J. DIFFERENTIAL GEOMETRY
23 (1986) 309–346

COLLAPSING RIEMANNIAN MANIFOLDS WHILE KEEPING THEIR CURVATURE BOUNDED. I.

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0. Introduction

Let Y^n be a complete connected riemannian manifold, and $p \in Y^n$. The injectivity radius, i_p, of the exponential map at p is defined to be the smallest r such that $\exp_p|\overline{B_r(p)}$ fails to be a diffeomorphism onto its image. The present paper is the first of two which are concerned with the situation in which the size of injectivity radius is "small" relative to the curvature.

In this part I, we show that if a smooth manifold X^n admits a certain topological structure called an *F-structure of positive rank*, then X^n also admits a family of metrics, g_δ, such that as $\delta \to 0$, i_p *converges uniformly to zero at all points*, p, but the curvature, K_α, stays bounded (independent of p and δ). Such a family of metrics is said to *collapse* with bounded curvature (by rescaling, one can assume $|K_\delta| \leqslant 1$).

In part II we prove a sort of strengthened converse to this collapsing result. A riemannian manifold Y^n is said to be *ε-collapsed* if $i_p < \varepsilon$ for all p. Intuitively, such a manifold appears to have dimension $< n$ if one examines it on a scale $\gg \varepsilon$. We show that in each dimension, there exists a *critical radius*, $\varepsilon(n)$, such that if Y^n is $\varepsilon(n)$-collapsed and $|K| \leqslant 1$, then Y^n admits an F-structure of positive rank. Thus, if Y^n admits a metric which is sufficiently collapsed, it actually admits a family of metrics which collapse with bounded curvature.

An F-structure on a space, X, is a natural generalization of a torus action. Different tori (possibly not all of the same dimension) act locally on finite covering spaces of subsets of X. These local actions satisfy a compatibility condition, which insures that X is partitioned into disjoint "orbits." The F-structure is said to have *positive rank* if all orbits are of positive dimension.

Received Oct. 15, 1985 and, in revised form, November 25, 1985. The author's research was supported in part by the National Science Foundation under Grant MCS 8102758 for the first author and Grant MCS 8203300 for the second author.

The existence of an F-structure of positive rank is a definite constraint on the topology of a space. In the *compact* case, it implies, for example, that the Euler characteristic is zero (compare however Example 1.6 of [4]). This vanishing phenomenon does *not* carry over to Pontrjagin numbers, except in the presence of further hypotheses; see Example 1.9 (we will show elsewhere that the Pontrjagin numbers of X^{4l} vanish if it admits a so-called "pure structure of positive rank, with amenable holonomy"). However, there is a strong interaction between F-structures, characteristic numbers, and secondary geometric invariants; see [3], [5], [13].

The collapsing family of metrics g_δ, associated to an F-structure of *positive* rank is obtained roughly as follows. Start with a metric g which is *invariant* for the structure in the sense that the local torus actions are isometric. Then shrink g in certain directions tangent to the orbits. In some cases, it is also necessary to expand g in directions orthogonal to the orbits, in order to keep the curvature bounded. Thus, the diameter, $\operatorname{diam}(Y_n, g_\delta)$, and volume, $\operatorname{Vol}(Y_n, g_\delta)$, may go to infinity as (Y_n, g_δ) collapses; it may also happen that they stay bounded or converge to zero.

The following examples (although they are presented informally) should serve to give some feeling for the concepts mentioned so far; see §1 for the precise definition of "F-structure" (which is somewhat technical).

Example 0.1 (*The Klein bottle*). View the Klein bottle as the total space of a circle bundle $S^1 \to K^2 \xrightarrow{p} S^1$. For each interval, I, in the base space, there are two canonical fiber preserving circle actions on $p^{-1}(I)$, which differ by the automorphism $x \to x^{-1}$. If one of these local actions is continued around the base circle, the opposite action is obtained. As a consequence of this *holonomy* phenomenon, no global action exists. (See Example 1.2 for further discussion.)

Example 0.2 (*Graph manifolds*). Take a finite collection of surfaces, Σ_i^2, with $\partial \Sigma_i^2 = \bigcup_{j=1}^{N(i)} S_{i,j}^1$, a disjoint union of circles. The product manifolds, $\Sigma_i^2 \times S_i^1$, have boundary components which are tori, $S_{i,j}^1 \times S_i^1$. Form a manifold with empty boundary, Y^3, by identifying these tori in pairs by elements of $\mathrm{SL}(2, Z)$.

On each piece $\Sigma_i^2 \times S_i^1 \subset Y^3$, S^1 acts by rotation of the factor S_i^1. At boundary components which have been identified, the corresponding circle actions need not agree. But if not, they generate an action of a 2-torus, which extends both of them. Thus, in this example, the torus which acts locally is of dimension 2 near such identified boundary components and of dimension 1 elsewhere.

Example 0.3 (*Compact flat manifolds*). If X^n is compact and flat, by the Bieberbach Theorem there is a finite normal covering $\tilde{X}^n$, which is isometric to a torus. Since the action of this torus on itself is transitive, the induced orbit structure on X^n consists of a single orbit, X^n.

Example 0.4 (*Collapse by scaling*). If (Y_n, g) is a complete manifold with injectivity radius uniformly bounded from above, then the family $(Y_n, \delta^2 g)$ collapses (intuitively, to a single point, in the compact case). However, for this collapse, the sectional curvature does not remain bounded unless Y_n is flat.

In view of Example 0.4, from now on the word "collapse" will be taken to mean "collapse with bounded curvature."

The following is the most transparent and in a sense the most basic collapse with bounded curvature.

