By 5.10 and 4.4.5, $\gamma_m'(p) \to \gamma'(p)$. Similar results can also be obtained for θ_m. By 4.1, $g_m \to 1$

uniformly as a quadratic form restricted to $U(M, g_0)$. Hence, $\forall\delta$, $0 < \delta \ll 1$, $\exists N = N(\delta)$ such that $\forall m \geqslant N(\delta)$,

$$\measuredangle_0(\gamma_m'(p), \theta_m'(p)) \leqslant (1+\delta)\measuredangle_0(\gamma'(p), \theta'(p))$$

and

$$\measuredangle_m(v, v') \leqslant (1+\delta)\measuredangle_0(v, v') \quad \forall v, v' \in TM_p - \{0\}.$$

By Toponogov's Theorem [7, p. 43]:

$$\begin{aligned} d_m(q_s, r_s) &\leqslant \rho\big(\measuredangle_m(\gamma_m'(p), \theta_m'(p)); d_m(p, q_s), d_m(p, r_s); \kappa\big) \\ &\leqslant \rho\big((1+\delta)^2 \measuredangle_0(\gamma'(p), \theta'(p)); d_m(p, q_s), d_m(p, r_s); \kappa\big). \end{aligned}$$

Hence $\forall\delta > 0$,

$$d_0(q_s, r_s) \leqslant \rho\big((1+\delta)^2 \measuredangle_0(\gamma'(p), \theta'(p)); d_0(p, q_s), d_0(p, r_s); \kappa\big),$$

$$d_0(q_s, r_s) \leqslant \rho\big(\measuredangle_0(\gamma'(p), \theta'(p)); d_0(p, q_s), d_0(p, r_s); \kappa\big).$$

Now let $s \to \infty$ to obtain

$$d_0(q, r) \leqslant \rho\big(\measuredangle_0(\gamma'(p), \theta'(p)); d_0(p, q), d_0(p, r); \kappa\big).$$

4.6.1. Corollary. *Let $p, q \in (M, g_0)$, γ be a* $\mathrm{mg}(q, p; g_0)$, *$v \in TM_q - \{0\}$ be such that $\measuredangle_0(v, \gamma'(q)) \leqslant \pi/2 - \varepsilon$, where $\varepsilon > 0$, and (M, g_0) be as in* 4.5. *There exists $0 < \delta = \delta(\varepsilon, \kappa) < \varepsilon_0$ such that $d_0(p, \exp_{q,g_0} tv) < d_0(p, q)$ $\forall t$, $0 < t < \delta$.*

4.6.2. Corollary. *Let $p, q \in (M, g_0)$ and q be a local maximum for $d_0(p, \cdot)$. Given any $v \in TM_q$, $\exists mg(q, p; g_0)$ γ such that $\measuredangle_0(\gamma'(q), v) \leqslant \pi/2$.*

One proves 4.6.2 by using 4.6.1 in [7, p. 107].

4.6.3. Remark. The first variation formula is valid on such (M, g_0) by [2]. We obtain the above results as a consequence of a stronger result, 4.5. In fact Toponogov Lemma 4.5 implies stronger results such as 4.7. A similar form of 4.7 can be proved by using a corollary of Rauch II, [7, p. 31], which is a second variational technique in the C^∞ case.

4.7. Lemma. *Let (M, g_0) be as in* 4.1 *and* 4.5 *with $\kappa = 1$. Let $p, q \in (M, g_0)$ and $v \in U(M, g_0)_p$ with $d_0(q, p) \leqslant d_0(q, \exp_{p,g_0} tv) \leqslant \pi/2$, $\forall t \in [-\delta, \delta]$ for some δ, $0 < \delta \ll d_0(q, p)$. Let $r = \exp_p \delta v$ and γ, θ be* $\mathrm{mg}(q, p; g_0)$ *and* $\mathrm{mg}(q, r; g_0)$ *respectively, such that $0 < \measuredangle_0(\gamma'(q), \theta'(q)) \ll \pi/2$. Define $w(s) \in U(M, g_0)_q$ to be the unique vector with $0 \leqslant s \leqslant \pi/2$, $\measuredangle_0(\gamma'(q), w(s)) = s$, and $\measuredangle_0(\theta'(q), w(s)) = |s - \measuredangle_0(\theta'(q), \gamma'(q))|$. Then $\forall s$, $0 \leqslant s \leqslant \pi/2$,*

$$d_0\big(\exp_q \delta w(s), \exp_p \delta v\big) < d_0(q, p).$$

4.7.1. *Proof.* By 4.6.1, $\measuredangle_0(v, \gamma'(p)) = \pi/2$. By 4.2.2, $\gamma'(q) \neq \theta'(q)$ and $w(s)$ is well defined. $d_0(p, q) \leqslant d_0(q, r) \leqslant \rho(\pi/2; \delta, d_0(p, q); 1) := a_0$ by 4.5.

$$\measuredangle_0\big(\theta'(q), \gamma'(q)\big) \geqslant \alpha\big(\delta, d_0(q, r), d_0(q, p)\big) \geqslant \alpha\big(\delta, d_0(q, p), a_0\big) := \alpha_0,$$

where the last inequality follows from $\alpha(d_0(p,q),\delta,a_0) < \pi/2$ and $d_0(q,r) \geqslant d_0(q,p)$. $\measuredangle_0(\theta'(q),w(s)) \leqslant |s-\alpha_0| < \pi/2$. By 4.5:

$$\begin{aligned} d_0\big(\exp_q \delta w(s), r\big) &\leqslant \rho\big(\measuredangle_0\big(\theta'(q),w(s)\big);\delta, d_0(q,r);1\big) \\ &\leqslant \rho\big(|s-\alpha_0|;\ \delta,\ d_0(q,r);1\big) \\ &\leqslant \rho\big(|s-\alpha_0|;\ \delta,\ a_0;1\big) \leqslant \rho\big(\tfrac{\pi}{2}-\alpha_0;\ \delta,\ a_0;1\big), \end{aligned}$$

since $0 < \delta \ll d_0(p,q) < a_0$ and $0 < \alpha_0 \ll \pi/2$. On $S^2(1)$ by the second variation formula, $\rho(\pi/2-\alpha_0,:\delta,a_0;1) < d_0(p,q)$.

5. Local properties of the limit metric

5.0. The main purpose of this section is to prove 5.9–5.12. 5.9 and 5.10 are used in §4, and 5.12 in §§6–7. In order to prove these one needs to have that the limit metric g_0 of 4.1 is C^1 in some differentiable coordinate charts. S. Peters obtained Gromov's result [20, 8.28], and showed that in fact g_0 is $C^{1,\alpha}$ in some $C^{2,\alpha}$ coordinate charts by applying the harmonic coordinates and estimates of Jost and Karcher [23] to his proof [31] of finiteness results in [32]. §5 can be read from two viewpoints. The first one assumes the work of Gromov [20] and obtains 5.1–5.5 which give the differentiability notion in a sufficient sense for the rest of the section. For the second viewpoint, it was mentioned in Greene and Wu [14] that the proof of Gromov [20] was unclear, since the equicontinuity of the g_m's in the normal coordinates (4.1.2) would seem to require a uniform bound on the covariant derivatives of the curvature tensor. For this viewpoint, either one repeats Gromov's proof 8.28 [20] (one may also use coordinate charts defined by distance functions) in harmonic coordinates by using the estimates of [23] explained in 5.2, or simply assumes the results of Peters [32] which imply 4.1, 5.1, and 5.5, then considers 5.2–5.4 as a preparation of the harmonic coordinates for 5.9–5.12. At this point, we emphasize that for an arbitrary $C^{1,\alpha}$ metric, the geometric results 5.9–5.12 and 4.4–4.7 may not be valid.

5.1. In the view of 5.0, we may assume that $g_m \to g_0$. Let $p_0 \in M$, and choose R_0 sufficiently small, m_1 sufficiently large so that $\forall m \geqslant m_1$,

$$B(p_0, R_0/2; g_m) \subseteq B(p_0, R_0; g_0) := U_1 \subseteq B(p_0, 2R_0; g_m) \subseteq U_0 \subseteq M,$$

U_0 is open, and there is a C^∞ coordinate chart $x : U_0 \to \mathbf{R}^n$. We may assume that the coordinate chart x (not necessarily normal) can be taken with $\|dg_m\|_{C^0,U_1} \leqslant C(\Lambda, R_0, n)$ by 5.0 and either (i) by [20, 8.28], $U_0 \subseteq B$ of 4.1, [23, p. 34], the relation of d_m and g_m, $|K(M,g_m)| \leqslant \Lambda^2$ (one may also use coordinate charts defined by distance functions), or (ii) by [23] as explained in 5.2, or (iii) by [32, Theorems 1.6 and 4.5]. Let g_{ij}^m be the components of g_m in the coordinates x. Hence $\|\partial(g_{ij}^m)/\partial x_k\|_{C^0,U_1} \leqslant C'(\Lambda, R_0, n)\ \forall m \geqslant m_1$.

5.2.1. Let m be fixed. One follows the construction of the harmonic coordinates in [23]. Almost linear coordinates L_m are constructed around p_0, for a given orthonormal frame at p_0. By Theorem 2.1 of [23, p. 62], $|dL_m - P_r^{-1}| < C(\Lambda, n, \varepsilon_0) \cdot d_m(x, p_0)^2$, where P_r is defined by parallel translation along radial geodesics from p_0. An averaging process gives canonical coordinates which do not depend on the choice of o.n. frame. One finds $h_i^m: B(p_0, R; g_m) \to \mathbf{R}$ with $\Delta(g_m)h_i^m = 0$, $h_i^m \mid \partial B(p_0, R; g_m) = l_i^m \mid \partial B(p_0, R; g_m)$ for $1 \leqslant i \leqslant n$, where $L_m = (l_1^m, l_2^m, \cdots, l_n^m)$, and $\Delta(g_m)$ is the laplacian for some R chosen small enough and independent of m ([23] and 5.2.4). One takes $H_m = (h_1^m, h_2^m, \cdots, h_n^m)$ and $\tilde{g}_m^{ik} = g_m(\operatorname{grad} h_i^m, \operatorname{grad} h_k^m)$. In [23, (5.5), p. 65] it is shown that $H_m: B(p_0, R, g_m) \to \mathbf{R}^n = TM_{p_0}$, $|dH_m - \mathrm{Id}| \leqslant c_7\sqrt{n}\,\Lambda^2 R^2$ on $B(p_0, R; g_m)$, and $\|d\tilde{g}_m\|_{C^{0,2/3}} \leqslant C(\Lambda R, n)\Lambda^2 R^2/\delta^2$ on $B(p_0, (1-\delta)R; g_m)$ [23, Theorem 5.2], where Id is defined by radial parallel translation of g_m.

5.2.2. As in [23, p. 62], $G_m = L_m \circ \exp_{p_0, g_m}: TM_{p_0} \to TM_{p_0}$, $G_m(0) = 0$, and $\forall \varepsilon > 0\, \exists \delta > 0$, where δ does not depend on m, such that $|dG_m - I|_v \leqslant \varepsilon$ if $v \in TM_{p_0}$ and $\|v\|_{g_m} \leqslant \delta(n, \Lambda, \varepsilon_0, \varepsilon)$. By 5.7 we can take I as the identity map of $\mathbf{R}^n$.

5.2.3. $R_1 > 0$ can be chosen sufficiently small and independent of m such that $G_m|B(0, R_1, TM_{p_0}, g_m)$ is 1-1. By taking $R_1 < \varepsilon_0$, $\exp_{p_0, g_m}$ is 1-1 on $B(0, R_1, M, g_m)$ and L_m is 1-1 on $B(p_0, R_1, M, g_m)$. The averaging process does not affect the uniform estimates on the differentials.

5.2.4. $H_m|\partial B(p_0, R, g_m) = L_m|\partial B(p_0, R, g_m)$. We choose $R < R_1$ independent of m in order to make $L_m|\partial B$ 1-1 and H_m of maximal rank (see (5.5) of [23]). Any map from an n-disc which is 1-1 at the boundary and of maximal rank in the interior is not only 1-1 locally but 1-1 on the whole disc. $H_m|B(p_0, R, M, g_m)$ is 1-1, an open map of maximal rank.

5.3.1. Choose R sufficiently small to satisfy the conditions of [23, 5.2] and $R < R_0/2$. If we consider h_i^m to be functions of the local coordinates x, then

$$\sum_{j,k} \frac{\partial}{\partial x_j}\left(\sqrt{g_m} \cdot g_m^{jk} \cdot \left(\frac{\partial h_i^m}{\partial x_k}\right)\right) = 0.$$

Let $R_2 > 0$ and $m_2 \geqslant m_1$ be such that $U_2 = B(p_0, R_2; g_0) \subseteq B(p_0, R, g_m)$ $\forall m \geqslant m_2$ and $R_2 \leqslant R$. Let $U_3 = B(p_0, R_2/2; g_0)$ and U_i' be the corresponding subsets of $\mathbf{R}^n$ via the coordinates $(x_1, x_2, \cdots, x_n)$, $i = 1, 2, 3$.

5.3.2. $\|h_i^m(x)\|_{C^{1,\alpha},U_3'}$ are uniformly bounded $\forall m \geqslant m_2$, independent of m, where $\alpha > 0$ can be chosen to be independent of m. This is an immediate consequence of Theorem 6.5 of [29, p. 284]; since $g_m \to g_0$, the linear equations in divergence form $\Delta(g_m)h_m^i = 0$ are uniformly elliptic, h_i^m are uniformly bounded on U_2, and $d_0(U_3, \partial U_2) \geqslant R_2/2$. This can also be proved by using Theorem 3.1 of [23] and Theorem 4.1 of [29, p. 399].

5.3.3. Hence for fixed i, $\{h_i^m(x)\}_{m=m_2}^{\infty}$ and $\{dh_i^m(x)\}_{m=m_2}^{\infty}$ form equicontinuous families on $\overline{U}_3$. By Arzela-Ascoli Theorem, one extracts a convergent subsequence of $h_i^m(x)$, which is also denoted by h_i^m, such that $h_i^m(x) \to h_i^0(x)$ in the C^1 sense, i.e. $h_i^m(x) \to h_i^0(x)$ and $dh_i^m(x) \to dh_i^0(x)$ uniformly on U_3, where $h_i^0(x)$ is a C^1 map. Hence $H_m \to H_0$ in the C^1 sense, where H_0 is a C^1 map from U_3 into $\mathbf{R}^n$ by using local coordinates $(x_1, x_2, \cdots, x_n)$ on U_3.

5.4.1. Define $H_m'\colon U_3 \to \mathbf{R}^n$ by $H_m'(p) = H_m(p) - H_m(p_0)$ for $m = 0$ or $m \geqslant m_2$. $H_m' \to H_0'$ in the C^1 sense and $H_m'(p_0) = 0$ all m. By (5.5) of [23], $|dH_m(p_0) - \mathrm{Id}| \leqslant C_7\sqrt{n}\,\Lambda^2R^2$. We can choose R in 5.2.4 small enough that $|\det dH_m(p_0)| \geqslant \delta > 0$, independent of m. So $dH_0'(p_0)$ is of maximal rank. Choose $0 < R_3 \leqslant R_2$ such that $H_0'\colon U_4 = B(p_0, R_3; g_0) \to \mathbf{R}^n$ is 1-1 and of maximal rank. $\exists m_3 \geqslant m_2$ such that $\bigcap_{m=m_3}^{\infty} H_m'(U_4)$ contains an open set $V \subseteq H_0'(U_4)$, containing 0. $\exists m_4 \geqslant m_3$ such that $\bigcap_{m=m_4}^{\infty}(H_m'\,|\,U_4)^{-1}(V)$ contains an open set $U \subseteq (H_0'\,|\,U_4)^{-1}(V)$, containing p_0.

5.4.2. Definition. $H_0'\colon U \to V$ is called a LHCS, limit harmonic coordinate system.

5.4.3. Let $g_m^* = ((H_m'\,|\,U_4)^{-1})^* g_m$. Obviously (U_4, g_m) is isometric to $(H_m'(U_4), g_m^*)$ $\forall m \geqslant m_4$. If $(y_1, y_2, \cdots, y_n)$ is the coordinate system for $\mathbf{R}^n$, then as in [23, pp. 60, 61],

$$g_m^*\,|\,H_m'(p) = \sum_{i,j} \tilde{g}_{ij}^m\big(H_m'(p)\big)\, dy_i \otimes dy_j,$$

where $\tilde{g}_m^{jk}(H_m'(p)) = g_m(\operatorname{grad} h_m^j, \operatorname{grad} h_m^k)(p)$ $\forall p \in U_4$. $U_4 \subseteq U_3 = B(p_0, R_2/2; g_0) \subseteq B(p_0, R_2, g_0) = U_2 \subseteq B(p_0, R; g_m)$ and $g_m \to g_0$, so $\exists m_5 \geqslant m_4$ such that $B(p_0, R_2/2, g_0) \subseteq B(p_0, 3R/4; g_m)$ $\forall m \geqslant m_5$. Hence

(i) By (5.8) of [23], $\|dg_m^*\|_{C^0} \leqslant C'(\Lambda, R, n)$ and

(ii) by Theorem 5.2 of [23], $\|dg_m^*\|_{C^{2/3}} \leqslant 16c(\Lambda R, n)\Lambda^2R^2$ on $H_m'(U_4)$ with respect to the distance function d_m^* on $H_m'(U_4)$, using parallel translation of g_m.

