J. DIFFERENTIAL GEOMETRY
25 (1987) 117–137

ONE-DIMENSIONAL GIBBS STATES AND AXIOM A DIFFEOMORPHISMS

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Abstract

We study the equilibrium statistical mechanics of one-dimensional classical lattice systems with exponentially decreasing interactions. For such systems, the Fournier transforms of pair correlation functions are meromorphic in a strip, and the residues of the poles can be expressed in terms of "Gibbs distributions." The latter are defined like Gibbs states, but without the positivity condition. Using symbolic dynamics, we can apply these results to Smale's Axiom A diffeomorphisms; the Gibbs distributions then become distributions in the sense of Schwartz on the manifold.

0. Introduction

The *Gibbs states* of equilibrium statistical mechanics are probability measures (on the space of configurations of an infinite system) which satisfy certain linear conditions.[1] We introduce here the more general concept of *Gibbs distributions*: they satisfy the same linear conditions as before, but are elements of a larger space than that of measures.[2] We determine a class of Gibbs distributions in the case of one-dimensional lattice systems with exponentially decreasing interactions. The interactions are allowed here to be complex. The main interest of this result is that it gives an expression for the residues of poles of the Fourier transform of the pair correlation function. While one-dimensional systems with short range interactions are in a sense trivial (they have no phase transition), they are useful in the analysis of certain differentiable dynamical systems on a manifold (Axiom A diffeomorphisms). We shall thus be able to study the correlation functions of these dynamical systems, and express the residues of the poles referred to, or *resonances*, in

Received December 30, 1985 and, in revised form, April 4, 1986.

[1] See Dobrushin [5]–[7], and Lanford and Ruelle [11].

[2] For an early study of a special example see Gallavotti and Lebowitz [10].

terms of Gibbs distributions on the manifold. Presumably, this analysis can be extended, as usual, to Axiom A flows.

The literature on statistical mechanics and its applications to Axiom A systems is extensive. For clarity, we reproduce in the present paper the details of a certain number of known results, referring to the original papers for the proofs (general references are Bowen [2], Ruelle [21]).

1. Subshifts of finite type, interactions, Gibbs distributions

Let J be a nonempty finite set and (t_{ij}) a square matrix indexed by $J \times J$, with elements 0 or 1. We define Ω to be the space of sequences $(j_k)_{k \in \mathbf{Z}}$ of elements of J such that $t_{j_k j_{k+1}} = 1$ for all k. Note that Ω is a closed subset of the compact product $J^{\mathbf{Z}}$ and thus compact. The shift $\tau : \Omega \to \Omega$ is defined by $(\tau\xi)_k = \xi_{k+1}$; τ is a homeomorphism. The pair (Ω, τ) is called a *subshift of finite type*.

We shall assume in what follows that all matrix elements of t^N are > 0 for sufficiently large N. This means that τ is topologically mixing on Ω. (The mixing condition simplifies the discussion to follow. The general situation can in some sense be reduced to the mixing case.)

Given any set $X \subset \mathbf{Z}$, let π_X be the restriction to Ω of the canonical projection $J^{\mathbf{Z}} \to J^X$, and let $\pi_X \Omega = \Omega_X$. Thus, if $\xi \in \Omega$, $\pi_X \xi = \xi \mid X \in \Omega_X$. The shift τ sends Ω_X to Ω_{X-1}. An *interaction* is a complex function Φ on the union $\bigcup_X^* \Omega_X$ over all finite nonempty intervals X of $\mathbf{Z}$, such that $\Phi \circ \tau = \Phi$ (translation invariance). Let $0 < \theta < 1$; we shall restrict our attention to the Banach space $\mathscr{B}_\theta$ of *exponentially decreasing* interactions for which

$$\theta^{-\operatorname{diam} X} \sup_{\xi \in \Omega_X} |\Phi(\xi)| \to 0 \quad \text{when } \operatorname{diam} X \to \infty .$$

(We have written $\operatorname{diam}[u, v] = v - u$.) We put on $\mathscr{B}_\theta$ the norm

$$\|\Phi\|_\theta = \sup_X \theta^{-\operatorname{diam} X} \sup_{\xi \in \Omega_X} |\Phi(\xi)| < +\infty .$$

Given $a, b \in \mathbf{Z}$ with $a < b$, write

$$V_{ab}(\xi) = \sum_{X : X \not\subset (-\infty,\ a],\, X \not\subset [b,\ \infty)} \Phi \circ \pi_X(\xi).$$

(If $b - a > 1$, the sum extends over those X which contain some X strictly between a and b; if $b - a = 1$, the sum extends over those X which contain both a and b.)

If $A \in \mathscr{C}(\Omega, \mathbf{C})$, we write

$$\|A\|_\infty = \max\{|A(\xi)| : \xi \in \Omega\},$$
$$\operatorname{var}_n A = \sup\{|A(\xi) - A(\xi')| : \xi_k = \xi'_k \text{ for } |k| < n\},$$
$$\|A\|_\theta = \sup_{n \geqslant 1} \theta^{-n} \operatorname{var}_n A, \qquad |||A|||_\theta = \|A\|_\infty + \|A\|_\theta.$$

We let $\mathscr{C}_\theta$ be the Banach space of those A for which $\lim_{n\to\infty} \theta^{-n} \operatorname{var}_n A = 0$, with the norm $|||\cdot|||_\theta$. Note that $\mathscr{C}_\theta$ is a Banach algebra (i.e. $|||AB|||_\theta \leqslant |||A|||_\theta |||B|||_\theta$) and that V_{ab}, $\exp \pm V_{ab} \in \mathscr{C}_\theta$. If $Y \subset \mathbf{Z}$, we let

$$\mathscr{C}_\theta(Y) = \{A \in \mathscr{C}_\theta(\Omega_Y, \mathbf{C}) = A \circ \pi_Y \in \mathscr{C}_\theta\}.$$

This is a Banach space with respect to the induced norm $A \to |||A \circ \pi_Y|||_\theta$. We denote by $\mathscr{C}_\theta^*, \mathscr{C}_\theta(Y)^*$ the duals of $\mathscr{C}_\theta, \mathscr{C}_\theta(Y)$. For $\sigma \in \mathscr{C}_\theta^*$ or $\mathscr{C}_\theta(Y)^*$ it will be convenient to write $\sigma(A) = \int \sigma(d\xi) A(\xi)$ as if σ were a measure.

We shall say that $\sigma \in \mathscr{C}_\theta^*$ is a *Gibbs distribution* if, for every choice of $a < b$, there is $\sigma_{ab} \in \mathscr{C}_\theta((-\infty, a] \cup [b, \infty))^*$ such that, for all $A \in \mathscr{C}_\theta$,

$$(1.1) \quad \int \sigma(d\xi) A(\xi) = \sum_{\eta \in \Omega_{(a,b)}} \int \sigma_{ab}(d\xi' d\xi'') e^{-V_{ab}(\xi' \vee \eta \vee \xi'')} A(\xi' \vee \eta \vee \xi'').$$

In this formula, $\xi' \vee \eta \vee \xi''$ denotes the element $\xi \in \Omega$ such that $\xi|(-\infty, a] = \xi'$, $\xi|(a, b) = \eta$, $\xi|[b, \infty) = \xi''$, and the integral is restricted to the set $\{(\xi', \xi'') : \xi' \vee \eta \vee \xi'' \text{ is defined}\}$. (If all the elements of (t_{ij}) are equal to 1, $\xi' \vee \eta \vee \xi''$ is always defined.)

Note that when Φ is real and σ is a probability measure, we recover the definition of a Gibbs state. Our conditions imply that there is one and only one Gibbs state in the present situation.

2. Determination of a class of Gibbs distributions

Define $V'_a \in \mathscr{C}_\theta((-\infty, a])$, $V''_b \in \mathscr{C}_\theta([b, \infty))$ by

$$V'_a(\xi') = \sum_{X : a \in X \subset (-\infty, a]} \Phi(\xi' | X), \qquad V''_b(\xi'') = \sum_{X : b \in X \subset [b, \infty)} \Phi(\xi'' | X).$$

Note that, if $a < b$,

$$(2.1) \qquad \begin{aligned} V'_a(\xi | (-\infty, a]) &= V_{(a-1)b}(\xi) - V_{ab}(\xi), \\ V''_b(\xi | [b, \infty)) &= V_{a(b+1)}(\xi) - V_{ab}(\xi). \end{aligned}$$

We shall look for Gibbs distributions such that (1.1) holds with $\sigma_{ab}(d\xi' d\xi'') = \sigma'_{ab}(d\xi') \sigma''_{ab}(d\xi'')$, where $\sigma'_{ab} \in \mathscr{C}_\theta((-\infty, a])^*$, $\sigma''_{ab} \in \mathscr{C}_\theta([b, \infty))^*$. In view of

(1.1), and using formulas (2.1), we can choose $\sigma'_{ab} = \sigma'_a$, $\sigma''_{ab} = \sigma''_b$ such that

$$\tag{2.2} \begin{aligned} \int \sigma'_a(d\xi')A(\xi') &= \sum_{\xi_a \in J} \int \sigma'_{a-1}(d\zeta) e^{-V'_a(\zeta \vee \xi_a)} A(\zeta \vee \xi'_a), \\ \int \sigma''_b(d\xi'')A(\xi'') &= \sum_{\xi_b \in J} \sigma''_{b+1}(d\zeta) e^{-V''_b(\xi_b \vee \zeta)} A(\xi_b \vee \zeta) \end{aligned}$$

for all $A \in \mathscr{C}_\theta((-\infty, \ a])$ or $A \in \mathscr{C}_\theta([b, \ \infty))$ respectively. Conversely, if the families (σ'_n), (σ''_n) satisfy (2.2), they determine a unique Gibbs distribution by (1.1).