Example 0.5 (*Generalized warped products*). Start with a surface of revolution, M^2, obtained by revolving an arc in the upper half plane about the x-axis. Thus, M^2 is diffeomorphic to $S^1 \times I$. The obvious isometric circle action on M^2 lifts to an isometric R action on the infinite cyclic covering $\tilde{M}^2 = R \times I$. Let $\{\delta Z\} \subset R$ denote the subgroup generated by a translation of size δ. Then the family $\tilde{M}^2/\{\delta Z\}$ collapses, but the curvature remains unchanged (we have unrolled M^2 and then rolled it up more tightly).

To extend the above example to higher dimensions, take $\tilde{M}^{k+l} = X^k \times R^l$, with (generalized) warped product metric

$$g = g_1(x) + \sum_{i,j=1}^{l} a_{ij}(x)\, dy_i\, dy_j. \tag{0.1}$$

Then $M^{k+l} = X^k \times R^l/\delta Z^l$ collapses (to X^k).

Note that the orbits of the F-structure on M^{k+l} have *constant dimension*, l. In such cases (as above) the collapse can always be performed so that the *diameter remains bounded*. In particular, the volume goes to zero.

Example 0.2 (*continued*). The collapse associated to the F-structure on the graph manifold, Y^3, is particularly easy to describe if the identifications of the boundaries simply interchange the roles of the two circles. In this case, choose a "cusp-like" metric on Σ_i^2 which near the boundary is isometric to the product of an interval and a circle, S_δ^1, of length δ. The curvature and volume can be chosen bounded independent of the size of δ for such a metric.

Now form $\Sigma_i^2 \times S_\delta^1$ with the product metric, and identify corresponding boundary components. The resulting manifold, (Y^3, g_δ), has injectivity radius everywhere $= \delta$. In fact, $\mathrm{Vol}(Y^3, g_\delta) \leqslant c\delta$. However, the orbits of the F-structure are not of constant dimension and $\mathrm{diam}(Y^3, g_\delta) \to \infty$.

As far as we are aware, the first example of collapse (apart from warped products and scaling) was discovered by M. Berger in about 1962. He considered the collapse of the unit sphere S^3, obtained by shrinking the circles of the Hopf fibration. It is clear that the "limit" of this collapse should be S^2 (in fact, with a metric of curvature $\equiv 4$). The notion of the limit of a collapse can be made precise by introducing the concept of *Hausdorff limit*: see [9] and §2. Berger's interest in the collapse of S^3 stemmed from his observation that it provides a counterexample to a specific conjecture concerning a lower bound for the injectivity radius on odd dimensional manifolds of positive curvature.

Another significant collapse (with variable topology) was discovered in the context of manifolds of positive curvature by Aloff and Wallach [1]. They exhibited an infinite sequence of pairwise nonhomeomorphic, homogeneous 7-manifolds with uniformly pinched positive curvature. By the finiteness theorem in riemannian geometry (see [2], [4], [9], [11]) such a sequence must collapse.

The remainder of this paper is divided into five sections and one appendix as follows.

1. $\tilde{\Gamma}$-structures and F-structures
2. Pure polarized collapses with bounded diameter
3. Polarized volume collapses
4. Nonpolarized collapses
   (a) Introduction
   (b) Main computation
   (c) Construction of slice polarizations
   (d) Collapse
5. F-structures and complete metrics on open manifolds
   (a) Introduction
   (b) Construction of a complete metric, $\mathscr{g}_0$
   (c) Expansion of $\mathscr{g}_0$
   (d) Collapse of the expanded metric

Appendix: Pure polarized structures on essential manifolds.

In §1, we define and give examples of generalized group actions called $\tilde{\mathscr{g}}$-structures. Essentially, an F-structure is a $\tilde{\mathscr{g}}$-structure for which all the groups which act locally are tori. In §2, we consider the case of a *pure* structure. Basically, this means that a *single* connected group acts locally, up to automorphism, on a finite covering space. We assume, moreover, that *all orbits are of the same positive dimension*; compare Examples 0.1, 0.4, and 0.5. This second condition defines what is called a *pure polarized structure*. For such structures, by shrinking a compatible metric in the direction of the orbits while leaving it unchanged in the orthogonal directions, we obtain a collapse for which the diameter stays uniformly bounded.

In §3 we consider the polarization for which the groups which act locally are not all of the same dimension. In this case, we can collapse in such a way that $\mathrm{Vol}(Y_n, g_\delta) \to 0$, but $\mathrm{diam}(Y_n, g_\delta) \to \infty$; compare Example 0.2.

As already indicated, there exist manifolds, M_F^{4l}, admitting F-structures (which are in fact pure) of positive rank, but for which some characteristic number is nonzero. By the Chern-Weil theory, these manifolds admit no collapse with bounded curvature, such that $\mathrm{Vol}(M_F^{4l}, g_\delta) \to 0$. In particular, M_F^{4l} does not carry any polarized F-structure. However, in §4, we show that any F-structure of positive rank admits what we call a *slice polarization*. This can be used to collapse in such a way that the volume behavior is controlled by the geometry of the orbit structure. For example, the manifolds M_F^4 can be collapsed so that the volume stays bounded. But it can also happen that the volume goes to infinity *or to zero* (even though the slice polarization is not an honest polarization: compare [10]).

In §5 we consider open manifolds which carry an F-structure outside a compact set. On such manifolds, we obtain complete metrics of bounded curvature with properties analogous to those of the metrics constructed in §2–4.

In the Appendix we exhibit a class of manifolds with the property that if a pure F-structure exists, it must be polarized. As a consequence, many of these manifolds can be shown to admit no pure F-structure of positive rank, although they do not admit such structures which are not pure.