5.4.4. Let $v_1, v_2 \in T\mathbf{R}_q^n$ for some $q \in V$. Then

$$g_m^*(v_1, v_2) = g_m\big(H_{m*}'^{-1}(v_1),\ H_{m*}'^{-1}(v_2)\big).$$

Since $H_{m*}'^{-1} \to H_{0*}'^{-1}$ and $g_m \to g_0$ uniformly,

$$\lim_{m\to\infty} g_m^*(v_1, v_2) = g_0\big(H_{0*}'^{-1}(v_1), H_{0*}'^{-1}(v_2)\big).$$

If we define $g_0^* = H_0'^* g_0$, then $g_m^* \to g_0^*$, at least pointwise. Since $(g_m^* | U)$ satisfy the conditions of 4.1, there exists a subsequence which is also denoted by g_m^* converging to g_0^* uniformly and $d_m^* \to d_0^*$ as in 4.1. For sufficiently large m, $d_m^*(p, q) \leqslant 2d_0^*(p, q)$ $\forall p, q \in V$. By [5, Chapter 6], or by 5.7 below and uniform bounds of 5.4.3 on the Christoffel symbols, one can compare the Euclidean and Riemannian parallel translations on V. Hence, there are uniform bounds on $C^{1,2/3}$ norms of g_m^* on V with respect to the fixed metric d_0^*. Finally, dg_m^* forms an equicontinuous and bounded family on V, and by Arzela-Ascoli Theorem one extracts a uniformly convergent subsequence of dg_m^* converging necessarily to dg_0^*. This also proves that g_0^* has to be C^1.

5.5. Theorem (*see* [32] *for a stronger version*). *Given a sequence g_m of C^∞-Riemannian metrics on a C^∞-manifold M^n with $|K(M, g_m)| \leqslant \Lambda^2$, $d(M, g_m) \leqslant D$ and $i(M, g_m) \geqslant \varepsilon_0$, there exists a subsequence g_k of g_m such that*

(i) *$g_k \to g_0$ in the sense of* 4.1.

(ii) *g_0 is C^1 in the following sense; $\forall p \in M$, $\exists$ an open set U containing p, and a C^1 local coordinate chart H_0' on U such that g_0 is C^1 with respect to this coordinate chart. H_0' is a C^1 limit of harmonic coordinate charts H_k' with respect to g_k. $\exists$ an open set $V \subseteq \bigcap_{k=1}^\infty H_k'(U)$ such that if g_k is considered as a metric on $H_k'(U) \subseteq \mathbf{R}^n$ as $g_k^* = H_k'^{-1*} g_k$, then $g_k^* \to g_0^*$ in the C^1 sense on V.*

Proof. See 5.1–5.4.

5.6. Remark. In [32], S. Peters has stronger results on the differentiability of the metric. Our main aim is to prove Proposition 5.12, and 5.5 is sufficient for that.

5.7.1. Let $X(t): \mathbf{R} \to \mathbf{R}^n$ be the solution of the first order linear ODE $X' = AX'$ with $X(0) = v_0$, where $A(t): \mathbf{R} \to \mathbf{R}^{n^2}$ is continuous and $\|A(t)\| \leqslant C$ $\forall t$. $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, |v_0|, C)$ independent of $X(t)$ and $A(t)$ such that $|X(t) - v_0| < \varepsilon$ if $0 \leqslant t < \delta$.

5.7.2. In local coordinates parallel translation is defined by first order linear ODE whose coefficients are Christoffel symbols for a C^∞ metric. The bounds on Christoffel symbols enables us to compare Euclidean and Riemannian parallel translation locally. Obviously the bounds on dg in a coordinate system determines the bounds on the Christoffel symbols of g.

5.8.1. Assume the hypotheses and notations of 5.1–5.5. For the smooth metric g_m^* on V define ${}^m\Gamma_{ij}^l$ as the Christoffel symbols, as usual in terms of partial derivatives of $\tilde{g}_{ij}^m$. Since $g_m^* \to g_0^*$ in the C^1 sense on V, ${}^k\Gamma_{ij}^l \to {}^0\Gamma_{ij}^l$ uniformly, where ${}^0\Gamma_{ij}^l$ is defined in the same way as ${}^k\Gamma_{ij}^l$. ${}^0\Gamma_{ij}^l$ are continuous on

V. Let $\gamma(t)$ be a C^1 curve in V and let $E(t)$ be a C^1 vector field along $\gamma(t)$. We say that $E(t)$ is in $P(V, H_0', \gamma)$ if $E(t) = \sum_i b^i(t)\partial/\partial y_i$ and

$$\frac{db^l}{dt} + \sum_{i,j} \left({}^0\Gamma_{ij}^l b^i\right)(\gamma(t)) \cdot \frac{d\gamma_i}{dt} = 0 \quad \forall l.$$

5.8.2. Remark. H_0' is not shown to be C^2 and in any change of coordinates the Christoffel symbols are involved with second order derivatives of the transition maps. Hence with this information one cannot construct a well-defined parallel translation. However, the transition maps are shown to be C^2 in [32], and hence there is a well-defined parallel translation on (M, g_0).

5.8.3. It follows from the theory of systems of linear ODE, [24, p. 137, Satz 1] or [9, Chapter 10], that for any given C^1 curve γ and $v_0 \in \mathbf{R}^n$, there exists unique $E \in P(V, H_0', \gamma)$ with $E(0) = v_0$.

5.8.4. Lemma. *If $E_1(t)$, $E_2(t) \in P(V, H_0', \gamma)$ then $g_0^*(E_1(t), E_2(t))$ is constant.*

5.8.5. *Proof.* Let $Y_i = \partial/\partial y_i$, $E_1(t) = \sum b^i(t)Y_i$, and $E_2(t) = \sum c^j(t)Y_j$. Then

$$\begin{aligned}\lim_{k\to\infty} \frac{d}{dt} g_k^*(E_1(t), E_2(t)) &= \lim_{k} \frac{d}{dt} \sum_{i,j} b^i c^j \tilde{g}_{ij}^k(\gamma(t)) \\ &= \frac{d}{dt} \sum_{i,j} b^i c^j \tilde{g}_{ij}^0(\gamma(t)) = \frac{d}{dt} g_0^*(E_1(t), E_2(t)),\end{aligned}$$

since $\tilde{g}_{ij}^k \to \tilde{g}_{ij}^0$ in the C^1 sense.

$$\nabla_{\dot\gamma}^{(k)} E_1(t) = \sum_{l=1}^n \left(\frac{db^l}{dt} + \sum_{i,j} {}^k\Gamma_{ij}^l(\gamma(t)) b^i(\gamma(t)) \frac{d\gamma^i}{dt} \right) Y_l$$

and hence

$$\lim_{k\to\infty} g_k^*\left(\nabla_{\dot\gamma}^{(k)} E_1(t), E_2(t)\right) = 0$$

since $E_1(t) \in P(V, H_0', \gamma)$, where $\nabla^{(k)}$ denotes the covariant derivative of the C^∞ metric g_k^*. Also

$$\frac{d}{dt} g_k^*(E_1(t), E_2(t)) = g_k^*\left(\nabla_{\dot\gamma}^{(k)} E_1(t), E_2(t)\right) + g_k^*\left(E_1(t), \nabla_{\dot\gamma}^{(k)} E_2(t)\right).$$

Consequently $(d/dt)g_0^*(E_1(t), E_2(t)) = 0$.

5.9. Let U, g_0, and g_k be as in 5.4 and 5.5, and let K be a compact subset of U with $\operatorname{int}(K) \neq \varnothing$. Consider a sequence of normal geodesics γ_k, where γ_k is with respect to g_k, $k > 0$, and $\gamma_k(0) \in \operatorname{int}(K)$. $\tilde\gamma_k = H_k'\gamma_k$ is a geodesic of the C^∞ metric g_k^*, and $\tilde\gamma_k = (\tilde\gamma_{k,1}, \tilde\gamma_{k,2}, \cdots, \tilde\gamma_{k,n})$ with

$$\frac{d^2}{dt^2}(\tilde\gamma_{k,l}) + \sum_{i,j} {}^k\Gamma_{ij}^l(\tilde\gamma_k(t)) \frac{d}{dt}(\tilde\gamma_{k,i}) \frac{d}{dt}(\tilde\gamma_{k,j}) = 0. \tag{5.9.1}$$

$g_k^*(\tilde{\gamma}_k'(t), \tilde{\gamma}_k'(t)) = 1$ and $\tilde{g}_{ij}^k \to \tilde{g}_{ij}^0$ uniformly, therefore $g_0^*(\tilde{\gamma}_k', \tilde{\gamma}_k')$ and $\|\tilde{\gamma}_k'(t)\|_{\mathbf{R}^n}$ are uniformly bounded. Since ${}^k\Gamma_{ij}^l$ are uniformly bounded independent of k, so are $d^2/dt^2(\tilde{\gamma}_{k,l})$. There exists a subsequence which is also denoted by $\tilde{\gamma}_k$ such that $\tilde{\gamma}_k(0) \to p_0 \in H_0'(K)$ and $\tilde{\gamma}_k'(0) \to T\mathbf{R}^n_{p_0}$ with $g_0^*(v_0) = 1$. Define $F_k(t) = (\tilde{\gamma}_k(t), \tilde{\gamma}_k'(t))$. $\{F_k(t)\}$ is an equicontinuous and bounded family with $F_k(0) \to (p_0, v_0)$. We extract another subsequence $F_k(t)$ such that $F_k(t)$ converges uniformly to $F_0(t)$ which is continuous.

5.10. Lemma. *Given a sequence of geodesics γ_m in (M, g_m), then there exists a subsequence converging to γ_0 in the C^1 sense where γ_0 is a geodesic of (M, g_0).*

5.10.1. *Proof.* By 5.9 and 5.3.3, it is sufficient to prove that γ_0 is a geodesic of g_0 (see 4.4.2). $\forall t_1, t_2$ $|t_1 - t_2| < \varepsilon_0$,

$$d_0^*(\tilde{\gamma}_0(t_1), \tilde{\gamma}_0(t_2)) = \lim_{k \to \infty} d_k^*(\tilde{\gamma}_k(t_1), \tilde{\gamma}_k(t_2)) = |t_1 - t_2|,$$

$$\|\tilde{\gamma}_0'(t)\|_{g_0^*} = \lim_{k \to \infty} \|\tilde{\gamma}_k'(t)\|_{g_k^*} = 1,$$

and hence $l(\tilde{\gamma}_0 | [t_1, t_2], g_0^*) = |t_1 - t_2|$. Therefore $\gamma_0 = H_0'^{-1}\tilde{\gamma}_0$ is a geodesic of g_0 in M.

5.10.2. If $(u_1, u_2, \cdots, u_n) \in V$ (5.5), then define $Z_k = (Z_k^1, Z_k^2, \cdots, Z_k^{2n})$ and

$$Z_k^l(u_1, u_2, \cdots, u_{2n}, t) = \begin{cases} u_{l+n} & \text{if } 1 \leqslant l \leqslant n, \\ -\sum\limits_{i,j}^{n} {}^k\Gamma_{ij}^l(u_1, \cdots, u_n) u_{i+n} \cdot u_{j+n} & \text{if } n+1 \leqslant l \leqslant 2n. \end{cases}$$

Using the notation of 5.9, for small t, (5.9.1) is equivalent to

$$F_k(t) = F_k(0) + \int_0^t Z_k(F_k(s), s)\, ds.$$

Since $F_k(t) \to F_0(t)$ and ${}^k\Gamma_{ij}^l \to {}^0\Gamma_{ij}^l$ uniformly, all functions are bounded and uniformly continuous; we have $Z_k(F_k(s), s) \to Z_0(F_0(s), s)$ uniformly and hence $F_0(t) = F_0(0) + \int_0^t Z_0(F_0(s), s)\, ds$.

5.10.3. Therefore, $\tilde{\gamma}_0(t)$ is C^2 and satisfies (5.9.1) for $k = 0$, which is equivalent to $\tilde{\gamma}_0'(t) \in P(V, H_0', \tilde{\gamma}_0)$. $\gamma_0(t) \subseteq M$ is not necessarily C^2, since H_0' is not necessarily C^2.

5.11. Lemma. *Let S be a totally geodesic 2-surface in (M, g_0), and let $p \in S$ be arbitrary. Choose U around p as in 5.4 and 5.5. Let $\gamma(t)$ be a geodesic of g_0 in S passing through p. Define $E(t)$ to be one of the continuous vector fields along $\tilde{\gamma}(t)$ with $E(t) \in TS_{\gamma(t)}$, $\|E(t)\|_{g_0} = 1$, $g_0(E(t), \gamma'(t)) = 0$. Then $dH_0'(E(t)) = \tilde{E}(t) \in P(V, H_0', \tilde{\gamma})$, where H_0' and V are as in 5.4 and 5.5 and $\tilde{\gamma} = H_0'\gamma$.*

5.11.1. *Proof.* Let $p' = H_0'(p)$ and $T(t) = \tilde{\gamma}'(t)$. As in 5.8.5,

$$0 = \frac{d}{dt}\left(g_0^*(\tilde{E}, \tilde{E})\right) = \frac{d}{dt}\lim_{k\to\infty} g_k^*(\tilde{E}, \tilde{E})$$
$$= \lim_{k\to\infty}\frac{d}{dt}g_k^*(\tilde{E}, \tilde{E}) = \lim_{k\to\infty} 2g_k^*\left(\nabla_T^{(k)}\tilde{E}, \tilde{E}\right),$$

where $\nabla^{(k)}$ is the connection of the C^∞ metric g_k^*, and $\tilde{E}$ is differentiable by using 5.10.2 and 5.10.3.

5.11.2.

$$0 = \frac{d}{dt}\left(g_0^*(\tilde{E}, T)\right) = \lim_{k\to\infty}\left[g_k^*\left(\nabla_T^{(k)}\tilde{E}, T\right) + g_k^*\left(\tilde{E}, \nabla_T^{(k)}T\right)\right]$$
$$= \lim_{k\to\infty} g_k^*\left(\nabla_T^{(k)}\tilde{E}, T\right)$$

since $\lim_{k\to\infty}\nabla_T^{(k)}T = 0$ by 5.10.2 and 5.10.3.

5.11.3. Let $\tilde{S} = H_0'(S)$, $N \in UN(\tilde{S}, g_0^*)_{p'} \subseteq T\mathbf{R}^n_{p'}$, and X and Y be differentiable vector fields in $T\tilde{S}$ around p'. Define

$$S_N(X, Y) = \lim_{k\to\infty} g_k^*\left(\nabla_X^{(k)}Y, N\right)(p').$$

This limit exists by ${}^k\Gamma_{ij}^l$ and g_k^* being convergent, and it only depends on $X_{p'}$ and $Y_{p'}$. $S_N(X,Y)$: $T\tilde{S}_{p'} \times T\tilde{S}_{p'} \to \mathbf{R}$ is a symmetric bilinear form. $S_N(x,x) = 0$ by 5.10.3. Hence $\forall N \in UN(\tilde{S}, g_0)_{p'}$, $S_N \equiv 0$ and $0 = S_N(T, \tilde{E}) = \lim_{k\to\infty} g_k^*(\nabla_T^{(k)}\tilde{E}, N)(p')$.

5.11.4. By 5.11.1–5.11.3, $\forall v \in T\mathbf{R}^n_{p'}$, $\lim_{k\to\infty} g_k^*(\nabla_T^{(k)}\tilde{E}, v)(p') = 0$. Hence $\lim_{k\to\infty}(\nabla_T^{(k)}\tilde{E})(p')$ exists and equals 0 in $T\mathbf{R}^n_{p'}$. In local terms, this is equivalent to $\tilde{E} \in P(V, H_0', \tilde{\gamma})$ (see 5.8.1).

5.12. Proposition. *Let S_1 and S_2 be totally geodesic 2-surfaces in (M, g_0) intersecting along a geodesic γ. Then the angle between the surfaces along γ is constant with respect to g_0. This is still true if S_i are totally geodesic surfaces with boundary and γ lies at the boundary of both.*

5.12.1. *Proof.* For any point p on γ, choose U and LHCS V around p as in 5.4 and 5.5. By 5.8.4 and 5.11, the angle between $H_0'(S_1)$ and $H_0'(S_2)$ along $H_0'(\gamma)$ is constant with respect to g_0^*. H_0' is C^1 and $g_0^* = H_0'g_0$, hence the result follows locally and then globally. If γ lies at the boundary of both S_1 and S_2, 5.11 can be proved by using a limit argument.

6. Proof of Theorem I

The main steps of this proof follow Gromoll-Grove [15], [17] closely, on a limit metric. On the other hand, since the limit metric is not necessarily smooth or even C^2, the arguments should be modified or changed. We will provide the proofs for the modified arguments, the rest will be stated only. Occasionally basic facts of C^∞-Riemannian geometry will have to be proved explicitly for the limit metric. In §6A dual convex sets A and B are constructed in a limit metric obtained in 6.1. A and B may have boundaries or not. Each case is investigated in §§6B, C, and D.

Proof of Theorem I. This is an immediate consequence of 6.1–6.3, 6.31, Theorems 6.10, 6.22, 6.23, 6.35 and 6.38, Hamilton [22], and the generalized sphere theorem, Grove and Shiohama [21].

6A. Main construction.

6.1.1. (Similarly as was done in [2].) Given $K \geqslant 4$, $n \geqslant 2$, and $\varepsilon_0 > 0$, by the Finiteness Theorems of Cheeger [6], [7], and [31], there are finitely many diffeomorphism classes of C^∞-Riemannian manifolds (M, g) with

$$1 \leqslant K(M, g) \leqslant K, \quad i(M, g) \geqslant \varepsilon_0, \quad d(M, g) \leqslant \frac{\pi}{2}, \quad \dim(M) = n.$$

Let $M_1, M_2, M_3, \cdots, M_s$ represent all such distinct classes, $s \geqslant 1$. Define $\inf\{\delta \mid \exists g$ on M_i with $C^\infty g$, $\pi/2 \geqslant d(M_i, g) \geqslant \pi/2 - \delta$, $i(M_i, g) \geqslant \varepsilon_0$, $1 \leqslant K(M_i, g) \leqslant K\}$ to be $\xi[M_i]$ and $\delta_0(K, n, \varepsilon_0) = \min(\{\xi[M_i] \mid \xi[M_i] \neq 0\} \cup \{\pi/2\})$. Obviously $\delta_0(K, n, \varepsilon_0) > 0$ and if n is even, then $\delta_0 = \delta_0(K, n)$ since $i(M, g) \geqslant \pi/2\sqrt{K}$.