We introduce now $\mathscr{T}'_a : \mathscr{C}_\theta((-\infty, \ a]) \to \mathscr{C}_\theta((-\infty, \ a-1])$, $\mathscr{T}''_b : \mathscr{C}_\theta([b, \ \infty)) \to \mathscr{C}_\theta([b+1, \ \infty))$ such that

$$(\mathscr{T}'_a A)(\zeta) = \sum_{\xi_a} e^{-V'_a(\zeta \vee \xi_a)} A(\zeta \vee \xi_a),$$

$$(\mathscr{T}''_b A)(\zeta) = \sum_{\xi_b} e^{-V''_b(\xi_b \vee \zeta)} A(\xi_b \vee \zeta).$$

If $\mathscr{T}'^*_a$, $\mathscr{T}''^*_b$ are the adjoints of these operators, we may rewrite (2.2) as

$$\tag{2.3} \mathscr{T}'^*_a \sigma'_{a-1} = \sigma'_a, \qquad \mathscr{T}''^*_b \sigma''_{b+1} = \sigma''_b.$$

If $a < b$, also let $\mathscr{K}_{ab} : \mathscr{C}_\theta((-\infty, \ a])^* \to \mathscr{C}_\theta([b, \ \infty))$ be defined by

$$(\mathscr{K}_{ab}\sigma')(\xi'') = \sum_{\eta \in \Omega_{(a,b)}} \int \sigma'(d\xi') e^{-V_{ab}(\xi' \vee \eta \vee \xi'')}.$$

(It has to be checked that if $\sigma' \in \mathscr{C}_\theta((-\infty, \ a])^*$, then $\mathscr{K}_{ab}\sigma' \in \mathscr{C}_\theta([b, \ \infty))$. We shall take $a = 0$, $b = 1$, and also suppose for simplicity that $t_{ij} = 1$ for all $i, j \in J$ (these restrictions are easily lifted). Write $f(\xi'')(\cdot) = V_{01}(\cdot \vee \eta \vee \xi'')$. Then $f(\xi'') \in \mathscr{C}_\theta((-\infty, \ a])$ and $|||f(\xi'')|||_\theta \leqslant 2\theta(1-\theta)^{-2}\|\Phi\|_\theta$. If ξ'' and $\xi^\times$ differ only by their components of index $\geqslant n$, then $|||f(\xi'') - f(\xi^\times)|||_\theta \leqslant 4\theta^n(1-\theta)^{-2}\|\Phi\|_\theta$. Using the fact that $\mathscr{C}_\theta((-\infty, \ a])$ is a Banach algebra, one has similar estimates for e^{-f}, from which it follows that $\mathscr{K}_{ab}\sigma' \in \mathscr{C}_\theta([b, \ \infty))$.) The adjoint $\mathscr{K}^*_{ab} : \mathscr{C}_\theta([a, \ \infty))^* \to \mathscr{C}_\theta((-\infty, \ a])$ satisfies

$$(\mathscr{K}^*_{ab}\sigma'')(\xi') = \sum_{\eta \in \Omega_{(a,b)}} \int \sigma''(d\xi'') e^{-V_{ab}}(\xi' \vee \eta \vee \xi'').$$

(It suffices to check that this formula implies $(\mathscr{K}^*_{ab}\sigma'', \sigma') = (\mathscr{K}_{ab}\sigma', \sigma'')$; this follows from a simple density argument.) We also have

$$\tag{2.4} \begin{aligned} \mathscr{K}_{ab}\mathscr{T}'^*_a &= \mathscr{K}_{a-1,b}, \qquad & \mathscr{T}''_b\mathscr{K}_{ab} &= \mathscr{K}_{a,b+1}, \\ \mathscr{T}'_a\mathscr{K}^*_{ab} &= \mathscr{K}^*_{a-b,b}, \qquad & \mathscr{K}^*_{ab}\mathscr{T}''^*_b &= \mathscr{K}^*_{a,b+1}. \end{aligned}$$

We define $\mathscr{L}' : \mathscr{C}_\theta((-\infty,0]) \to \mathscr{C}_\theta((-\infty,0])$, $\mathscr{L}'' : \mathscr{C}_\theta[1,\infty)) \to \mathscr{C}_\theta([1,\infty))$, and $\mathscr{K}: \mathscr{C}_\theta((-\infty,0])^* \to \mathscr{C}_\theta([1,\infty))$ by $\mathscr{L}'A = (\mathscr{T}_0'A)\circ\tau$, $\mathscr{L}''A = (\mathscr{T}_1''A)\circ\tau^{-1}$, and $\mathscr{K} = \mathscr{K}_{01}$. We may also write

$$\mathscr{L}'^* = \mathscr{T}_0'^*\tau, \qquad \mathscr{L}''^* = \mathscr{T}_1''^*\tau^{-1},$$

where $\tau : \mathscr{C}_\theta(X)^* \to \mathscr{C}_\theta(X-1)^*$ is the adjoint of $A \to A\circ\tau$. We then have, in view of (2.4), the formulas

$$\text{(2.5)} \qquad \mathscr{K}\mathscr{L}'^* = \mathscr{L}''\mathscr{K}, \qquad \mathscr{K}^*\mathscr{L}''^* = \mathscr{L}'\mathscr{K}^*.$$

Whenever $a < b$ we also have the relations

$$\text{(2.6)} \qquad (\mathscr{L}'^*)^{b-a}\tau^a\sigma_a' = \tau^b\sigma_b', \qquad (\mathscr{L}''^*)^{b-a}\tau^{b-1}\sigma_b'' = \tau^{a-1}\sigma_a''$$

equivalent to (2.2) or (2.3).

It is known (see [19] or [21, Chapter 5]) that the spectral radius of $\mathscr{L}'$ is bounded by e^P, where $P = P_{\mathrm{Re}\,\Phi}$ is the "pressure" associated with the real part of the interaction Φ. The part of the spectrum of $\mathscr{L}'$ (resp. $\mathscr{L}'^*$) in the region $\{z : |z| > \theta e^P\}$ is discrete and consists of eigenvalues with finite multiplicity as noted by Pollicott [17].[3] The generalized eigenspaces E_λ' of $\mathscr{L}'$ and $E_\lambda'^*$ of $\mathscr{L}'^*$ corresponding to an eigenvalue λ have the same dimension and are naturally dual to each other. If $\lambda \neq \mu$, then $(E_\mu'^*, E_\lambda') = 0$. Similar properties hold for $\mathscr{L}''$, and E_λ'', $E_\lambda''^*$ are defined by analogy with E_λ', $E_\lambda'^*$. In view of (2.5), $\mathscr{K}$ sends $E_\lambda'^*$ into E_λ'' and $\mathscr{K}^*$ sends $E_\lambda''^*$ into E_λ'; we shall see below that these maps are bijective.

2.1. Proposition. *If $\sigma' \in E_\lambda'^*$ and $\sigma'' \in E_\mu''^*$, then the element σ of $\mathscr{C}_\theta^*$ defined by*

$$\text{(2.7)} \qquad \sigma(d\xi' \vee d\xi'') = \sigma'(d\xi')\sigma''(d\xi'')e^{-V_{01}(\xi'\vee\xi'')}$$

is a Gibbs distribution.

Since $\mathscr{L}'^*$ is a linear isomorphism of $E_\lambda'^*$ and $\mathscr{L}''^*$ is a linear isomorphism of $E_\mu''^*$, there are measures σ_n', σ_n'' for all $n \in \mathbf{Z}$ such that $\sigma_0' = \sigma'$, $\sigma_1'' = \sigma''$ and the identities (2.6) hold, therefore (2.7) defines a Gibbs distribution.

Note that, if $\sigma' \neq 0$ and $\sigma'' \neq 0$, (1.1) implies $\sigma \neq 0$. (Here we use the assumption that all matrix elements of t^N are > 0 for sufficiently large N.) Because of our definition of $\mathscr{C}_\theta$, we can choose a, b, A, with $a < b$, and $A : \Omega_{(a,b)} \to \mathbf{C}$ such that

$$\sum_{\eta\in\Omega_{(a,b)}} \int \sigma_a'(d\xi')\sigma_b''(d\xi'')e^{-V_{ab}(\xi'\vee\eta\vee\xi'')}A(\eta) \neq 0,$$

[3] This important remark by Pollicott is based on Nussbaum's essential spectral radius formula [16].

and therefore $\mathscr{K}_{a,a+1}\sigma'_a \neq 0$, $\mathscr{K}^*_{b-1,b}\sigma''_b \neq 0$. The first inequality is equivalent to

$$0 \neq \left(\mathscr{K}_{a,a+1}\sigma'_a\right) \circ \tau^{-a} = \mathscr{K}\left(\tau^a \sigma'_a\right).$$

If $a \geqslant 0$, we have $\tau^a \sigma'_a = (\mathscr{L}'^*)^a \sigma'_0$, hence

$$0 \neq \mathscr{K}\left(\mathscr{L}'^*\right)^a \sigma'_0 = \left(\mathscr{L}''\right)^a \mathscr{K}\sigma'_0,$$

hence $\mathscr{K}\sigma'_0 \neq 0$. If $a < 0$, we have $(\mathscr{L}'^*)^{-a}\tau^a\sigma'_a = \sigma'_0$; since $\mathscr{K}(\tau^a\sigma'_a) \in E''_\lambda$ we have

$$0 \neq \mathscr{L}''^{-a}\mathscr{K}\left(\tau^a\sigma'_a\right) = \mathscr{K}\left(\mathscr{L}'^*\right)^{-a}\tau^a\sigma'_a = \mathscr{K}\sigma'_0.$$

Therefore, we have always $\mathscr{K}\sigma' \neq 0$, and similarly $\mathscr{K}^*\sigma'' \neq 0$.

2.2. Proposition. *The map $\mathscr{K}$ restricted to E'^*_λ is a bijection to E''_λ, and $\mathscr{K}^*$ restricted to E''^*_λ is a bijection to E'_λ.*

We have just seen that these maps are injective. We thus have

$$\dim E'^*_\lambda \leqslant \dim E''_\lambda = \dim E''^*_\lambda \leqslant \dim E'_\lambda = \dim E'^*_\lambda$$

which forces $\dim E'^*_\lambda = \dim E''_\lambda$, $\dim E''^*_\lambda = \dim E'_\lambda$. Therefore, the maps must be onto.

