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GEOMETRY OF MAPS BETWEEN GENERALIZED FLAG MANIFOLDS

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A classical problem in Differential Geometry is the construction and classification of *minimal immersions* between Riemannian manifolds. A recently studied variant of this problem is obtained on replacing minimal immersions by *harmonic maps*. If the manifolds are homogeneous with respect to the actions of certain Lie groups, and are equipped with homogeneous Riemannian metrics, one naturally expects the theory of Lie groups to play a role. We shall use this principle, in a very specific situation, to produce new examples of harmonic maps.

In §1 we summarize well-known facts concerning homogeneous geometric structures on homogeneous spaces G/H, and in §2 we discuss the second fundamental form of a map $G/H \to G'/H'$ which is induced by a homomorphism $\theta: G \to G'$. Our basic result appears in §3 (Theorem 3.4), this being a necessary and sufficient algebraic condition for such a map to be harmonic, when G/H and G'/H' are generalized flag manifolds and G' is the unitary group. An important example is discussed in §4, namely that of the "higher order Gauss maps" of the maximal projective weight orbit of an irreducible representation of G. The harmonic maps which arise this way are very special, being "homogeneous," but in §5 we show how they may be modified to produce large families of "nonhomogeneous" examples.

Lest this program seem too uninspiring, we shall attempt to give some justification and to point out connections with other problems of current interest. The principal motivation was provided by the paper [15], in particular the results concerning harmonic maps from $\mathbf{C}P^1$ to $\mathbf{C}P^n$, and the author is grateful to Professor James Eells and John Wood for discussing their results. If $f: \mathbf{C}P^1 \to \mathbf{C}P^n$ is holomorphic, there are the well-known holomorphic "associated curves" $f_0, f_1, f_2, \cdots$ (see [18, Chapter 2, §4]); f_i is the map from $\mathbf{C}P^1$

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into the Grassmannian $\mathrm{Gr}_{i+1}(\mathbf{C}^{n+1})$ of $(i+1)$-planes in $\mathbf{C}^{n+1}$ defined locally (using homogeneous coordinates) by the formula

$$f_i(z) = f(z) \wedge f'(z) \wedge \cdots \wedge f^{(i)}(z).$$

In [15] it is shown that the map

$$f_{(i)} : \mathbf{C}P^1 \to \mathbf{C}P^n, \qquad f_{(i)}(z) = f_{i-1}(z)^{\perp} \cap f_i(z),$$

is harmonic with respect to the standard Kähler metrics of $\mathbf{C}P^1$ and $\mathbf{C}P^n$, and (what is more surprising) conversely, that *any* harmonic map of $\mathbf{C}P^1$ into $\mathbf{C}P^n$ must arise by this construction. Thus one has a description of harmonic maps in terms of holomorphic maps. Since the appearance of [15], further constructions of harmonic maps from Riemann surfaces into Hermitian symmetric spaces have been given by various authors, and some idea of these developments may be obtained from the papers [9], [10], [11], [25]. (A classification of harmonic maps from $\mathbf{C}P^1$ into a complex Grassmannian has recently been achieved by S. S. Chern and J. G. Wolfson and by F. E. Burstall and J. C. Wood, for example.) The relevance of the present article is that it provides a generalization in a different direction: the constructions of §§4 and 5, when applied to the Lie group $G = SU_2$, give (without introducing local coordinates) a description of all harmonic maps from $\mathbf{C}P^1$ to $\mathbf{C}P^n$. This group-theoretic approach also provides a connection between [15] and earlier work of W.-Y. Hsiang, H. B. Lawson, M. do Carmo, and N. R. Wallach on minimal immersions of homogeneous spaces (see [28], for example).

A second reason for wishing to have explicit examples of harmonic maps comes from Mathematical Physics. The problem of studying the critical points of some functional on a space of maps is difficult to treat by general methods, yet considerable progress has been made in recent years in particular cases, for example with the Yang-Mills functional and the Yang-Mills-Higgs functional (see [3], [4], [27]). The energy functional (defined on the space of smooth maps between two Kähler manifolds, say), whose critical points are precisely the harmonic maps, is more tractable from the point of view of calculations than these two examples, but appears to share some of their significant properties. For example, the maps of minimum energy (in a fixed homotopy class) are distinguished geometrically—they are the holomorphic (or antiholomorphic) maps. Moreover, they are characterized as the solutions to the Cauchy-Riemann equations, which are first order, whereas a general critical point is a solution of the "harmonic map equation" $\operatorname{tr} \nabla df = 0$, which is second order. The minima may often be used to construct nonminimal critical points, and as we remarked above, *all* nonminimal critical points are obtained in the case of maps from $\mathbf{C}P^1$ to $\mathbf{C}P^n$; a quite different situation where this happens (for the

Yang-Mills functional) is described in [3]. Further discussion of these features of the energy functional may be found in [7], [8], [19], while a deeper connection with the Yang-Mills and Yang-Mills-Higgs functionals is given in [2].

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1. Invariant geometric structures on flag manifolds

A *flag manifold* is a homogeneous space G/T, where G is a compact simple Lie group and T is a maximal torus. For example, if G is the special unitary group SU_n, and T consists of the subgroup of diagonal matrices, G/T may be identified with the set of "full flags" $\{0\} = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \mathbf{C}^n$, where E_i is a subspace of $\mathbf{C}^n$ of dimension i. This will be denoted $F(1,2,\cdots,n)$ or simply F_n. A *generalized flag manifold* is a homogeneous space $G/C(S)$ where $C(S)$ is a centralizer of a (not necessarily maximal) torus S. For example, if $G = SU_n$, $C(S)$ must be conjugate to a subgroup of the form $S(U_{n_1} \times \cdots \times U_{n_k})$, where the positive integers $n_1,\cdots,n_k$ satisfy $n_1 + \cdots + n_k = n$. If $m_i = n_1 + \cdots + n_i$, the quotient $SU_n/S(U_{n_1} \times \cdots \times U_{n_k})$ may be identified with the set $F(m_1,\cdots,m_k)$ of "partial flags" $\{0\} = E_0 \subset E_{m_1} \subset \cdots \subset E_{m_{k-1}} \subset E_{m_k} = \mathbf{C}^n$. The complex Grassmannians ($\mathrm{Gr}_r(\mathbf{C}^n) = F(r,n)$) and projective spaces ($\mathbf{C}P^{n-1} = F(1,n)$) are familiar examples; these turn out to be the only spaces, amongst those of the form $F(m_1,\cdots,m_k)$, which are Hermitian symmetric. In this paper we shall exploit the richer geometrical properties of the spaces F_n (or, more generally, G/T) which may be thought of as at the "opposite extreme" to a symmetric space. Our basic reference on homogeneous spaces is [5].

The negative of the Killing form of G is a positive definite inner product $\langle\ ,\ \rangle$ on the Lie algebra $L(G) = \mathfrak{g}$, and if H is a Lie subgroup with Lie algebra $L(H) = \mathfrak{h}$, one has an orthogonal decomposition

$$L(G) = L(G/H) \oplus L(H)$$

which is in fact an H-module decomposition for the restriction to H of the adjoint representation of G. The tangent bundle $T(G/H)$ of G/H is the homogeneous bundle associated to the principal bundle $G \to G/H$ via the H-module $L(G/H)$. We shall sometimes write $G/H = M$, $L(G/H) = L(M) = \mathfrak{m}$, to simplify notation. An "H-invariant geometric structure" of $T(G/H)$

is thus obtained by specifying one for $L(G/H)$, and it is with such geometric structures that we shall be concerned. We assume that $H = C(S)$ for some subtorus of the maximal torus T; there is a T-module decomposition

$$L(G/T) \otimes \mathbf{C} = \sum_{\alpha \in \Delta} E_\alpha,$$

where $\Delta \subset L(T)^*$ is the set of roots and E_α is the root space corresponding to $\alpha \in \Delta$. We refer to [1], [21] for elementary properties of roots, and for Lie theory in general. One then has

$$L(G/H) \otimes \mathbf{C} = \sum_{\alpha \in \Delta_{G/H}} E_\alpha,$$

where $\Delta_{G/H} \subset \Delta$ is the subset of complementary roots (see §1 of [5]).

An H-invariant *almost complex structure* on G/H corresponds to an H-invariant endomorphism J of $L(G/H)$ with $J^2 = -I$. If $H = T$, such endomorphisms correspond to decompositions $\Delta = \Delta^+ \cup \Delta^-$ with the property $\Delta^- = \{-\alpha \mid \alpha \in \Delta^+\}$, whereby the decomposition $L(G/T) \otimes \mathbf{C} = L(G/T)_{1,0} \oplus L(G/T)_{0,1}$ into $(1,0)$ and $(0,1)$ parts is given by

$$L(G/T) \otimes \mathbf{C} = \left(\sum_{\alpha \in \Delta^+} E_\alpha \right) \oplus \left(\sum_{\alpha \in \Delta^-} E_\alpha \right).$$

The almost complex structure is integrable precisely when Δ^+ is the set of positive roots with respect to a choice of fundamental Weyl chamber in $L(T)$ [5]. Throughout this article, we fix a fundamental Weyl chamber D in $L(T)$, with dual chamber D^* in $L(T)^*$ and set of positive roots Δ^+, and we denote by J the corresponding complex structure on G/T. The decomposition of $L(G/T)$ as a real T-module may conveniently be written as

$$L(G/T) = \sum_{\alpha \in \Delta^+} V_\alpha,$$

where $V_\alpha \otimes \mathbf{C} = E_\alpha \oplus E_{-\alpha}$. A general ($T$-invariant) almost complex structure is specified by whether or not it agrees with J on each V_α, so there are $2^{|\Delta^+|}$ possibilities. Note that if $L(G/H)$ is an irreducible H-module, and G/H admits an H-invariant almost complex structure, then it admits only two such (and these are conjugate).