Portions of this paper were written while the first author enjoyed the hospitality of the Institute des Hautes Etudes Scientifique, Bûres-sur-Yvette and the Mathematical Sciences Research Institute, Berkeley.

We are indebted to Ofer Gabber for conversations which helped us to formulate the definition of a $\tilde{\mathscr{G}}$-structure in the language of sheaves.

1. $\tilde{\mathscr{G}}$-structures and F-structures

In this section, we discuss certain generalizations of the concept of a group action.

A *partial action*, A, of a topological group, G, on a Hausdorff space, X, is given by the following data.

(i) A neighborhood $\mathscr{D} \subset G \times X$ of $e \times X$, where $e \in G$ is the identity element. This $\mathscr{D}$ is called the *domain* (of definition) of the action.

(ii) A continuous map A: $\mathscr{D} \to X$, also, written $(g, x) \to gx$, such that $(g_1 g_2)x = g_1(g_2 x)$ whenever $(g_1, g_2 x)$ and $(g_1 g_2, x)$ lie in $\mathscr{D}$, and such that $ex = x$ for all $x \in X$.

Two partial actions $(A, \mathscr{D}_1)$, $(A_2, \mathscr{D}_2)$ are called (locally) *equivalent* if there is a domain $\mathscr{D} \subset \mathscr{D}_1 \subset \mathscr{D}_2$, containing $e \times X$, such that $A_1|\mathscr{D}_2 = A_2|\mathscr{D}_2$. A *local action*, $\{A\}$, is defined as the equivalence class of a partial action A of G on X. Every global action A defines an obvious local action A_{loc}. A local action which can be obtained in this way is called *complete*.

Remark 1.1. An elementary connectedness argument shows that A_{loc} determines A uniquely, in case G is connected.

In the smooth case, the category of local actions is equivalent to the category of *infinitesimal actions*: these are continuous homomorphisms of the Lie algebra of G to the Lie algebra of vector fields on X. For example, if $G \simeq R$, then a local action is given by a vector field on X, and completeness amounts to the integrability of the field.

From now on, we assume that G is *connected*.

A subset $X_0 \subset X$ is called (locally) $\{A\}$-*invariant* if for some representative $(A, \mathscr{D}) \in \{A\}$, one has $ga \in X_0$ for all $(g, a) \in \mathscr{D}$. Since the intersection of $\{A\}$-invariant sets is $\{A\}$-invariant, it follows that each point $x \in X$ is contained in a unique minimal $\{A\}$-invariant subset called the *orbit* $\mathscr{O} = \mathscr{O}_x \subset X$, and that the orbits partition the space X. Moreover, if A_{loc} is complete, the orbits of A_{loc} and A coincide.

A local action, $\{A\}$, on X can be restricted to any open subset, $U \subset X$, by taking an open subset $\mathscr{D}' \subset G \times X$, which contains $e \times U$ and such that $gx \in U$ for all $(g, x) \in \mathscr{D}'$ with $x \in U$. Furthermore, if $Y \to X$ is a *local homeomorphism*, then $\{A\}$ pulls back to a local action, $f^*\{A\}$, on Y, in a similar way.

Now consider a sheaf, $\mathscr{g}$, of connected topological groups over X. Let $g(U)$ denote the group of sections over U. An *action* of $\mathscr{g}$ on X is given by a local action of the group $\mathscr{g}(U)$ on U for every connected open set $U \subset X$, such that the structure homomorphisms $\mathscr{g}(U) \to \mathscr{g}(U')$ (for $U' \subset U$) agree with the restrictions of the local actions from U to U'.

A set S is called *invariant* if for all open sets U, the intersection $S \cap U$ is invariant for $\mathscr{g}(U)$. Again, X is partitioned into minimal invariant sets called *orbits*. A set which is the disjoint union of orbits is called *saturated*.

Example 1.1. In the smooth case, an action of $\mathscr{g}$ on X amounts to a homomorphism of the Lie algebra sheaf associated to $\mathscr{g}$ into the sheaf of germs of vector fields on X. As a specific example, let X be an affine flat manifold, infinitesimally (and hence locally) acted on by the Lie algebra sheaf of *parallel* vector fields.

Let G_x denote the stalk of $\mathscr{g}$ at x. If $f \colon Y \to X$ is a locally homeomorphic map, let $f^*(\mathscr{g})$ denote the pullback sheaf.

The following is a significant generalization of the concept of completeness introduced previously.

Definition 1.1. An action of a sheaf $\mathscr{g}$ of connected groups on X is called *complete* if for all $x \in X$, there exists an open neighborhood $V(x)$ and a locally homeomorphic map π: $\tilde{V}(x) \to V(x)$ ($\tilde{V}(x)$ Hausdorff) such that

(i) If $\pi(\tilde{x}) = x$, then for any open neighborhood $W \subset \tilde{V}(x)$ of $\tilde{x}$, the structure homomorphism $\pi^*(\mathscr{g})(W) \to G_{\tilde{x}} \overset{\text{def}}{=} G_x$ is an isomorphism.

(ii) The local action of $\pi^*(\mathscr{g})$ on $\tilde{V}(x)$ is complete.

Example 1.1 (*continued*). For affine flat manifolds, this agrees with the usual definition of completeness.

Note that the orbits of $\pi^*(\mathscr{g})$ on $\tilde{V}(x)$ project to orbits of $\mathscr{g}$ on $V(x)$.