The class of Gibbs distributions in which we shall be interested is that determined by Proposition 2.1. We shall denote by $\mathscr{G}_{\lambda\mu}$ the linear space spanned by distributions of the form (2.7) with $\sigma' \in E'^*_\lambda$ and $\sigma'' \in E''^*_\mu$. (We might of course consider linear combinations between different $\mathscr{G}_{\lambda\mu}$.)

2.3. Proposition. $\tau\mathscr{G}_{\lambda\mu} = \mathscr{G}_{\lambda\mu}$, *and the spectrum of* $\tau|\mathscr{G}_{\lambda\mu}$ *is* $\{\lambda\mu^{-1}\}$. *In particular, if* $\dim E'_\lambda = \dim E''_\lambda = 1$, *then* $\tau|\mathscr{G}_{\lambda\mu} = \lambda\mu^{-1}$.

We have indeed

$$(2.8)\qquad \begin{aligned}(\tau\sigma)(d\xi' \vee d\xi'') &= \left(\tau\sigma'_1\right)\left(d\xi'\right)\left(\tau\sigma''_2\right)\left(d\xi''\right)e^{-V_{01}(\xi' \vee \xi'')}\\ &= \left(\mathscr{L}'^*\sigma'\right)\left(d\xi'\right)\left(\left(\mathscr{L}''^* \mid E''_\mu\right)^{-1}\sigma''\right)e^{-V_{01}(\xi' \vee \xi'')}\end{aligned}$$

so that $\tau\mathscr{G}_{\lambda\mu} \subset \mathscr{G}_{\lambda\mu}$. Similarly $\tau^{-1}\mathscr{G}_{\lambda\mu} \subset \mathscr{G}_{\lambda\mu}$ so that $\tau\mathscr{G}_{\lambda\mu} = \mathscr{G}_{\lambda\mu}$. The statements about the spectrum also follow readily from (2.8).

If $\tilde\sigma \in \mathscr{G}_{\tilde\lambda\tilde\mu}$, where $\tilde\sigma$ is associated with $\tilde\sigma' \in E'^*_{\tilde\lambda}$, $\tilde\sigma'' \in E''^*_{\tilde\mu}$, we may define

$$(2.9)\qquad (\sigma, \tilde\sigma) = \sigma'(\mathscr{K}^*\tilde\sigma'') \cdot \sigma''(\mathscr{K}\tilde\sigma')$$

which extends to a bilinear form on $\mathscr{G}_{\lambda\mu} \times \mathscr{G}_{\tilde\lambda\tilde\mu}$.

2.4. Proposition. *The bilinear form defined by* (2.9) *is invariant under the translation* $\sigma \to \tau\sigma$, $\tilde\sigma \to \tau\tilde\sigma$, *and vanishes unless* $\tilde\lambda = \mu$, $\tilde\mu = \lambda$.

The proof follows readily from (2.8).

3. Use of the functions $A_\Phi, A'_\Phi, A''_\Phi$

Instead of basing statistical mechanics on the "interaction" Φ, we may use functions on $\Omega, \Omega_{(-\infty,0]}$, or $\Omega_{[1,\infty)}$ as we now indicate (see for instance Ruelle [21]). If $\Phi \in \mathscr{B}_\theta$, we define $A_\Phi \in \mathscr{C}_{\theta^2}$, $A'_\Phi \in \mathscr{C}_\theta((-\infty,0])$, and $A''_\Phi \in \mathscr{C}_\theta([1,\infty))$ by

$$-A_\Phi(\xi) = \Phi(\xi|\{0\}) + \sum_{n=1}^{\infty} [\Phi(\xi|[-n+1,n]) + \Phi(\xi|[-n,n])],$$

$$-A'_\Phi(\xi') = \sum_{n=0}^{\infty} (\xi'|[-n,0]) = V'_0(\xi'),$$

$$-A''_\Phi(\xi'') = \sum_{n=1}^{\infty} \Phi(\xi''|[1,n]) = V''_1(\xi'').$$

Let us say that $A \in \mathscr{C}_\theta$ [or $A' \in \mathscr{C}_\theta((-\infty, 0])$, or $A'' \in \mathscr{C}_\theta([1, \infty))$] is *cohomologous to zero in* $\mathscr{C}_\theta$ [or $\mathscr{C}_\theta((-\infty, 0])$, or $\mathscr{C}_\theta([1, \infty))$] if there is $C \in \mathscr{C}_\theta$ [or $C' \in \mathscr{C}_\theta((-\infty, 0])$, or $C'' \in \mathscr{C}_\theta([1, \infty))$] such that $A = C - C \circ \tau$ [or $A'_\Phi = C' - C' \circ \pi' \circ \tau^{-1}$, or $A''_\Phi = C'' - C'' \circ \pi'' \circ \tau$, where π' denotes the projection of $\Omega_{(-\infty, 1]}$ to $\Omega_{(-\infty, 0]}$ and π'' the projection of $\Omega_{[0, \infty)}$ to $\Omega_{[1, \infty)}$]. We then write $A \sim 0$ [or $A' \sim 0$, or $A'' \sim 0$], and similarly $A \sim B$ if $A - B \sim 0$ (see Livšic [13], [14] and Sinai [26] for this cohomology). The following results hold.

3.1. Proposition. (a) *There are continuous linear maps* $\varphi, \varphi', \varphi''$ *of* $\mathscr{C}_{\theta^2}$, $\mathscr{C}_\theta((-\infty, 0])$, $\mathscr{C}_\theta([1, \infty))$ *to* $\mathscr{B}_\theta$ *such that* $A_{\varphi B} = B$, $A'_{\varphi' B'} = B'$, *and* $A''_{\varphi'' B''} = B''$.

(b) *The linear maps* $\Phi \to A_\Phi$, A'_Φ, A''_Φ *are continuous* $\mathscr{B}_\theta \to \mathscr{C}_{\theta^2}$, $\mathscr{C}_\theta((-\infty, 0])$, $\mathscr{C}_\theta([1, \infty))$ (*and surjective by* (a)). *Furthermore* $A_\Phi \sim A'_\Phi \circ \pi_{(-\infty,0]} \sim A''_\Phi \circ \pi_{[1,\infty)}$ *in* $\mathscr{C}_\theta$.

(c) *The conditions* $A_\Phi \sim 0$ *in* $\mathscr{C}_{\theta^2}$, $A'_\Phi \sim 0$ *in* $\mathscr{C}_\theta((-\infty, 0])$, *and* $A''_\Phi \sim 0$ *in* $\mathscr{C}_\theta([1, \infty))$ *are all equivalent. We express them by writing* $\Phi \sim 0$.

(d) *If* $\Phi, \Phi' \in \mathscr{B}_\theta$ *and* $\Phi' \sim \Phi$ (*i.e.* $\Phi' - \Phi \sim 0$), *the spaces* $\mathscr{G}_{\lambda\mu}$ *of Gibbs distributions for the interactions* Φ *and* Φ' *are the same*.

(e) *The pressure* P_Φ *associated with a real interaction* $\Phi \in \mathscr{B}_\theta$ *depends only on the cohomology class of* A_Φ (*or* A'_Φ *or* A''_Φ) *and can in particular be denoted by* $P(A_\Phi)$.

In order to construct φ in (a) one writes A as a sum $\sum_{n \geqslant 0} A_n$, where $A_n(\xi)$ depends only on $\pi_{[-n,n]}\xi$ and $\|A - \sum_0^{n-1} A_k\|_\infty \leqslant \operatorname{var}_n A$. Then one takes $\Phi \circ \pi_{[-n,n]} = -A_n$ and $\Phi \circ \pi_X = 0$ if diam X is odd. Similarly, for φ' and φ''.

In the proof of (b) one writes $A_\Phi - A''_\Phi \circ \pi_{(-\infty,0]} = C - C \circ \tau$. Suppose that $\Phi(\xi \mid X)$ vanishes unless $\dim X = 2n - 1$ [resp. $\operatorname{diam} X = 2n$]. Then one writes

$$C(\xi) = \Phi(\xi \mid [-2n+1,0]) + \cdots + \Phi(\xi \mid [-n, n-1])$$
$$\left[\text{resp. } C(\xi) = \Phi(\xi \mid [-2n,0]) + \cdots + \Phi(\xi \mid [-n-1, n-1])\right].$$

This definition extends by linearity and continuity to all $\Phi \in \mathscr{B}_\theta$, yielding $C \in \mathscr{C}_\theta$.

In view of (b), the proof of (c) reduces to checking the following:

$$\left(A'_\Phi \circ \pi_{(-\infty,0]} \sim 0 \text{ in } \mathscr{C}_\theta\right) \Rightarrow \left(A'_\Phi \sim 0 \text{ in } \mathscr{C}_\theta((-\infty,0])\right),$$
$$\left(A''_\Phi \circ \pi_{[1,\infty)} \sim 0 \text{ in } \mathscr{C}_\theta\right) \Rightarrow \left(A''_\Phi \sim 0 \text{ in } \mathscr{C}_\theta([1,\infty))\right).$$

By assumption, if ξ' is a periodic orbit of period n for $\pi'\tau^{-1}$ (remember that π' is the projection $\Omega_{(-\infty,1]} \to \Omega_{(-\infty,0]}$), then

$$\sum_{l=0}^{n-1} A'_\Phi\left((\pi'\tau^{-1})^l \xi'\right) = 0.$$

One may thus construct C' such that $A'_\Phi = C' - C' \circ \pi' \circ \tau^{-1}$ by first choosing a dense orbit $\{(\pi'\tau^{-1})^k \eta : k \geqslant 0\}$ in $\Omega_{(-\infty,0]}$, then writing

$$C'\left((\pi'\tau^{-1})^k \eta\right) = -\sum_{l=0}^{k-1} A'_\Phi\left((\pi'\tau^{-1})^l \eta\right)$$

and extending the definition of C' by continuity. It is easy to verify that $C' \in \mathscr{C}_\theta((-\infty, 0])$. Similarly for the study of A''_Φ. (See Livšic [13], [14] and Sinai [26] for the construction of C', and Ruelle [21, Theorem 5.2], for an application close to the present one.)