6.1.2. Proposition. *Any C^∞-Riemannian manifold (M, g) with $1 \leqslant K(M, g) \leqslant K$, $i(M, g) \geqslant \varepsilon_0$, and $\pi/2 - \delta_0(K, n, \varepsilon_0) < d(M, g)$ has either $d(M, g) > \pi/2$, or $d(M, g) \leqslant \pi/2$ with $\xi[M] = 0$. Hence M either satisfies any common property of the diffeomorphism classes $[M_i]$ with $\xi[M_i] = 0$, or is homeomorphic to a sphere by* [21].

6.1.3. Let (M, g) be as in 6.1.2 with $d(M, g) \leqslant \pi/2$ with $\xi[M] = 0$. There exists a sequence of C^∞-Riemannian metrics g_m on M such that

$$1 \leqslant K(M, g_m) \leqslant K, \quad i(M, g_m) \geqslant \varepsilon_0, \quad \pi/2 - 1/m \leqslant d(M, g) \leqslant \pi/2 \qquad \forall m \in \mathbf{N}^+.$$

By Gromov's Compactness Theorem 4.1.0 [20], *we extract a subsequence which we denote also by g_m, such that $g_m \to g_0$* (4.1), *where g_0 has the properties obtained in* §§4 *and* 5. *Unless otherwise stated, in all of the following M or (M, g_0) denotes this limit metric with the distance function d_0.*

6.1.4. Since $\forall m\ \exists p_m, p_m' \in M$ such that $\pi/2 \geqslant d_m(p_m, p_m') \geqslant \pi/2 - 1/m$, and M is compact, $\exists p_0, p_0' \in M$ with $d_0(p_0, p_0') = \pi/2$. $\forall p, q \in M$, $d_m(p, q) \leqslant \pi/2$, hence $d_0(p, q) \leqslant \pi/2$ and $d(M, g_0) = \pi/2$.

6.1.5. Dual sets as in [15], [17]: For $X \subseteq (M, g_0)$ define $X' = \{x \in M \mid d_0(x, X) = \pi/2\}$. $X \subseteq X''$ and $X' = X'''$. By 6.1.4, there exists a pair of dual compact sets A and B in (M, g_0) with $A' = B$ and $B' = A$.

6.2.1. Definition. In this text, a set $X \subseteq (M, g)$ is said to be convex if $\forall p, q \in X$, any $\mathrm{mg}(p, q; g) \subseteq X$. X is said to be r-convex if any minimal geodesic of length $< r$ with endpoints in X lies in X [15].

6.2.2. A and B are convex sets. Given any $p, q \in A$ and any $\mathrm{mg}(p, q)$ γ, the closest point on γ to B cannot have distance $< \pi/2$ to B, by 4.2.2, 4.6.1, and 4.5 (see [15], [17] and [2]).

6.3. Both A and B are totally geodesic C^1 submanifolds of M without or with boundary which may not be C^1. For the proof, see [2, third sublemma, p. 144]. Also one can modify the arguments of [8, pp. 417–418] for this case. In this text, the interior or boundary of a convex submanifold are taken with respect to the topology of the submanifold. for convention $\partial\{\text{point}\} \neq \varnothing$. *In the following A and B always denote such compact convex dual submanifolds of (M, g_0).*

6B. The case of $\partial A \neq \varnothing$ and $\partial B \neq \varnothing$.

6.4. Definitions. 1. ($\forall m \geqslant 0$. See [19].) Let $p, q \in M$. q is called a nontrivial critical point for the function $d_m(p, \cdot)$ if $q \neq p$ and $\forall v \in TM_q - \{0\}$ there exists a $\mathrm{mg}(p, q; g_m)$ γ_v such that $\measuredangle_m(\gamma_v'(q), v) \leqslant \pi/2$.

2. Let X be a convex set in (M, g) with $\partial X \neq \varnothing$. For any $p \in X$, define $C_pX = \{v \in TM_p \mid v = 0, \text{or } \exists\delta = \delta(v) > 0, \exp_p v[0, \delta(v)] \subseteq \mathrm{int}(X)\}$.

3. Let $U \subseteq S^n(1)$ be any subset. Define $CH(U)$ to be the smallest subset of S^n with (i) $U \subseteq CH(U)$ and (ii) for any nonantipodal pair $x, y \in CH(U)$, the shortest arc joining x to y lies in $CH(U)$.

4. Let $p, q \in (M, g)$, $X \subseteq M$. The link from p to q $(\neq p)$ is defined to be $L(p, q; g) = \{\gamma'(p) \in U(M, g)_p \mid \gamma \text{ is a } \mathrm{mg}(p, q; g)\}$. The link from p $(\notin \overline{X})$ to X is $L(p, X; g) = \{v \in U(M, g) \mid \exp_{p,g} v d(p, X; g) \in \overline{X}\}$.

6.5.1. Combining 6.4.1–6.4.3, q is a nontrivial critical point for $d(p, \cdot)$ if and only if $CHL(q, p)$ contains an antipodal pair.

6.5.2. For convex A, $A - \partial A$ is a totally geodesic a-dimensional submanifold and, $\forall p \in A - \partial A$, C_pA is an a-dimensional subspace of TM_p. If $p \in \partial A$, then one can show that $\overline{C}_p$ is an a-dimensional convex cone contained in a closed half of an a-dimensional subspace $\hat{C}_p$ in TM_p. (see [8, pp. 419–420, Proposition 1.8])

6.6. Lemma. *Let A_1 be a closed convex set in (M, g_0) with $\partial A_1 \neq \varnothing$, and $p \in A_1 - \partial A_1$. Then $d_0(p, \cdot)$ has no nontrivial critical points in ∂A_1.*

6.6.1. *Proof.* For any $q \in \partial A_1$, $\exists\delta > 0$ such that $\{v \in \hat{C}_qA_1 \mid \|v\|_0 < \delta$ and $\exp_q v \in B(q, \delta, \mathrm{int}(A_1))\}$ is an open subset of $\hat{C}_qA_1$.

6.6.2. Let $q \in \partial A_1$ be any point. Suppose that $CHL(q, p)$ contains a pair of antipodal points. Define $S_1 = \hat{C}_pA \cap U(M, g_0)_q$ and let D_1 be a closed hemisphere in S_1 such that $CHL(q, p) \subseteq \overline{C}_q \cap UM_q \subseteq D_1 \subseteq S_1$ and $S_2 = \partial D_1$

in S_1. If $L(q,p) \cap S_2 = \varnothing$, then $L(q,p) \subseteq \operatorname{int}(D_1)$ in S_1, and $CHL(q,p) \subseteq \operatorname{int}(D_1)$. But, $\operatorname{int}(D_1)$ contains no pairs of antipodal points; so, $L(q,p) \cap S_2 \neq \varnothing$. Let $v \in L(q,p) \cap S_2$. Then $v \in S_2 \cap \bar{C}_q \cap UM_q \subseteq \partial\bar{C}_q \cap UM_q$ in S_1. $\exp_q(v \cdot [0, d_0(q,p)]) \subseteq A_1$, and by 6.6.1, $\exists \varepsilon_1 > 0$ such that $\exp_q(v[0,\varepsilon_1]) \subseteq \partial A_1$. Let $q' = \exp_q \varepsilon_1 v$. Then $\bar{C}_{q'}$ contains both $\pm((d/dt)(\exp_q tv)|_{t=0}) := \pm w$. $\bar{C}_{q'}$ is contained in a half-space, so $w \in \partial(\bar{C}_{q'} \cap UM_{q'})$ in $\hat{C}_{q'} \cap UM_{q'}$. By a similar argument $\exists \varepsilon_2 > \varepsilon_1$ such that $\exp_q([0,\varepsilon_2]v) \subseteq \partial A_1$. By the connectedness of an interval in $\mathbf{R}$, one obtains $p \in \partial A_1$ which is not the case. Hence, $CHL(q,p)$ contains no pairs of antipodal points and recall 6.5.1.

6.7. Lemma. *Let A_1 be a closed convex set in (M, g_0), $q \in \partial A_1$, $p \in \operatorname{int}(A_1)$, γ be a* $\mathrm{mg}(p,q;g_0)$, *and $d_0(p,q) = d_0(p,\partial A_1)$. Then*

$$C_q - \{0\} = \left\{ v \in \hat{C}_q \mid \measuredangle\left(v, -\gamma'(q)\right) < \pi/2 \right\}.$$

6.7.1. *Proof.* See [8, Lemma 1.7, p. 419] together with 4.6.1.

6.8. Lemma. *Let $p', q' \in \operatorname{int}(A_1)$, $\partial A_1 \neq \varnothing$, where A_1 is a closed convex set in (M, g_0) of* 6.1.3, *and γ_0 be any* $\mathrm{mg}(p',q';g_0)$. *Then the function $f(t) = d_0(\gamma_0(t), \partial A_1) : [0, d_0(p',q')] \to \mathbf{R}$ cannot have any local minimum at $t_0 \in (0, d_0(p',q'))$.*

6.8.1. *Proof.* Suppose that $\gamma \cap \partial A \neq \varnothing$. Let $p \in \partial A$ with

$$\gamma_0\left(\left[0, d_0\left(p_0, p'\right)\right)\right) \subseteq \operatorname{int}(A_1).$$

Then $\gamma_0'(p_0) \in C_{p_0}$, which is open, and $\gamma_0'(p_0) \in \bar{C}_{p_0}$. This is not possible since $\bar{C}_{p_0}$ is a closed cone contained in a closed half of $\hat{C}_{p_0}$. So, $\gamma_0 \subseteq \operatorname{int}(A_1)$ and $f > 0$.

6.8.2. Suppose $\exists t_0$ and δ such that $(t_0 - \delta, t_0 + \delta) \subseteq (0, d_0(p',q'))$ and $\forall t \in (t_0 - \delta, t_0 + \delta)$, $f(t) \geqslant f(t_0)$. Let $\gamma_0(t_0) = p$. $\exists q \in \partial A_1$ such that $d_0(p,q) = d_0(p, \partial A_1) = f(t_0)$. Choose a sequence t_n, $n \in \mathbf{N}^+$, with $t_0 + \delta \geqslant t_n \geqslant t_{n+1}$, $t_n \to t_0$, and $\{\gamma_n\}_{n=1}^{\infty}$, where γ_n is a $\mathrm{mg}(q, \gamma(t_n); g_0)$ and $\gamma_n \to \gamma$ uniformly where γ is a $\mathrm{mg}(q,p;g_0)$. Then $\gamma \subseteq \operatorname{int}(A_1) \cup \{q\}$. $\measuredangle_0(\gamma'(p), \gamma_0'(t_0)) = \pi/2$ by 4.6.1, and $\gamma_n'(q) \to \gamma'(q)$ by 4.2.2, 4.4, and 5.10. For sufficiently large N, let $\theta = \gamma_N$ so that $\theta \subseteq \operatorname{int}(A_1)$, $\measuredangle_0(\theta'(q), \gamma'(q)) \ll \pi/2$, and $\delta' = t_N - t_0 \ll \min(\varepsilon_0, d_0(q,p))$. $\theta \neq \gamma$, so one defines $w(s)$: $[0, \pi/2] \to \hat{C}_q \cap UM_q$ as in Lemma 4.7. By 6.7, if $s < \pi/2$, $w(s) \in C_q$; so, $\exists \eta(s)$, $0 < \eta(s) \leqslant \infty$, such that $\exp_q w(s) \cdot (0, \eta(s)) \subseteq \operatorname{int}(A_1)$ and choose $\eta(s)$ to be maximal. Define $r(s) = \exp_q \eta(s) w(s)$ if $\eta(s) < \infty$. Then $r(s) \in \partial A_1$. Define $\beta_0 = \measuredangle_0(\theta'(q), \gamma'(q))$.

6.8.3. Claim. $\exists s_0 \in (\beta_0, \pi/2]$ *such that* $\exp_q \delta' w(s_0) \in \partial A_1$. *Clearly* $\eta(\beta_0) \geqslant d_0(q, \gamma(t_N)) > \delta'$. *Let* $\eta_0 = \inf\{\eta(s) \mid \beta_0 \leqslant s < \pi/2\}$.

6.8.3.1. If $\eta_0 \geqslant \delta'$, then $\exp_q\{w(s)t \mid 0 < t < \delta'$ and $\beta_0 \leqslant s < \pi/2\} \subseteq \operatorname{int}(A_1)$ and $\exp_q(0,\delta']w(\pi/2) \subseteq \bar{A}_1$. $w(\pi/2) \perp \gamma'(q)$, hence $w(\pi/2) \in \bar{C}_q - C_q$ and $\exp_q(0,\delta']w(\pi/2) \subseteq \partial A_1$ by 6.7.

6.8.3.2. If $\eta_0 < \delta'$, then define $I_1 = \{s \in [\beta, \pi/2) \mid \eta(s) > \delta'\}$ and $I_2 = \{s \in [\beta_0, \pi/2) \mid \eta(s) < \delta'\}$. We may assume that $I_1 \cup I_2 = [\beta_0, \pi/2)$, since otherwise 6.8.3 holds. $I_1 \neq \varnothing$, $I_2 \neq \varnothing$, so $\exists s_0 \in \bar{I}_1 \cap \bar{I}_2$. Let $s_n \in I_1$, $s_n' \in I_2$, $n \in \mathbf{N}^+$, such that $s_n \to s_0$ and $s_n' \to s_0$. Let $\eta_1 = \underline{\lim}\,\eta(s_n)$ and $\eta_2 = \overline{\lim}\,\eta(s_n')$, $\eta_2 \leqslant \delta' \leqslant \eta_1$. $\exp_q w(s_0)\eta_2$ is a limit point of $\{\exp_q w(s_n')\eta(s_n') \mid n \in \mathbf{N}^+\} \subseteq \partial A_1$ which is closed, and hence $\exp_q w(s_0)\eta_2 \in \partial A_1$. If $\eta_1 < \infty$, then similarly $\exp_q w(s_0)\eta_1 \in \partial A_1$. If either $\eta_1 = \delta'$ or $\eta_2 = \delta'$, then 6.8.3 holds. If $0 < \eta_2 < \delta' < \eta_1 < \infty$, then $\exp_q w(s_0)[0, \eta_1] \subseteq \bar{A}_1$, and $\exp_q w(s_0)\{0, \eta_1, \eta_2\} \subseteq \partial A_1$. By a similar argument to 6.6.2, $\exp_q w(s_0)[0, \eta_1] \subseteq \partial A_1$, particularly $\exp_q w(s_0)\delta' \in \partial A_1$. If $\eta_2 = 0$, then $s_0 = \pi/2$ and $\exp_q[0, \eta_1]w(\pi/2) \subseteq \partial A_1$ by a similar argument to 6.8.3.1. If $\eta_1 = \infty$ and $\eta_2 > 0$, then $\exp_q[0, \infty]w(s_0) \subseteq A_1$ and $\exp_q \eta_2 w(s_0)$, $q \subseteq \partial A_1$; by an argument similar to 6.6.2 $\exp_q[0, \infty]w(s_0) \subseteq \partial A_1$. So, 6.8.3 holds.

6.8.4. By Lemma 4.7 and Claim 6.8.3, there exists a point $\exp_q w(s_0)\delta' \in \partial A_1$ with

$$\begin{aligned} f(t_N) = d_0(\gamma_0(t_N), \partial A_1) &\leqslant d_0\big(\gamma_0(t_0 + \delta'), \exp_q\left(w(s_0)\delta'\right)\big) \\ &< d_0(q, p) = f(t_0), \end{aligned}$$

where $t_N = t_0 + \delta' \in (t_0 - \delta, t_0 + \delta)$. By obtaining a contradiction to the assumption of 6.8.2, one proves Lemma 6.8.

6.9. Proposition. *Let A and B be dual, compact, convex sets in (M, g_0) as in 6.1–6.3. If $\partial A \neq \varnothing$, then there exists a unique $p_0 \in A$ with $d_0(p_0, \partial A) = \max\{d_0(p, \partial A) \mid p \in A\}$ and $d_0(p_0, \cdot)$: $M \to \mathbf{R}$ has no nontrivial critical points in $M - \{p_0\}'$, where $B \subseteq \{p_0\}' = \{p \in M \mid d(p, p_0) = \pi/2\}$.*

6.9.1. *Proof.* Clearly p_0 exists. Suppose $\exists p_0' \in A$ with $p_0 \neq p_0'$ and $d_0(p_0, \partial A) = d_0(p_0', \partial A)$. Let γ be any mg$(p_0, p_0'; g_0)$ and $f(t) = d_0(\gamma(t), \partial A)$, $f \leqslant d_0(p_0, \partial A)$. By 6.8, such f does not exist and hence p_0 is unique.

6.9.2. Let $q \in A - \{p_0\}$ be any point and $c = d_0(q, \partial A)$. Define $A^c = \{q' \in A \mid d_0(q', \partial A) \geqslant c\}$. Given $q_1, q_2 \in A^c$, and any mg$(q_1, q_2; g_0)$ γ, the function $f(t) = d_0(\gamma(t), \partial A)$ has to attain its minimum at the end points by 6.8. Hence $\gamma \subseteq A^c$ and A^c is convex. $p_0 \in \operatorname{int}(A^c)$ in A, $d(p_0, \partial A) > c$ by 6.9.1, and $q \in \partial A^c$ in A. By 6.6, q cannot be a critical point for $d_0(p_0, \cdot)$.

6.9.3. Let $q \in M - (A \cup \{p_0\}')$ be any point, and let γ_1 and γ_2 be mg(q, p_0) and mg(q, q_0), respectively, where $q_0 \in B$ and $d_0(q, q_0) = d_0(q, B)$. $d_0(q, q_0) < \pi/2$, $d_0(q, p_0) < \pi/2$, and $d_0(p_0, q_0) = \pi/2$. By 4.5 and 6.1.5, $\measuredangle_0(\gamma_1'(q), \gamma_2'(q)) > \pi/2$. $\forall v \in L(q, p_0)$ and hence $\forall v \in CHL(q, p_0)$, we have $\measuredangle_0(v, \gamma_2'(q)) > \pi/2$. $CHL(q, p_0)$ cannot contain a pair of antipodal points, and by 6.5.1 the proposition follows.