Suppose π: $\tilde{V}(x) \to V(x)$ is a normal covering. Then the group, Γ, of covering transformations of π: $\tilde{V}(x) \to V(x)$ has a natural (holonomy) action on $\pi^*(\mathscr{g})$. It follows that there is a sheaf $\mathscr{S}$ on $\tilde{V}(x)$ such that the stalk of $\pi^*(\mathscr{S})$ at $\tilde{y} \in \tilde{V}(x)$ is the image of the structure homomorphism $\pi^*(\mathscr{g})(\tilde{V}(x))$ and by Remark 1.1, for $\gamma \in \Gamma$, $g \in \pi^*(\mathscr{g})(\tilde{V}(x))$, $\tilde{y} \in \tilde{V}(x)$, we have

$$\gamma(g\tilde{y}) = \gamma(g)\gamma(\tilde{y}). \tag{1.1}$$

Definition 1.2. A $\tilde{\mathscr{g}}$-structure, $\mathscr{G}$, on X is a sheaf, $\mathscr{g}$, of connected topological groups on X and a complete local action of $\mathscr{g}$ on X such that the sets $V(x)$ can be chosen to satisfy the following conditions.

(i) π: $\tilde{V}(x) \to V(x)$ is a normal covering.

(ii) For all x, $V(x)$ is saturated.

(iii) For all $\mathscr{O}$, if $x, y \in \bar{\mathscr{O}}$, then $V(x) = V(y)$.

It follows from (iii) that $\mathscr{g}|\bar{\mathscr{O}}$ is a *locally constant* sheaf, i.e. $\mathscr{g}|\bar{\mathscr{O}}$ is locally isomorphic to the sheaf of locally constant maps of $\bar{\mathscr{O}}$ to the group G_x. Put otherwise, $\mathscr{g}|\bar{\mathscr{O}}$ is a flat bundle such that each fiber is a group and the holonomy acts by automorphisms of the fiber. However, it need not be the case that $\mathscr{g}$ is locally constant on some neighborhood of $\bar{\mathscr{O}}$, since the structure homomorphisms need not be injective; see Example 1.4 and Remark 1.2.

Definition 1.3. If $\mathscr{g}$ is locally constant on $V(x)$ for all x, then $\mathscr{G}$ is called *pure*.

Suppose $\mathscr{G}$ is pure. Let $x \in X$, and fix $\tilde{x} \in \tilde{V}(x)$ with $\pi(\tilde{x}) = x$. Since $\mathscr{g}(\tilde{V}(x)) = G_x$, it follows that $\tilde{V}(x) \xrightarrow{\tau} \tilde{V}^E(x) \to V(x)$, where $\tilde{V}^E(x)$ is the *holonomy covering* of $\mathscr{g}|V(x)$ (with base point $\tau(\tilde{x})$). As a consequence of (1.1), the action of G_x descends to $\tilde{V}^E(x)$ and (1.1) continues to hold there.

For $x \in X$, let $(\tilde{X}^E, \tilde{x})$ denote the holonomy covering of the locally constant sheaf $\mathscr{g}$ with canonical basepoint. Let π: $\tilde{X}^E \to X$ be the projection. Given $y \in X$ and a curve, c, from x to y, the space $(\tilde{V}^E(y), \tilde{y})$ is naturally identified with a component of $\pi^{-1}(V(y)) \subset (\tilde{V}^E, x)$. Also parallel translation along c includes an isomorphism $G_x \to G_y$. Thus, the action of G_y on $(\tilde{V}^E(y), \tilde{y})$

induces an action of G_x on the corresponding component of $\pi^{-1}(V(y))$ and it is immediate from (1.1) and the definition of $(\tilde{X}^E, \tilde{x})$ that these actions give rise to a global action of G_x on $(\tilde{X}^E, \tilde{x})$. Thus,

Proposition 1.1. *If a $\tilde{\mathscr{g}}$-structure is pure, then the local action of $\tilde{\mathscr{g}}(\tilde{X}^E, \tilde{x})$ is complete.*

Example 1.2 (*Flat bundles*). A basic example of a pure $\tilde{\mathscr{g}}$-structure is the following. Let E be the total space of a locally constant sheaf, $\mathscr{g}$, of connected groups, and ρ: $E \to x$ the projection. Then for $x \in X$ and $y \in \rho^{-1}(x)$, there is an obvious action of the stalk G_y of $\rho^*(\mathscr{g})$ on $\rho^{-1}(U)$, provided U is chosen so that $\mathscr{g}|U$ is trivial. In particular there is a pure $\tilde{\mathscr{g}}$-structure on E, where the sheaf which acts is $\rho^*(\mathscr{g})$.

Definition 1.4. A $\tilde{\mathscr{g}}$-structure is called an *F-structure* if for all x, the group G_x is isomorphic to a torus, and the sets $V(x)$ (of Definition 1.2) can be chosen so that the coverings $\tilde{V}(x)$ are finite.

Definition 1.5. If one can always choose $\tilde{V}(x) = V(x)$, then the F-structure is called a T-structure.

Example 1.3 (*Example* 0.3 *reformulated*). Let X^n be a compact flat riemannian manifold. By the Bieberbach Theorem, for each $x \in X$ the holonomy covering $(\tilde{X}^n, \tilde{x})$ has the natural structure of a torus, T_x^n. The torus T_x^n acts on itself by the left translation. Hence, it acts canonically on any $(\tilde{X}^n, \tilde{y})$ as well. The holonomy transformations act on T_x^n by conjugation. The set $\bigcup_x T_x^n$ has the natural structure of a locally constant sheaf $\mathscr{g}$ (with stalk T_x^n) and the action of T_x^n on any fixed $(\tilde{X}^n, \tilde{x}_0)$ induces the local action of $\mathscr{g}$ on X.