To prove (d), it suffices to show that $\pi_{(-\infty,0]}\mathscr{G}_{\lambda\mu}$ remains the same under the replacement $\Phi \to \Phi'$ [use translation invariance of $\mathscr{G}_{\lambda\mu}$ and density of $\bigcup_{n \geqslant 0} \mathscr{C}_\theta((-\infty, 0]) \circ \tau^n$ in $\mathscr{C}_\theta$]. The space $\pi_{(-\infty,0]}\mathscr{G}_{\lambda\mu}$ is spanned by the products $S'\sigma'$, with $S' \in E'_\mu$, $\sigma' \in E'^*_\lambda$. Since $V'_0 = A'_\Phi$, we have

$$\begin{aligned}\left(\mathscr{L}'_{\Phi'} B\right)(\xi') &= \sum_{\eta \in J} B(\tau\xi' \vee \eta) \exp A'_{\Phi'}(\tau\xi' \vee \eta)\\ &= \sum_{\eta \in J} B(\tau\xi' \vee \eta) \exp\left[A'_\Phi(\tau\xi' \vee \eta) + C'(\tau\xi' \vee \eta) - C'(\xi')\right]\end{aligned}$$

with $C' \in \mathscr{C}_\theta((-\infty, 0])$. Therefore

$$e^{C'}\mathscr{L}'_{\Phi'} = \mathscr{L}'_\Phi e^{C'} \tag{3.1}$$

so that when $\Phi \to \Phi'$, $E'_\mu \to e^{-C'}E'_\mu$ and $E'^*_\lambda \to e^{C'}E'^*_\lambda$. The products $S'\sigma'$ are thus unchanged, proving (d).

If Φ is real, Proposition 3.5 below identifies e^P as the largest eigenvalue of $\mathscr{L}'$ in absolute value. In view of (3.1) this is unchanged when $\Phi \to \Phi'$ with $A'_\Phi \sim A'_{\Phi'}$, proving (e).

3.2. Proposition. (a) *If $\Phi' \sim \Phi$, the eigenvalues of $\mathscr{L}'$ and their multiplicities are unchanged by the replacement $\Phi \to \Phi'$.*

(b) *Where λ is a simple eigenvalue of $\mathscr{L}'$, it depends analytically on $A = A_\Phi \in \mathscr{C}_{\theta^2}$. We then have*

$$D_A\lambda = \lambda\sigma_{\lambda\lambda}|_{\mathscr{C}_{\theta^2}}, \tag{3.2}$$

where $D_A\lambda \in \mathscr{C}^*_{\theta^2}$ is the derivative of λ at A, and $\sigma_{\lambda\lambda} \in \mathscr{C}^*_\theta$ is the only element of $\mathscr{G}_{\lambda\lambda}$ such that $\sigma_{\lambda\lambda}(1) = 1$. ($\sigma_{\lambda\lambda}$ is also τ-invariant.) Note that $D_A\lambda$ extends uniquely to an element of $\mathscr{C}'_\theta$ (viz. $\lambda\sigma_{\lambda\lambda}$) because $\mathscr{C}_{\theta^2}$ is dense in $\mathscr{C}_\theta$. (There are similar results using $A'_\Phi \in \mathscr{C}_\theta((-\infty, 0])$ or $A''_\Phi \in \mathscr{C}_\theta([1, \infty))$ instead of A_Φ.)

Part (a) results from the proof of (d) in the previous proposition. Since λ is a simple eigenvalue of $\mathscr{L}'_{\varphi A}$, it depends analytically on $A'_{\varphi A} \in \mathscr{C}_\theta((-\infty, 0])$, and since $A \to A'_{\varphi A}$ is a continuous linear map, the analyticity statement in (b) is verified. Note that $\pi_{(-\infty,0]}\mathscr{G}_{\lambda\lambda}$ is one-dimensional and consists of the multiples of $S'\sigma'$, where $S' \in E'_\lambda$, $\sigma' \in E'^*_\lambda$, and we may assume that $\sigma'(S') = 1$. The only $\sigma_{\lambda\lambda} \in \mathscr{G}_{\lambda\lambda}$ mapped to $S'\sigma'$ by $\pi_{(-\infty,0]}$ is π-invariant and satisfies $\sigma_{\lambda\lambda}(1) = 1$.

To prove (3.2) it suffices to show that

$$(D_A\lambda)B = \lambda\sigma_{\lambda\lambda}(B)$$

for A and B in a dense subset of $\mathscr{C}_{\theta^2}$. In particular, we may take A of the form $A' \circ \pi_{(-\infty,0]} \circ \pi^n$ and B of the form $B' \circ \pi_{(-\infty,0]} \circ \tau^n$. Therefore it suffices to show that

$$(D_{A'}\lambda)B' = \lambda\sigma'(S'B'), \tag{3.3}$$

where $D_{A'}\lambda$ is now the derivative at A' of λ considered as an analytic function on $\mathscr{C}_\theta((-\infty, 0])$. Taking σ' and S' as before, we have

$$\begin{aligned}(D_{A'}\lambda)B' &= \left[D_{A'}\left[\sigma'(\mathscr{L}'S')\right]\right]B' \\ &= \left[(D_{A'}\sigma')(\mathscr{L}'S') + \sigma'(\mathscr{L}'D_{A'}S')\right]B' + \left[\sigma'((D_{A'}\mathscr{L}')S')\right]B' \\ &= \lambda\left[(D_{A'}\sigma')(S') + \sigma'(D_{A'}S')\right]B' + \left[\sigma'((D_{A'}\mathscr{L}')S')\right]B'.\end{aligned}$$

The first term vanishes because $\sigma'(S') \equiv 1$. Since $((D_{A'}\mathscr{L}')S')B' = \mathscr{L}'(S'B')$ we obtain

$$(D_{A'}\lambda)B' = \sigma'(\mathscr{L}'(S'B')) = \lambda\sigma'(S'B')$$

and (3.3) is verified.

We introduce now an analytic function d which will be useful later.

3.3. Proposition. *If $A \in \mathscr{C}_{\theta^2}$, $A' \in \mathscr{C}_\theta((-\infty,\ 0])$, and $A'' \in \mathscr{C}_\theta([1,\ \infty))$, we write*

$$Z_n = \sum_{\xi : \tau^n\xi = \xi} \exp\left[A(\xi) + A(\tau\xi) + \cdots A(\tau^{n-1}\xi)\right],$$

$$Z_n' = \sum_{\xi : \tau^n\xi = \xi} \exp\left[B'(\xi) + B'(\tau\xi) + \cdots B'(\tau^{n-1}\xi)\right],$$

$$Z_n'' = \sum_{\xi : \tau^n\xi = \xi} \exp\left[B''(\xi) + B''(\tau\xi) + \cdots B''(\tau^{n-1}\xi)\right],$$

where $B' = A' \circ \pi_{(-\infty,\ 0]}$ and $B'' = A'' \circ \pi_{[1,\ \infty)}$. We define

$$d(ze^A) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n\right),$$

$$\det(1 - ze^{A'}) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n'\right),$$

$$\det(1 - ze^{A''}) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n''\right)$$

as power series. There is some $\bar\theta < 1$ such that $\det(1 - ze^{A'})$ *converges when $|z|\bar\theta e^{P'} < 1$, with $P' = P(\mathrm{Re}A' \circ \pi_{(-\infty,0]})$, and* $\det(1 - ze^{A''})$ *converges when $|z|\bar\theta e^{P''} < 1$, with $P'' = P(\mathrm{Re}A'' \circ \pi_{[1,\infty)})$.*

If $\Phi \in \mathscr{B}_\theta$, we have

$$d(z\exp A_\Phi) = \det\left(1 - z\exp A_\Phi'\right) = \det\left(1 - z\exp A_\Phi''\right).$$

These quantities do not change if Φ is replaced by $\Psi \sim \Phi$. In the region $\{z : |z|\bar\theta e^P < 1\}$, with $P = P(\mathrm{Re}A_\Phi)$, the zeros of $z \to d(z\exp A_\Phi)$ coincide with the inverses λ^{-1} of the eigenvalues of $\mathscr{L}'$ such that $\lambda > \bar\theta e^P$, and have the same multiplicity.

There are analytic functions $z \to \mathscr{N}'(ze^{A'})$, $\mathscr{N}''(ze^{A''})$ defined on $\{z : |z|\bar\theta e^P < 1\}$ and with values in the bounded operators in $\mathscr{C}_\theta((-\infty,\ 0])$ or $\mathscr{C}_\theta([1,\ \infty))$ such that

$$(1 - z\mathscr{L}')^{-1} = \frac{\mathscr{N}'\left(z\exp A_\Phi'\right)}{\det\left(1 - z\exp A_\Phi'\right)},$$
$$(1 - z\mathscr{L}'')^{-1} = \frac{\mathscr{N}''\left(z\exp A_\Phi''\right)}{\det\left(1 - z\exp A_\Phi''\right)}. \tag{3.4}$$

All these results are essentially contained in Pollicott [17].[4] (The statement about multiplicities of the zeros, and about the existence of $\mathcal{N}'$, $\mathcal{N}''$ are not in there, but follow readily.) The "determinant" notation is suggested both by the definitions and by (3.4).