6.9.4. Let $\partial A \neq \varnothing$ and p_0 be as above. $\exists$ dual convex sets A_1 and B_1 such that $p_0 \in A_1 \subseteq A$ and $B \subseteq \{p_0\}' \subseteq B_1$. $\partial A_1 \neq \varnothing$ by 4.6.2, 6.4.1, 6.4.2, and 6.9.2. $d_0(p_0, \partial A_1)$ may not be maximal. However, by replacing A, B with A_1, B_1 we may assume that $\exists p_0 \in A$ such that $\{p_0\}' \subseteq B$ and $d(p_0, \cdot\,)$ has no nontrivial critical points on $M - B$. One can proceed similarly if $\partial B \neq \phi$, but not simultaneously for both.

6.10. Theorem. *Let A and B be dual, convex sets in (M, g_0) as in* 6.1–6.3, *and* 6.9.4. *If $\partial A \neq \varnothing$ and $\partial B \neq \varnothing$, then M is homeomorphic to a sphere.*

Proof. The main idea of our proof is similar to [15], [17], and [21].

6.10.1. Let g_0, g_m be as in 6.1.3. Choose $p_0 \in A$ as in 6.9.4 and $q_0 \in B$ with $d_0(q_0, \partial B) = \max\{d_0(q, \partial B) \mid q \in B\}$. $\exists \delta$, $0 < \delta < \pi/2$, such that $M = N_1 \cup N_2$ where $N_1 = B(p_0, \delta; g_0)$ and $N_2 = B(q_0, \delta; g_0)$. Otherwise, by compactness $\exists p \in M$ with $d_0(p, q_0) = d_0(p, p_0) = \pi/2$, which is not possible by 4.6.2, 6.4.1, 2, 6.9.2 for B, q_0, and 6.9.4. $\forall m \geqslant 0$, $\forall p \in (M, g_m)$ define $\delta_A^m(p) = \min\{r \mid \text{for some } v \in U(M, g_m)_p, CHL(p, p_0; g_m) \subseteq \overline{B}(v, r; U(M, g_m)_p, \measuredangle_m)\}$, $\delta_A^m(N_1) = \sup\{\delta_A^m(p) \mid p \in \overline{N}_1\}$, and $\delta_B^m(p)$, $\delta_B^m(N_2)$ in a similar way. Let $\eta_p^m = \min\{\measuredangle_m(v, w) \mid v \in L(p, p_0; g_m)$ and $w \in L(p, q_0; g_m)\}$ and $\eta^m(X) = \inf\{\eta_p^m \mid p \in \overline{X}\}$ where $X \subseteq M$. By 6.9.3, $\delta_A^0(p) < \pi/2$ $\forall p \in M - B$, $\delta_B^0(p) < \pi/2$ $\forall p \in M - \{q_0\}'$, and $\eta_p^0 > \pi/2$ $\forall p \in M - (\{q_0\}' \cup B)$. For any $p_n \to p$ and any $q \in M$, the limit set of $L(p_n, q; g_m)$ is a subset of $L(p, q, g_m)$ for a fixed $m \geqslant 0$. Hence $\exists \delta_2 > 0$ such that $\delta_A^0(N_1) \leqslant \pi/2 - 2\delta_2$, $\delta_B^0(N_2) \leqslant \pi/2 - 2\delta_2$, and $\eta^0(N_1 \cap N_2) \geqslant \pi/2 + 2\delta_2$.

6.10.2 $(M, g_m) \to (M, g_0)$ as in 4.1. By 4.5.2 and 5.10, if γ_m is $\mathrm{mg}(p, q; g_m)$ $\forall m \geqslant 1$, then the limit set of γ_m's is the subset of $\mathrm{MG}(p, q; g_0)$. Hence, $\exists m_0 \neq 0$ and δ_3 such that $\delta_A^{m_0}(N_1) \leqslant \pi/2 - \delta_2$, $\delta_B^{m_0}(N_2) \leqslant \pi/2 - \delta_2$, $\eta^{m_0}(N_1 \cap N_2) \geqslant \pi/2 + \delta_2$, $M = N_3 \cup N_4$, $N_3 \subseteq N_1$, and $N_4 \subseteq N_2$, where $N_3 = B(p_0, \delta_3; g_{m_0})$ and $N_4 = B(q_0, \delta_3; g_{m_0})$.

6.10.3. By applying the mollifier techniques of [21] to $d_{m_0}(p_0, \cdot)$ on N_3 and $d_{m_0}(q_0, \cdot)$ on N_4 we can obtain two smooth functions $f_i: N_{i+2} \to [0, \pi/2]$, $i = 1, 2$, such that $f_1(p_0) = f_2(q_0) = 0$, $\nabla f_i \neq 0$, and $f_i > 0$ on $N_{i+2} - \{p_0, q_0\}$; ∇f_i is transversal to both ∂N_3 and ∂N_4 for $i = 1, 2$. One observes that 6.10.2 has the essential information of [21, Lemma 1.3 and Proposition 1.5, pp. 204–205]. Now it is straightforward to show that M is homeomorphic to a sphere, following [21] for the C^∞ metric g_{m_0}.

6C. The case of $\partial A = \varnothing$ and $\partial B \neq \varnothing$.

6.11. By 6.10, there is no loss of generality in assuming that $\partial A = \varnothing$ in this section.

6.12. Lemma. *Let (M, g_0) be as in* 6.1, *p_1, p_2, $p_3 \in (M, g_0)$, and p_1', p_2', $p_3' \in S^2(1)$ with $0 < d_0(p_i, p_j) = d(p_i', p_j') \leqslant \pi/2$, $1 \leqslant i < j \leqslant 3$. Let*

θ_i and η_i be $\mathrm{mg}(p_{i+1}, p_{i+2}; g_0)$ *and* $\mathrm{umg}(p'_{i+1}, p'_{i+2}; \textit{standard})$ *for* $i = 1,2,3$, *indices* mod 3, *respectively. If*

$$0 < \measuredangle_0\big(-\theta_2'(p_1), \theta_3'(p_1)\big) = \measuredangle\big(-\eta_2'(p_1), \eta_3'(p_1)\big) := \alpha_0 < \pi,$$

then

$$d_0(p_3, \theta_3(t)) = d\big(p_3', \eta_3(t)\big) \quad \forall t \in \big[0, d(p_1, p_2)\big].$$

6.12.1. *Proof.* Let $a_i = d_0(p_i, p_{i+1})$, indices mod 3. Fix $t_0 \in (0, a_1)$. Let $\mu(s)$ be a $\mathrm{mg}(\theta_3(t_0), p_3; g_0)$, $\alpha_1 := \measuredangle_0(-\theta_3'(t_0), \mu'(0))$, and $l = d(\eta_3(t_0), p_3')$. By 4.5, $\exists u_0 \geqslant 0$ with $d_0(\theta_3(t_0), p_3) = l - u_0$. For

$$u \in \big[0, l - \max(|t_0 - a_3|, |a_1 - a_2 - t_0|)\big]$$

define $\beta_1(u)$, $\beta_2(u)$ with $0 \leqslant \beta_i \leqslant \pi$,

$$\cos\beta_1(u) \cdot \sin(l-u) = \big(\cos a_3 - \cos t_0 \cdot \cos(l-u)\big)/\sin t_0,$$

$$\cos\beta_2(u) \cdot \sin(l-u) = \big(\cos a_2 - \cos(a_1 - t_0) \cdot \cos(l-u)\big)/\sin(a_1 - t_0).$$

By 4.5, $\alpha_1 \geqslant \beta_1(u_0)$ and $\pi - \alpha_1 \geqslant \beta_2(u_0)$. Since $0 < a_1$, $a_1 - t_0 < \pi/2$,

$$(d/du)\big((\sin(l-u))(\cos\beta_1(u) + \cos\beta_2(u))\big) < 0 \quad \text{for } u > 0.$$

$\beta_1(0) + \beta_2(0) = \pi$, hence $\cos\beta_1(0) + \cos\beta_2(0) = 0$. For $u > 0$, $\sin(l-u) \cdot (\cos\beta_1(u) + \cos\beta_2(u)) < 0$, so, $\beta_1(u) + \beta_2(u) > \pi$. Since $\pi = \alpha_1 + (\pi - \alpha_1) \geqslant \beta_1(u_0) + \beta_2(u_0)$, we conclude that $u_0 = 0$, which proves the lemma by t_0 being arbitrary.

6.13. Proposition. *If* p, $r \in A$, $q \in B$, γ_1 *is a* $\mathrm{mg}(p, r; g_0)$ *of length* $\alpha \leqslant \pi/2$, *and* γ_2 *is a* $\mathrm{mg}(q, p)$, *where A and B are as in* 6.1–6.3, 6.11, *then there exists a unique* $\mathrm{mg}(q, r)$ γ_3 *and* 2*-surface L bounded by* γ_1, γ_2, *and* γ_3, *where L is totally geodesic and isometric to the inside of a triangle in* $S^2(1)$ *with the side lengths* α, $\pi/2$, $\pi/2$.

6.13.1. Remark. If we compare Lemma 8 of [2] with 6.13, in our case $\varepsilon_0 \leqslant i(M, g_0) \leqslant \pi/2$.

Proof. We first prove for $\alpha \leqslant \varepsilon_0/2$.

6.13.2. $\measuredangle_0(\gamma_2'(p), \gamma_1'(p)) = \pi/2$ by 4.6.1, 6.1.5, and 6.11. On $S^2(1)$, choose p', q', r' with $d(p', q') = d(q', r') = \pi/2$ and $d(p', r') = \alpha$. Let η_1, η_2 and η_3 be $\mathrm{umg}(p', r')$, $\mathrm{umg}(q', p')$, and $\mathrm{umg}(q', r')$, respectively, and let $L' \subseteq S^2(1)$ be the region bounded by η_i and which has area α. Let $f = \exp_p \circ \phi \circ (\exp_{p'})^{-1} : L' \to L := f(L')$, where $(\exp_{p'})^{-1} := B(p', 3\pi/4) \to B(0, 3\pi/4)$, ϕ is an isometric imbedding of $TS^2(1)_{p'}$ into (TM_p, g_0) with $\phi(\exp_{p'}^{-1} q') = -\pi\gamma_2'(p)/2$, and $\phi(\exp_{p'}^{-1} r') = \alpha\gamma_1'(p)$.

6.13.3. For $0 \leqslant s \leqslant \pi/2$, define $r_s' = \eta_3(\pi/2 - s)$, $v_s' = (\exp_{p'}^{-1})(r_s')$, $\eta_4^s(t) = \exp_{p'} t v_s'/\|v_s'\|$, $f(r_s') = r_s$, and $\gamma_4^s(t) = f(\eta_4^s(t))$. For $0 \leqslant s \leqslant \varepsilon_0/2$, $d_0(p, r_s) \leqslant \varepsilon_0$ and $\gamma_4^s(t)$ is $\mathrm{mg}(p, r_s)$. By 4.5, $d_0(r_s, r) \leqslant d(r_s', r')$ and $d_0(r_s, q) \leqslant d(r_s', q')$.

$$\pi/2 = d_0(q, r) \leqslant d_0(q, r_s) + d_0(r_s, r) \leqslant d\big(q', r_s'\big) + d\big(r_s', r'\big) = \pi/2;$$

so, $d_0(r_s, r) = d(r_s', r') = s$. Define $\gamma_3(t) = f(\eta_3(t))$, $0 \leqslant t \leqslant \pi/2$. By Lemma 3 of [2, p. 138], $l(\gamma_3 | [a, b], g_0) \leqslant b - a$, if $(\pi - \varepsilon_0)/2 \leqslant a \leqslant b \leqslant \pi/2$. γ_3 is a $\mathrm{mg}(r_{\varepsilon_0/2}, r)$. Let $L_1' \subseteq L'$ be bounded by η_1, $\eta_4^{\varepsilon_0/2}$, and $\eta_3 | [(\pi - \varepsilon_0)/2, \pi/2]$.

6.13.4. Claim. $f: L_1' \to L_1 := f(L_1')$ is an isometry. Let $q_1', q_2' \in L_1'$. Choose q_3', q_4' on $\eta_3(t)$ with $q_i \in \mathrm{umg}(p', q_{i+2}')$, $i = 1, 2$. Let $q_i = f(q_i')$, $1 \leqslant i \leqslant 4$. Then $d_0(q_3, q_4) = d(q_3', q_4')$ by 6.13.3, $d(q_2', q_3') = d_0(q_2, q_3)$ by 6.12, and similarly $d(q_2', q_1') = d_0(q_2, q_1)$.

6.13.5. L_1 is totally geodesic since it is the image of a Riemannian manifold under a distance preserving map, locally.

6.13.6. One proves the following similarly to 6.13.3–6.13.5. For $0 \leqslant s \leqslant \pi/2$, define $p_s' = \eta_2(\pi/2 - s)$ and $p_s = f(p_s')$. Then $d(p_s', r') = d_0(p_s, r)$ by 6.12. Let η_5^s be the $\mathrm{umg}(r', p_s')$ and $\gamma_5^s(t) = f(\eta_5^s(t))$. For $0 \leqslant s \leqslant \varepsilon_0/2$, $\gamma_5^s(t) \in B(p, \varepsilon_0; g_0)$ and they are $\mathrm{mg}(r, p_s)$ by 6.12, [2, Lemme 3, p. 138], and arguments similar to 6.13.3. Let $L_2' \subseteq L'$ be bounded by $\eta_1, \eta_5^{\varepsilon_0/2}$, and $\eta_2 | [(\pi - \varepsilon_0)/2, \pi/2]$. Then $f | L_2': L_2' \to L_2 := f(L_2')$ is an isometry and L_2 is totally geodesic.

6.13.7. $L_1 \cup L_2$ is totally geodesic. $\exists s_1 > \varepsilon_0/8$ such that $\mathrm{umg}(p_{s_1}', r_{s_1}')$ $\theta_{s_1} \subseteq \mathrm{int}(L_1' \cup L_2')$. Let $R_1' \subseteq L_1' \cup L_2'$ be bounded by η_1, η_2, η_3, and θ_{s_1}. Then $f | R_1': R_1' \to R_1 := f(R_1')$ is a local isometry and R_1 is totally geodesic. Since γ_2 and $\gamma_4^{s_1}$ are minimal, $d_0(p_{s_1}, r_{s_1}) = d(p_{s_1}', r_{s_1}')$ by 6.12 and hence $f(\theta_{s_1})$ is $\mathrm{mg}(p_{s_1}, r_{s_1})$.

$$\measuredangle_0\left((f \circ \theta_{s_1})'(p_{s_1}), -\gamma_2'(p_{s_1})\right) = \measuredangle\left(\theta_{s_1}'(p_{s_1}'), -\eta_2'(p_{s_1}')\right) = \beta(s_1) \leqslant \pi/2$$

and $d_0(p_{s_1}, r_{s_1}) := \alpha(s_1) \leqslant \alpha$.

6.13.8. Replace $p, r, p', r', \gamma_1, \pi/2, L', \alpha$, and f with $p_{s_1}, r_{s_1}, p_{s_1}', r_{s_1}', f(\theta_{s_1})$, $\beta(s_1)$, $L' - R_1'$, $\alpha(s_1)$, and $f_{s_1} = \exp_{p_{s_1}} \circ \phi_{s_1} \circ (\exp_{p_{s_1}'})^{-1}$ which is defined similarly, respectively. By repeating 6.13.3–6.13.7 one obtains $R_2' \subseteq L' - R_1'$ and $s_2 > \varepsilon_0/4$, replacing R_1' and s_1. $R_2 = f_{s_1}(R_2')$ is totally geodesic and locally isometric to $S^2(1)$ by f_{s_1}. $R_1 \cup R_2$ is totally geodesic since $(L_1 \cup L_2) \cap R_2$ is open in $(L_1 \cup L_2) \cup R_2$ by $\theta_{s_1} \subseteq \mathrm{int}(L_1' \cup L_2')$. Hence $f := R_1' \cup R_2' \to R_1 \cup R_2$ is a well-defined local isometry. By induction, one obtains that $f := L' \to L$ is a local isometry, and L is totally geodesic. For any $t \leqslant \alpha$, the image μ_t of the minimal geodesic from q' to $\eta_1(t)$ in L' under f, is a geodesic of length $\pi/2$ from q to $\gamma_1(t)$, so it is minimal and lies in L.

6.13.9. Now let $\alpha \leqslant \pi/2$. Apply 6.13.2–6.13.8 to $p, q, \gamma_1(\varepsilon_0/2)$, γ_1, and γ_2 to obtain $L^{(1)}$ as above. Then apply 6.13.2–6.13.8 to $\gamma_1(\varepsilon_0/4)$, q, $\gamma_1(3\varepsilon_0/4)$, γ_1, and $\mu_{\varepsilon_0/4}$ to obtain $L^{(2)}$. $L^{(1)} \cap L^{(2)}$ is open in $L^{(1)} \cup L^{(2)}$. So $L^{(1)} \cup L^{(2)}$ is totally geodesic and locally isometric to $S^2(1)$. Inductively one obtains L which is totally geodesic, and $f := L' \to L$ defined for $\alpha \leqslant \pi/2$ is a local isometry. Since μ_α is minimal one repeats 6.13.4 to see that f is an isometry.

6.13.10. Uniqueness of L and $\gamma_3 = \mu_\alpha$ follows 5.12.

6.14. Corollary. *Let A and B as in* 6.1–6.3, 6.11. $\forall p, r \in A$ *and* $\forall q \in B$,

(1) $\forall v \in L(q, p; g_0)\ \exists w \in L(q, r; g_0)$ *such that* $\measuredangle_0(v, w) = d_0(p, r)$;

(2) *There is a natural bijection between* $L(q, p; g_0)$ *and,* $L(q, r; g_0)$ *locally.*

6.15. Definition. Let A and B be convex sets in (M, g_0) as in 6.1–6.3, 6.11. For any $p, r \in A$, $q \in B$, $v \in L(p, q; g_0)$, and any mg(p, r) γ, we define $P(\gamma, q)(v)$ to be the unique vector in $L(r, q)$ such that in 6.13 $-\gamma_2'(p) = v$, $\gamma_1 = \gamma$, and $-\gamma_3'(r) = P(\gamma, q)(v)$.