Similar comments apply to *invariant Riemannian metrics* on G/H, which we shall identify with invariant inner products on $L(G/H)$. A T-invariant metric $\langle\langle\ ,\ \rangle\rangle$ is specified by how it differs from $\langle\ ,\ \rangle$ (minus the Killing form) on each V_α. Since an irreducible module has a one-dimensional family of invariant metrics, $\langle\langle\ ,\ \rangle\rangle = \langle\ ,\ \rangle_r$ for some function $r : \Delta^+ \to \mathbf{R}^+$, where $\langle\ ,\ \rangle_r$ is the metric defined by $\langle x, y\rangle_r = r(\alpha)\langle x, y\rangle$ for all $x, y \in V_\alpha$. (The same notation will be used for an inner product on $L(G/H)$ and the induced inner product on $L(G/H)^*$.) Clearly $\langle\ ,\ \rangle_r$ is Hermitian with respect to J, hence one has a nondegenerate 2-form ω_r defined by

$$\omega_r(X,Y) = \langle X, JY\rangle_r, \qquad X,Y \in L(G/T).$$

If $\langle\ ,\ \rangle_r$ is a Kähler metric, ω_r is given by

$$\omega_r(X,Y) = \gamma([X,Y]_{L(T)}), \qquad X,Y \in L(G/T),$$

for some $\gamma \in D^*$, because each class in $H^2(G/T; \mathbf{R}) = L(T)^*$ contains a unique invariant 2-form (see [17]). In this case, $r(\alpha) = \langle \gamma, \alpha\rangle$.

More generally one may consider an H-invariant *affine connection* on G/H, i.e., an H-invariant covariant derivative operator

$$\nabla : \Gamma(T(G/H)) \to \Gamma(T^*(G/H) \otimes T(G/H)).$$

This corresponds to an H-module transformation

$$\Lambda : L(G/H) \otimes L(G/H) \to L(G/H)$$

as we shall explain in some detail, following [23], since this provides an opportunity to establish notation for later use. First, $L(G)$ refers to the Lie algebra of *left-invariant* vector fields on G, and G acts naturally on the coset space $G/H = \{gH \mid g \in G\}$ on the left. If $X \in L(G)$, we denote by X^* the vector field on G/H defined by $X^*_{gH} = (d/dt)(\exp tX \cdot gH)|_{t=0}$. The reader should beware of sign errors in the literature stemming from the fact that $[X^*, Y^*] = -[X,Y]^*$. If L_Z denotes the Lie derivative with respect to the vector field $Z \in \Gamma(T(G/H))$, the operator

$$\nabla_Z - L_Z : \Gamma(T(G/H)) \to \Gamma(T(G/H))$$

is linear over functions (i.e., it defines a $(1,1)$-tensor). With $Z = X^*$ for $X \in L(G)$, homogeneity of ∇ implies that this $(1,1)$-tensor is determined by its behavior at the identity coset $o \in G/H$. By definition, $(\nabla_Z)_m$ depends only on Z_m (for any $m \in G/H$). These remarks show that the operator ∇ is determined by the H-invariant linear map Λ defined by

$$\Lambda(X,Y) = \left(\nabla_{X^*}Y^* - L_{X^*}Y^*\right)_o, \qquad X,Y \in L(G/H).$$

Conversely, any such map Λ defines an H-invariant connection (see [23, Volume II, Chapter 10, Theorem 2.1]). This is the required correspondence.

The *canonical connection* is given by $\Lambda = 0$; this is characterized by the property that its geodesics through $o \in G/H$ are given by exponentiating one-parameter subgroups of G, and parallel translation is induced by the natural action of G on G/H. (Alternatively, the splitting $L(G) = L(G/H) \oplus L(H)$ defines a connection in the principal bundle $G \to G/H$, in the sense of

[23], from which the canonical connection is obtained via the standard procedure for passing from principal bundles to associated vector bundles.) The Riemannian (or Levi-Cività) connection associated to the metric $\langle\ ,\ \rangle$ (introduced above) will be called the *Killing connection*; this is given by $\Lambda(X,Y) = \frac{1}{2}[X,Y]_{L(G/H)}$ (the component of $\frac{1}{2}[X,Y]$ lying in $L(G/H)$). To prove this, first use the conditions $W\cdot\langle U,V\rangle = \langle\nabla_W U,V\rangle + \langle U,\nabla_W V\rangle$ ("compatibility with the metric") and $\nabla_U V = \nabla_V U + [U,V]$ ("torsion free") to obtain the identity

$$2\langle\nabla_U V,W\rangle = U\cdot\langle V,W\rangle + V\cdot\langle W,U\rangle - W\cdot\langle U,V\rangle + \langle W,[U,V]\rangle + \langle V,[W,U]\rangle - \langle U,[V,W]\rangle.$$

Now take $U = X^*$, $V = Y^*$, $W = Z^*$ for $X,Y,Z\in L(G/H)$. Since $\langle Y^*,Z^*\rangle$ is constant, $X^*\cdot\langle Y^*,Z^*\rangle = 0$ (etc.), and since $\langle\ ,\ \rangle$ is bi-invariant, $\langle Y^*,[Z^*,X^*]\rangle + \langle[Z^*,Y^*],X^*\rangle = 0$. The formula for Λ follows.

The Riemannian connection associated to an invariant Kähler metric on G/H will be referred to as a *Kähler connection*; the covariant derivative operator here is obtained by "realifying" a complex operator

$$D:\Gamma(T(G/H)_{1,0})\to\Gamma\big((T(G/H)\otimes\mathbf{C})^*\otimes T(G/H)_{1,0}\big)$$

which may be written as $D = D_{1,0}\oplus D_{0,1}$ using the decomposition $T(G/H)\otimes\mathbf{C} = T(G/H)_{1,0}\oplus T(G/H)_{0,1}$. Hence the connection is determined by complex H-module transformations

$$\Lambda_{1,0}:L(G/H)_{1,0}\otimes L(G/H)_{1,0}\to L(G/H)_{1,0},$$
$$\Lambda_{0,1}:L(G/H)_{0,1}\otimes L(G/H)_{1,0}\to L(G/H)_{1,0},$$

which are defined in analogy with Λ. To determine these (following [17]), recall [18, Chapter 0, §5] that D is characterized in terms of the complex structure J and the Hermitian metric h of G/H by the properties:

(1) $D_{0,1} = \bar\partial$ (the "$\bar\partial$-operator" of the holomorphic bundle $T(G/H)_{1,0}$),

(2) $dh(X,Y) = h(DX,Y) + h(X,DY)$, $X,Y\in\Gamma(T(G/H)_{1,0})$.

If $X,Y\in L(G/H)_{1,0}$, one defines in the obvious way $X^*,Y^*\in\Gamma(T(G/H)_{1,0})$ and these are holomorphic vector fields. By (1), $(D_{0,1})_{\bar X^*}Y^* = \bar\partial_{\bar X^*}Y^* = 0$. Hence

$$\Lambda_{0,1}(\bar X,Y) = -\big(L_{\bar X^*}Y^*\big)_o = [\bar X,Y]_{L(G/H)_{1,0}},$$

which, it should be noted, depends only on the complex structure and not on h. By (2), $h(D_{X^*}Y^*,Z^*) + h(Y^*,D_{X^*}Z^*) = 0$ for $X,Y,Z\in L(G/H)_{1,0}$, which in terms of the underlying Riemannian metric gives $\langle D_{X^*}Y^*,\bar Z^*\rangle_r + \langle Y^*,D_{\bar X^*}\bar Z^*\rangle_r = 0$, hence

$$(D_{1,0})_{X^*}Y^* = -\big((D_{0,1})_{X^*}\big)^*Y^*,$$

where the adjoint is taken with respect to h.

Although the Riemannian connections described above are those of primary interest, several results in §3 will be stated for the Riemannian connection associated to the general T-invariant metric $\langle\ ,\ \rangle_r$ on G/T, so we shall need to know the corresponding map Λ. As in the calculation for the Killing connection, we find

$$2\langle\nabla_{X^*}Y^*, Z^*\rangle_r = \langle[X^*, Y^*], Z^*\rangle_r - \langle Y^*, [X^*, Z^*]\rangle_r - \langle X^*, [Y^*, Z^*]\rangle_r$$

from which it follows (see [23, Volume II, Chapter 10, §3]) that Λ may be written in the form $\Lambda(X, Y) = \frac{1}{2}[X, Y]_{L(G/T)} + U(X, Y)$. Here, $U \in (S^2L(G/T)^*) \otimes L(G/T)$ is determined by the condition

$$2\langle U(X,Y), Z\rangle_r = \langle X, [Z, Y]_{L(G/T)}\rangle_r + \langle Y, [Z, X]_{L(G/T)}\rangle_r.$$

An explicit formula for U will be given in §2.

Finally, we make some remarks on the particular flag manifold F_n, in the light of the preceding discussion. The description of F_n as the space of flags in $\mathbf{C}^n$ endows it naturally with the structure of a complex manifold; indeed it is a homogeneous space of the complex group $\mathrm{Gl}_n\mathbf{C}$. This agrees with the invariant complex structure defined by the choice of positive roots $\{x_i - x_j\}_{i>j}$ for SU_n (with respect to the standard maximal torus $S(U_1 \times \cdots \times U_1)$), and thus is determined in practice by choosing the standard ordered basis of $\mathbf{C}^n$. It would therefore have been more logical (if less conventional) to define F_n as the space of ordered n-tuples of lines in $\mathbf{C}^n$, which are mutually orthogonal for the standard Hermitian inner product (this space being naturally isomorphic to $SU_n/S(U_1 \times \cdots \times U_1)$). However, since we shall always consider F_n with the complex structure just defined, the original definition is satisfactory. There are tautologously defined holomorphic vector bundles on F_n whose fibers over a flag $\{0\} = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \mathbf{C}^n$ are $E_0, E_1, \cdots, E_n$ respectively. We denote these bundles by the same letters $E_0, E_1, \cdots, E_n$. There are corresponding line bundles $L_1, L_2, \cdots, L_n$, where L_i $(= (E_{i-1})^\perp \cap E_i)$ is the homogeneous bundle induced by the representation $\lambda_i: S(U_1 \times \cdots \times U_1) \to U_1$ given by projecting to the ith factor. One has $T_{1,0}F_n \cong \sum_{i>j} L_i \otimes L_j^*$. The corresponding $S(U_1 \times \cdots \times U_1)$-module decomposition is $L(F_n) \otimes \mathbf{C} = \sum_{i \neq j} \lambda_i \otimes \lambda_j^*$. As a subbundle of $F_n \times \mathbf{C}^{n+1}$, each bundle L_i acquires a Hermitian metric, hence so does $\sum_{i>j} L_i \otimes L_j^*$; the corresponding Riemannian metric on F_n may be described alternatively as that which is induced via the natural embedding $F_n \to \mathbf{C}P^{n-1} \times \cdots \times \mathbf{C}P^{n-1}$, where each $\mathbf{C}P^{n-1} \cong SU_n/S(U_1 \times U_{n-1})$ has the Riemannian metric $\langle\ ,\ \rangle$. From this one sees that the metric on F_n is $2\langle\ ,\ \rangle$. (Note that a quite different metric is obtained on F_n using the holomorphic embedding $F_n \to \mathrm{Gr}_1(\mathbf{C}^n) \times \mathrm{Gr}_2(\mathbf{C}^n) \times \cdots \times \mathrm{Gr}_{n-1}(\mathbf{C}^n)$; if each Grassmannian has the metric $\langle\ ,\ \rangle$, F_n acquires the Kähler metric corresponding to the weight

$$\gamma = \tfrac{1}{2}((n-1)x_n + (n-3)x_{n-1} + \cdots - (n-1)x_1),$$

i.e., half the sum of the positive roots. In general, $\gamma = \frac{1}{2}\Sigma_{\alpha \in \Delta^+} \alpha$ gives perhaps the most natural invariant Kähler metric on G/T, this being the essentially unique invariant Kähler-Einstein metric [17].)