Example 1.4 (*Structure homomorphisms not injective*). Let X be the space formed as follows. Take $S^1 \times [0,1]$ and attach $S^1 \times 1$ to $S^1 \times 0$ by a covering map of degree 2. The image of $S^1 \times [1-\varepsilon, 1]$ in X is a Möbius band B_ε. The image $S^1 \times [0, \varepsilon]$ in X is a half open cylinder, C_ε. Moreover, $\overline{C}_\varepsilon \cap B_\varepsilon = S$, where S is the central circle in B_ε (and $\overline{C}_\varepsilon$ is the closure of C_ε).

The orbits, $\mathcal{O}$, in X will be the images of circles, $S^1 \times A$. For each connected open set $U \subset X$, put

$$T(U) = \{\bigcup \mathcal{O} | \mathcal{O} \cap U \leq \varnothing\}. \tag{1.2}$$

Let $\mathscr{g}$ be the sheaf associated to the presheaf which assigns to each U the identity component of the isometry group of $T(U)$. By definition, there is a complete action of $\mathscr{g}$ on X, which defines a T-structure.

Note that for all $T(U) \neq X$, $\mathscr{g}(U)$ is isomorphic to a circle. However, if, for example, $T(U_1) = B_\varepsilon \cup C_\varepsilon$ and $T(U_2) = C_\varepsilon$, then the restriction map $\mathscr{g}(U_1) \to \mathscr{g}(U_2)$ is a 2-fold covering. As a consequence the total space of $\mathscr{g}$ is not Hausdorff at points lying over S, and the local action of $\mathscr{g}$ is not locally isomorphic to a pure structure in a neighborhood of S.

Remark 1.2. Observe that since a nontrivial local isometry defined on a connected subset of a riemannian manifold is not equal to the identity on any nonempty open subset, examples like the one above do *not* occur for *effective* local actions of *compact* groups in the smooth category. From now on, *we will restrict attention to $\tilde{\mathscr{g}}$-structures of this type*. For such structures, the restriction maps $\mathscr{g}(U_1) \to \mathscr{g}(U_2)$ are *injective*.

If the action of $\mathscr{g}$ on X defines a $\tilde{\mathscr{g}}$-structure, $\mathscr{G}$, and $\mathscr{g}' \subset \mathscr{g}$ is a subsheaf, then $\mathscr{g}'$ defines a $\tilde{\mathscr{g}}$-structure, $\mathscr{G}'$, called a *substructure*. We write $\mathscr{G}' \subset \mathscr{G}$. Note that the stalks of $\mathscr{G}'$ are *not* required to be closed subgroups. The subsheaf whose stalks are the closures of those of $\mathscr{G}'$ is written $\overline{\mathscr{G}}'$, the *closure* of $\mathscr{G}'$.

For a $\tilde{\mathscr{g}}$-structure as in Remark 1.2 above, for each $x \in X$ the neighborhood $V(x)$ of Definition 1.2 can be chosen such that there is a (unique) *pure substructure*, $\mathscr{G}_\alpha$, of $\mathscr{G}|V(x)$, with stalk $G_{x,\alpha} = G_\alpha$.

The *rank* of $\mathscr{G}$ at x is defined as $\dim \mathscr{O}_x$. We say that $\mathscr{G}$ has *positive rank* if the rank is positive for all $x \in X$.

Let $\mathscr{G}' \subset \mathscr{G}$ (with $\mathscr{G}$ as above). Let $X = \bigcup U_\alpha$ be a *locally finite* covering by connected open sets U_α. For each α, let $\mathscr{G}'_\alpha \subset \mathscr{G}'$ be a pure substructure with stalk $\mathscr{G}'_{x,\alpha} \subseteq \mathscr{G}'_x$ at $x \in U_\alpha$.

Definition 1.6. The collection $\{(U_\alpha, \mathscr{G}'_\alpha)\}$ is called an *atlas* for $\mathscr{G}'$ if

(i) Each U_α is saturated for the orbits of $\overline{\mathscr{G}'}$.

(ii) For each x, there exists $U_\alpha \ni x$, such that $G'_{x,\alpha} = G'_x$.

Definition 1.7. A substructure $\mathscr{P} \subset \mathscr{G}$ is called a *polarization* if it has an atlas, $\mathscr{A}$, such that for all α, the rank of $\mathscr{G}'_\alpha$ is the same positive number at all $x \in \mathscr{G}'_\alpha$ (although rank $\mathscr{G}'_\alpha$ might depend on α).

A polarization is called *pure* if $\mathscr{G}'$ is a pure substructure (in which case it suffices to take a single $U_\alpha = X$). The notions of atlas and polarization play an important role in the collapsing constructions of §§2, 3, and 4.

If $\mathscr{G}$ has positive rank, one way to find a substructure, $\mathscr{G}' \subset \mathscr{G}$, of positive rank which possesses an atlas is the following. Take a locally finite open covering by sets $U_\alpha = V(x_\alpha)$ and pure a substructure $\mathscr{G}_\alpha$ on each U_α, such that $G_{x_\alpha,\alpha} = G_{x_\alpha}$ and the rank of $\mathscr{G}_\alpha$ is positive. Enlarge this covering by adding all nonempty intersections, $U_{(\alpha)} = U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ and assign to $U_{(\alpha)}$ the pure substructure whose stalk at $x \in U_{(\alpha)}$ is the smallest subgroup $G_{(\alpha)}$ containing $\bigcup_1^k G_{x,\alpha_j}$. Then $\{(U_{(\alpha)}, G(\alpha))\}$ is an atlas for the substructure $\mathscr{G}'$, determined by the condition $G'_x = G_{x,(\alpha)}$, where $U_{(\alpha)}$ is the intersection of all those U_α containing x. The rank of $\mathscr{G}'$ is positive and it is easy to see that in fact we can choose $\mathscr{G}'$ such that rank $\mathscr{G}' =$ rank $\mathscr{G}$.