3.4. Proposition. *Let $\theta' \in (\theta, 1)$. For each eigenvalue λ of $\mathcal{L}'$, with $\theta' e^P < |\lambda| \leqslant e^P$, we choose a basis $(S'_{\lambda\alpha})$ of E'_λ and a basis $(\sigma'_{\lambda\alpha})$ of E'^*_λ such that $\sigma'_{\lambda\alpha}(S'_{\lambda\beta}) = \delta_{\alpha\beta}$ and $\sigma'_{\lambda\alpha}(\mathcal{L}' S'_{\lambda\beta}) = \lambda(L'_\lambda)_{\alpha\beta}$, where the matrix L'_λ is in Jordan normal form. Then*

$$\mathcal{L}' \cdot = \sum_\lambda \lambda \sum_{\alpha\beta} S'_{\lambda\alpha}\left(L'_\lambda\right)_{\alpha\beta} \sigma'_{\lambda\beta}(\cdot) + e^P \mathcal{R}' \cdot$$

where $\mathcal{R}'$ has spectral radius $\leqslant \theta'$, hence

$$(1 - ze^{-P}\mathcal{L}')^{-1} \cdot = \sum_\lambda \sum_{\alpha\beta} S'_{\lambda\alpha}\left((1 - ze^{-P}\lambda L'_\lambda)^{-1}\right)_{\alpha\beta} \sigma'_{\lambda\beta}(\cdot) + U'(z) \cdot$$

where $U'(z)$ is an analytic function of z for $|z| < \theta^{-1}$. In particular, if the eigenvalues of $\mathcal{L}'$ are simple, we have

$$(1 - ze^{-P}\mathcal{L}')^{-1} \cdot = \sum_\lambda \frac{S'_\lambda \sigma'_\lambda(\cdot)}{1 - ze^{-P}\lambda} + U'(z) \cdot \tag{3.5}$$

In the general case, for each eigenvalue λ of $\mathcal{L}'$ there is p, with $1 \leqslant p \leqslant$ multiplicity of λ, such that $(1 - ze^{-P}\mathcal{L}')^{-1}$ has a pole of order p at λ with leading term

$$\frac{\sum_q S'^q_\lambda \sigma'^q_\lambda(\cdot)}{(1 - ze^{-P}\lambda)^p}, \tag{3.6}$$

where S'^q_λ, σ'^q_λ are eigenvectors of $\mathcal{L}'$, $\mathcal{L}'^*$ to the eigenvalue λ.

Note that there are similar results with $\mathcal{L}''$ replacing $\mathcal{L}'$. We check only (3.6), the rest is immediate. Let L'_λ contain the $p \times p$ "normal" block

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}, \tag{3.7}$$

[4] Pollicott shows that the part of the spectrum of $\mathcal{L}'$ and $\mathcal{L}''$ in $\{\lambda : |\lambda| > \theta e^P\}$ consists of eigenvalues of finite multiplicity. But the analyticity of $\det(1 - ze^{A'})$ and $\det(1 - ze^{A''})$ is proved only when $|z|\bar{\theta}e^{P'} < 1$ or $|z|\bar{\theta}e^{P''} < 1$ with some $\bar{\theta} > \theta$. (I am indebted to the referee for reminding me of this).

corresponding to values $q+1,\cdots,q+p$ of the indices α, β of $(L'_\lambda)_{\alpha\beta}$. Using the formula

$$\begin{pmatrix} 1-u & -u & & \\ & 1-u & \ddots & \\ & & \ddots & -u \\ & & & 1-u \end{pmatrix}^{-1} = \begin{pmatrix} (1-u)^{-1} & u(1-u)^{-2} & \cdots & u^{p-1}(1-u)^{-p} \\ & (1-u)^{-1} & & \\ & & \ddots & u(1-u)^{-2} \\ & & & (1-u)^{-1} \end{pmatrix}$$

we see that the block (3.7) contributes to $(1 - ze^{-P}\mathscr{L}')^{-1}$ a pole of order p at $e^P\lambda^{-1}$. The leading term is

$$\frac{S'_{\lambda,q+1}\sigma'_{\lambda,q+p}(\cdot)}{(1-ze^{-P}\lambda)^p}.$$

We write $S'^q_\lambda = S'_{\lambda,q+1}$, $\sigma'^q_\lambda = \sigma'_{\lambda,q+p}$, and note that $\mathscr{L}'S'^q_\lambda = \lambda S'^q_\lambda$, $\mathscr{L}'^*\sigma'^q_\lambda = \lambda\sigma'^q_\lambda$; (3.6) follows readily.

3.5. Proposition. *If the interaction Φ is real, then e^P, with $P = P_\Phi$, is a simple eigenvalue of $\mathscr{L}'$, and the rest of the spectrum of $\mathscr{L}'$ is contained in the open disk $\{z : |z| < e^P\}$.*

For the proof, see Ruelle [19] or [21].

4. Correlation functions

In this section, Φ will be real, so that Proposition 3.5 applies. The Gibbs distribution $\sigma_{e^Pe^P}$ (in the notation of Proposition 3.2(b)) is just the ordinary Gibbs state for Φ, and we shall denote it by ρ. The correlation function $\rho_{BC}(\cdot)$ is defined (for $B, C \in \mathscr{C}_\theta$) by

$$\rho_{BC}(k) = \rho((B \circ \tau^k)C) - \rho(B)\rho(C), \tag{4.1}$$

and we shall be interested in its Fourier transform

$$\hat{\rho}_{BC}(\alpha) = \sum_{k=-\infty}^{\infty} e^{k\alpha}\rho_{BC}(k). \tag{4.2}$$

Let $\sigma' \in E'^*_{e^P}$, $S' \in E'_{e^P}$, where σ' and S' are positive and satisfy $\sigma'(S') = 1$. By Proposition 3.2(b) a probability measure $\tilde{\rho}$ on $\Omega_{(-\infty,0]}$ is defined equivalently by

$$\tilde{\rho} = \pi_{(-\infty,0]}\rho, \quad \text{or} \quad \tilde{\rho}(\cdot) = \sigma'(S'\,\cdot). \tag{4.3}$$

Before studying (4.1) and (4.2), we take $B', C' \in \mathscr{C}_\theta((-\infty, 0])$ and discuss $\tilde{\rho}((B' \circ \tau'^n)C')$, where

$$\tau' = \pi' \circ \tau^{-1}, \qquad \pi' : \Omega_{(-\infty,1]} \mapsto \Omega_{(-\infty,0]}. \tag{4.4}$$

We have

$$\tilde{\rho}\big((B'\circ\tau'^n)C'\big)=\sigma'\big((B'\circ\tau'^n)S'C'\big)=e^{-nP}\sigma'(B'\mathscr{L}'^nS'C')$$

and therefore, for $|z|\leqslant 1$,

$$\begin{aligned}\sum_{n\geqslant 0} z^n\tilde{\rho}\big((B'\circ\tau'^n)C'\big)&=\sigma'\Big(B'\sum_{n\geqslant 0}(ze^{-P}\mathscr{L}')^nS'C'\Big)\\ &=\tilde{\rho}\Big(B'\sum_{n\geqslant 0}(ze^{-P}S'^{-1}\mathscr{L}'S')^nC'\Big)\\ &=\tilde{\rho}\big(B'(1-ze^{-P}S'^{-1}\mathscr{L}'S')^{-1}C'\big).\end{aligned}\tag{4.5}$$

This expression extends meromorphically to $\{z:|z|<\theta^{-1}\}$ in view of the following result.

4.1. Proposition. (a) *With the notation* (4.3), (4.4) *we may write*

$$\sum_{n\geqslant 0} z^n\tilde{\rho}\big((B'\circ\tau'^n)C'\big)=\frac{\tilde{\rho}\big(B'\mathscr{N}'(z\exp(-P-A'_\Psi))C'\big)}{\det\big(1-z\exp(-P-A'_\Psi)\big)},$$

where the numerator is analytic in $\{z:|z|<\bar{\theta}^{-1}\}$, *and bilinear continuous in* B', C'. *The operator* $\mathscr{N}'$ *is given by Proposition* 3.3, *with* $A'_\Psi=A'_\Phi+\log S'-\log S'\circ\tau'$.

(b) *To avoid complications let us assume that the eigenvalues of* $\mathscr{L}'$ *are simple, so that* E'_λ *is spanned by* S'_λ, E'^*_λ *by* σ'_λ. *We may also assume that* $\sigma'_\lambda(S'_\lambda)=1$. *Then, choosing* θ' *as in Proposition* 3.4, *we have*

$$\sum_{n\geqslant 0} z^n\tilde{\rho}\big((B'\circ\tau'^n)C'\big)=\sum_\lambda\frac{\sigma'(S'_\lambda B')\sigma'_\lambda(S'C')}{1-z\lambda e^{-P}}+\tilde{\rho}\big(B'\mathscr{V}'(z)C'\big),$$

where $\mathscr{V}'(z)=S'^{-1}U'(z)S'$ *and* U' *is given by Proposition* 3.4; *the last term is thus analytic in* z *for* $|z|<\theta'^{-1}$ *and bilinear continuous in* B', C'.

The proof results from application of Propositions 3.3 and 3.4 to (4.5).

We come now to the study of (4.1), (4.2), and write

$$B-\rho(B)=\sum_{m=0}^{\infty}B_m\circ\pi_{(-\infty,0]}\circ\tau^m,\qquad C-\rho(C)=\sum_{m=0}^{\infty}C_m\circ\pi_{(-\infty,0]}\circ\tau^m,$$

where $B_m, C_m\in\mathscr{C}_\theta((-\infty,\ 0])$, and

$$\operatorname{var}_{2m+1}B_m=\operatorname{var}_{2m+1}C_m=0,$$

$$\|B_m\|_\infty\leqslant\theta^m|||B|||_\theta,\qquad \|C_m\|\leqslant\theta^m|||C|||_\theta,$$

$$\lim_{m\to\infty}\theta^{-m}\|B_m\|_\infty=\lim_{m\to\infty}\theta^{-m}\|C_m\|_\infty=0.$$