The notation introduced here will be extended to the generalized flag manifold $F(m_1, \cdots, m_k)$, i.e., the tautologous holomorphic vector bundles will be denoted $E_0, E_{m_1}, \cdots, E_{m_k}$, and we shall write $L_i = (E_{m_{i-1}})^\perp \cap E_{m_i}$, this being a homogeneous vector bundle induced by the representation $\lambda_i : S(U_{n_1} \times \cdots \times U_{n_k}) \to U_{n_i}$.

2. The fundamental forms of homogeneous maps

Let $\theta : G \to G'$ be a homomorphism of compact Lie groups G, G' such that $\theta(H) \subset H'$ for certain subgroups H, H'. We write $L(G) = \mathfrak{g}$, $L(H) = \mathfrak{h}$, $L(G/H) = \mathfrak{m}$, with similar definitions for $\mathfrak{g}'$, $\mathfrak{h}'$, $\mathfrak{m}'$. Denote by $f_\theta : G/H \to G'/H'$ the induced map of homogeneous spaces. The derivative df_θ is determined by its restriction to $o \in G/H$, which is the H-module transformation $df_\theta : \mathfrak{m} \to \mathfrak{m}'$ given in terms of the Lie algebra homomorphism $\theta : \mathfrak{g} \to \mathfrak{g}'$ by

$$df_\theta(X) = \theta(X)_{\mathfrak{m}'}, \qquad X \in \mathfrak{m}.$$

We fix invariant connections on G/H, G'/H' given by Λ, Λ'. The iterated covariant derivative $\nabla^{i-1}(df_\theta) \in \Gamma(\otimes^i T^*(G/H) \otimes f_\theta^{-1} T(G'/H'))$ will be called the ith fundamental form of df_θ ($i \geqslant 2$). We use the same notation for the corresponding H-module transformation

$$\nabla^{i-1}(df_\theta) : \otimes^i \mathfrak{m} \to \mathfrak{m}'.$$

Lemma 2.1. *For X, $Y \in \mathfrak{m}$,*

$$\begin{aligned}\nabla(df_\theta)(X, Y) &= [\theta(X)_{\mathfrak{h}'}, \theta(Y)_{\mathfrak{m}'}] \\ &\quad + \Lambda'(\theta(X)_{\mathfrak{m}'}, \theta(Y)_{\mathfrak{m}'}) - \theta(\Lambda(X, Y))_{\mathfrak{m}'} \\ &= [\theta(X)_{\mathfrak{h}'}, df_\theta(Y)] + \Lambda'(df_\theta(X), df_\theta(Y)) - df_\theta(\Lambda(X, Y)).\end{aligned}$$

Proof. By definition, $\nabla(df_\theta)(X^*, Y^*) = \nabla_{X^*} df_\theta(Y^*) - df_\theta(\nabla_{X^*} Y^*)$. Note that $df_\theta(Y^*) = \theta(Y)^* \circ f_\theta$, so $\nabla_{X^*} df_\theta(Y^*) = (\nabla_{df_\theta(X^*)} \theta(Y)^*) \circ f_\theta$ (using the formula for the pull-back connection in §1 of [13]) $= (\nabla_{\theta(X)^* \circ f_\theta} \theta(Y)^*) \circ f_\theta$. Hence

$$\nabla(df_\theta)(X, Y) = \left(\nabla_{\theta(X)^*} \theta(Y)^*\right)_o - df_\theta\left(\nabla_{X^*} Y^*\right)_o.$$

The second term is equal to $\theta([X,Y]_{\mathfrak{m}})_{\mathfrak{m}'} - \theta(\Lambda(X,Y))_{\mathfrak{m}'}$, because of the formula for Λ in §1 and that for df_θ above. To evaluate the first term we need an expression for $(\nabla_{U^*}V^*)_o$ when $U, V \in \mathfrak{g}'$. Writing $U = U_{\mathfrak{h}'} + U_{\mathfrak{m}'}$, we may replace U^* by $(U_{\mathfrak{m}'})^*$ since $\nabla_A B$ is tensorial in A. Writing $V = V_{\mathfrak{h}'} + V_{\mathfrak{m}'}$, we obtain

$$(\nabla_{U^*}V^*)_o = \left(\nabla_{(U_{\mathfrak{m}'})^*}(V_{\mathfrak{h}'})^*\right)_o - [U_{\mathfrak{m}'}, V_{\mathfrak{m}'}]_{\mathfrak{m}'} + \Lambda'(U_{\mathfrak{m}'}, V_{\mathfrak{m}'})$$

on applying the formula for Λ' given in §1. Since $\nabla_A B - L_A B$ is tensorial in B, and $(V_{\mathfrak{h}'})^*_o = 0$, we have

$$\left(\nabla_{(U_{\mathfrak{m}'})^*}(V_{\mathfrak{h}'})^*\right)_o = \left[(U_{\mathfrak{m}'})^*, (V_{\mathfrak{h}'})^*\right]_o = -[U_{\mathfrak{m}'}, V_{\mathfrak{h}'}].$$

Thus $(\nabla_{U^*}V^*)_o = -[U_{\mathfrak{m}'}, V]_{\mathfrak{m}'} + \Lambda^*(U_{\mathfrak{m}'}, V_{\mathfrak{m}'})$. Setting $U = \theta(X)$, $V = \theta(Y)$, we obtain the required formula after a short calculation.

Examples. (1) For the canonical connections on G/H and G'/H',

$$\nabla(df_\theta)(X,Y) = [\theta(X)_{\mathfrak{h}'}, \theta(Y)_{\mathfrak{m}'}] = [\theta(X)_{\mathfrak{h}'}, df_\theta(Y)].$$

(2) For the Killing connections on G/H and G'/H',

$$\begin{aligned}\nabla(df_\theta)(X,Y) &= \tfrac{1}{2}([\theta(X)_{\mathfrak{h}'}, \theta(Y)_{\mathfrak{m}'}] - [\theta(X)_{\mathfrak{m}'}, \theta(Y)_{\mathfrak{h}'}]) \\ &= \tfrac{1}{2}([\theta(X)_{\mathfrak{h}'}, df_\theta(Y)] + [\theta(Y)_{\mathfrak{h}'}, df_\theta(X)]).\end{aligned}$$

Note that this is symmetric in X and Y, and that it coincides with the formula of the previous example when G/H, G'/H' are symmetric spaces.

It will be convenient to use "complex" notation in future. With this in mind, we fix a basis $\{e_\alpha\}_{\alpha\in\Delta}$ of $L(G/T)\otimes\mathbf{C}$ which satisfies:

(1) $e_\alpha \in E_\alpha$, $\alpha \in \Delta$,

(2) $\langle e_\alpha, e_\beta\rangle = -1$ if $\alpha + \beta = 0$, $= 0$ otherwise.

Here, $\langle\ ,\ \rangle$ denotes both minus the Killing form and its complex linear extension to $L(G)\otimes\mathbf{C}$. (In general, maps will be extended by complex linearity from now on, without change of notation or further comment.) Such a basis of $L(G/T)\otimes\mathbf{C}$ has the additional well-known properties:

(3) $[e_\alpha, e_{-\alpha}] = h_\alpha \in L(T)\otimes\mathbf{C}$,

(4) $[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}$ if $\alpha+\beta\in\Delta$, $= 0$ if $0 \neq \alpha+\beta \notin \Delta$.

In terms of this basis of $L(G/T)\otimes\mathbf{C}$ the map U, for which the Riemannian connection with respect to the metric $\langle\ ,\ \rangle_r$ is given by $\Lambda(X,Y) = \frac{1}{2}[X,Y]_{L(G/T)\otimes\mathbf{C}} + U(X,Y)$ (see §1), is readily computed as

$$U(e_\alpha, e_\beta) = \sum_{\gamma\in\Delta} c^\gamma_{\alpha,\beta}e_\gamma,$$

where

$$c^{\gamma}_{\alpha,\beta} = \begin{cases} 0 & \text{if } \gamma \neq \alpha + \beta, \\ \frac{1}{2}\big(r(|\beta|) - r(|\alpha|)\big)N_{\alpha,\beta}/r(|\alpha + \beta|) & \text{if } \gamma = \alpha + \beta, \end{cases}$$

(and where $|\alpha| = \alpha$ if $\alpha \in \Delta^+$, $|\alpha| = -\alpha$ if $-\alpha \in \Delta^+$). Thus, if $\langle\ ,\ \rangle_r$ is Kähler and $\alpha, \beta \in \Delta^+$, $U(e_{-\alpha}, e_\beta) = \frac{1}{2}[e_{-\alpha}, e_\beta]$. Hence $\Lambda_{0,1}(e_{-\alpha}, e_\beta) = [e_{-\alpha}, e_\beta]$, in agreement with the general results of §1. As an example, if $G/T = F_n$ and if for $\alpha = x_i - x_j$ we write $e_\alpha = e_{i,j}$, $r(|\alpha|) = r_{i,j}$, then Λ is given by

$$\Lambda(e_{i,j}, e_{k,l}) = \begin{cases} \frac{1}{2}\big((r_{i,l} - r_{i,j} + r_{j,l})/r_{i,l}\big)e_{i,l} & \text{if } j = k, \\ \frac{1}{2}\big((-r_{k,j} + r_{i,j} - r_{k,i})/r_{k,j}\big)e_{k,j} & \text{if } i = l, \\ 0 \quad \text{otherwise.} \end{cases}$$

If $\langle\ ,\ \rangle_r$ is Kähler, this simplifies further to

$$\Lambda_{1,0}(e_{i,j}, e_{k,l}) = \begin{cases} \big(r_{j,l}/r_{i,l}\big)e_{i,l} & \text{if } j = k, \\ \big(r_{k,i}/r_{k,j}\big)e_{k,j} & \text{if } i = l, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Lambda_{0,1}(e_{i,j}, e_{k,l}) = \begin{cases} e_{i,l} & \text{if } j = k,\ i > l, \\ -e_{k,j} & \text{if } i = l,\ k > j, \\ 0 & \text{otherwise.} \end{cases}$$

These formulas may now be used in conjunction with Lemma 2.1, to give further examples.