The following lemma is convenient for the constructions of §5 and provides a simple picture of structures which possess an atlas.

Lemma 1.2. *Let $\{(U_\alpha, \mathscr{G}_\alpha)\}$ be an atlas for $\mathscr{G}$ on (a possibly open) manifold X. Then there is an atlas $\{(\underline{U}_\alpha, \underline{\mathscr{G}}_\alpha)\}$ for $\mathscr{G}$ with the following properties:*

(1) the sets $\underline{U}_\alpha$ have compact closure.

(2) If $x \in \underline{U}_{\alpha_1} \cap \cdots \cap \underline{U}_{\alpha_k}$, then (for some ordering) $G_{x,\alpha_1} \subseteq G_{x,\alpha_2} \subseteq \cdots \subseteq G_{x,\alpha_k}$.

(3) For all $\underline{U}_\alpha$ and all $x \in \underline{U}_\alpha$, there is at most one $\underline{U}_\beta$ $(\beta \neq \alpha)$ with $x \in \underline{U}_\beta$ and $G_{x,\alpha} = G_{x,\beta}$.

Proof. (1) Clearly, we can assume $\{U_\alpha\}$ itself has this property.

(2) Let U_β, U_γ satisfy $x \in U_\beta \cap U_\gamma$ but neither $G_{x,\beta} \subseteq G_{x,\gamma}$ nor $G_{x,\gamma} \subseteq G_{x,\beta}$. Take saturated open sets $\hat{U}_\beta$, $\hat{U}_\gamma$ with $\hat{U}_\beta \cap \hat{U}_\gamma = \varnothing$ and $U_\beta \setminus \overline{U}_\gamma \subset \hat{U}_\beta \subset U_\beta$, $U_\gamma \setminus \overline{U}_\beta \subset \hat{U}_\gamma \subset U_\gamma$. Since, clearly, $G_{x,\beta} \neq G_x \neq G_{x,\gamma}$, it follows that $\hat{U}_\beta$, $\hat{U}_\gamma$ together with the remaining $\{U_\alpha\}$ cover X. We can now construct a covering $\{\underline{\underline{U}}_\alpha\}$ by induction, such that $\underline{\underline{U}}_\alpha \subset U_\alpha$, and if we put $\underline{\underline{\mathscr{G}}}_\alpha = \mathscr{G}_\alpha | \underline{\underline{U}}_\alpha$, then $\{\underline{\underline{U}}_\alpha, \underline{\underline{\mathscr{G}}}_\alpha\}$ satisfies (1) and (2).

(3) Let $\underline{\underline{U}}_{\alpha_1}, \underline{\underline{U}}_{\alpha_2}, \ldots$ be a maximal subcollection of $\{\underline{\underline{U}}_\alpha\}$ such that $\bigcup \underline{\underline{U}}_{\alpha_j}$ is connected and $G_{x,\alpha_i} = G_{x,\alpha_j}$ whenever $x \in \underline{\underline{U}}_{\alpha_i} \cap \underline{\underline{U}}_{\alpha_j}$. Put $\underline{U}_{\alpha_1} = \underline{\underline{U}}_{\alpha_1}$. Let, say, $\underline{\underline{U}}_{\alpha_2} \cdots \underline{\underline{U}}_{\alpha_{k_2}}$ be those $\underline{\underline{U}}_{\alpha_j}$ whose intersection with $\underline{\underline{U}}_{\alpha_1}$ is nonempty. Let $\underline{U}_{\alpha_2}, \ldots, \underline{U}_{\alpha_{t_2}}$ be the connected components of $\underline{\underline{U}}_{\alpha_2} \cup \cdots \cup \underline{\underline{U}}_{\alpha_{k_2}}$. By proceeding in this way and repeating the process for all subcollections as above, we obtain the required covering.

An atlas satisfying the properties of Lemma 1.2 is called *regular*.

Remark 1.3. Let $\mathscr{A}$ be a regular atlas for $\mathscr{G}$. Let $\{U'_\alpha\}$ be an open covering by saturated subsets, $U'_\alpha \subset U_\alpha$. Then by restricting $\mathscr{G}_\alpha$ to U'_α we obtain a regular atlas, $\mathscr{A}'$, for a substructure $\mathscr{G}' \subset \mathscr{G}$. We write $\mathscr{A}' \subset \mathscr{A}$.

Remark 1.4. If the atlas $\{(U_\alpha, \mathscr{G}_\alpha)\}$ in Lemma 1.2 is a polarization, then the regular atlas $\{(\underline{U}_\alpha, \mathscr{G}_\alpha)\}$ is a polarization as well.

Remark 1.5. By dropping (1) in Lemma 1.2, we can also drop (3) and strengthen (2) to read $G_{x,\alpha_1} \not\subseteq G_{x,\alpha_2} \not\subseteq G_{x,\alpha_k}$.

A riemannian metric g is called *invariant* for $\mathscr{G}$ if the local action of the sheaf g is isometric.