Define also

$$B(z)=\sum_{m=0}^{\infty}z^mB_m,\qquad C(z)=\sum_{m=0}^{\infty}z^mC_m.$$

We thus have

$$\hat{\rho}_{BC}(\alpha) = \sum_{k=-\infty}^{\infty} e^{k\alpha}\rho\left[\left(B\circ\tau^{k} - \rho(B)\right)\left(C - \rho(C)\right)\right]$$

$$= \sum_{k=-\infty}^{\infty}\sum_{m,n\geqslant 0} e^{k\alpha}\rho\left[\left(B_m\circ\pi_{(-\infty,0]}\circ\tau^{k+m}\right)\left(C_n\circ\pi_{(-\infty,0]}\circ\tau^{n}\right)\right]$$

$$= \sum_{m,n\geqslant 0} e^{(n-m)\alpha}\Bigg[\sum_{k\leqslant n-m} e^{(k+m-n)\alpha}\tilde{\rho}\left(\left(B_m\circ\tau'^{(n-m-k)}\right)C_n\right)$$

$$+\sum_{k\geqslant n-m} e^{(k+m-n)\alpha}\tilde{\rho}\left(B_m\left(C_n\circ\tau'^{(k+m-n)}\right)\right) - \tilde{\rho}\left(B_mC_n\right)\Bigg] \tag{4.6}$$

$$= \sum_{l\geqslant 0} e^{-l\alpha}\tilde{\rho}\left(\left(B(e^{-\alpha})\circ\tau'^{l}\right)C(e^{\alpha})\right)$$

$$+\sum_{l\geqslant 0} e^{l\alpha}\tilde{\rho}\left(B(e^{-\alpha})\left(C(e^{\alpha})\circ\tau'^{l}\right)\right) - \tilde{\rho}\left(B(e^{-\alpha})C(e^{\alpha})\right)$$

$$= \tilde{\rho}\left(B(e^{-\alpha})(1 - e^{-\alpha}e^{-P}S'^{-1}\mathscr{L}'S')^{-1}C(e^{\alpha})\right)$$

$$+\tilde{\rho}\left(C(e^{\alpha})(1 - e^{\alpha}e^{-P}S'^{-1}\mathscr{L}'S')^{-1}B(e^{-\alpha})\right)$$

$$-\tilde{\rho}\left(B(e^{-\alpha})C(e^{\alpha})\right).$$

Note that our way of writing $\hat{\rho}_{BC}(\alpha)$ is asymmetric in the sense L' rather than L'' plays a priviledged role. We could of course write $\hat{\rho}_{BC}(\alpha)$ in other ways.

4.2. Proposition. *We use definitions* (4.1), (4.2) *with* $B, C \in \mathscr{C}_\theta$.

(a) *We may write*

$$\hat{\rho}_{BC}(\alpha) = \frac{N_{BC}(e^{-\alpha})}{\det\left(1 - \exp\left(-\alpha - P + A'_\Phi\right)\right)} + \frac{N_{CB}(e^{\alpha})}{\det\left(1 - \exp\left(\alpha - P + A'_\Phi\right)\right)}$$
$$-\tilde{\rho}\left(B(e^{-\alpha})C(e^{\alpha})\right),$$

where

$$N_{BC}(e^{\alpha}) = \tilde{\rho}\left(B(e^{-\alpha})\mathscr{N}'\left(\exp\left(-\alpha - P + A'_\Psi\right)\right)C(e^{\alpha})\right)$$

and $A'_\Psi = A'_\Phi + \log S' - \log S'\circ\tau^*$. *Therefore* $\hat{\rho}_{BC}(\alpha)$ *is meromorphic for* $|\mathrm{Re}\,\alpha| < \log\,\theta^{-1}$, *with poles located at* $\pm(\log\lambda - P)$ *where the* λ *are the eigenvalues of* $\mathscr{L}'$ *with the exception of* e^P.

(b) *To avoid complications, let us first assume that the eigenvalues of* $\mathscr{L}'$ *are simple, so that* $\mathscr{G}_{e^P\lambda}$ *and* $\mathscr{G}_{\lambda e^P}$ *are one-dimensional, spanned by* $\sigma_{e^P\lambda}$ *and* $\sigma_{\lambda e^P}$ *respectively. We may also assume that* $(\sigma_{e^P\lambda}, \sigma_{\lambda e^P}) = 1$ (*see* (2.9)). *Finally, we choose* $\theta' \in (\theta, 1)$. *Then*

$$\hat{\rho}_{BC}(\alpha) = \sum_{\lambda}{}^{*}\left[\frac{\sigma_{e^P\lambda}(B)\sigma_{\lambda e^P}(C)}{1 - e^{-\alpha-P}\lambda} + \frac{\sigma_{e^P\lambda}(C)\sigma_{\lambda e^P}(B)}{1 - e^{\alpha-P}\lambda}\right] + \mathscr{W}_{BC}(\alpha),$$

where the sum Σ^* *extends over those eigenvalues* λ *of* $\mathscr{L}'$ *(in finite number) such that* $|\lambda| > \theta' e^P$ *and* $\lambda \neq e^P$; $\mathscr{W}_{BC}(\alpha)$ *is analytic for* $|\mathrm{Re}\,\alpha| < \log \theta'^{-1}$, *and bilinear continuous in* B, C.

(c) *In the general case, for each eigenvalue* $\lambda \neq e^P$ *of* $\mathscr{L}'$ *there is p such that* $\hat{\rho}_{BC}(\alpha)$ *has poles of order p at* $e^{\pm\alpha} = e^P \lambda^{-1}$. *These poles have an expression of the form*

$$(4.7) \qquad \sum_{r=1}^{p} \left[\frac{\Sigma_s \sigma_{rs}^-(B) \sigma_{rs}^+(C)}{(1 - e^{-\alpha - P}\lambda)^r} + \frac{\Sigma_s \sigma_{rs}^-(C) \sigma_{rs}^+(B)}{(1 - e^{\alpha - P}\lambda)^r} \right],$$

where $\sigma_{rs}^- \in \mathscr{G}_{e^P\lambda}$, $\sigma_{rs}^+ \in \mathscr{G}_{\lambda e^P}$. *For the leading order terms we have* $\tau\sigma_{ps}^{\pm} = (e^{-P}\lambda)^{\pm 1}\sigma_{ps}^{\pm}$.

This proposition is obtained by inserting (3.4) or (3.5) into (4.6). Concerning (a), notice that $\det(1 - \exp(-\alpha - P - A'_\Phi))$ is analytic when $\mathrm{Re}\,\alpha > \log\theta$, vanishing when $e^{\alpha+P}$ is an eigenvalue of $\mathscr{L}'$. If $\theta < \theta^* < 1$, we have $C(e^\alpha) \in \mathscr{C}_{\theta^*}((-\infty, 0])$ provided $\mathrm{Re}\,\alpha + \log\theta \leqslant 2\log\theta^*$, and $\mathscr{N}'(e^{-\alpha-P}A'_\Phi)$ is analytic as an operator on $\mathscr{C}_{\theta^*}((-\infty, 0])$ provided $\mathrm{Re}\,\alpha > \log\theta^*$. Therefore $N_{BC}(e^{-\alpha})$ is analytic when $\log\theta < \mathrm{Re}\,\alpha < |\log\theta|$, and bilinear continuous in B, C. Similarly for $N_{CB}(e^\alpha)$, $\tilde{\rho}(B(e^{-\alpha})C(e^\alpha))$.

To obtain (b) we write

$$\hat{\rho}_{BC}(\alpha) = \sum_{\lambda}{}^{*} \left[\frac{\sigma'(S'_\lambda B(e^{-\alpha}))\sigma'_\lambda(S'C(e^\alpha))}{1 - e^{-\alpha-P}\lambda} + \frac{\sigma'(S'_\lambda C(e^\alpha))\sigma'_\lambda(S'B(e^{-\alpha}))}{1 - e^{\alpha-P}\lambda} \right]$$
$$+\tilde{\rho}(B(e^{-\alpha})\mathscr{V}'(e^{-\alpha})C(e^\alpha)) + \tilde{\rho}(C(e^\alpha)\mathscr{V}'(e^\alpha)B(e^{-\alpha}))$$
$$-\tilde{\rho}(B(e^{-\alpha})C(e^\alpha))$$

and replace in Σ^* the numerators by their values at the zeros of the denominators.

To prove (c) we consider the contribution to the poles of $\hat{\rho}_{BC}$ coming from one $p \times p$ block (3.7) in L'_λ. This contribution has the form

$$(4.8) \qquad \sum_{1 \leqslant i \leqslant j \leqslant p} \left[\sigma'(S'_{\lambda,q+i}B(e^{-\alpha}))\sigma'_{\lambda,q+j}(S'C(e^\alpha)) \frac{(e^{-\alpha-P}\lambda)^{j-i}}{(1 - e^{-\alpha-P}\lambda)^{j-i+1}} + \text{other term} \right],$$

where the "other term" is obtained from the first by the exchange of $B(e^{-\alpha})$, $C(e^\alpha)$ and the replacement of $e^{-\alpha}$ by e^α in the denominator. To proceed we write

$$e^{-\alpha} = e^P\lambda^{-1}(1 - (1 - e^{-\alpha-P}\lambda))$$

and expand the first term of (4.8) in powers of $(1 - e^{-\alpha - P}\lambda)$; only the terms of total degree < 0 are kept. Let us provisionally assume that there is only one term in the definitions of $B(z)$ and $C(z)$, i.e.,

$$\begin{aligned} B(e^{P}\lambda^{-1}) &= (e^{P}\lambda^{-1})^{m} B_m, & C(e^{-P}\lambda) &= (e^{-P}\lambda)^{n} C_n, \\ B - \rho(B) &= B_m \circ \pi_{(-\infty,0]} \circ \tau^{m}, & C - \rho(C) &= C_n \circ \pi_{(-\infty,0]} \circ \tau^{n}. \end{aligned} \tag{4.9}$$

If we express $\sigma'(S'_{\lambda,q+i}B_m)$ and $\sigma'_{\lambda,q+j}(S'C_n)$ in terms of $\sigma_k^-(B)$ and $\sigma_l^+(C)$ with $\sigma_k^- \in \mathcal{G}_{e^P\lambda}$ and $\sigma_l^+ \in \mathcal{G}_{\lambda e^P}$, $1 \leqslant k,\ l \leqslant p$, we obtain an expression of the form

$$\sum_{r=1}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} P_{rkl}\left[\frac{\sigma_k^-(B)\sigma_l^+(C)}{(1 - e^{-\alpha - P}\lambda)^{r}} + \frac{\sigma_k^-(C)\sigma_l^+(B)}{(1 - e^{\alpha - P}\lambda)^{r}}\right], \tag{4.10}$$

where the P_{rkl} are polynomials in m, n arising from the calculation. These polynomials must be constants because if B and C have a representation of type (4.9) with given m, n, they have a similar representation for all larger m and n. Since the P_{rkl} are constants, the case where B is a sum over m and C a sum over n is easy to handle: the expression (4.10) remains valid.