Recall that a map $f: M \to N$ of manifolds with connections is said to be *totally geodesic* if $\nabla(df) = 0$, and that if the connection on M is that associated to a Riemannian metric, f is said to be *harmonic* if $\operatorname{tr}\nabla(df) = 0$, where the trace is taken with respect to the metric on M. For general information on harmonic maps see [12], [13].

Lemma 2.2. *Let $H = T$. If the connection on G/T is that associated to the Riemannian metric $\langle\ ,\ \rangle_r$, then f_θ is harmonic if and only if the element $\sum_{\alpha\in\Delta}(1/r(|\alpha|))(e_\alpha \otimes e_{-\alpha})$ is in the kernel of the linear transformation $\nabla(df_\theta): L(G/T) \otimes L(G/T) \to L(G'/H')$.*

Proof. As a basis over $\mathbf{R}$ for $L(G/T) \subset L(G/T) \otimes \mathbf{C}$ one may take the elements

$$\big(1/\sqrt{2r(\alpha)}\big)(e_\alpha - e_{-\alpha}), \qquad \big(i/\sqrt{2r(\alpha)}\big)(e_\alpha + e_{-\alpha})$$

for $\alpha \in \Delta^+$. This is orthonormal with respect to $\langle\ ,\ \rangle_r$. Hence f_θ is harmonic if and only if the element

$$\sum_{\alpha\in\Delta^+} [(e_\alpha - e_{-\alpha})\otimes(e_\alpha - e_{-\alpha}) - (e_\alpha + e_{-\alpha})\otimes(e_\alpha + e_{-\alpha})]/r(\alpha)$$

$$= -\sum_{\alpha\in\Delta^+} (e_\alpha \otimes e_{-\alpha} + e_{-\alpha}\otimes e_\alpha)/r(\alpha)$$

is in the kernel of $\nabla(df_\theta)$. q.e.d.

We conclude this section with some generalities concerning osculating flags. If $E \subset F$ are bundles over a manifold M with connections ∇^E, ∇^F, the formula

$$(X, s) \to \nabla^F_X s - \nabla^E_X s, \qquad X \in \Gamma(TM),\ s \in \Gamma(E),$$

defines an element $B_2 \in \Gamma(\mathrm{Hom}(TM \otimes E, F))$ called the second fundamental form of E in F. Writing $E = E^{(1)}$, we define $E^{(2)} \subset F$ by

$$E^{(2)}_m = E^{(1)}_m + \mathrm{Im}(B_2)_m$$

for each $m \in M$; this is a bundle over the open subset of M consisting of points m for which $E^{(2)}_m$ has maximal dimension. The procedure may be repeated: given a flag $E^{(1)} \subset E^{(2)} \subset \cdots \subset E^{(i)} \subset F$, replace E by $E^{(i)}$ in the formula above to obtain $B_{i+1} \in \Gamma(\mathrm{Hom}(TM \otimes E^{(i)}, F))$, then define

$$E^{(i+1)}_m = E^{(i)}_m + \mathrm{Im}(B_{i+1})_m.$$

The construction terminates after a finite number of steps, and we call the resulting flag of bundles $E^{(1)} \subset E^{(2)} \subset \cdots$ (defined over an open subset of M) the osculating flag of E in F. More generally, a similar construction is possible if the inclusion $E \subset F$ is replaced by any bundle map. For example, if $f: M \to N$ is a map of manifolds with connections, we may take $E = TM$, $F = f^{-1}(TN)$, in which case the subbundle $E^{(i)}$ of $f^{-1}(TN)$ is generated by df and the fundamental forms $\nabla(df), \cdots, \nabla^{i-1}(df)$. For the special case $f_\theta: G/H \to G'/H'$, the bundles $E^{(i)}$ are defined everywhere, by homogeneity, and the osculating flag is determined by the corresponding flag of H-modules

$$df_\theta(\mathfrak{m}) = \mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \cdots \subset \mathfrak{m}'.$$

Proposition 2.3. *If G'/H' is a symmetric space with its canonical connection, the osculating flag satisfies*

$$\begin{aligned}\mathfrak{m}_i &= \mathfrak{m}_{i-1} + [\theta(\mathfrak{m}), \mathfrak{m}_{i-1}]_{\mathfrak{m}'} \\ &= \mathfrak{m}_{i-1} + [\theta(\mathfrak{m})_{\mathfrak{h}'}, \mathfrak{m}_{i-1}] \qquad (i \geqslant 2)\end{aligned}$$

(*and in particular is independent of the invariant connection on G/H*).

Proof. First, $\mathfrak{m}_2$ is spanned by $\mathfrak{m}_1$ and vectors of the form $(\nabla_{\theta(X)^*}\theta(Y)^*)_o$ for $X, Y \in \mathfrak{m}$ (see the proof of Lemma 2.1). As in Lemma 2.1,

$$\begin{aligned}\left(\nabla_{\theta(X)^*}\theta(Y)^*\right)_o &= -[\theta(X)_{\mathfrak{m}'}, \theta(Y)]_{\mathfrak{m}'} + \Lambda'(\theta(X)_{\mathfrak{m}'}, \theta(Y)_{\mathfrak{m}'}) \\ &= -[\theta(X)_{\mathfrak{m}'}, \theta(Y)]_{\mathfrak{m}'}\end{aligned}$$

(as $\Lambda' = 0$). This gives the first stated expression for $\mathfrak{m}_2$; the second follows as $[\mathfrak{m}', \mathfrak{m}'] \subset \mathfrak{h}'$ (since G'/H' is symmetric). The general case may be obtained by induction. q.e.d.

If now G'/H' (and hence $\mathfrak{m}'$) has an invariant metric, one obtains an H-module decomposition

$$\mathfrak{m}' = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \cdots$$

by writing $\mathfrak{l}_1 = df_\theta(\mathfrak{m})$, $\mathfrak{l}_i = (\mathfrak{m}_{i-1})^\perp \cap \mathfrak{m}_i$. This (or the corresponding bundle decomposition) will be referred to as the *osculating space decomposition* for f_θ. (In classical terms, $\mathfrak{l}_i$ is the ith osculating space at $o \in G/H$. For more details of this, see §11 of [28].)

3. Projective weight orbits of unitary representations

Let $\theta : G \to U_{n+1}$ be a homomorphism, i.e., a unitary representation of G on $\mathbf{C}^{n+1}$. As usual, θ will also denote the corresponding Lie algebra representation $\theta : L(G) \otimes \mathbf{C} \to L(U_{n+1}) \otimes \mathbf{C}$. To simplify notation, however, the operator on $\mathbf{C}^{n+1}$ corresponding to $\theta(x) \in L(U_{n+1}) \otimes \mathbf{C} = \mathrm{End}(\mathbf{C}^{n+1})$ will be indicated by $W \mapsto x \cdot W$. Let $V_0, V_1, \cdots, V_n$ be any ordered basis of $\mathbf{C}^{n+1}$, orthonormal with respect to the standard Hermitian inner product, consisting of weight vectors of θ. Let the corresponding weights be $\Lambda_0, \Lambda_1, \cdots, \Lambda_n \in L(T)^*$. The lines $[V_i] \in \mathbf{CP}^n$ are fixed points for the induced action of T, and we shall be interested in the geometrical properties of the orbits $G \cdot [V_i] = \{[\theta(g)V_i] \mid g \in G\} \subset \mathbf{C}P^n$. It will be convenient to consider $G \cdot [V_i]$ as the image of the map

$$f_{V_i} : G/T \to U_{n+1}/C[V_i] \cong \mathbf{C}P^n,$$

where $C[V_i]$ is the centralizer in U_{n+1} of $[V_i]$. This in turn will be considered as the composition

$$G/T \xrightarrow{f_\theta} F_{n+1} \xrightarrow{\pi_i} \mathbf{C}P^n,$$

where f_θ is the map of §2 and π_i is the natural projection $SU_{n+1}/S(U_1 \times \cdots \times U_1) \to U_{n+1}/C[V_i]$ induced by the inclusion of the maximal torus $S(U_1 \times \cdots \times U_1) = \bigcap_{i=0}^n C[V_i]$ in $C[V_i]$. In this section we shall give necessary and sufficient conditions for f_{V_i} to be harmonic (Corollary 3.6).

The map f_{V_i} was obtained by selecting the weight vector V_i. More generally, one may select a flag $\{0\} = E_0 \subset E_{m_1} \subset \cdots \subset E_{m_{k-1}} \subset E_{m_k} = \mathbf{C}^{n+1}$ (denoted σ), where the m_i-plane E_{m_i} is spanned by a subset of $\{V_0, \cdots, V_n\}$, and this defines a map

$$f_\sigma : G/T \to F(m_1, \cdots, m_k)$$

which may be expressed as a composition in a similar way:

$$G/T \xrightarrow{f_\theta} F_{n+1} \xrightarrow{\pi_\sigma} F(m_1, \cdots, m_k).$$

Our main result, Theorem 3.4, is a necessary and sufficient condition for f_σ to be harmonic, from which Corollary 3.6 will follow.

We begin with some remarks on complex structures.

Lemma 3.1. *The vector $e_\alpha \cdot V_i$ is either zero or a weight vector with weight $\Lambda_i + \alpha$.*

Proof. This is an elementary fact, which we state only because it will be used several times. It follows from the equations $he_\alpha - e_\alpha h = [h, e_\alpha] = 2\pi\sqrt{-1}\,\alpha(h)e_\alpha$, $h \cdot V_i = 2\pi\sqrt{-1}\,\Lambda_i(h)V_i$, $(h \in L(T) \otimes \mathbf{C})$. q.e.d.