Lemma 1.3. *Let $\mathscr{A} = \{(U_\alpha, \mathscr{G}_\alpha)\}$ be a regular atlas for $\mathscr{G}$ and let $\mathscr{A}' = \{(U'_\alpha, \mathscr{G}_\alpha)\}$, where $\overline{U}'_\alpha \subset U_\alpha$. Suppose $\mathscr{G}$ has the property that all coverings $\tilde{V}(x) \to V(x)$ (in Definition 1.2) can be chosen finite. Then there is an invariant metric for $\mathscr{G}'$.*

Proof. The relation of Lemma 1.2(2) induces a natural partial ordering on the $\{U_\alpha\}$. Start with some maximal U_α and take a finite covering of $\overline{U}'_\alpha$ by sets of the form $V(x_1) \cdots V(x_k)$ with $\overline{V(x_j)} \subset U_\alpha$. Take any metric g_0 on $V(x_1)$, pull it back to $\tilde{V}(x_1)$, and average over the action of G_x and over the finite group of covering transformations. Extend the resulting metric to any smooth metric g_1 on $V_1(x) \cup V_2(x)$. Repeat the process for $g_1 | V(x_2)$. By continuing in this way we get a metric on $\bigcup U'_\alpha$ with U'_α maximal and in the same way we get the required invariant metric for $\mathscr{G}'$.

By using (invariant smoothings of) distance functions for an invariant metric, we can construct invariant smooth functions, say f_α: $U'_\alpha \to [1/2, 1]$, with $\bigcup_\alpha f_\alpha^{-1}(1/2) = X$. Then a standard application of Sard's Theorem gives

Lemma 1.4. *For almost all $c_\alpha \in (1/2, 1)$ the sets $f_\alpha^{-1}(c_\alpha)$ are smooth closed codimension 1 submanifolds which intersect transversally (and define an atlas $\mathscr{A}'' \subset \mathscr{A}$).*

We can now verify the result on the vanishing of the Euler characteristic mentioned

Proposition 1.5. *Let X be a compact manifold which carries an F-structure of positive rank, then $\chi(X) = 0$.*

Proof. Let $\mathscr{A}$ be an atlas for a substructure of positive rank with $U_\alpha = V(x_\alpha)$. Let $\mathscr{A}'' \subset \mathscr{A}$ be as in Lemma 1.4. Then for every intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ there is a finite covering, say $\tilde{U}_{(\alpha)} \subset \tilde{V}_{(x_{\alpha_1})}$, on which a torus G_{x_α}, acts with no common fixed points. By a well-known argument almost all elements of $G_{x_{\alpha_1}}$ are fixed point free. Thus, by the Lefschetz fixed point theorem, $\chi(\tilde{U}_{(\alpha)}) = \chi(U_{(\alpha)}) = 0$. Then $\chi(X) = 0$ as well.

Remark 1.6. In Proposition 1.5 it is not essential that X is a manifold.

Here are some further examples of F-structures.

Example 1.5 (*S^3 and R^4*). View R^4 as $\mathbf{C}^2 = (z_1, z_2)$. There is an obvious T^2-action ($T^2 = (\theta_1, \theta_2)$) given by

$$(1.3) \qquad (\theta_1, \theta_2) \cdot (z_1, z_2) = \left(e^{i\theta_1} z_1, e^{i\theta_2} z_2\right),$$

with orbits of dimensions 0, 1, 2. Since there exist orbits of dimension 0, the corresponding T-structure admits no polarization.

There is also an induced T-structure on the unit sphere, S^3. All orbits are of dimension 2, with the exception of the circles $S^3 \cap \{(z_1, 0)\}$ and $S^3 \cap \{(0, z_2)\}$. Any choice of 1-parameter subgroup, S^1_γ, with $0 < \theta_1/\theta_2 = \gamma < \infty$, gives rise to a pure polarization, $\mathscr{P}_\gamma$, for which all orbits are 1-dimensional.

We can define another T-structure on S^3 which is not pure by picking η, with $1/\sqrt{2} < \eta < 1$, and setting

$$(1.4) \qquad U_j = \left\{(z_1, z_2) \in S^3 \,\middle|\, |z_j| < \eta\right\}, \qquad j = 1, 2.$$

For $x \in U_1 \setminus U_2$ ($U_2 \setminus U_1$) we let $G_x = S^1_{\gamma_1}$ ($G_x = S^1_{\gamma_2}$) and $V(x) = U_1$ ($V(x) = U_2$). For $x \in U_1 \cap U_2$, $G_x = T^2$ and $V(x) = U_1 \cap U_2$.

Note that for $\gamma \neq 1$, the orbits $S^3 \cap \{(z_1, 0)\}$ and $S^3 \cap \{(0, z_2)\}$ are never principle orbits of S^1_γ, i.e., their isotropy groups, while discrete, are not minimal.

Finally, observe that this example generalizes to higher dimensions.

Example 1.2 (*continued, solvemanifolds*). Let $A \in \mathrm{Sl}(2, Z)$ be an automorphism of the torus T^2 with two real distinct eigenvalues. Thus, if

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \tag{1.5}$$

then $|a + d| \geqslant 2$. The mapping torus, M^3, of A, is by definition the affine flat bundle $T^2 \to M^3 \to S^1$, with holonomy A (as is well known, M^3 is a solvemanifold).

As above, there is a pure T-structure on M^3 whose orbits are the fibers. This T-structure has a natural pure polarization (of rank 2). It also admits exactly two pure polarizations of rank 1, the orbits of which correspond to the eigen-directions of A (and are not closed).

Example 1.6 (*The flat bundle* $\mathscr{E}_\theta^3$). Let $R^2 \to \mathscr{E}_\theta^3 \to S^1$ denote the trivial R^2 bundle over S^1, equipped with the connection whose holonomy is given by rotation through an angle $2\pi\theta$. A point in $\mathscr{E}_\theta^3$ is denoted by (t, w) where $t \in R/Z$ $(= S^1)$ and $w \in R^2$. Then parallel translation v units along the base is given by

$$P(v)(t, w) = (t + v, R(v\theta)w), \tag{1.6v}$$

where $R(v\theta)$ denotes rotation through an angle $2\pi v\theta$.