The order p term (leading order term) in (4.8) corresponds to $i = 1$, $j = p$, yielding the expression

$$\begin{aligned} &\sigma'\big(S'_{\lambda,q+1}B(e^{P}\lambda^{-1})\big)\sigma'_{\lambda,q+p}\big(S'C(e^{-P}\lambda)\big)(1 - e^{-\alpha - P}\lambda)^{-p} + \text{other term} \\ &\qquad = \frac{\sigma_1^-(B)\sigma_p^+(C)}{(1 - e^{-\alpha - P}\lambda)^{p}} + \frac{\sigma_1^-(C)\sigma_p^+(B)}{(1 - e^{\alpha - P}\lambda)^{p}}, \end{aligned} \tag{4.11}$$

where $\tau\sigma_1^- = e^{P}\lambda^{-1}\sigma_1^-$, $\tau\sigma_p^+ = e^{-P}\lambda\sigma_p^+$. Summing (4.10) over blocks yields (4.7), and since the leading order terms are of the form (4.11), the proof of (c) is complete.

5. Gibbs distributions for Axiom A diffeomorphisms

We refer the reader to Smale [27] or Bowen [3] for the definition of Axiom A diffeomorphisms and of basic sets. We shall be concerned with the ergodic theory of Axiom A systems, as discussed by Bowen [2]. (We could also work in the more abstract framework of "Smale spaces" as discussed in Ruelle [21].) The connection with the statistical mechanics discussed earlier is via *symbolic dynamics* as we now indicate. Given are a compact manifold M, which we may take as C^{∞}, with a diffeomorphism f of class $C^{1+\epsilon}$, $\varepsilon > 0$, and a compact invariant subset Λ. If f satisfies Axiom A, and Λ is a basic set, it is possible to define a subshift of finite type (Ω, τ) and a continuous map $\omega : \Omega \to M$ such that $\omega\Omega = \Lambda$ and $\omega\tau = f\omega$, and ω has other nice properties which we shall make explicit only to the extent that we need them. The definition of (Ω, τ) and ω depends on the choice of a *Markov partition* on Λ (see Sinai [24], [25] and Bowen [1], [2]). The representation of points of Λ by points of Ω (which are sequences of *symbols* $j \in J$), and the replacement of the diffeomorphisms f by the shift τ constitute "symbolic dynamics."

Let $\mathscr{C}^\alpha(M)$ denote the Banach space of complex Hölder continuous functions of exponent α on M. The functions which occur naturally in the study of Axiom A systems are in $\mathscr{C}^\alpha(M)$ for suitable α. Having fixed $\alpha \in (0,1)$ one shows that there exists $\theta \in (0,1)$ such that

(5.1) $$\mathbf{A} \circ \omega \in \mathscr{C}_{\theta^2} \quad \text{if } \mathbf{A} \in \mathscr{C}^\alpha(M).$$

Furthermore, one can choose $\beta < \alpha$ such that

(5.2) $$\mathbf{B} \circ \omega \in \mathscr{C}_\theta \quad \text{if } \mathbf{B} \in \mathscr{C}^\beta(M).$$

The linear maps $\mathscr{C}^\alpha(M) \mapsto \mathscr{C}_{\theta^2}$ and $\mathscr{C}^\beta(M) \mapsto \mathscr{C}_\theta$ thus defined are continuous. The numbers α, θ, β may be chosen independent of the symbolic dynamics (i.e., they may be taken to the same "for all sufficiently fine Markov partitions").

Given $\mathbf{A} \in \mathscr{C}^\alpha(M)$ we may, in view of (5.1) and Proposition 3.1(a), choose $\Phi \in \mathscr{B}_\theta$ such that

$$\mathbf{A} \circ \omega = A_\Phi.$$

We let $\mathscr{G}_{\lambda\mu}$ be the spaces of Gibbs distributions on Ω with respect to Φ and define subspaces $\mathbf{G}_{\lambda\mu}$ of $\mathscr{C}^\alpha(M)^*$ by

$$\mathbf{G}_{\lambda\mu} = \omega\mathscr{G}_{\lambda\mu}, \quad \text{where } (\omega\sigma)(h) = \sigma(h \circ \omega).$$

If $\boldsymbol{\sigma} \in \bigcup \mathbf{G}_{\lambda\mu}$, we call $\boldsymbol{\sigma}$ a *Gibbs distribution* on Λ. Since smooth functions on M are Hölder continuous, $\boldsymbol{\sigma}$ also defines a distribution on M in the sense of Schwartz, and the support of $\boldsymbol{\sigma}$ is contained in Λ. The next proposition shows that the knowledge of $\boldsymbol{\sigma}$ as a Schwartz distribution determines it as an element of $\mathscr{C}^\alpha(M)^*$.

5.1. Proposition. *Let $\boldsymbol{\sigma}$ be a Gibbs distribution on Λ and $h \in \mathscr{C}^\alpha(M)$. There are smooth functions h_i such that $h_i \to h$ in the topology of $\mathscr{C}^\beta(M)$, and therefore $\boldsymbol{\sigma}(h_i) \to \boldsymbol{\sigma}(h)$.*

By regularization (see Schwartz [22]) one obtains a sequence h_i of smooth functions on h such that $h_i - h \to 0$ uniformly, and

$$|h_i(x) - h_i(y)| \leqslant K(\operatorname{dist}(x,y))^\alpha,$$

where dist denotes a Riemann distance on M, and the constant K is independent of i. Therefore we have also

$$|h_i(x) - h(x) - (h_i(y) - h(y))| \leqslant 2K(\mathrm{dist}(x,y))^{\alpha} \leqslant \left[2K(\mathrm{dist}(x,y))^{\alpha-\beta}\right](\mathrm{dist}(x,y))^{\beta}.$$

Given $\varepsilon > 0$ we can thus find $\delta > 0$ such that, if $\mathrm{dist}(x, y) \leqslant \delta$, then

$$|h_i(x) - h(x) - (h_i(y) - h(y))| \leqslant \varepsilon(\mathrm{dist}(x,y))^{\beta}.$$

If $\mathrm{dist}(x, y) \geqslant \delta$ we have, for sufficiently large i,

$$|h_i(x) - h(x) - (h_i(y) - h(y))| \leqslant 2\max_x|h_i(x) - h(x)| \leqslant 2\max_x|h_i(x) - h(x)| \cdot \frac{(\mathrm{dist}(x,y))^{\beta}}{\delta^{\beta}} \leqslant (\mathrm{dist}(x,y))^{\beta}$$

which proves that $h_i \to h$ in the topology of $\mathscr{C}^{\beta}(M)$. But then $h_i \circ \omega \to h \circ \omega$ in the topology of $\mathscr{C}_{\theta}$ and therefore

$$\sigma(h_i) = (\omega\sigma)h_i = \sigma(h_i \circ \omega) \to \sigma(h \circ \omega) = (\omega\sigma)h = \sigma(h).$$

Our definition of the spaces $\mathbf{G}_{\lambda\mu}$ depends on the choice of a Markov partition. We have not been able to eliminate this dependence in general, but Propositions 5.2, 5.3 and Corollary 5.5 yield partial results on the problem.

5.2. Proposition. *If* $\mathbf{A}$ *is real, the largest eigenvalue* e^P *of* $\mathscr{L}'$ *and the Gibbs measure* $\boldsymbol{\sigma} = \omega\rho$ *are independent of the choice of Markov partition. We write* $\mathbf{P} = \mathbf{P}(\mathbf{A})$.

This is a well-known fact: If I is the set of f-invariant probability measures on Λ, and if $h(\boldsymbol{\sigma})$ denotes the entropy of $\boldsymbol{\sigma} \in I$, we have

$$P = \max_{\boldsymbol{\sigma} \in I}(h(\boldsymbol{\sigma}) + \boldsymbol{\sigma}(\mathbf{A})) \tag{5.3}$$

and the unique maximum is reached for $\boldsymbol{\sigma} = \boldsymbol{\rho}$ (see [2] or [21]).

5.3. Proposition. *For any* $\mathbf{A} \in \mathscr{C}^{\alpha}(M)$, *define*

$$\zeta_{\mathbf{A}}(z) = \exp\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{x\in \mathrm{Fix} f^n|\Lambda}\exp\sum_{k=0}^{n-1}\mathbf{A}(f^k x).$$

This series is convergent for $|z| < \exp(-\mathbf{P}(\mathrm{Re}\,\mathbf{A}))$ *and* $\zeta_{\mathbf{A}}$ *extends meromorphically to the region* $|z| < \bar{\theta}^{-1}\exp(-\mathbf{P}(\mathrm{Re}\,\mathbf{A}))$.

We may define $\mathbf{P}^* < \mathbf{P}$ *such that if*

$$|z| < \left\{\bar{\theta}^{-1}\exp(-\mathbf{P}(\mathrm{Re}\,\mathbf{A})), \exp(-\mathbf{P}^*(\mathrm{Re}\,\mathbf{A}))\right\},$$

we have

$$\zeta_{\mathbf{A}}(z) = \frac{\Phi(z)}{d(z\exp A_{\Phi})}, \tag{5.4}$$

where Φ *is holomorphic and does not vanish.*

The proof of this proposition is based on the counting of periodic points for a finite collection of subshifts of finite type (see Manning [15] and Bowen [3]). Here we follow [21, §7.23] which expresses $\zeta_{\mathbf{A}}$ in terms of functions of the form $d(ze^{A})$ for subshifts of finite type. Using Pollicott's results [17] which we have reproduced above as Proposition 3.3 one obtains the asserted meromorphic extension of $\zeta_{\mathbf{A}}$. From [21, §7.23] one also obtains (5.4), where P^* is the "pressure" for a suitable closed f-invariant set Λ^* strictly contained in Λ. We have $\mathbf{P}^* < \mathbf{P}$ because it is known that the unique measure ρ which yields the maximum in (5.3) has support Λ and cannot therefore have support in Λ^*.