On G/T we retain the complex structure J of §1 determined by a fixed choice Δ^+ of positive roots, and on F_{n+1} we take the standard complex structure J_{n+1} given by the choice of positive roots,

$$x_i - x_j \in L(S(U_1 \times \cdots \times U_1))^*, \qquad 0 \leqslant j < i \leqslant n.$$

Definition 3.2. *Let S be any torus. If $\lambda \in L(S)^*$, the associated one-dimensional complex S-module, whose character $\chi : S \to S^1$ is given by $\chi(\exp(s)) = \exp(2\pi\sqrt{-1}\,\lambda(s))$, will be denoted ρ^λ.*

With this convention, the invariant complex structures J, J_{n+1} are described by the following decompositions:

$$L(G/T) \otimes \mathbf{C} \cong \Big(\sum_{\alpha \in \Delta^+} \rho^\alpha\Big) \oplus \Big(\sum_{\alpha \in \Delta^-} \rho^\alpha\Big),$$

$$L(F_{n+1}) \otimes \mathbf{C} \cong \Big(\sum_{i>j} \rho^{x_i - x_j}\Big) \oplus \Big(\sum_{i<j} \rho^{x_i - x_j}\Big),$$

where the first summand is the $(1,0)$ component in each case.

Proposition 3.3. *The map f_θ is holomorphic with respect to J, J_{n+1} if for each weight Λ_j and positive root α, $\Lambda_j + \alpha$ either is not a weight, or is a weight Λ_i with $i > j$.*

Proof. Assume the condition on the weights holds. If f_θ is not holomorphic, $df_\theta(e_\alpha)$ has nonzero component in $\rho^{x_i - x_j}$ for some $i < j$. Hence $e_\alpha \cdot V_j$ has nonzero component in the line spanned by V_i. By Lemma 3.1, $e_\alpha \cdot V_j$

(being nonzero) is a weight vector with weight $\Lambda_j + \alpha$, so its component in the line spanned by V_i is too. Hence $\Lambda_j + \alpha = \Lambda_i$ with $i < j$, which is a contradiction. q.e.d.

On the generalized flag manifold $F(m_1, \cdots, m_k)$ we shall take the standard complex structure of §1, i.e., that for which

$$L(F)_{1,0} = \sum_{i>j} \lambda_i \otimes \lambda_j^*.$$

As an $S(U_1 \times \cdots \times U_1)$-module,

$$\lambda_i \otimes \lambda_j^* \cong \sum_{s \in A_i,\, t \in A_j} \rho^{x_s - x_t},$$

where $A_i = \{r \mid V_r \in E_{m_{i-1}}^{\perp} \cap E_{m_i}\}$. Thus, if the flag σ is the standard full flag with $E_i = [V_0, \cdots, V_{i-1}]$ (brackets denote linear span), then π_σ is the identity map and the complex structure on F is J_{n+1}. At the opposite extreme, if the flag σ is $\{0\} \subset [V_i] \subset \mathbf{C}^{n+1}$, then $\pi_\sigma = \pi_i$. The complex structure on $F = \mathbf{C}P^n$ is that for which

$$L(\mathbf{C}P^n)_{1,0} \cong \sum_{j=0,\, j \neq i}^{n} \rho^{x_i - x_j}.$$

Thus, π_i is holomorphic if and only if $i = n$. (More generally, π_σ is holomorphic if and only if $s > t$ for all $s \in A_i$, $t \in A_j$ with $i > j$.)

From now on, we shall fix the metric $\langle\ ,\ \rangle$ on $F(m_1, \cdots, m_k)$. When $F = \mathbf{C}P^n$, this is a multiple of the Fubini-Study metric, and is Kähler with respect to the chosen complex structure. We write p_i for orthogonal projection onto the subspace $E_{m_{i-1}}^{\perp} \cap E_{m_i} = [V_r \mid r \in A_i]$ of $\mathbf{C}^{n+1}$.

Theorem 3.4. *Let G/T have the metric $\langle\ ,\ \rangle_r$. The map $f_\sigma : G/T \to F(m_1, \cdots, m_k)$ is harmonic if and only if the operators*

$$(*) \qquad \sum_{\alpha \in \Delta} p_i e_\alpha (p_i - p_j) e_{-\alpha} p_j / r(|\alpha|)$$

on $\mathbf{C}^{n+1}$ are all zero for $0 \leqslant i \neq j \leqslant k$.

Proof. Applying 2.1 and 2.2 to f_σ, we find the condition for f_σ to be harmonic to be

$$(**) \qquad \sum_{\alpha \in \Delta} (1/r(|\alpha|)) \Big\{ \big[\theta(e_\alpha)_{\mathfrak{h}'}, \theta(e_{-\alpha})_{\mathfrak{m}'}\big] + \tfrac{1}{2}\big[\theta(e_\alpha)_{\mathfrak{m}'}, \theta(e_{-\alpha})_{\mathfrak{m}'}\big]_{\mathfrak{m}'} - \big(\theta(\Lambda(e_\alpha, e_{-\alpha}))\big)_{\mathfrak{m}'} \Big\} = 0,$$

where the decomposition $L(SU_{n+1}) \otimes \mathbf{C} = \mathfrak{m}' \oplus \mathfrak{h}'$ is that given by $F(m_1, \cdots, m_k)$, and Λ is the linear map defining the Riemannian connection for $\langle\ ,\ \rangle_r$. We have used the fact that the map Λ' defining the connection on

$F(m_1,\cdots,m_k)$ is given by $\Lambda'(X,Y)=\frac{1}{2}[X,Y]_{\mathfrak{m}'}$ (see §1). Now, from the formulas for Λ in §§1 and 2 we see that $\Lambda(e_\alpha,e_{-\alpha})=0$ for all $\alpha\in\Delta$, so the third term in (**) vanishes. As the bracket is skew symmetric, the second term vanishes (on summing over Δ). Thus, the condition for f_σ to be harmonic is that $\sum_{\alpha\in\Delta}(1/r(|\alpha|))\theta(e_\alpha)\otimes\theta(e_{-\alpha})$ is in the kernel of the map

$$\otimes^2(L(SU_{n+1})\otimes\mathbf{C})\to\mathfrak{m}',$$
$$(X,Y)\mapsto[X_{\mathfrak{h}'},Y_{\mathfrak{m}'}],$$

the bracket being that of $L(SU_{n+1})\otimes\mathbf{C}$. Recall that $\mathfrak{m}'=\sum_{i\neq j}\lambda_i\otimes\lambda_j^*$. We shall prove that the component in $\lambda_i\otimes\lambda_j^*$ of the image of $\sum_{\alpha\in\Delta}(1/r(|\alpha|))\theta(e_\alpha)\otimes\theta(e_{-\alpha})$ is precisely the operator (*). Let us write $\theta(e_\alpha)=X=\sum_{a\neq b}X^{a,b}$, where $X^{a,b}$ is the component of $X\in L(F_{n+1})\otimes\mathbf{C}$ in $\rho^{x_a-x_b}$, and similarly $\theta(e_{-\alpha})=Y=\sum_{a\neq b}Y^{a,b}$. If $s\in A_i$, $t\in A_j$, the component of $[X_{\mathfrak{h}'},Y_{\mathfrak{m}'}]$ in $\rho^{x_s-x_t}$ is

$$\sum_{b\in A_i}[X^{s,b},Y^{b,t}]+\sum_{a\in A_j}[X^{a,t},Y^{s,a}],$$

which may be written as

$$\left[\sum_{b\in A_i}X^{s,b},\sum_{b\in A_i}Y^{b,t}\right]+\left[\sum_{a\in A_j}X^{a,t},\sum_{a\in A_j}Y^{s,a}\right].$$

This is the element of $\rho^{x_s-x_t}\cong\operatorname{Hom}(\rho^{x_t},\rho^{x_s})$ represented by the operator $e_\alpha p_i e_{-\alpha}+e_{-\alpha}p_je_\alpha$. Summing over $s\in A_i$, $t\in A_j$ gives the operator

$$p_i(e_\alpha p_ie_{-\alpha}-e_{-\alpha}p_je_\alpha)p_j,$$

and summing finally over $\alpha\in\Delta$ gives (*), as required.

Remarks. (1) Since $p_i^*=p_i$ and $e_\alpha^*=e_{-\alpha}$ (as operators on $\mathbf{C}^{n+1}$), the theorem remains true if the operator (*) is required to be zero only for $i<j$.

(2) An alternative expression for the operator (*) occurs in the proof, namely:

$$\sum_{\alpha\in\Delta}\big((p_ie_\alpha)(p_ie_{-\alpha})-(p_ie_{-\alpha})(p_je_\alpha)\big)p_j/r(|\alpha|).$$

(3) The theorem remains true if the connection associated to $\langle\ ,\ \rangle$ is replaced by the canonical connection on $F(m_1,\cdots,m_k)$; in this case $\Lambda'=0$ and the second term in (**) is replaced by zero. When $k=2$, both these connections are essentially the same as the standard Kähler connection.

Corollary 3.5. *Let σ be the flag $\{0\}\subset[V_{i_1},\cdots,V_{i_k}]\subset\mathbf{C}^{n+1}$, and let p, $p^\perp$ denote the orthogonal projections onto $[V_{i_1},\cdots,V_{i_k}]$, $[V_{i_1},\cdots,V_{i_k}]^\perp$ respectively. If G/T has the metric $\langle\ ,\ \rangle_r$, and $\mathrm{Gr}_k(\mathbf{C}^{n+1})$ has the metric $\langle\ ,\ \rangle$ (the essentially unique invariant Kähler metric), the map*

$$f_\sigma:G/T\to\mathrm{Gr}_k(\mathbf{C}^{n+1})$$

is harmonic if and only if $[V_{i_1},\cdots,V_{i_k}]$ *is preserved by the operator*

$$\text{(i)}\quad \sum_{\alpha\in\Delta} e_\alpha(p^\perp - p)e_{-\alpha}/r(|\alpha|)$$

or, equivalently, by the operator

$$\text{(ii)}\quad \sum_{\alpha\in\Delta^+} \left(e_\alpha p^\perp e_{-\alpha} - e_{-\alpha} p e_\alpha\right)/r(\alpha).$$

Proof. From Theorem 3.4 and Remark (1) above, the condition for f_σ to be harmonic is that the operator

$$\sum_{\alpha\in\Delta} p^\perp e_\alpha(p^\perp - p)e_{-\alpha}p/r(|\alpha|)$$

be zero. This proves the assertion concerning the operator (i). Rewriting this condition as in Remark (2), we have

$$\sum_{\alpha\in\Delta} p^\perp\left(e_\alpha p^\perp e_{-\alpha} - e_{-\alpha}pe_\alpha\right)p/r(|\alpha|) = 0.$$

Since

$$\begin{aligned}\left(e_\alpha p^\perp e_{-\alpha} - e_{-\alpha}pe_\alpha\right) &- \left(e_{-\alpha}p^\perp e_\alpha - e_\alpha p e_{-\alpha}\right)\\ &= e_\alpha(p^\perp + p)e_{-\alpha} - e_{-\alpha}(p + p^\perp)e_\alpha = e_\alpha e_{-\alpha} - e_{-\alpha}e_\alpha\end{aligned}$$

acts as a scalar on each $[V_i]$, it preserves $[V_{i_1},\cdots,V_{i_k}]$, so we may replace the sum over $\alpha\in\Delta$ by the sum over $\alpha\in\Delta^+$. The assertion concerning the operator (ii) now follows.