6.16. $\forall p \in (M, g_0)$, there is a natural metric on $U(M, g_0)_p$, namely $\measuredangle_0(w_1, w_2)$ $\forall w_1, w_2 \in U(M, g_0)$. With this metric, $U(M, g_0)_p$ is isometric to $S^{n-1}(1)$.

6.17.1. Fix $q \in B$, and define $N(p, q) = (\operatorname{Span} L(p, q)) \cap U(M, g_0)_p$ $\forall p \in A$. By 5.12, $P(\gamma, q)$ is an isometry from $L(p, q)$ onto $L(r, q)$ with 6.16, where γ is any mg(p, r). Hence $\dim N(p, q) = \dim N(r, q)$ $\forall p, r \in A$. Let $\dim N(p, q) = \lambda' - 1$, $1 \leqslant \lambda' \leqslant n$. $\exists$ unique extension $\overline{P}(\gamma, q)\colon N(p, q) \to N(r, q)$ such that $\overline{P}$ is an isometry (6.16). For any $p, r \in A$, let $\mathscr{C}(p, r)$ be the collection of all curves from p to r in A which are geodesics of A except at a finite number of points. $\overline{P}(\theta, q)$ is defined for $\theta \in \mathscr{C}(p, r)$. Let $G(p, q) = \{\overline{P}(\theta, q) \mid \theta \in \mathscr{C}(p, p)\}$. G is a subgroup of the isometry group of $N(p, q) = S^{\lambda' - 1}(1)$. G is an algebraic subgroup of $\mathrm{O}(\lambda')$. Let $v_0 \in L(p, q)$ be arbitrary. $G(p, q)v_0 \subseteq L(p, q)$ which is closed. $\overline{G}v_0 = \overline{Gv_0} \subseteq L(p, q)$, where $\overline{G}$ is the closure of G in $\mathrm{O}(\lambda')$. $\overline{G}$ is a lie subgroup of $\mathrm{O}(\lambda')$ and the orbit $\overline{G}(v_0)$ is a compact smooth submanifold of $U(M, g_0)_p$. Let $E_q = \{\overline{P}(\theta, q)v_0 \mid r \in A, \theta \in \mathscr{C}(p, r)\} \subseteq UN(A, g_0)$. Then

$$\begin{aligned} E_q &= \{\overline{P}(\theta, q)(Gv_0) \mid r \in A, \theta \in \mathscr{C}(p, r)\} \\ &= \{\overline{P}(\gamma, q)(Gv_0) \mid r \in A, \gamma \in \mathrm{MG}(p, r)\}, \end{aligned}$$

$\{\overline{P}(\gamma, q)(\overline{G}v_0) \mid r \in A, \gamma \in \mathrm{MG}(p, r)\}$ is a subfiber bundle of $UN(A, g_0)$ and equal to $\overline{E}_q$.

6.17.2. The fibers $\sigma^{-1}(r)$ of the fiber bundle $\sigma\colon \overline{E}_q \to A$ are smooth compact submanifolds $\overline{P}(\gamma, q)(\overline{G}v_0)$ of $UN(A, g_0)_r$ for any mg(p, r) γ in A. Obviously, $\sigma^{-1}(r) \subseteq L(r, q; g_0)$.

6.18. Remark. 6.17 is quite similar to the proof of Proposition 3.4 in [17] in which parallel translation and holonomy are used (see 5.8.2).

6.19. Lemma [17]. *Let* $F \hookrightarrow E \xrightarrow{\sigma} B_0$ *be a fiber bundle where F is a closed manifold and E is homeomorphic to S^N. Let $E_0 \subseteq E$ be a subset such that $\sigma \mid E_0$: $E_0 \to B_0$ has a structure of a fiber bundle*: $F_0 \hookrightarrow E_0 \to B_0$ *where F_0 is a closed submanifold of F. Then $E_0 = E$.*

Proof. See [17, Proposition 3.4] and also 6.27.5.

6.20. Let $F \hookrightarrow S^N \to B_0$ be a fiber bundle where F and B_0 are compact manifolds with F and $B_0 \neq$ point, and $N \in \mathbf{N}^+$.

6.20.1. If $N = 1$, then this has to be a finite covering of S^1 by S^1. So we may assume that $N \geqslant 2$.

6.20.2. Let F be connected, hence B_0 be simply-connected. By [4], F has the homotopy type of S^1, S^3, or S^7. If $F \simeq S^1$, then B_0 has the homotopy type of $\mathbf{C}P^k$. If $F \simeq S^3$, then B_0 has the integral cohomology ring isomorphic to those of $\mathbf{H}P^k$. If $F \sim S^7$, then B_0 is homeomorphic to S^8 (also by [33]). N has to be odd, by $\chi(S^N) = \chi(B_0)\chi(F)$.

6.20.3. If F is not connected, then $\sigma: S^N \to B_0$ lifts to $\tilde{\sigma}: S^N \to \tilde{B}_0$ and one obtains $F_0 \hookrightarrow S^N \to \tilde{B}_0$, where F_0 is any connected component of F. If F is discrete, then $S^N \to B_0$ is a covering map.

6.20.4. If N is even, then $S^N \to B_0$ is a covering map and $\pi_1(B_0) = \mathbf{Z}_2$.

6.20.5. If N is odd and F is not connected, then $\tilde{B}_0$ is as in 6.20.2.

6.20.6. In all cases, $\dim F + 1$ divides $N + 1$.

6.21. Proposition [17]. *Let A and B be as in* 6.1–6.3, 6.9.4. *If $\partial A = \varnothing$ and $\partial B \neq \varnothing$, then $B = \{q_0\}$, $A =$* cutlocus(q_0), *$B =$* normalcutlocus(A), *and UNA is homeomorphic to S^{n-1}.*

6.21.1. *Proof.* This can be proved by using 6.9, passing to an appropriate C^∞-metric g_{m_0} as in 6.10.1 and 6.10.2, and obtaining a smooth function f from $d_{m_0}(q_0, \cdot)$ by techniques of [21], where $f > 0$ and $\|\nabla f\| \neq 0$ on $M - (\{q_0\} \cup N(A, \varepsilon, g_0))$ for small ε and ∇f is transversal to $\partial N(A, \varepsilon; g_0)$, to show that UNA is homeomorphic to S^{n-1}. The rest follows as in Proposition 3.4 of [17] by using 6.17 and 6.19. $\bigcup_{p \in A} L(p, q_0; g_0) = UN(A, g_0)$ and $\bigcup_{p \in A} L(q_0, p; g_0) = U(M, g_0)_{q_0}$ by $\partial A = \varnothing$.

6.22. Theorem. *Let A and B be convex sets in (M, g_0) as in* 6.1–6.3, 6.9.4. *If $\partial A = \varnothing$, $\partial B \neq \varnothing$, and $\pi_1(M, p) \neq 0$, then (M, g_0) is isometric to* $\mathbf{R}P^n(1)$.

6.22.1. *Proof.* If $n = 2$, then A is a closed geodesic of length π by 4.6.2, 6.13, and 6.21. M is locally isometric to $S^2(1)$ except possibly on $\{q_0\}$ and A by 6.13. By convexity and $\dim(A) = 1$, any geodesic in A of length $\pi/2$ is minimal. For any $p \in A$, the dual set of $\{p\}$ contains at least two points and cannot have boundary (6.10). Hence there are other pairs of dual sets A and B as in the hypothesis. Hence M is locally isometric to $S^2(1)$, and therefore isometric to $\mathbf{R}P^2(1)$.

If $n \geqslant 3$, then $\pi_1(A, p_1) = \pi_1(M - \{p_0\}, p_1) - \pi_1(M, p_1)$ for some $p_1 \in A$ by 6.21. In the fiber bundle $S^{\lambda'-1} = UNA_p \hookrightarrow S^{n-1} = UNA \to A$, $\lambda' = 1$ by $\pi_1(A, p_1) \neq 0$ and 6.20. So, $\dim A = n - 1$ and $L(p, q_0) = UNA_p$ is a pair of antipodal points for all $p \in A$. $L(q_0, p)$ is a pair of antipodal points by 4.6.2. Let $f: S^{n-1}(1) = UM_{q_0} \to A^{n-1}$ be given by $f(v) = \exp_{q_0} \pi v/2$. By 4.5 and 6.16, f is distance decreasing, locally 1-1, and hence a local isometry by 6.14.1.

The $L(q_0, p)$'s being antipodal pairs implies that $(A, g_0|A)$ is isometric to $\mathbf{R}P^{n-1}(1)$. Let $q_1, q_2 \in M$ be arbitrary, and $p_1, p_2 \in A$ such that $d_0(q_0, q_i) + d_0(q_i, p_i) = \pi/2$ for $i = 1,2$. Choose $p_0 \in A$ with $d_0(p_0, p_i) = \pi/2$, $i = 1,2$. The sets $A_1 = \{p \in M \mid d_0(p, p_0) = \pi/2\}$ and $B_1 = \{p_0\}$ are convex dual sets by $A = \mathbf{R}P^{n-1}(1)$, and $\partial A_1 = \varnothing$ by 6.10. A_1 is isometric to $\mathbf{R}P^{n-1}(1)$. $q_0, q_1, q_2 \in A_1$. Hence, $\forall v_1, v_2 \in TM_{q_0}$ with $\|v_1\|_0$, $\|v_2\|_0 \leqslant \pi/4$, $d_0(\exp_{q_0} v_1, \exp_{q_0} v_2) = \rho(\measuredangle_0(v_1, v_2), \|v_1\|_0, \|v_2\|_0; 1)$ (4.3.1). (M, g_0) is locally isometric to $S^n(1)$ around q_0. The same is true for $p_0 \in A$ by using A_1 and B_1, and hence for $q_1 \in A_1$. $q_1 \in M$ was arbitrary, hence (M, g_0) is locally isometric to $S^n(1)$. $\pi_1(M, p_1) = \pi_1(A, p_1) = \mathbf{Z}_2$. Using $A =$ cutlocus (q_0), one constructs an isometry from $\mathbf{R}P^n(1)$ onto (M, g_0).

6.23. Theorem. *Let A and B be convex sets in (M, g_0) as in* 6.1–6.3, 6.9.4, *and with $\partial A = \varnothing$, $\partial B \neq \varnothing$, and $\pi_1(M, p) = 0$. We define $a = \dim A$ and $\lambda = n - a$. Then, we have the following*: $\lambda = 2$, 4, *or* 8. $n = k\lambda$ *for* $k \in \mathbf{N}^+$, $k \geqslant 2$. *If* $\lambda = 2$, *then M^n has the homotopy type of $\mathbf{C}P^k$. If* $\lambda = 4$ *or* 8, *then* $H^*(M, \mathbf{Z}) \cong \mathbf{Z}[x]/x^{k+1}$ *where* $x \in H^\lambda(M, \mathbf{Z})$. *If* $\lambda = 8$ *then* $k = 2$ *and* $n =$ 16. *That is if* $\lambda = 4$ *or* 8 *then M has the cohomology ring structure of* $\mathbf{H}P^k$ *or* $\mathrm{Ca}P^2$.

6.23.1. *Proof.* If $n = 2$, then A has to be a closed geodesic, and by 6.22.1, M is locally isometric to $S^n(1)$ which has diameter π. So, $n \geqslant 3$. $0 = \pi_1(M, p) = \pi_1(A, p)$ by 6.21. The fiber bundle $UNA_p = S^{\lambda-1} \hookrightarrow UNA = S^{n-1} \to A^a$ and 6.20 will give $\lambda = 2, 4$, or 8. $H^*(A, \mathbf{Z}) \cong \mathbf{Z}[x]/x^k$, where $a = (k-1)\lambda$, $x \in H^\lambda(A, \mathbf{Z})$, $k \geqslant 2$, $a \geqslant 2$, $n \geqslant 4$, by 6.20. If $\lambda = 8$ then $k = 2$ and A is homeomorphic to S^8. By 6.21, A is a strong deformation retract of $M - \{q_0\}$. For the inclusion i: $A \hookrightarrow M - \{q_0\}$, i^*: $H^*(M - \{q_0\}, \mathbf{Z}) \to H^*(A, \mathbf{Z})$ is an isomorphism. The cohomology exact sequence for the pair $(M, M - \{q_0\})$ with $\mathbf{Z}$ coefficients has the following part:

$$H^q(M, M - \{q_0\}) \to H^q(M) \xrightarrow{j^*} H^q(M - \{q_0\}) \to H^{q+1}(M, M - \{q_0\}),$$

where j: $M - \{q_0\} \hookrightarrow M$ is the inclusion map. If $1 \leqslant q \leqslant n-1$, then $H^q(M, M - \{q_0\}) = 0$. So, $I = i^*j^*$: $H^q(M, \mathbf{Z}) \to H^q(A, \mathbf{Z})$ is an isomorphism for $0 \leqslant q \leqslant n-2$. $\pi_1(M, p) = 0$; so, $H^{n-1}(M, \mathbf{Z}) = 0$ and $H^n(M, \mathbf{Z}) = \mathbf{Z}$. Therefore, $H^q(M, \mathbf{Z}) = \mathbf{Z}$ if $\lambda \mid q$ and $0 \leqslant q \leqslant n$; $= 0$ otherwise. Let y be the generator of $H^\lambda(M, \mathbf{Z})$. $\lambda \leqslant n - 2$, and $x = I(y)$ generates $H^\lambda(A, \mathbf{Z})$. $I(y^l) = (I(y))^l = x^l \neq 0$, and hence $y^l \neq 0$ for $1 \leqslant l \leqslant k-1$. $y^k \neq 0$ since there is no torsion in $H^*(M, \mathbf{Z})$ and the pairing $H^q \otimes H^{n-q} \to \mathbf{Z}$ is nonsingular [36, p. 159, 5.27]. Hence y^l generates $H^{l\lambda}(M, \mathbf{Z})$, $0 \leqslant l \leqslant k$, and $H^*(M, \mathbf{Z}) \cong \mathbf{Z}[y]/y^{k+1}$. If $\lambda = 2$, then by [4], [3, pp. 189, 190] and [27] M^n has the homotopy type of $\mathbf{C}P^k$.

6D. The case of $\partial A = \partial B = \varnothing$.

6.24. By 6.10, 6.22, and 6.23, there is no loss of generality in assuming that $\partial A = \partial B = \varnothing$ in this section. We define $a = \dim A$ and $b = \dim B$, $a, b > 0$.

6.25. Definition. For any $p, q \in (M, g_0)$ we define $T(p,q)$: $L(p,q) \to L(q,p)$ by $T(p,q)\gamma'(p) = -\gamma'(q)$ for any $\mathrm{mg}(p,q)$ γ.

6.26. Let $q_0 \in B$. Construct $E = \overline{E}_{q_0}$ as in 6.17. then $F \hookrightarrow E \to A$ is a fiber bundle, where F, E, and A are closed manifolds with the possibility that F has many components or is discrete. Let E' be $\{v \in UN(B, g_0)_{q_0} \mid p \in A, w \in E, v = T(p, q_0)(w)\}$. Then $F' \hookrightarrow E' \xrightarrow{\sigma'} A$ is a fiber bundle with F' and E' being homeomorphic to F and E respectively, where $\sigma'(v) = \exp_{q_0} \pi v/2$.

6.27. Proposition [17]. *Let A, B, E' be as in* 6.24 *and* 6.26. *Then $E' = UN(B, g_0)_{q_0} = S^{n-b-1}$. Consequently, $\bigcup_{p \in A} L(q_0, p) = UN(B, g_0)_{q_0}$, $M = \exp_{g_0}[0, \pi/2]UN(B, g_0)$, and the normal cutlocus of B is A. By symmetry, the similar statements are true if A and B are interchanged and q_0 is replaced by $p_0 \in A$.*

Proof. See [17] for a slightly different proof for the C^∞ case.

6.27.1. Let $\varepsilon_0 > 0$ be as in 4.1. Let $S = \{\exp_{q_0} \varepsilon_0 v/2 \mid v \in UNB_{q_0}\}$, $S \cap B = \varnothing$, $0 < d_0(S, B) = \varepsilon_1 < \varepsilon_0$. Let $q' \in S^2(1)$, $v_0 \in US^2(1)_{q'}$, and $p' = \exp_{q'} v_0\pi/2$. $\exists \varepsilon_2 > 0$ such that $d(p', \exp_{q'} tw) \leqslant \pi/2 - \varepsilon_1/2$ if $w \in US^2(1)_{q'}$, $\measuredangle (w, v_0) \leqslant \varepsilon_2$, and $\varepsilon_0/2 \leqslant t \leqslant \pi/2$.

6.27.2. Choose $\varepsilon_3 < \varepsilon_1/2$ such that $N_2 = N(A, \varepsilon_3; g_0)$ and ∂N_2 are homeomorphic to the unit normal disc bundle of A in M and UNA, respectively, ∂N_2 is a differentiable submanifold of M, and similarly for B with $N_4 = N(B, \varepsilon_3; g_0)$. $\exists \varepsilon_4 > 0$ with $N_2 \cup N_3 = N_1 \cup N_4 = M$, where

$$N_1 = N(A, \pi/2 - \varepsilon_4; g_0) \text{ and } N_3 = N(B, \pi/2 - \varepsilon_4, g_0).$$

Let $N = \overline{N}_1 - N_2$. Then $\forall p \in N$, $d_0(p, A)$ and $d_0(p, B)$ are in $[\varepsilon_4, \pi/2 - \varepsilon_4]$. By 4.5, $\exists \varepsilon_5 > 0$ such that $\forall p \in N$, $\forall v \in L(p, A; g_0)$, $\forall w \in L(p, B; g_0)$, $\measuredangle_0(v, w) \geqslant \pi/2 + 2\varepsilon_5$. $\forall p \in N$ any $\mathrm{mg}(p, A; g_0)$ cuts ∂N_2 orthogonally. $\forall p \in \partial N_4$, any $\mathrm{mg}(p, A)$ makes an angle $\geqslant 2\varepsilon_5$ with ∂N_4. Any sequence of $\mathrm{mg}(p, A; g_m)$ γ_m, $m \in \mathbf{N}^+$, has a C^1 convergent subsequence converging to γ_0, a $\mathrm{mg}(p, A; g_0)$ (see 5.10), and similarly for B. Hence $\exists m_0$ such that

(i) $\forall p \in N, \forall v \in L(p, A; g_{m_0}), \forall w \in L(p, B; g_{m_0}), \measuredangle_{m_0}(v, w) \geqslant \pi/2 + \varepsilon_5$,

(ii) $\forall p \in N$ any $\mathrm{mg}(p, A; g_{m_0})$ cuts ∂N_2 transversally;

(iii) $\forall p \in \partial N_4$ any $\mathrm{mg}(p, A; g_{m_0})$ cuts ∂N_4 transversally of an angle $\geqslant \varepsilon_5$.