5.4. Theorem. *Let ρ be the Gibbs measure corresponding to the real function $\mathbf{A} \in \mathscr{C}^\alpha(M)$. For $\mathbf{B}, \mathbf{C} \in \mathscr{C}^\alpha(M)$, let*

$$\hat{\rho}_{\mathbf{BC}}(\alpha) = \sum_{k=-\infty}^{\infty} e^{k\alpha}\left[\rho\left((\mathbf{B}\circ f^k)\mathbf{C}\right) - \rho(\mathbf{B})\rho(\mathbf{C})\right].$$

This extends to a meromorphic function of α for $|\mathrm{Re}\,\alpha| < \log\theta^{-1}$, regular at $\alpha = 0$, and the position of the poles is independent of $\mathbf{B}$, $\mathbf{C}$.

We may write

$$\hat{\rho}_{\mathbf{BC}}(\alpha) = \mathbf{N}_{\mathbf{BC}}(e^{-\alpha})\zeta_{\mathbf{A}}(e^{-\alpha-P}) + \mathbf{N}_{\mathbf{CB}}(e^{\alpha})\zeta_{\mathbf{A}}(e^{\alpha-P}), \tag{5.5}$$

where $\mathbf{N}_{\mathbf{BC}}(e^{-\alpha})$ is holomorphic for $|\mathrm{Re}\,\alpha| < \min\{\log\bar{\theta}^{-1}, P - P^*\}$ with $P =$ $\mathbf{P}(\mathrm{Re}\,\mathbf{A})$, $P^* = \mathbf{P}^*(\mathrm{Re}\,\mathbf{A})$.

If $0 < \theta' < \theta$ we may also write

$$\hat{\rho}_{\mathbf{BC}}(\alpha) = \mathscr{W}_{\mathbf{BC}}(\alpha) + \sum_{\lambda}{}^{*}\sum_{p=1}^{p(\lambda)}\left[\frac{b_{\lambda r}(\sigma_\lambda^-(\mathbf{B}), \sigma_\lambda^+(\mathbf{C}))}{(1 - e^{-\alpha-P}\lambda)^r} + \frac{b_{\lambda r}(\sigma_\lambda^-(\mathbf{C}), \sigma_\lambda^+(\mathbf{B}))}{(1 - e^{\alpha-P}\lambda)^r}\right].$$

In this formula $\mathscr{W}_{\mathbf{BC}}$ is analytic for $|\mathrm{Re}\,\alpha| < \log\theta'^{-1}$ and bilinear continuous in $\mathbf{B}$, $\mathbf{C}$, *the sum Σ^* extends over a finite set of numbers λ with $\theta' e^P < |\lambda| < e^P$, the $b_{\lambda r}$ are bilinear forms on $(\mathbf{G}_{e^P\lambda})^* \times (\mathbf{G}_{\lambda e^P})^*$, and $\sigma_\lambda^-(\mathbf{B})$, $\sigma_\lambda^+(\mathbf{C})$ are the elements of $(\mathbf{G}_{e^P\lambda})^*$, $(\mathbf{G}_{\lambda e^P})^*$ determined by* $\mathbf{B}$ *and* $\mathbf{C}$.

All of this results directly from Proposition 4.2, with the help of (5.4). (Note that the term $\tilde{\rho}(B(e^{-\alpha})C(e^{\alpha}))$ of Proposition 4.2(a) can be distributed between the two terms of (5.5).)

5.5. Corollary. *Define*

$$Z_\lambda^- = \left\{\zeta \in (\mathbf{G}_{e^P\lambda})^* : b_{\lambda r}(\zeta, \sigma_\lambda^+(\mathbf{C})) = 0 \text{ for all } r, \mathbf{C}\right\}$$

and let H_λ^- be the orthogonal of Z_λ^- in $\mathbf{G}_{e^P\lambda}$. Then the subspace H_λ^- is independent of the choice of a Markov partition. A subspace H_λ^+ of $\mathbf{G}_{\lambda e^P}$ is similarly defined, and is also independent of the choice of a Markov partition.

6. Further remarks

The discussion which we have given above for Axiom A diffeomorphisms can be made to apply to expanding maps of manifolds (see Shub [23], and [21]). As indicated in the introduction, our discussion of Gibbs distributions and correlation functions also probably extends to Axiom A flows (Pollicott [18] has given a partial proof of the fact that the Fourier transform of the correlation function is meromorphic in a strip, its poles being related to those of the zeta-function). A possible extension where $\exp \mathbf{A}$ acts on a linear bundle over M instead of being scalar (see Fried [8]) has not been investigated.[5]

For an Axiom A attractor Λ there is a particularly interesting Gibbs measure ρ corresponding to $\mathbf{A} = -\log \mathbf{J}$, where $\mathbf{J}$ is the Jacobian in expanding directions (see Sinai [26], Ruelle [20], Bowen and Ruelle [4]). We have here $\mathbf{P}(\mathbf{A}) = 0$ and ρ is *smooth along unstable directions* so that the Gibbs distributions $\sigma^+ \in \mathbf{G}_{\lambda 1}$ are also smooth along unstable directions (in a sense to be made precise).

Axiom A diffeomorphisms and flows constitute a good class of examples on which to test conjectures about more general dynamical systems. Instead of a Gibbs measure, let us consider a probability measure ρ which is smooth along unstable directions (this is known to be equivalent to "*entropy* = Σ *positive characteristic exponents*," a property which generalizes $\mathbf{P}(\mathbf{A}) = 0$; see Ledrappier and Young [12]). For a diffeomorphism, we may also assume that ρ has no zero characteristic exponent. It is then natural to ask if $\hat{\rho}_{\mathbf{BC}}(\alpha)$ defined as in (4.1), (4.2) is meromorphic in a strip. One can be more specific and speculate that poles (close to the real axis) are located at those values of α such that $e^{\pm\alpha} = \lambda$, where $\tau\sigma = \lambda\sigma$ and the distribution σ is smooth along unstable directions (again in a sense to be made precise). Similarly for flows.

Finally, let us mention that Frisch and Morf [9] have discussed the existence of poles in the time correlation functions of dynamical systems (not their Fourier transforms!). The methods which they use are not related to those described in the present paper.

References

[1] R. Bowen, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math. **92** (1970) 725–747.

[2] ______, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math., Vol. 470, Springer, Berlin, 1975.

[5] See also T. Adachi & T. Sunada, *L-functions of dynamical systems and topological graphs*, Preprint.

[3] ______, *On Axiom* A *diffeomorphisms*, CBMS Regional Conf. Ser. in Math., No. 35, Amer. Math. Soc., Providence, RI, 1978.

[4] R. Bowen & D. Ruelle, *The ergodic theory of Axiom* A *flows*, Invent. Math. **29** (1975) 181–202.

[5] R. L. Dobrushin, *The description of a random field by means of conditional probabilities and conditions of its regularity*, Teor. Verojatnost. i Primenen **13** (1968) 201–229; English transl., Theory Probab. Appl. **13** (1968) 197–224.

[6] ______, *Gibbsian random fields for lattice systems with pairwise interactions*, Funktsional. Anal. i Prilozhen. **2** (1968) No. 4, 31–43; English transl., Functional Anal. Appl. **2** (1968) 292–301.

[7] ______, *The problem of uniqueness of a Gibbsian random field and the problem of phase transitions*, Funktsional. Anal. i Prilozhen. **2** (1968) No. 4, 44–57; English transl., Functional Anal. Appl. **2** (1968) 302–312.

[8] D. Fried, *The zeta functions of Ruelle and Selberg.* I, Preprint.

[9] U. Frisch & R. Morf, *Intermittency in nonlinear dynamics and singularities at complex times*, Phys. Rev. A **23** (1981) 2673–2705.

[10] G. Gallavotti & J. L. Lebowitz, *Some remarks on Ising-spin systems*, Phys. A **70** (1973) 219–221.

[11] O. Lanford & D. Ruelle, *Observables at infinity and states with short range correlations in statistical mechanics*, Comm. Math. Phys. **13** (1969) 194–215.

[12] F. Ledrappier & L.-S. Young, *The metric entropy of diffeomorphisms, Part* I. *Characterization of measures satisfying Pesin's formula. Part* II. *Relations between entropy, exponents and dimension*, Annals of Math.

[13] A. N. Livšic, *Homology properties of Y-systems*, Mat. Zametki **10** no. 5, 555–564; English transl., Math. Notes **10** (1971) 758–763.

[14] ______, *Cohomology of dynamical systems*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972) No. 6, 1296–1320; English transl., Math. USSR Izv. **6** (1972) 1276–1301.

[15] A. Manning, *Axiom* A *diffeomorphisms have rational zeta functions*, Bull. London Math. Soc. **3** (1971) 215–220.

[16] R. D. Nussbaum, *The radius of the essential spectrum*, Duke Math. J. **37** (1970) 473–478.

[17] M. Pollicott, *Meromorphic extensions of generalized zeta functions*, Preprint.

[18] ______, *On the rate of mixing of Axiom* A *flows*, Preprint.

[19] D. Ruelle, *Statistical mechanics of a one-dimensional lattice gas*, Comm. Math. Phys. **9** (1968) 267–278.

[20] ______, *A measure associated with Axiom* A *attractors*, Amer. J. Math. **98** (1976) 619–654.

[21] ______, *Thermodynamic formalism*, Encyclopedia of Mathematics and its Applications, Vol. 5, Addison-Wesley, Reading, MA, 1978.

[22] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.

[23] M. Shub, *Endomorphisms of compact differentiable manifold*, Amer. J. Math. **91** (1969) 175–199.

[24] Ia. G. Sinai, *Markov partitions and C-diffeomorphisms*, Funktsional. Anal. i Prilozhen. **2** (1968) No. 1, 64–89; English transl., Functional Anal. Appl. **2** (1968) 61–82.

[25] ______, *Construction of Markov partitions*, Funktsional. Anal. i Prilozhen. **2** (1968) No. 3, 70–80; English transl., Functional Anal. Appl. **2** (1968) 245–253.

[26] ______, *Gibbs measures in ergodic theory*, Uspekhi Mat. Nauk **27** (1972) No. 4, 21–64; English transl., Russian Math. Surveys **27** (1972) No. 4, 21–69.

[27] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967) 747–817.

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