Corollary 3.6. *Let V be a weight vector of the representation θ. If G/T has the metric $\langle\ ,\ \rangle_r$, and $\mathbf{C}P^n$ has the metric $\langle\ ,\ \rangle$, the map $f_V\colon G/T\to\mathbf{C}P^n$ is harmonic if and only if V is an eigenvector of the operator*

$$\theta_r = \sum_{\alpha\in\Delta^+} e_\alpha e_{-\alpha}/r(\alpha).$$

Proof. Without loss of generality we may assume $V = V_i$ for some i and apply Corollary 3.5. Since $e_\alpha\cdot V_i$ and V_i are weight vectors for distinct weights (see Lemma 3.1), they are orthogonal, hence the operator (ii) of Corollary 3.5 reduces to θ_r. q.e.d.

In the following theorem we list some situations where Corollary 3.6 applies.

Theorem 3.7. *Let θ be a representation as in Corollary 3.6.*

(1) *Weight vectors $V_0,\cdots,V_n$ may be chosen so that each map f_{V_i} is harmonic with respect to the metrics $\langle\ ,\ \rangle_r$ of G/T and $\langle\ ,\ \rangle$ of $\mathbf{C}P^n$.*

(2) *If V is a weight vector of multiplicity one, f_V is harmonic with respect to the metric $\langle\ ,\ \rangle_r$ of G/T and $\langle\ ,\ \rangle$ of $\mathbf{C}P^n$.*

(3) *If θ is irreducible, and V is any weight vector, f_V is harmonic with respect to the metrics $\langle\ ,\ \rangle$ of G/T and $\langle\ ,\ \rangle$ of $\mathbf{C}P^n$.*

Proof. (1) The operator θ_r of Corollary 3.6 is Hermitian and preserves weight spaces, hence one may take a basis of its eigenvectors in each weight space.

(2) The weight space $[V]$ is preserved by θ_r.

(3) The weight vector V is an eigenvector of θ_r if and only if it is an eigenvector of $\Sigma_{\alpha\in\Delta}e_\alpha e_{-\alpha}$ (this is (i) of Corollary 3.5). This is the case if and only if V is an eigenvector of the Casimir operator (with respect to a suitable basis of $L(G)\otimes\mathbf{C}$ extending $\{e_\alpha\}_{\alpha\in\Delta}$) of θ. As θ is irreducible, the Casimir operator acts as a scalar. q.e.d.

A version of part (2) of the theorem exists for the case $f_\sigma\colon G/T\to \mathrm{Gr}_k(\mathbf{C}^{n+1})$: if the subspace $[V_{i_1},\cdots,V_{i_k}]$ is a single weight space of θ, then f_σ is harmonic. This is because the operator $\Sigma_{\alpha\in\Delta}(1/r(|\alpha|))e_\alpha(p^\perp-p)e_{-\alpha}$ of Corollary 3.5 preserves weight spaces.

It may be appropriate to make some remarks on the relations between harmonic maps and minimal immersions at this point. Let $f\colon M^m\to N^n$ be an immersion of Riemannian manifolds, whose dimensions are m, n and whose metrics are g, h respectively. It is said to be isometric if $g=f^{-1}h$. For such an immersion f, the "mean curvature normal; at $x\in M^m$ is defined classically to be $(1/m)\mathrm{tr}\,\nabla df_x$. (The fact that $g=f^{-1}h$ allows one to show [13] that ∇df_x is in the normal space $df(T_xM)^\perp$.) If the mean curvature normal is zero for all $x\in M^m$, f is said to be *minimal*. Hence an isometric immersion is minimal if and only if it is a harmonic map (i.e., a solution of $\mathrm{tr}\,\nabla df=0$). In variational terms, an isometric immersion is minimal if and only if it is a (local) extremum for the volume functional

$$f\mapsto\int_M |\det f^{-1}h|^{1/2},$$

whereas a map is harmonic if and only if it is a (local) extremum for the energy functional

$$f\mapsto\tfrac{1}{2}\int_M \mathrm{tr}\,f^{-1}h.$$

The previous remarks assert that, if one restricts attention to the space of isometric immersions, the critical points of these two functionals coincide (see §2 of [14]).

Proposition 3.8. *Assume that the action of the isotropy subgroup G_i of $[V_i]$ on $L(G\cdot[V_i])$ is irreducible. If G/T has the metric $\langle\ ,\ \rangle$ (minus the Killing form), f_{V_i} is harmonic if and only if the orbit $G\cdot[V_i]$ is minimally embedded with respect to the induced metric.*

Proof. The map f_{V_i} may be factored as the composition

$$G/T\xrightarrow{a_i}G\cdot[V_i]\xrightarrow{b_i}\mathbf{C}P^n$$

of the projection a_i of G/T onto the orbit $G \cdot [V_i] \cong G/G_i$, followed by the inclusion map b_i. With respect to the metrics $\langle\ ,\ \rangle$ on G/T and G/G_i, a_i is a Riemannian submersion (i.e., a submersion which is an isometry on horizontal tangent vectors), and (see §4 of [12]) $b_i \circ a_i$ is harmonic if and only if b_i is. The induced metric on $G \cdot [V_i]$ is invariant under the action of G_i, hence it must be $\alpha\langle\ ,\ \rangle$ for some $\alpha > 0$, as this action of G_i is irreducible. Thus, $G \cdot [V_i]$ is minimally embedded if and only if f_{V_i} is harmonic with respect to $\alpha\langle\ ,\ \rangle$. From the form of the energy functional one sees that this is true if and only if f_{V_i} is harmonic with respect to $\langle\ ,\ \rangle$. q.e.d.

In [22], W.-Y. Hsiang observed that, if a Lie group G acts on a Riemannian manifold M, the orbit of a point is minimally embedded if and only if its volume is extremal, where the volume functional is now restricted to the space of orbits of the given type. In particular, an orbit of "isolated type" is minimally embedded. This provides another proof of Theorem 3.7 (2) (and of the assertion following Theorem 3.7). On the other hand, Theorem 3.7 (3) should be compared with a result of N. R. Wallach (Proposition 8.1 of [28]) which says that if $\theta : G \to SO_{n+1}$ is an irreducible real representation, then any isotropy irreducible orbit of G in the sphere $S^n \subset \mathbf{R}^{n+1}$ is minimally embedded.

4. Example: the Gauss maps of a projective weight orbit

In this section we shall describe a situation of particular geometrical significance where Corollary 3.5 applies. Let $\theta : G \to U_{n+1}$ be an irreducible representation, with weights $\Lambda_0, \cdots, \Lambda_n$ arranged to satisfy the condition of Proposition 3.3, and choose weight vectors $V_0, \cdots, V_n$. Thus Λ_0 is the maximal weight, and $f_{V_n} = \pi_n \circ f_\theta$ is holomorphic. Since $e_\alpha \cdot V_n = 0$ for all $\alpha \in \Delta^+$, it follows from Corollary 3.6 that f_{V_n} is harmonic, whatever metric $\langle\ ,\ \rangle_r$ is assigned to G/T. The roots $\alpha \in \Delta^+$ for which $e_{-\alpha} \cdot V_n \neq 0$ define a set of positive complementary roots $\Delta^+_{G \cdot [V_n]}$ for the homogeneous space $G \cdot [V_n]$, and hence an invariant complex structure. With respect to this, the embedding $G \cdot [V_n] \to \mathbf{C}P^n$ is holomorphic, in accordance with Borel-Weil theory [6].

There is a "Gauss map" $G/T \to \mathrm{Gr}_k(\mathbf{C}^{n+1})$ whose value at $o \in G/T$ is the k-plane $[e_{-\alpha} \cdot V_n^- \mid \alpha \in \Delta^+_{G \cdot [V_n]}]$. Since the vectors $e_{-\alpha} \cdot V_n$ here are weight vectors for distinct weights, we can assume (by rechoosing $V_0, \cdots, V_{n-1}$) that each $e_{-\alpha} \cdot V_n$ is some V_i; with the notation of Corollary 3.5, this Gauss map is $f_\sigma : G/T \to \mathrm{Gr}_k(\mathbf{C}^{n+1})$. We shall verify presently the condition of Corollary 3.5, hence f_σ is harmonic. This is a special case of Theorem 4.3 below, concerning all "higher order Gauss maps" of f_{V_n}. To define these maps, we use the holomorphic osculating flag of f_{V_n} as introduced in §2, which is a flag of complex G_n-modules

$$df_\theta(\mathfrak{m}_{1,0}) = \mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \cdots \subset L(\mathbf{C}P^n)_{1,0}.$$

With the conventions of §3,

$$L(\mathbf{C}P^n)_{1,0} = \sum_{i=0}^{n-1} \rho^{x_n - x_i} \cong \left(\sum_{i=0}^{n-1} \rho^{x_i} \right)^* \otimes \rho^{x_n},$$

hence there is an associated flag of submodules of $\mathbf{C}^{n+1} = (\sum_{i=0}^{n-1} \rho^{x_i}) \oplus \rho^{x_n}$ which we write as

$$\mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathbf{C}^{n+1}$$

($\mathfrak{n}_0 = \rho^{x_n}$, and if $i > 0$, $\mathfrak{n}_i = \rho^{x_n} \oplus (\rho^{-x_n} \otimes \mathfrak{m}_i)^*$). Finally we define $\mathfrak{k}_0 = \mathfrak{n}_0$, $\mathfrak{k}_i = (\mathfrak{n}_{i-1})^\perp \cap \mathfrak{n}_i$ $(i > 0)$.