One applies the mollifier techniques of [21] to the function $d_{m_0}(\cdot, A)$ of the C^∞ metric g_{m_0} to obtain a smooth function f with $|\nabla f| \neq 0$ on N with ∇f transversal to ∂N_2 and ∂N_4.

6.27.3. N_2 is homeomorphic to the unit normal disc bundle of A in M. Using the integral curves of ∇f one constructs h: $[0,1] \times (M - N_4) \to M - N_4$ with $h(0,p) = p$, $h(1,p) \in A$ $\forall p \in M - N_4$, and $h(p,t) = p$ $\forall p \in A$, $\forall t \in [0,1]$. Hence A is a strong deformation retract of $M - N_4$.

6.27.4. Let ϕ: $[0,\pi] \to [\varepsilon_0/2, \pi/2]$ be continuous with $\phi(0) = \pi/2$ and $\phi([\varepsilon_2, \pi]) = \varepsilon_0/2$. Let f_1: $UN(B, g_0)_{q_0} \to M$ by $f_1(v) = \exp_{q_0, g_0} v\phi(d(v, E'))$ (see 6.16, 6.26). If $d(v, E') \geqslant \varepsilon_2$ then $d_0(f_1(v), B) \geqslant \varepsilon_1$. If $d(v, E') \leqslant \varepsilon_2$ then, by 6.27.1 and 4.5, $d_0(f_1(v), A) \leqslant \pi/2 - \varepsilon_1/2$. Hence, f_1: $UN(B, g_0)_{q_0} \to M - N_4$ and $f_2 = h(1, f_1(v))$: $UN(B, g_0)_{q_0} \to A$ with $f_1(v) \in A$ and $f_2(v) = h(1, f_1(v)) = f_1(v) = \exp_{q_0} \pi v/2 = \sigma'(v)$ $\forall v \in E'$ (see 6.26).

6.27.5. (See [17, Proposition 3.4].) Suppose $E' \neq UN(B, g_0)_{q_0} = S^{n-b-1}$. $\exists H$: $[0,1] \times E' \to UNB_{q_0}$ with $H(0,v) = v$ and $H(1,v) = v_0 \in E'$ $\forall v \in E'$. $f_2 H$: $[0,1] \times E' \to A$ with $f_2 H(0,v) = \sigma'(v)$ and $f_2 H(1,v) = f_2(v_0) = p_0$. By the homotopy covering theorem [34, p. 54], $\exists \tilde{H}$: $[0,1] \times E' \to E'$ with $\sigma' \tilde{H} = f_2 H$ and $\tilde{H}(0,v) = v$ $\forall v \in E'$. $\tilde{H}(1, E') \subseteq \sigma'^{-1}(p_0) \simeq F'$. $\dim F' < \dim E'$ and both F' and E' are closed $\mathbf{Z}_2$-oriented manifolds. The identity map of E' cannot be homotopic to a map which sends the top homology class to 0. Hence $E' = UN(B, g_0)_{q_0}$.

6.27.6. $E' \subseteq L(q_0, A; g_0) \subseteq UN(B, g_0)_{q_0}$, and hence all are equal. $q_0 \in B$ is arbitrary. A is the normal cutlocus of B and vice versa. The rest follows.

6.28. Let $p, p_0 \in A$ and $q_0, q \in B$. We have the fiber bundles $F \hookrightarrow E \xrightarrow{\sigma} A$, and $F' \hookrightarrow E' \xrightarrow{\sigma'} A$ as in 6.17, 6.26. $\sigma'^{-1}(p) \subseteq L(q_0, p)$. $E' = UNB_{q_0} = S^{n-b-1}$ and hence $F' = \sigma'^{-1}(p) = L(q_0, p)$ by 6.27. So $L(p, q_0) = \sigma^{-1}(p) = F$ which is a compact smooth submanifold of UNA_p by 6.17. By symmetry, $\forall p, q$, $L(p,q)$, $L(q,p)$ are smooth compact submanifolds of UNA_p and UNB_q respectively. E is homeomorphic to S^{n-b-1}.

6.29. By 4.6.2, F' is not a point. If F' is connected, then $\pi_1(A) = 0$ and $F' \sim S^{\lambda-1}$, where $\lambda = 2$, 4, or 8 by 6.20. Clearly $\dim F' = \dim F = \lambda - 1$. If F' is not connected, then either $\tilde{A}$ is E' itself with $\lambda = 1$ or $\exists$ a fiber bundle $F_0' \hookrightarrow E' \to \tilde{A}$, where F_0' is any connected component of F' with $F_0' \simeq S^{\lambda-1}$, $\lambda = 2$, 4, or 8, and whenever $\dim E' > 1$. If $\dim E' = 1$, then $E' \to A$ is a finite covering of S^1 by S^1. $\dim E' = n - b - 1$, so $a + b + \lambda = n$, λ divides all a, b, and n. Since $L(p,q)$ is homeomorphic to $L(q,p)$ via T of 6.25, obtaining the above bundles for B results with the same fiber, but the total spaces of the bundles might be different spheres.

6.30. Remark. In the case of $\partial A = \varnothing$ and $B = \{q_0\}$, the $L(p, q_0)$ are equal to UNA_p, but one cannot conclude that the $L(q_0, p)$ are smooth submanifolds of $U(M, g_0)_{q_0}$ since $\overline{P}$ is not defined on $\{q_0\}$.

6.31. Under the conditions of 6.24, $n = a + b + \lambda \geqslant 3$. By Hamilton's results [22], any compact Riemannian 3-manifold of positive Ricci curvature admits a metric of constant sectional curvature 1. Hence, there is nothing to prove in Theorems I and II in the simply connected case when $n = 3$. *In the following we assume that* $n \geqslant 4$ *when* M *is simply connected. Also we take* A *and* B *with* $\dim A \leqslant \dim B$. $a \leqslant n - 3$, since $0 < \lambda \leqslant a \leqslant b$. $\pi_1(M, q) = \pi_1(M - A, q) = \pi_1(B, q)$ by 6.27 $\forall q \in B$.

6.32. If $\pi_1(M,q) = 0$, then the fibration $S^{b+\lambda-1} = UN(A, g_0)_p \to B^b$ gives $b \geqslant 2$ and the fiber $L(p,q)$ is connected. So, $L(q,p)$ is connected $\forall q \in B$, $p \in A$. $\pi_1(A, P) = 0$ by $\lambda \geqslant 2$ (see 6.29), and the fibration $S^{a+\lambda-1} = UN(B, g_0)_q \to A^a$ with connected fibers.

6.33. Proposition. *If* A *and* B *are convex sets in* (M, g_0), *as in* 6.1–6.3, 6.24, *and* $\pi_1(M, p) = 0$, *then* $\lambda \neq 8$, *where* λ *is given by* 6.29.

6.33.1. *Proof.* Suppose that $\lambda = 8$. Then $a = b = \lambda = 8$ by 6.20.2. A and B are homeomorphic to S^8. Let $C = \{\exp_{q_0} tv \mid v \in UN(B, g_0)_{q_0}, t \in [0, \pi/2]\}$ for some $q_0 \in B$. C is a topological submanifold of M since $L(p, q_0) \cong S^7$ in $UN(A, g_0)_p$ $\forall p \in A$, and A is a strong deformation retract of $C - \{q_0\}$ by 6.27. By a similar proof to 6.23.1, $H^*(C, \mathbf{Z}) = \mathbf{Z}[x]/x^3$, $x \in H^8(C, \mathbf{Z})$. $B - \{q_0\}$ is homeomorphic to D^8. $M - C = \{\exp_q tv \mid q \in B - \{q_0\}, v \in UN(B, g_0)_q, t \in [0, \pi/2)\}$ and is homeomorphic to D^{24}; this map can be extended to a continuous map from $\overline{D}^{24}$ onto M by 6.27. So C is a strong deformation retract of $M - \{p_0\}$, $p_0 \notin C$. By a similar proof to 6.23.1, $H^*(M, \mathbf{Z}) = \mathbf{Z}[x]/x^4$, where $x \in H^8(M, \mathbf{Z})$. Such a manifold does not exist by [35].

6.34. Proposition. *Let* A *and* B *be convex sets in* (M, g_0) *as in* 6.1–6.3 *and* 6.24, *and* $\pi_1(A, p) = \pi_1(B, q) = 0$. *Let* $a = a'\lambda$ *and* $b = b'\lambda$ *where* λ *is given in* 6.29. *Then* $\lambda = 2$ *or* 4. *If* $\lambda = 2$ *then* A *and* B *are isometric to* $\mathbf{C}P^{a'}$ *and* $\mathbf{C}P^{b'}$, *respectively. If* $\lambda = 4$, *then* A *and* B *are isometric to* $\mathbf{H}P^{a'}$ *and* $\mathbf{H}P^{b'}$ *respectively.*

Remark. A is a C^1 submanifold and σ': $E' \to A$ *is* C^0; *neither is known to be* C^∞ *at this point.*

6.34.1. *Proof.* Let $q_0 \in B$ be arbitrary and fixed. Consider $E' = L(q_0, A) = UN(B, g_0)_{q_0} = S^{n-b-1}(1)$ as an abstract manifold, with the C^∞ metric d of 6.16, by 6.27. Recall 6.28, 6.29: σ': $E' \to A$, $\sigma'^{-1}(p) = L(q_0, p)$ is a compact smooth submanifold of E'. Let $p_1, p_2 \in A$, $p_1 \neq p_2$. Let $v \in \sigma'^{-1}(p_1)$, γ be any $\mathrm{mg}(p_1, p_2; g_0)$, and $w = T(p_2, q_0) \circ P(\gamma, q_0) \circ T(q_0, p_1)(v)$. By 6.13, $\measuredangle_0(w, v) = d_0(p_1, p_2)$, $d(v, \sigma'^{-1}(p_2)) \leqslant d_0(p_1, p_2)$. $\forall u \in \sigma'^{-1}(p_2)$,

$$\begin{aligned} d_0(p_1, p_2) &= d_0\left(\exp_{q_0} \pi v/2, \exp_{q_0} \pi u/2\right) \\ &\leqslant \rho(\measuredangle_0(u, v), \pi/2, \pi/2; 1) = \measuredangle_0(u, v) = d(u, v), \end{aligned}$$

by 4.5. Hence the fibers of σ': $E' \to A$ are equidistant: $\forall p_1, p_2 \in A$, $\forall v \in \sigma'^{-1}(p_1)$, $d(v, \sigma'^{-1}(p_2)) = d_0(p_1, p_2)$.

6.34.2. The fibration of the smooth manifold $(E', d) = S^{n-b-1}(1)$ has smooth equidistant fibers S^1 or S^3 (6.28, 6.29, 6.32, 6.33). We will show that this is a smooth fibration.

6.34.3. Let $p_0 \in A$, $F_0' = \sigma'^{-1}(p_0)$, $v_0 \in F_0'$ be fixed. Define $D = B(p_0, \varepsilon_0/2; A, g_0)$. $\forall p \in D$, $\exists$umg(p_0, p) γ_p. Let $v_0' = T(q_0, p_0)(v_0)$. By the uniqueness of the surfaces obtained in 6.13, $f_{v_0}(p) = T(p, q_0) \circ \overline{P}(\gamma_p, q_0)(v_0')$: $D \to E'$ is C^0. For $w \in UTA_{p_0}$, $t \in [0, \varepsilon_0/2]$, $f_{v_0}(\exp_{p_0} tw)$ is a geodesic arc in E' starting from v_0, which is normal to F_0' at v_0 by 6.34.1 and 4.5, and f_{v_0} is 1-1. ϕ_{v_0}: $UTA_{p_0} \to UN(F_0')_{v_0}$, defined by $\phi_{v_0}(w) = (d/dt)(f_{v_0}(\exp_{p_0} tw))(0)$, is 1-1 and continuous. $\dim UN(F_0') = n - b - 1 - \lambda = a - 1 = \dim UTA_{p_0}$. Hence ϕ_{v_0} is a homeomorphism, and so is f_{v_0}: $D \to \{\exp_{v_0} tw \mid t \in [0, \varepsilon_0/2], w \in UN(F_0')_{v_0}\}$. Any geodesic arc of length $\rho \leqslant \varepsilon_0/2$, normal to F_0' at v_0, corresponds to a umg(p_0, p) of length ρ for a unique $p \in D$ and vice versa by 6.13, 5.12.

6.34.4. Claim. $N(F_0', \varepsilon_0/2, d, E') \cap$ Normal cutlocus$(F_0') = \varnothing$. Suppose $\exists \varepsilon_i$, $0 < \varepsilon_i \leqslant \varepsilon_0/2$, $v_i \in F_0'$, $w_i \in UN(F_0')_{v_i}$, $i = 1, 2$, with $v_3 = \exp_{v_1} \varepsilon_1 w_1 = \exp_{v_2} \varepsilon_2 w_2$. Let $\gamma_i(t) = \exp_{q_0}(\exp_{v_i} tw_i)$, $i = 1, 2$. Both γ_1 and γ_2 are geodesics of lengths ε_1 and ε_2 starting at p_0 ending at $p_1 = \exp_{q_0} v_3$. $i(M, g_0) \geqslant \varepsilon_0$, so $\gamma_1 = \gamma_2$ and $\varepsilon_1 = \varepsilon_2 = d_0(p_0, p_1)$. Both $\exp_{v_i} tw_i$, $i = 1, 2$, are normal to $\sigma'^{-1}(p_1)$ since $\varepsilon_1 = d_0(\sigma'^{-1}(p_1), F_0') = d_0(p_0, p_1)$, $w_1 = w_2$, and $v_1 = v_2$ by ϕ_{v_3} being 1-1 and $\gamma_1 = \gamma_2$, (E', d) is a smooth Riemannian manifold and F_0' is a smooth submanifold. Hence the claim follows from the structure of the normal cutlocus in the C^∞ category. In fact the focal points of F_0' correspond to the cutlocus of p_0 in A.

6.34.5. $d(\cdot, F_0')$: $N(F_0', \varepsilon_0/2, d) - F_0' \to (0, \varepsilon_0/2)$ is smooth, $\{v \in E' \mid d(v, F_0') = r\}$ is a smooth submanifold of E', $0 < r < \varepsilon_0/2$, and it is the union of all fibers $\sigma'^{-1}(p)$ which has $d(\sigma'^{-1}(p), \sigma'^{-1}(p_0)) = d_0(p_0, p) = r$.

6.34.6. One repeats the proofs of Lemma 6.2 and Proposition 6.1 of [13, pp. 12–15], to prove that the fibration of E' by $\sigma'^{-1}(p)$, $p \in A$, is a smooth fibration with compact fibers $\simeq S^1, S^3$; that is $\exists$ a smooth map σ_0 and a smooth manifold A_0^a such that $F_0' \hookrightarrow E' \xrightarrow{\sigma_0} A_0$ is a smooth fiber bundle and σ_0 is a C^∞ submersion. By Proposition 2 of [13, p. 6] and since the fibers are equidistant (parallel in the terminology of [13]), there exists a C^∞-Riemannian metric g' on A_0 such that σ_0: $(E', d) \to (A_0, g')$ is a C^∞-Riemannian submersion.

If $\lambda = 2$, then by [16, Corollary 2.2], the smooth metric fibration of $(E', d) = S^{n-b-1}(1)$ by S^1 is congruent to the Hopf fibration $S^{n-b-1}(1) \to \mathbf{C}P^{a'}$, and hence the simply connected (A_0, g') is isometric to $\mathbf{C}P^{a'}$.

If $\lambda = 4$, then in [18, Corollary 5.4] all Riemannian submersions $S^{n-b-1}(1) \to A_0^a$ by the smooth fibers $\simeq S^3$ are classified to be the Hopf fibration $S^{n-b-1}(1) \to \mathbf{H}P^{a'}$, and (A_0^a, g') is isometric to $\mathbf{H}P^{a'}$.

Clearly $\lambda = 2, 4$ are the only possibilities by 6.20.2 and 6.33.

6.34.7. Define I: $(A_0, g') \to (A, g_0 | A)$ by $I(x) = \sigma'(\sigma_0^{-1}(x))$. I is well defined, 1-1, and onto. By [7, pp. 65, 66, 68], for any C^∞-Riemannian submersion the distance between two fibers is equal to the distance between their images under the projection map. By 6.34.1, I is an isometry.

6.35. Theorem. *Let A and B be dual convex sets in (M, g_0) as in* 6.1–6.3, *such that both have positive dimension and no boundary. If $n \geqslant 4$ and $\pi_1(M, p) = 0$, then (M^n, g_0) is isometric to $\mathbf{C}P^{n/2}$ or $\mathbf{H}P^{n/4}$. In fact $k \geqslant 3$, where $k\lambda = n$ and $\lambda = 2$ or 4 for $\mathbf{C}$ or $\mathbf{H}$, respectively. Hence, (M, g_0) is a C^∞-Riemannian manifold.*

Proof. $k \geqslant 3$ follows from 6.29. In this proof we only use g_0 on M.

6.35.1. By 6.26–6.34 we have the following. A and B are totally geodesic simply connected submanifolds of (M, g_0) at a distance $\pi/2$ from each other. A is the normal cutlocus of B and vice versa. $\forall p \in A$, $q \in B$, $UN(A, g_0)_p \xrightarrow{\sigma} B$ is a fiber bundle with fibers $\sigma^{-1}(q) = L(p, q) = S^{\lambda-1}$, a great sphere in $UNA_p = S^{n-a-1}(1)$, and $\lambda = 2$ or 4. $a + b + \lambda = n$, $\lambda \,|\, a, b, n$. A and B are isometric to $\mathbf{C}P^{a/2}$ and $\mathbf{C}P^{b/2}$ respectively if $\lambda = 2$; or to $\mathbf{H}P^{a/4}$ and $\mathbf{H}P^{b/4}$ respectively if $\lambda = 4$.

6.35.2. $H^\lambda(M, \mathbf{Z}) \neq 0$ and hence M is not homeomorphic to a sphere. This follows from the long exact sequence for cohomology for the pair $(M, M - A)$, $H^\lambda(M - A) = H^\lambda(B)$ by A being the normal cutlocus of B and B being a strong deformation retract of $M - A$, and $H^i(M, M - A) = H^i(N(A, \varepsilon), N(A, \varepsilon) - A) = 0$ for $i = \lambda$, $\lambda + 1$ by $N_\varepsilon(A)$ being homeomorphic to the $n - a$ dimensional normal disc bundle of A in M, Thom Isomorphism Theorem [30], and $\lambda + 1 \leqslant n - a - 1$, $b \geqslant 2$.

6.35.3. Claim. $\forall p_1, p_2 \in M$, we can choose A and B as above and $p_1, p_2 \in A$.

Proof. Let A_1, B_1 satisfy 6.35.1. Let $p_1 \notin A_1$ and γ be $\mathrm{mg}(p_3, p_1)$ with $p_3 \in A_1$, $l(\gamma) = d_0(p_1, A_1)$. $\gamma'(p_3) \in UNA_1$. $\gamma(\pi/2) = q_1 \in B_1$. Let $q_2 \in B_1$ and $p_4 \in A_1$ with $d_0(q_1, q_2) = d_0(p_3, p_4) = \pi/2$. Find dual convex sets A_2, B_2 with $p_3, q_1 \in A_2$ and $p_4, q_2 \in B_2$. $p_1 \in A_2$ by convexity. $\partial A_2 = \partial B_2 = \varnothing$ by 6.35.3.1. Obviously A_2, B_2 satisfy 6.35.1, as above. Let $p_2 \notin A_2$, since otherwise the claim holds. Let θ be $\mathrm{mg}(p_1, p_2)$. First, assume $\theta'(p_1) \in UNA_2$. $\theta(\pi/2) = q_3 \in B_2$. Pick $q_4 \in B_2$ and $p_5 \in A_2$ with $d_0(p_1, p_5) = d_0(q_3, q_4) = \pi/2$. Find convex dual sets A_3, B_3 with $p_1, q_3 \in A_3$ and $p_5, p_4 \in B_3$. $\partial A_3 = \partial B_3 = \varnothing$ (6.35.3.1) and $p_1, p_2 \in A_3$. A_3 and B_3 satisfy 6.35.1. Second, $\theta'(p_1) \notin UTA_2$ since A_2 is totally geodesic and $p_2 \notin A_2$. Third, assume that

$\theta'(p_1)=\mu_1 v_1+\mu_2 v_2$, where $0<\mu_i<1$, $v_1 \in UN(A_2)_{p_1}$, $v_2 \in UT(A_2)_{p_1}$. Define $\gamma_i(t) = \exp_{p_1} tv_i$, $i = 1,2$. Let $p_6 = \gamma_2(\pi/2) \in A_2$, $q_5 = \gamma_1(\pi/2) \in B_2$. $d_0(p_1, p_6) = \pi/2$ by A_2 being isometric to $\mathbf{C}P^l$ for some l. By the proof of 6.13 $\theta(\pi/2) = r_1$ lies on a mg(p_6, q_5) γ_3. Let $q_6 \in B_2$ with $d_0(q_5, q_6) = \pi/2$, and construct dual convex sets A_4 and B_4 with $p_1, q_6 \in A_4$ and $p_6, q_5 \in B_4$. Clearly $\partial A_4 = \partial B_4 = \varnothing$ and 6.35.1 holds for A_4, B_4. $r_1 \in B_4$ by convexity. $d_0(p_1, r_1) = \pi/2$, and hence $\theta'(p_1) \in UNA_4$; this reduces to the previous case.

6.35.3.1. Suppose $\partial A_2 \neq \varnothing$ and $p_0 \in A_2$ is at maximal distance from ∂A_2. p_0 cannot lie on a closed geodesic by 6.8, 6.8.1, 6.9. $A_2 \cap A_1$ and $B_2 \cap A_1$ form a dual convex pair in A_1 ($= \mathbf{C}P^{a'}$ or $\mathbf{H}P^{a'}$), so each is a submanifold of A_1 without boundary or is a point. Let p_0' be the closest point of $A_2 \cap A_1$ to p_0. $p_0 \neq p_0'$ by $d(p_0', A_2 \cap B_1) = \pi/2$, 4.6.2, 6.4.1, and 6.9.2. Let γ be a mg(p_0', p_0). $\gamma'(0) \in UN(A_1 \cap A_2)$. $\gamma'(0)$ is normal to $UN(A_2 \cap A_1) \cap UTA_1$ by 4.5, $d(p_0, A_1 \cap B_2) = \pi/2$ and 6.35.1. $\gamma'(0) \in UNA_1$, $\gamma(\pi/2) = p_0'' \in B_1$, $\gamma'(\pi/2) \in UNB_1$. By 4.5, $d(p_0'', B_2 \cap B_1) = \pi/2$ and so $p_0'' \in B_1 \cap A_2$. By 6.34.6, $L(p_0'', p_0')$ and $L(p_0', p_0'')$ are great spheres in $UM_{p_0''}$ and $UM_{p_0'}$, respectively. Hence $\gamma(k\pi) = p_0'$ and $\gamma(\pi/2 + k\pi) = p_0''$ $\forall k \in \mathbf{Z}$. $\gamma(\mathbf{R}) \subseteq A_2$. One obtains a contradiction by 6.8, and hence $\partial A_2 = \varnothing$.

6.35.4. Any two points of (M, g_0) are contained in a totally geodesic convex set A which is isometric to either $\mathbf{C}P^{a/2}$ or $\mathbf{H}P^{a/4}$. Hence $i(M, g_0) = d(M, g_0) = \pi/2$ and $\forall p_1, p_2 \in M$ with $d_0(p_1, p_2) = \pi/2$, $L(p_1, p_2)$ is a great sphere $S^{\lambda-1}$ in UM_{p_1}.

6.35.5. Claim. $\forall p_1, p_2, p_3 \in M$, $\exists$ a totally geodesic convex submanifold C^c of M which is isometric to $\mathbf{C}P^{c/2}$ or $\mathbf{H}P^{c/4}$ and $p_i \in C$, $i = 1,2,3$.

By 6.35.3 we may assume that $p_1, p_2 \in A$, $p_3 \notin A$, and p_i are distinct. Let $p_4, p_5 \in B$ with $d_0(p_4, p_5) = \pi/2 = d_0(p_4, p_3) + d_0(p_3, A)$. Construct dual convex sets A_1 and B_1 with $\{p_3, p_4\} \cup A \subseteq A_1$, and $p_5 \in B_1$. Similar to 6.35.3.1: $\partial A_1 = \varnothing$. If $\partial B_1 = \varnothing$, then the claim holds. If $\partial B_1 \neq \varnothing$, then $B_1 = \{p_5\}$, by 6.21, 6.35.1. The fiber bundle of 6.23.1, $U(M, g_0)_{p_5} \to A_1$, has totally geodesic equidistant fibers $S^{\lambda-1}$ by 6.35.4. By the proof of 6.34, $A_1 = C^c$ is isometric to $\mathbf{C}P^{c/2}$ or $\mathbf{H}P^{c/4}$ (also see [12], [13]), C is totally geodesic and convex.

6.35.6. Given $p \in M$, consider σ_P: $(M, g_0) \to (M, g_0)$ defined by $\sigma_p(\exp_p tv) = \exp_p - tv$ $\forall t \in [0, \pi/2]$. $\forall q, r \in M$, we choose C of 6.35.5 containing p, q, and r. Since C is isometric to a symmetric space, σ_p is well defined and $d_0(q, r) = d_0(\sigma_p(q), \sigma_p(r))$ in C and hence in M. (M, g_0) is a symmetric space. As in [2], each σ_p is C^1 [28, Theorem IV.3.10], the group of isometries G of (M, g_0) is a Lie group [28, Theorem I. 4.6], G is transitive, and (M, g_0) is a homogenous space which has to be a C^∞-Riemannian manifold. By

6.35.5 (M, g_0) is a C^∞, simply connected, symmetric space of rank 1. $\mathrm{Ca}P^2$ does not admit dual convex sets A and B with $\partial A = \partial B = \varnothing$. Hence (M, g_0) is isometric to $\mathbf{C}P^{n/2}$ or $\mathbf{H}P^{n/4}$.

6.36. [17, §5]. Let (M, g_0), A, B be as in 6.1–6.3 and 6.24, and let $\pi_1(M, p) \neq 0$ *in the rest of this section.* Let $(\tilde{M}, \tilde{g}_0)$ be the Riemannian universal cover of (M, g_0). $K(M, g_m) \geqslant 1$ implies that $d(\tilde{M}, \tilde{g}_m) \leqslant \pi$, and hence $\pi/2 \leqslant d(\tilde{M}, \tilde{g}_0) \leqslant \pi$. Let η: $(\tilde{M}, \tilde{g}_0) \to (M, g_0)$ be the Riemannian covering map, i.e. $\tilde{g}_0 = \eta^* g_0$, $\tilde{A} = \eta^{-1}(A)$, and $\tilde{B} = \eta^{-1}(B)$. $\tilde{A}$ and $\tilde{B}$ are totally geodesic. Given $p \in \tilde{M}$ with $\tilde{d}_0(p, \tilde{B}) = \pi/2$, let γ be $\mathrm{mg}(p, \tilde{B})$. $\gamma'(\pi/2) \in UN\tilde{B}$, $\eta_* \gamma'(\pi/2) \in UNB$, $(\eta\gamma)(0) \in A$, and $\gamma(0) = p \in \tilde{A}$. $\forall p \in \tilde{A}$, $\tilde{d}_0(p, \tilde{B}) = \pi/2$. Let $p, r \in \tilde{A}$, $q \in \tilde{B}$ with $\tilde{d}_0(p, q) = \pi/2$ and $\exists \mathrm{mg}(p, r)$ $\gamma_1 \subseteq \tilde{A}$. Any $\mathrm{mg}(p, q)$ is normal to $\tilde{A}$, hence by 4.5 and above $\tilde{d}_0(q, r) = \pi/2$. If $\tilde{A}_0$ and $\tilde{B}_0$ are the connected components of $\tilde{A}$ and $\tilde{B}$ containing p and q respectively, then $\forall p' \in \tilde{A}_0$, $\forall q' \in \tilde{B}_0$, $\tilde{d}_0(p', q') = \pi/2$. By 6.31, $n \geqslant 3$; and if $n \geqslant 4$, then $\pi_1(M, q) = \pi_1(B, q)$, $\tilde{B}$ is connected and so is $\tilde{A}$ since $\mathrm{codim}(B) > 1$ and A is the normal cutlocus of B. The following also takes care of $n = 3$ and $d(\tilde{M}, \tilde{g}_0) > \pi/2$.

6.36.1. Let $p \in \tilde{M}$ with $\tilde{d}_0(p, \tilde{A}_0) = l$ and let γ be a $\mathrm{mg}(A_0, p)$. Then $\gamma'(0) \in UN\tilde{A}$, $\gamma(\pi/2) \in \tilde{B}$, $\gamma'(\pi/2) \in UN\tilde{B}$, $\gamma(\pi) \in \tilde{A}$, and hence $d(p, \tilde{A}) \leqslant \min(l, \pi - l)$. $\tilde{M} = N(\tilde{A}, \pi/2)$ and if $p \in \tilde{B}$ then $d(p, \tilde{A}_0) = \pi/2$. $\forall p' \in \tilde{A}$, $\forall q' \in \tilde{B}$, $\tilde{d}_0(p', q') = \pi/2$. As in 6.2.2 and [17], $\forall p', q' \in \tilde{A}$ with $\tilde{d}_0(p', q') < \pi$, any $\mathrm{mg}(p, q) \subseteq \tilde{A}$ (see 4.5, 4.6.1). If $\exists p_1, p_2, p_3 \in \tilde{M}$ with $d(p_1, p_i) = \pi$ for $i = 2, 3$ then $p_2 = p_3$ by 4.5. Since $\dim \tilde{A} \geqslant 1$, $\tilde{A}$ is connected and π-convex [15], [17] and so is $\tilde{B}$.

6.37. Lemma. *Assume that* 6.36 *holds. If* $n = 3$, *then* $a = b = \lambda = 1$, *and* $\tilde{A}$ *and* $\tilde{B}$ *are closed geodesics of shortest period* 2π *and* $d(\tilde{M}, \tilde{g}_0) = \pi$.

6.37.1. *Proof.* $a = b = \lambda = 1$ follows 6.28 and 6.29. Let $p' \in \tilde{A}$ be fixed and $p = \eta(p')$. Since $\tilde{B}$ is connected, the maps σ: $UNA_p = S^1 \to B = S^1$ and σ_1: $UN\tilde{A}_{p'} = S^1 \to \tilde{B} = S^1$ are l and l' fold covering maps respectively, $l' \cdot |\pi_1(M, p)| = l$, where $\sigma(v) = \exp_p \pi v/2$, $\sigma_1(v') = \exp_{p'} \pi v'/2$. $\tilde{A}$ and $\tilde{B}$ are normal cutloci of each other since A and B are. $UN\tilde{A}$ and $UN\tilde{B}$ are oriented in $\tilde{M}$. $\tilde{M}$ is the union of the two solid tori $\overline{N}(\tilde{A}, \pi/4)$ and $\overline{N}(\tilde{B}, \pi/4)$, attached along their boundaries by a diffeomorphism of T^2. Let $C = \{\exp_{p'} tv \mid v \in UN\tilde{A}_{p'}, 0 \leqslant t \leqslant \pi/2\}$. $\tilde{M} - C$ is homeomorphic to a 3-disc and hence $0 = \pi_1(\tilde{M}, p_0') = \pi_1(C, p_0')$. C is obtained by attaching a 2-disc to $\tilde{B} = S^1$ along its boundary by a l'-fold covering map. By Van Kampen's Theorem $\pi(C, p_0') = \mathbf{Z}/l'\mathbf{Z}$, and hence $l' = 1$. $(UN\tilde{A}_{p'}, d) \to (\tilde{B}, \tilde{g}_0 | \tilde{B})$ is a Riemannian covering map (see 6.13 and 6.16). $\tilde{B}$ is a closed geodesic of smallest period 2π and any part of length π is minimal by π-convexity of $\tilde{B}$; hence $d(\tilde{M}, \tilde{g}_0) = \pi$.

6.38. Theorem. *Let A and B be dual convex sets in (M, g_0) as in* 6.1–6.3 *such that both have positive dimension, no boundary, and $\pi_1(M, p) \neq 0$.*

(i) *If $d(\tilde{M}, \tilde{g}_0) = \pi/2$, then $(\tilde{M}, \tilde{g}_0)$ is isometric to $\mathbf{C}P^{n/2}$, $\pi_1(M, p) = \mathbf{Z}_2$, $n/2$ is odd, and $n \geqslant 6$. $\tilde{g}_0$ and hence g_0 is a C^∞-Riemannian metric. In fact (M, g_0) is unique up to isometry,* [17, *Theorem* 5.3] *and* [37, *p.* 304].

(ii) *If $d(\tilde{M}, \tilde{g}_0) > \pi/2$, then $d(\tilde{M}, \tilde{g}_0) = \pi$, $(\tilde{M}, \tilde{g}_0)$ is isometric to $S^n(1)$, and $\tilde{g}_0$ and hence g_0 is a C^∞-Riemannian metric. See* [17, *Theorem* 5.2] *and* [37] *for the classification of such (M, g_0).*

6.38.1. *Proof.* (i) It is the same as [17, Theorem 5.3], by using 6.31, 6.35–6.37. Smoothness of $\tilde{g}_0$ follows 6.35.6, and it is a local property.

6.38.2. (ii) $d(\tilde{M}, \tilde{g}_m) > \pi/2$ for some $m \in \mathbf{N}^+$, and hence $\tilde{M}$ is homeomorphic to S^n by [21]. $\lambda = 1$, since $\tilde{A}$ and $\tilde{B}$ are normal cutloci of each other (6.36), one repeats the proofs of 6.33 and 6.35.2. Let $p \in \tilde{A}$, $q \in \tilde{B}$. σ': $UNA_{\eta(p)} = E' \to B$ is a covering map (6.26, 6.29). σ': $(E', d) \to (B, g_0 \,|\, B)$ is a distance decreasing map (4.5, 6.16), and it is a local isometry by 6.13. $\tilde{\sigma}'$: $(E', d) = (UN\tilde{A}_p, \measuredangle_0) \to (\tilde{B}, \tilde{g}_0 \,|\, \tilde{B})$ is a local isometry by 6.13, where $\tilde{\sigma}'(v) = \exp_p \pi v/2$. If $n \geqslant 4$, then $\pi_1(B, q') = \pi_1(M, q')$ by 6.31, $\pi_1(\tilde{B}, q) = 0$, and $\tilde{\sigma}'$: $(E', d) \to (\tilde{B}, \tilde{g}_0 \,|\, \tilde{B})$ is an isometry since $\tilde{B}$ is π-convex. If $n = 3$, then see 6.37.1. Hence $L(p, q)$ contains only one vector, so does $L(q, p)$. So, $UN\tilde{B}_q \to \tilde{A}$ is an isometry. Consequently $d(\tilde{M}, \tilde{g}_0) = \pi$.