Definition 4.1. *If the weight vectors $V_0, \cdots, V_n$ are chosen to be compatible with this decomposition, in the sense that $\mathfrak{k}_i = [V_j \mid j \in A_i]$ for some subset $A_i \subset \{0, 1, \cdots, n\}$, then the ith Gauss map is the map*

$$f_{\sigma_i} : G/T \to \mathrm{Gr}_{n_i}(\mathbf{C}^{n+1}) \qquad (n_i = |A_i|)$$

defined by the flag $\{0\} \subset \mathfrak{k}_i \subset \mathbf{C}^{n+1}$.

Proposition 4.2. *The flag $\mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathbf{C}^{n+1}$ may be identified as follows*:

$$\mathfrak{n}_i = \left[e_{-\alpha_1} \cdots e_{-\alpha_k} \cdot V_n \mid \alpha_1, \cdots, \alpha_k \in \Delta^+, 0 \leqslant k \leqslant i \right]$$

with the understanding that $e_{-\alpha_1} \cdots e_{-\alpha_k} \cdot V_n = V_n$ if $k = 0$.

Proof. By definition, $\mathfrak{n}_0 = [V_n]$. The remarks above concerning $G \cdot [V_n]$ give the case $i = 1$. According to Proposition 2.3 we have $\mathfrak{m}_i = \mathfrak{m}_{i-1} + [\theta(\mathfrak{m}_{1,0}), \mathfrak{m}_{i-1}]_{\mathfrak{m}'_{1,0}}$ for $i \geqslant 2$, the bracket being that of $L(SU_{n+1}) \otimes \mathbf{C} \subset \mathrm{End}(\mathbf{C}^{n+1})$, where we identify $\mathfrak{m}'_{1,0} = L(\mathbf{C}P^n)_{1,0} \cong (\sum_{i=0}^{n-1} \rho^{x_n})^* \otimes \rho^{x_n}$ with a subpace of $L(F_{n+1})_{1,0} \subset \mathrm{End}(\mathbf{C}^{n+1})$. We shall show that this translates into the formula

$$\mathfrak{n}_i = \mathfrak{n}_{i-1} + [e_{-\alpha} \cdot X \mid X \in \mathfrak{n}_{i-1}, \alpha \in \Delta^+]$$

from which the proposition follows by induction. First, note that the action of e_α on $(\sum_{i=0}^{n-1} \rho^{x_i})^* \otimes \rho^{x_n}$ (via Lie bracket) corresponds to the natural action of $e_{-\alpha}$ on $\mathbf{C}^{n+1} = (\sum_{i=0}^{n-1} \rho^{x_i}) \oplus \rho^{x_n}$. Hence

$$\mathfrak{n}_i = \mathfrak{n}_{i-1} + \left[e_{-\alpha} \cdot X \mid X \in \mathfrak{n}_{i-1}, \alpha \in \Delta^+_{G \cdot [V_n]} \right].$$

An induction argument now shows that one may replace $\Delta^+_{G \cdot [V_n]}$ by Δ^+. To start this, take $X \in \mathfrak{n}_1$ and $\beta \in \Delta^+$; we have to show that $e_{-\beta} \cdot X \in \mathfrak{n}_1 + [e_{-\alpha} \cdot X \mid X \in \mathfrak{n}_1, \alpha \in \Delta^+_{G \cdot [V_n]}]$. It is sufficient to take $X = e_{-\gamma} \cdot V_n$ and to assume $\beta \notin \Delta^+_{G \cdot [V_n]}$, so that $e_{-\beta} \cdot V_n = 0$. Then

$$e_{-\beta} e_{-\gamma} \cdot V_n = e_{-\gamma} e_{-\beta} \cdot V_n + [e_{-\beta}, e_{-\gamma}] \cdot V_n = [e_{-\beta}, e_{-\gamma}] \cdot V_n \in \mathfrak{n}_1.$$

The inductive step is similar.

Theorem 4.3. *Let $\theta : G \to U_{n+1}$ be an irreducible representation, with weights $\Lambda_0, \cdots, \Lambda_n$ ordered so that for all $\alpha \in \Delta^+$, $\Lambda_i + \alpha$ either is not a weight or is Λ_j for some $j > i$. Let $\mathbf{C}^{n+1} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots$ be the decomposition described above corresponding to the osculating space decomposition for the projective maximal weight orbit (i.e., for f_{V_n}). Then the Gauss maps*

$$f_{\sigma_i} : G/T \to \mathrm{Gr}_{n_i}(\mathbf{C}^{n+1}) \qquad \left(\mathfrak{k}_i = \left[V_j \mid j \in A_i\right], \dim \mathfrak{k}_i = n_i\right)$$

are harmonic, with respect to the metric $\langle\ ,\ \rangle_r$ on G/T and the metric $\langle\ ,\ \rangle$ on $\mathrm{Gr}_{n_i}(\mathbf{C}^{n+1})$.

Proof. Note that

$$e_{-\alpha} \cdot \mathfrak{n}_i \subset \mathfrak{n}_{i+1} \quad (\alpha \in \Delta^+, i \geqslant 0),$$
$$e_{\alpha} \cdot \mathfrak{n}_i \subset \mathfrak{n}_i \quad (\alpha \in \Delta^+, i \geqslant 0).$$

(The first is true by Proposition 4.2, the second may be proved by induction on i on making use of the relation $e_\alpha e_\beta - e_\beta e_\alpha = [e_\alpha, e_\beta]$ and the fact that $e_\alpha \cdot V_n = 0$.) Since $(e_\alpha)^* = e_{-\alpha}$ (as operators on $\mathbf{C}^{n+1}$), these give

$$e_\alpha \cdot (\mathfrak{n}_i)^\perp \subset (\mathfrak{n}_{i-1})^\perp \quad (\alpha \in \Delta^+, i \geqslant 1),$$
$$e_{-\alpha} \cdot (\mathfrak{n}_i)^\perp \subset (\mathfrak{n}_i)^\perp \quad (\alpha \in \Delta^+, i \geqslant 0).$$

It follows that

$$(*) \qquad \begin{aligned} e_{-\alpha} \cdot \mathfrak{k}_i &\subset \mathfrak{k}_i \oplus \mathfrak{k}_{i+1} \quad (\alpha \in \Delta^+, i \geqslant 0),\\ e_{\alpha} \cdot \mathfrak{k}_i &\subset \mathfrak{k}_i \oplus \mathfrak{k}_{i-1} \quad (\alpha \in \Delta^+, i \geqslant 1). \end{aligned}$$

We shall show that $\mathfrak{k}_i$ is invariant under the operators $e_\alpha p e_{-\alpha} - e_{-\alpha} p^\perp e_\alpha$ and $e_\alpha p^\perp e_{-\alpha} - e_{-\alpha} p e_\alpha$ for $\alpha \in \Delta^+$, hence f_{σ_i} is harmonic by Corollary 3.5 (p denotes orthogonal projection onto $\mathfrak{k}_i$). If the identities

$$e_{-\alpha} \cdot X = p e_{-\alpha} \cdot X + p^\perp e_{-\alpha} \cdot X,$$
$$e_{\alpha} \cdot X = p e_{\alpha} \cdot X + p^\perp e_{\alpha} \cdot X$$

are multiplied by e_α, $e_{-\alpha}$ respectively and then subtracted, we obtain

$$(e_\alpha e_{-\alpha} - e_{-\alpha} e_\alpha) \cdot X = \left(e_\alpha p e_{-\alpha} - e_{-\alpha} p^\perp e_\alpha\right) \cdot X + \left(e_\alpha p^\perp e_{-\alpha} - e_{-\alpha} p e_\alpha\right) \cdot X.$$

If $X \in \mathfrak{k}_i$, we see from the formulas $(*)$ that

$$\left(e_\alpha p e_{-\alpha} - e_{-\alpha} p^\perp e_\alpha\right) \cdot X \in \mathfrak{k}_i \oplus \mathfrak{k}_{i-1},$$
$$\left(e_\alpha p^\perp e_{-\alpha} - e_{-\alpha} p e_\alpha\right) \cdot X \in \mathfrak{k}_i \oplus \mathfrak{k}_{i+1}.$$

Since $(e_\alpha e_{-\alpha} - e_{-\alpha} e_\alpha) \cdot X \in \mathfrak{k}_i$ ($\mathfrak{k}_i$ has a basis consisting of weight vectors, with respect to which $e_\alpha e_{-\alpha} - e_{-\alpha} e_\alpha$ acts diagonally), we deduce that in fact

$$\left(e_\alpha p e_{-\alpha} - e_{-\alpha} p^\perp e_\alpha\right) \cdot X \in \mathfrak{k}_i, \qquad \left(e_\alpha p^\perp e_{-\alpha} - e_{-\alpha} p^\perp e_\alpha\right) \cdot X \in \mathfrak{k}_i$$

as required. q.e.d.

More geometrically, this result may be phrased as follows. The flag of osculating bundles for the holomorhic embedding $f: G \cdot [V_n] \mapsto \mathbf{C}P^n$ defines in a tautologous way a holomorphic map

$$G \cdot [V_n] \to F(m_1, m_2, \cdots)(T(\mathbf{C}P^n)_{1,0})$$

into the (partial) flag bundle of $T(\mathbf{C}P^n)_{1,0}$ ($m_i = \dim \mathfrak{m}_i$). But there is a natural equivalence

$$F(m_1, m_2, \cdots)(T(\mathbf{C}P^n)_{1,0}) \cong F(1, m_1 + 1, m_2 + 1, \cdots, n + 1)$$

so one obtains a holomorphic map from $G \cdot [V_n]$ (and hence from G/T) into the generalized flag manifold $F(1, m_1 + 1, m_2 + 1, \cdots, n + 1)$. Theorem 4.3 asserts that the maps defined by composing with the natural projections onto the Grassmannians $\mathrm{Gr}_{m_i - m_{i-1}}(\mathbf{C}^{n+1})$ are all harmonic.