6.38.3. $\tilde{g}_0$ is C^1 a priori, so Toponogov's maximal diameter theorem is not applicable. $\tilde{M} = \overline{N}(\tilde{A}, \pi/2) = \overline{N}(\tilde{B}, \pi/2)$ (6.36). Pick $p_1, p_2 \in \tilde{A}$ with $\tilde{d}_0(p_1, p_2) = \pi$. $\tilde{A} \subseteq \overline{N}(\{p_1, p_2\}, \pi/2)$. $\forall q \in \tilde{M}$, $\exists q_1 \in \tilde{A}$ with $\tilde{d}_0(q, q_1) = \tilde{d}_0(q, \tilde{A}) \leqslant \pi/2$. $\exists \mathrm{mg}(q_1, q)$ γ_1, $\mathrm{mg}(q_1, \{p_1, p_2\})$ γ_2, both with length $\leqslant \pi/2$. $\gamma_1'(q_1) \in UN\tilde{A}$, $\gamma_2 \subseteq \tilde{A}$; so, by 4.5, $d(q, \{p_1, p_2\}) \leqslant \pi/2$, and $\overline{N}(\{p_1, p_2\}, \pi/2) = \tilde{M}$. Let $C = \{q \in \tilde{M} \,|\, \tilde{d}_0(p_i, q) = \pi/2$ for $i = 1, 2\}$. $\tilde{B} \subseteq C$ and $C_1 = C \cap \tilde{A}$ is a great $(a-1)$ sphere in $\tilde{A} = S^a(1)$. C is π-convex and the union of all minimal geodesics of length $\pi/2$ between C_1 and $\tilde{B}$. It is a connected totally geodesic, $b + (a-1) + 1 = n - 1$ dimensional submanifold of $\tilde{M}$. $\partial C = \varnothing$, by proof similar to 6.35.3.1 and C being the union of closed geodesics by using 6.38.2. $\{p_1, p_2\} = \{q \in M \,|\, \tilde{d}_0(q, C) = \pi/2\}$ since $\tilde{B} \cup C_1 \subseteq C$ and 6.36. Define μ: $L(p_1, C) \to C$ by $\mu(v) = \exp_{p_1} \pi v/2$. One can apply 6.13, 5.12, 6.34.3 to C and p_1 to obtain the following:

(i) μ is 1-1 on $\mu^{-1}(C_1 \cup \tilde{B})$ by 6.38.2, so it is 1-1 on $L(p_1, C)$.

(ii) μ is a local isometry and $L(p_1, C)$ is complete in UM_{p_1}.

(iii) $L(p_1, C)$ is totally geodesic $n-1$ dimensional submanifold of UM_{p_1}.

Hence μ is an isometry from $(UM_{p_1}, d) = S^{n-1}(1)$ onto $(C, \tilde{g}_0 \,|\, C)$.

6.38.4. For any $q_1, q_2 \in \tilde{M}$, pick mg(p_1, p_2)'s γ_1 and γ_2 with $q_i = \gamma_i(r_i)$, $i = 1, 2$.

$$\tilde{d}_0(\gamma_1(\pi/2), \gamma_2(\pi/2)) = \measuredangle_0(\gamma_1'(0), \gamma_2'(0)) = \measuredangle_0(\gamma_1'(\pi), \gamma_2'(\pi)) := \alpha,$$

by C and $\{p_1, p_2\}$ being π-convex dual pair. $r_3 := d_0(q_1, q_2) \leqslant \rho(\alpha, r_1, r_2; 1)$.

6.38.5. Claim. $r_3 = \rho(\alpha, r_1, r_2; 1)$.

Case (i). $r_1, r_2, \alpha \leqslant \pi/2$. This follows by applying 6.12 twice, starting with the triangle $p_1, \gamma_1(\pi/2), \gamma_2(\pi/2)$.

Case (ii). $r_1, r_2 \leqslant \pi/2 \leqslant \alpha \leqslant \pi$. Choose θ any mg(q_1, q_2) in $\overline{B}(p_1, \pi/2)$ by the convexity of C. $\forall q \in \tilde{M}$, there exists a unique mg(p_1, p_2) γ_q which contains q. Pick $0 = t_0 < t_1 < \cdots < t_l = d(q_1, q_2)$, $s_i = \theta(t_i)$, $\alpha_i = \measuredangle_0(\gamma_{s_i}'(0), \gamma_{s_{i+1}}'(0))$ such that $\alpha_i \leqslant \pi/2$.

$$\pi \geqslant r_3 = \sum d_0(s_i, s_{i+1}) = \sum \rho(\alpha_i, d_0(p_1, s_i), d_0(p_1, s_{i+1}); 1)$$
$$\geqslant \rho\left(\sum \alpha_i, r_1, r_2; 1\right) \geqslant \rho(\alpha, r_1, r_2; 1),$$

by (i), $r_1 = d_0(p_1, s_0)$, $r_2 = d_0(p_1, s_l)$, $\pi \geqslant \sum \alpha_i \geqslant \alpha$.

Case (iii). $r_1, r_2 \geqslant \pi/2$, $0 \leqslant \alpha \leqslant \pi$. Using (i) and (ii) for p_2: $r_3 = \rho(\alpha, \pi - r_1, \pi - r_2; 1) = \rho(\alpha, r_1, r_2; 1)$.

Hence both $\overline{B}(p_1, \pi/2)$ and $\overline{B}(p_2, \pi/2)$ are isometric to hemispheres in $S^n(1)$. C separates $\tilde{M}$ into two open connected sets.

Case (iv). $r_1 > \pi/2$, $r_2 < \pi/2$. Choose any mg(q_1, q_2) θ and let $\{q_3\} = \theta \cap C$. Using (i) and (ii) for each piece of θ in $\overline{B}(p_i, \pi/2)$, and using the inequalities of (ii) for $l = 2$ one obtains the claim.

6.38.6. Hence $(\tilde{M}, \tilde{g}_0)$ is locally isometric to $S^n(1)$ and homeomorphic to S^n. One constructs an isometry from $S^n(1)$ onto $(\tilde{M}, \tilde{g}_0)$, using $\exp_{p_1}$ and 6.38.5. So $\tilde{g}_0$ is C^∞ and so is g_0.

7. Proofs of Theorems IIA and IIB

They will be proved together.

7.1. Let $K \geqslant 4$, $n \geqslant 2$, $\delta > 0$ be given. If a smooth n-dimensional manifold M admits a C^∞-Riemannian metric g which satisfies (i)–(v) below then we say that M satisfies condition (K, n, δ).

(i) $1 \leqslant K(M, g) \leqslant K$.

(ii) $\pi_1(M, p) = 0$. $H^*(M, \mathbf{Z}) = \mathbf{Z}[x]/x^{k+1}$, $x \in H^\lambda(M, \mathbf{Z})$, $n = k\lambda$, $\lambda = 2$, 4, or 8, $k = k[M] \geqslant 2$, n even; if $\lambda = 8$ then $k = 2$ and $n = 16$.

(iii) $\pi/\sqrt{K} \leqslant i(M, g) \leqslant d(M, g) \leqslant \pi/2$.

(iv) If $k[M] \geqslant 2$, then $\exists p_1, p_2, p_3 \in M$ with $d(p_i, p_j; g) \geqslant \pi/2 - \delta$ for $1 \leqslant i < j \leqslant 3$.

(v) If $k[M]=2$, then $\forall p_1, p_2\ \exists p_3 \in M$ with $d(p_1, p_2; g) \geqslant \pi/2 - \delta$ implies that $d(p_i, p_3) \geqslant \pi/2 - \delta$ for $i = 1$ and 2.

7.2. Let K and n be fixed. There are finitely many diffeomorphism types of manifolds satisfying condition $(K, n, \pi/2)$ by [6], [7], [31]. Clearly there exists such diffeomorphism classes. Let $M_1, M_2, \cdots, M_l$ represent all such distinct classes. Define $\xi_i = \xi[M_i] = \inf\{\delta \mid M_i$ satisfies condition $(K, n, \delta)\}$ for $1 \leqslant i \leqslant l$. Also define $\delta_1(K, n) = \min(\{\xi_i \mid \xi_i \neq 0,\ 1 \leqslant i \leqslant l\} \cup \{\delta_0(K, n)\}$. Then $\delta_1(K, n) > 0$.

7.3. Let (M, g) be a C^∞-Riemannian manifold satisfying the hypothesis. $H^\lambda(M, \mathbf{Z}) \neq 0$, so $d(M, g) \leqslant \pi/2$ by [21]. M satisfies condition (K, n, δ) for $\delta < \delta_1$, hence $\xi[M] = 0$. Let g_m be a sequence of C^∞ metrics with (M, g_m) satisfying condition $(K, n, 1/m)$. One extracts a convergent subsequence of g_m converging to a limit metric in the sense of 4.1. g_0 satisfies all properties obtained in §§4–5. Let d be distance function for g_0.

7.4. Claim. (M, g_0) is isometric to $\mathbf{C}P^k$, $\mathbf{H}P^k$, or $\mathrm{Ca}P^2$ with their standard metrics, and g_0 is a C^∞-Riemannian metric.

7.4.1. By compactness and $g_m \to g_0$, for $k \geqslant 2$. $\exists p_1, p_2, p_3 \in M$ with $d(p_i, p_j) = \pi/2$ for $1 \leqslant i < j \leqslant 3$, and for $k = 2$ $\forall p_1, p_2$, $\exists p_3 \in M$ with $d(p_1, p_2) = \pi/2$ implies that $d(p_1, p_3) = d(p_2, p_3) = \pi/2$. Obviously the hypothesis of Theorem I is satisfied. Let $D = \{p_1, p_2\}'$ and $C = D'$, be dual convex sets as in 6.1.5. If $\partial C = \partial D = \varnothing$. then 7.4 holds by 6.35. So we may assume that one has boundary. Apply 6.9.4 to C, D to obtain C_1, D_1. By 6.10, 6.9.4, only one has boundary. Recall 6.21. If $\partial C_1 = \varnothing$ and $D_1 = \{p_0\}$, then let $C_1 = A$, $D_1 = B$ and replace p_3 with p_0. If $\partial D_1 = \varnothing$ and $C_1 = \{p_0\}$, then (i) $p_0 \notin \{p_1, p_2\}$ and $\{p_1, p_2\} \cap \{p_0\}' = \varnothing$ (4.6.2, 6.4.1, 6.9.2), (ii) let γ be $\mathrm{umg}(p_0, p_1)$, and $p_4 = \gamma(\pi/2) \in D_1$, (iii) $d(p_4, p_1) < \pi/2$, $d(p_1, p_3) = \pi/2$, $p_3 \in D_1$, $\gamma'(\pi/2) \in UND_1$, so $d(p_4, p_3) = \pi/2$ by 4.5, (iv) let $A = D_1$, $B = C_1$, and replace p_1, p_2, p_3 with p_3, p_4, p_0 respectively. Hence, one may assume that $\exists p_1, p_2, p_3 \in M$ with $p_1, p_2 \in A$, $\partial A = \varnothing$, $\{p_3\} = B$, $d(p_i, p_j) = \pi/2$, $1 \leqslant i < j \leqslant 3$. One constructs dual convex sets A_1 and A_2 in A with $p_1 \in A_1$, $p_2 \in A_2$ satisfying 6.9.4. Let $B_2 = \{q \in M \mid d(q, A_1) = \pi/2\}$ and $B_1 = \{q \in M \mid d(q, B_2) = \pi/2\}$. B_1 and B_2 are dual convex sets in M, $A_1 = B_1$, and $\{p_3\} \cup A_2 \subseteq B_2$.

7.4.2. Case for $\partial A_1 = \varnothing$. Suppose $\partial B_2 \neq \varnothing$. Let q be at maximal distance from ∂B_2. $q \notin A_2 \cup \{p_3\}$ by 4.6.2, 6.4.1, 6.9.2. Let γ be a normal geodesic with $\gamma(0) = \gamma(\pi) = p_3$, $\gamma(c) = q$, $\gamma(\pi/2 + c) = q''$, and $\gamma(\pi/2) = q' \in A_2 \subseteq A \cap B_2$, by 6.22. Apply 6.9.4 to A_1, B_2 to obtain the dual convex pair D_1, D_2 in M, with $D_1 \supseteq A_1$, $D_2 \subseteq B_2$, $\partial D_2 \neq \varnothing$. Then $\partial D_1 = \varnothing$ by 6.10, $D_2 = \{q\}$, and $q'' \in D_1$ by 6.22. So $d(q, q'') = \pi/2$, and $\gamma'(0) = \gamma'(\pi)$. Hence B_2

contains the closed geodesic γ, which is not possible by 6.8, 6.8.1, 6.9 (similarly in 6.35.3.1). So $\partial B_2 = \varnothing$, by a proof by contradiction. Claim 7.4 holds and $k \geqslant 3$ by 6.35. Similarly if $\partial A_2 = \varnothing$.

7.4.3. Case for $\partial A_1 \neq \varnothing$ and $\partial A_2 \neq \varnothing$. By 6.10, A is homeomorphic to a sphere, so to S^λ by 6.22. $k = 2$. Apply 6.9.4 to A_1, B_2 to obtain the dual convex pair C_1, C_2 in M with $C_1 \subseteq A_1$, $\partial C_1 \neq \varnothing$, $B_2 \subseteq C_2$. Then $\partial C_2 = \varnothing$ by 6.10, $C_1 = \{q_0\}$, and C_2 is a homotopy λ-sphere and C^1 submanifold by 6.20.2, 6.21, and 6.22. $C_3 = \exp_{p_2}[0, \pi/2] \cdot L(p_2, p_3) \subseteq B_2$ and C_3 is homeomorphic to S^λ. Hence $C_3 = B_2 = C_2$ and $A_2 = \{p_2\}$. Similarly, $A_1 = \{p_1\}$, $\forall q \in A$, $L(q, p_3) = UNA_q$ by 6.21. $\exists q' \in A$ with $d(q, q') = \pi/2$ by 7.4.1. Construct dual convex sets $\{q\}$ and $A_3 \supseteq \{p_3, q'\}$. Hence $L(p_3, q) = UN(A_3)_{p_3} \cong S^{\lambda-1}(1)$. The fiber bundle $E' = UM_{p_3} = L(p_3, A) = S^{2\lambda-1}(1) \xrightarrow{\sigma} A^\lambda$ constructed as in 6.26 with $\dim B = 0$ has fibers of great spheres. Equidistancy follows 6.34.1. By a similar proof of 6.34.3–6.34.6, and using [12] or [13], this equidistant fibration of $S^{2\lambda-1}(1)$ by great spheres $S^{\lambda-1}(1)$ is congruent to a Hopf fibration: $S^{\lambda-1}(1) \hookrightarrow S^{2\lambda-1}(1) \to S^\lambda(4)$, where $\lambda = 2, 4$, or 8. A is isometric to $S^\lambda(4)$ as in 6.34.

A is the cutlocus of p_3 by 6.21. Given any $q_1 \in M$, $\exists q_2, q_3 \in M$ with $d(q_i, q_j) = \pi/2$ for $1 \leqslant i < j \leqslant 3$. If $q_1 = p_i$, then there is nothing to prove. If $p_3 \neq q_1$, then let $q \in A$ be with $d(p_3, q_1) + d(q_1, q) = \pi/2$. $\exists q_2 \in A$ with $d(q_2, q) = \pi/2$. $d(q, p_3) = \pi/2$, $d(q_2, q_1) = \pi/2$ by 4.5, and q_3 exists by 7.4.1. Repeat 7.4.1 for $q_i : D = \{q_1, q_2\}'$, $C = D'$. $\partial C = \partial D = \varnothing$ cannot occur by $k = 2$. If $\partial C_1 = \varnothing$, then $\{q_1, q_2\}$ is an antipodal pair in $C_1 = S^\lambda(4)$ and $q_0 = q_3$ since otherwise one would obtain $q_5 \in C_1$ with $d(q_5, q_3) = d(q_3, C_1) < \pi/2$, and $d(q_3, \{q_1, q_2\}) < \pi/2$ by 4.5 and $d(q_5, \{q_1, q_2\}) \leqslant \pi/4$. The case of $\partial D_1 = \varnothing$ cannot occur; since otherwise: q_4, q_3 would be an antipodal pair in $D_1 = S^\lambda(4)$, similarly $d(q_4, q_2) = d(q_2, D_1)$, q_0, q_4, q_1, q_2, lie on a closed geodesic by $d(q_1, q_2) = \pi/2$, $q_4, q_0 \in C$, $D = \{q_3\}$ which is contradictory with itself: $D_1 = D$. Hence given q_1, q_2, q_3 with $d(q_i, q_j) = \pi(1 - \delta_{ij})/2$, we can choose A with $q_1, q_2 \in A$, and $\{q_3\}$, A form a dual convex pair. Using this one can prove that:

(i) $i(M, g_0) = \pi/2 = d(M, g_0)$.

(ii) $\forall q_1 \in M$, $C(q_1) =$ cutlocus of q_1 with respect to g_0 is a totally geodesic submanifold of M, isometric to $S^\lambda(4)$.

(iii) $\forall q_1 \in M$, $\forall q_2 \in C(q_1)$, the union of all $\mathrm{mg}(q_1, q_2)$ forms a convex set with no boundary isometric to $S^\lambda(4)$ in which q_1 and q_2 are antipodal.

(iv) Any geodesic of M is a closed geodesic of least period π.

(v) $\forall q_1 \in M$, $\forall q_2 \in C(q_1)$ $\forall \mathrm{mg}(q_1, q_2)$ γ_1, $\forall \gamma_2$ a geodesic in $C(q_1)$ passing through q_2, $\exists$ a unique totally geodesic 2-surface L containing q_1, q_2, γ_1, and

γ_2, locally isometric to $S^2(1)$. L is isometric to $\mathbf{R}P^2(1)$ by using (iv) and 6.13.

One follows [2, pp. 148–150] to show that (M, g_0) is a compact symmetric space of rank 1 with smooth metric g_0. The rest follows from the classification of such spaces ([1], [2], [7]).

7.5. Theorems IIA and IIB follow 7.4.

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