For example, let θ be the mth symmetric power $S^m\lambda$ of the standard representation $\lambda: SU_{n+1} \to SU_{n+1}$ of $G = SU_{n+1}$. This is irreducible, with maximal projective weight orbit $G \cdot [V_N] \cong G/H \cong \mathbf{C}P^n$ ($N + 1 = \dim S^m\lambda$, $H = S(U_1 \times U_n)$). With the notation of §1, $\lambda|_H$ decomposes as $\lambda_1 \oplus \lambda_n$, and the decomposition $\mathbf{C}^{n+1} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots$ turns out to coincide with the decomposition

$$\theta|_H = \lambda_1^m \oplus \left(\lambda_1^{m-1} \otimes \lambda_n\right) \oplus \left(\lambda_1^{m-2} \otimes S^2\lambda_n\right) \oplus \cdots$$

of $\theta|_H = S^m(\lambda_1 \oplus \lambda_n)$ into irreducible H-modules. The corresponding decomposition $L(\mathbf{C}P^N)_{1,0} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \cdots$ is therefore

$$\left(\lambda_1^* \otimes \lambda_n\right) \oplus \left(\left(\lambda_1^*\right)^2 \otimes S^2\lambda_n\right) \oplus \cdots$$

which is also equal to

$$L(G/H)_{1,0} \oplus S^2L(G/H)_{1,0} \oplus \cdots$$

(recall from §2 that $\mathfrak{l}_i = (\mathfrak{m}_{i-1})^\perp \cap \mathfrak{m}_i$ and $\mathfrak{l}_1 = L(G/H)_{1,0}$). For further details and examples see [20]. It should be pointed out that this example is very special; for a general irreducible representation θ, the H-module $\mathfrak{k}_i$ will not in general be irreducible, and the H-module $\mathfrak{l}_i$ will in general be a proper submodule of $S^iL(G/H)_{1,0}$ ($i > 1$).

5. Nonhomogeneous maps

The main theorem of this section expresses the relation between Theorem 4.3 and the results of [15]; further developments of Theorem 4.3 will be treated in a subsequent article. We begin with an alternative approach to Theorem 4.3, when G/T has a *Kähler* metric, based on [15].

Lemma 5.1. *Let M be a Kähler manifold, and let the flag manifold $F = F(a_1, a_2, n+1)$ have any invariant Kähler metic. If*

(1) f is holomorphic, and

(2) f is horizontal with respect to the projection $\pi : F \to \mathrm{Gr}_{a_2 - a_1}(\mathbf{C}^{n+1})$, then $\pi \circ f$ is harmonic (with respect to the given metric on M and the metric $\langle\ ,\ \rangle$ on $\mathrm{Gr}_{a_2-a_1}(\mathbf{C}^{n+1})$).

Note. As in §1 we decompose $T_{1,0}F$ as $L_2 \otimes L_1^* + L_3 \otimes L_1^* + L_3 \otimes L_2^*$; the subbundle $L_2 \otimes L_1^* + L_3 \otimes L_2^*$ is called the horizontal subbundle. To say that f is horizontal with respect to π means $df(T_{1,0}M) \subset L_2 \otimes L_1^* + L_3 \otimes L_2^*$.

Proof. The lemma is a version of the "fundamental composition principle" (Lemma 3.5 of [15]). One has

$$\nabla d(\pi \circ f)(X, Y) = d\pi(\nabla df(X, Y)) + \nabla d\pi(df(X), df(Y)),$$

hence it suffices to show that

(a) f is harmonic, and

(b) $\nabla d\pi$ is zero on vectors of the form $df(e_{-\alpha}) \otimes df(e_\alpha)$, $\alpha \in \Delta^+$.

Part (a) follows from a well-known result of A. Lichnerowicz (see [26]), since f is a holomorphic map between Kähler manifolds. To verify (b), note that by Lemma 2.1,

$$\nabla d\pi(X, Y) = [X_{\mathfrak{h}'}, Y_{\mathfrak{m}'}] - \Lambda(X, Y)_{\mathfrak{m}'},$$

where $G'/H' = \mathrm{Gr}_{a_2-a_1}(\mathbf{C}^{n+1})$ and Λ gives the connection on F. If X is horizontal, $X_{\mathfrak{h}'} = 0$, so certainly the first term $[X_{\mathfrak{h}'}, Y_{\mathfrak{m}'}]$ is zero when $X = df(e_{-\alpha})$, $Y = df(e_\alpha)$. Since f is holomorphic,

$$\Lambda(df(e_{-\alpha}), df(e_\alpha))_{\mathfrak{m}'} = \Lambda_{0,1}(df(e_{-\alpha}), df(e_\alpha))_{\mathfrak{m}'} = [df(e_{-\alpha}), df(e_\alpha)]_{\mathfrak{m}'}$$

(using the formula for $\Lambda_{0,1}$ in §1), and this is zero as $[\mathfrak{m}', \mathfrak{m}'] \subset \mathfrak{h}'$. Hence the second term is also zero when $X = df(e_{-\alpha})$, $Y = df(e_\alpha)$.

Alternative proof of Theorem 4.3. Let σ be the flag $\{0\} \subset \mathfrak{n}_{i-1} \subset \mathfrak{n}_i \subset \mathbf{C}^{n+1}$. We shall apply Lemma 5.1 with $M = G/T$, $f = f_\sigma$; note that the map f_{σ_i} of Theorem 4.3 is $\pi \circ f_\sigma$. Certainly f_σ is holomorphic, and it is horizontal with respect to π because of the formulas

$$(*) \qquad e_{-\alpha} \cdot \mathfrak{k}_i \subset \mathfrak{k}_i \oplus \mathfrak{k}_{i+1}, \qquad e_\alpha \cdot \mathfrak{k}_i \subset \mathfrak{k}_i \oplus \mathfrak{k}_{i-1}$$

derived at the beginning of the proof of Theorem 4.3. Hence f_{σ_i} is harmonic. q.e.d.

This alternative method allows us to prove the following generalization of Theorem 4.3. A nonzero $(n+1) \times (n+1)$ complex matrix $P \in \mathrm{End}(\mathbf{C}^{n+1})$ induces a transformation

$$[P] : \mathbf{C}P^n - [\ker P] \to \mathbf{C}P^n$$

which defines an element of $PGl_{n+1}\mathbf{C}$ when P is nonsingular. As in projective geometry we shall refer to $[P]$ as a projectivity. If $\sigma \in F(m_1, \cdots, m_k)$ is a flag, $[P] \cdot \sigma$ (when defined) shall mean the flag obtained by applying P to the subspaces defining σ; thus $[P] \cdot \sigma \in F(n_1, \cdots, n_j)$ for some $n_1, \cdots, n_j$. Now consider the map $f_{\sigma_i} = \pi \circ f_\sigma$ of Theorem 4.3. We define

$$f_{\sigma_i}^P : G/T \to \mathrm{Gr}_{a_2 - a_1}(\mathbf{C}^{n+1})$$

by the formula

$$f_{\sigma_i}^P(m) = \pi([P] \cdot f_\sigma(m))$$

whenever this makes sense.

Theorem 5.2. *For any P (for which $f_{\sigma_i}^P$ is defined), $f_{\sigma_i}^P$ is harmonic with respect to any Kähler metric on G/T and the metric $\langle\ ,\ \rangle$ on $\mathrm{Gr}_{a_2-a_1}(\mathbf{C}^{n+1})$.*

Proof. The map $m \mapsto [P] \cdot f_\sigma(m)$ is holomorphic, since f_σ is. It is horizontal with respect to π, since f_σ is (the horizontally condition is a linear condition in TF). Hence $f_{\sigma_i}^P$ is harmonic, by Lemma 5.1. q.e.d.

As we have remarked in the introduction, this is of interest even in the simplest case $G = SU_2$: the harmonic maps produced on applying Theorem 5.2 to the irreducible representations $\theta = S^m\lambda$ of SU_2 $(m = 0, 1, 2, \cdots)$ give all harmonic maps $\mathbf{C}P^1 \to \mathbf{C}P^n$ (with respect to standard Kähler metrics). To explain this, we continue the discussion of the example at the end of §4 in the case $n = 1$ (here $N = m$). The holomorphic map $f_{V_m} : \mathbf{C}P^1 \to \mathbf{C}P^m$ is a rational normal curve of degree m, and the decomposition $\mathbf{C}^{m+1} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots$ is just the decomposition of $\mathbf{C}^{m+1}$ into weight spaces:

$$\mathbf{C}^{m+1} = [V_m] \oplus [V_{m-1}] \oplus \cdots .$$

Hence the ith Gauss map f_{σ_i} is $f_{V_{m-i}} : \mathbf{C}P^1 \to \mathbf{C}P^m$, i.e., the embedding of the ith projective weight orbit. The sequence of maps $f_{V_m}, f_{V_{m-1}}, \cdots$ has already been considered by other authors, in greater generality [15], [24]. Indeed, if $g : \mathbf{C}P^1 \to \mathbf{C}P^m$ is any holomorphic map, one defines the ith "transform" $g_{(i)} : \mathbf{C}P^1 \to \mathbf{C}P^m$ (locally, using homogeneous coordinates) by

$$g_{(i)}(z) = g(z) \wedge g'(z) \wedge \cdots \wedge g^{(i-1)}(z)^\perp \cap g(z) \wedge g'(z) \wedge \cdots \wedge g^{(i)}(z).$$

One of the main results of [15] is that $g_{(i)}$ is harmonic and, conversely, any harmonic map $\mathbf{C}P^1 \to \mathbf{C}P^1$ is of the form $g_{(i)}$ for suitable g and i. Now, it is well known that g may be obtained by applying a projectivity $[P]$ to some rational normal curve $f : \mathbf{C}P^1 \to \mathbf{C}P^n$ $(n \geqslant m)$. To be explicit, g (being algebraic) may be written as

$$g(z) = [p_0(z); \cdots; p_n(z)],$$

where $p_i(z) = \sum_{j=0}^{n} p_{ij} z^j$ is a polynomial, so $g = [P] \cdot f$ where $P = (p_{ij})$ and $f(z) = [1; z; \cdots; z^n]$. (We suppress trivial identifications such as the inclusion of $\mathbf{C}P^m$ in $\mathbf{C}P^n$.) Since the rational normal curve f is projectively equivalent to any other such curve, we may as well assume that $g = [P] \cdot f_{V_n}$, i.e. $g = [P] \cdot f_{\sigma_0}$ with the notation of Theorem 4.3. By linearity, $g_{(i)} = f^P_{\sigma_i}$, hence the most general harmonic map $g_{(i)}$ is of the type occurring in 5.2, as asserted.

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