Observe that $\mathscr{E}_\theta^3$ carries the structure of a complete flat riemannian manifold whose isometry group $I(\mathscr{E}_\theta^3)$ is the torus, $S^1 \times S^1$, generated by

$$T(u)(t, w) = (t + u, w), \qquad R(v)(t, w) = (t, R(v)w). \tag{1.7}$$

The full group $I(\mathscr{E}_\theta^3)$ defines a pure T-structure which is of rank 2 everywhere except along the zero section of $\mathscr{E}_\theta^3$. Any 1-parameter subgroup other than $R(v)$ defines a pure polarization of rank 1.

Example 1.7 (*The space* $\mathscr{M}^4 = \bigcup_\theta \mathscr{E}_\theta^3$). Consider the family $[0, 1] \times \mathscr{E}_\theta^3$ consisting of pairs $(\theta, \mathscr{E}_\theta^3)$. The spaces $\mathscr{E}_0^3$ and $\mathscr{E}_1^3$ are abstractly isometric. Their parallel translations are given by

$$P(v)(t, w) = \begin{cases} ((t + v), w), & \theta = 0, \\ (t + v, R(v)w), & \theta = 1. \end{cases} \tag{1.8}$$

The map

$$f(0, t, w) = (1, t, R(t)w) \tag{1.9}$$

provides an isometry between $\mathscr{E}_0^3$ and $\mathscr{E}_1^3$. It induces the isomorphism of isometry groups given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1.10}$$

The space formed from $[0, 1] \times \mathscr{E}_\theta^3$ by identifying $(0, E_0^3)$ with $(1, E_1^3)$ via f will be denoted by $\mathscr{M}^4$. The flat T^2-bundle (locally constant sheaf) over S^1 with holonomy given by (1.10) is a nilmanifold. Its pullback to $\mathscr{M}^4$ defines a pure T-structure on $\mathscr{M}^4$. As above, this structure has rank 2 everywhere except along the zero sections of the various $\mathscr{E}_\theta^3$, where it has rank 1. Moreover, unlike the matrix A in (1.5), the matrix in (1.10) has only a single eigenvector. It corresponds to the circle $R(v)$, the action of which fixes the zero sections of the $\mathscr{E}_\theta^3$. From this it is clear that *the pure T-structure on $\mathscr{M}^4$ has positive rank but admits no polarization*. In fact, it is the most basic example of an F-structure with this property. There are no such examples in dimension 3 and any example in dimension 4 contains an orbit, a neighborhood of which looks like (perhaps a finite covering of) this example.

Example 1.8 (*A nonpolarizable structure on $T^2 \times R^4$*). The space $\mathscr{M}^4$ in Example 1.7 can be regarded as the total space of the complex line bundle with first Chern number 1, over T^2. If we take the Whitney sum of this bundle with the bundle of Chern number -1, we obtain the trivial bundle with total space $T^2 \times R^4$. Now, by a simple modification of the previous example, we find a pure T-structure on $T^2 \times R^4$ which is of rank 2 except at $T^2 \times 0$ where it is of rank 1. Moreover, this structure admits no polarization.

Example 1.9 (*Pure T-structure on M_F^{4l} with $\sigma(M_F^{4l}) = 2$*). The previous example of a pure T-structure of positive rank which admits no polarizations can be sharpened. There exist closed manifolds which carry a pure T-structure of positive rank, but which have nonzero signature. Hence, as noted in §0, they admit no polarized T-structure whatsoever.

The following particularly nice family of such examples is due, essentially, to T. Januszkiewicz. To describe them. Let

$$T^{2l+1} = (e^{i\theta_1}, \cdots, e^{i\theta_{2l+1}}), \qquad D = (e^{i\theta}, \cdots, e^{i\theta}) \tag{1.11}$$

and let S_j denote the image of

$$(1, \cdots, e^{i\theta}, 1 \cdots) \tag{1.12}$$

in T^{2l+1}/D. Then T^{2l+1}/D acts on

$$\mathbf{C}P(2l) = (z_1, \cdots, z_{2l+1})/D, \qquad \sum |z_i|^2 = 1, \tag{1.13}$$

with $2l + 1$ fixed points,

$$p_j = \left(0, \cdots, \overset{j}{1}, 0, \cdots\right). \tag{1.14}$$

If we use the product structure

$$S^1 \times \cdots \times \hat{S}_j \times \cdots \times S_{2l+1} \tag{1.15}$$

on T^{2l+1}/D and identify the tangent space to $\mathbf{C}P(2l)$ at p_j, with

$$(z_1, \cdots, \hat{z}_j, \cdots, z_{2l+1}), \tag{1.16}$$

then T^{2l+1}/D acts by the standard representation of T^{2l}. Let D_j denote the diagonal of T^{2l+1}/D with respect to the product structure in (1.15). Then the action of S_j on the tangent space at p_k is given by

$$S_j(z_1,\cdots,\hat{z}_k,\cdots,z_{2l+1}) = D_k(\bar{z}_1,\cdots,\hat{z}_k,\cdots,\bar{z}_{2l+1}). \tag{1.17}$$

Now choose normal coordinate systems at the points p_j and from each of these delete a ball, B_j^{4l}, about the origin. Take two copies Σ_1^{4l}, Σ_2^{4l} of the resulting manifold with boundary and form a closed manifold, M_F^{4l}, as follows. Let

$$f_j\colon \{1,\cdots,2l+1\} \to \{0,1\} \tag{1.18}$$

be any function which takes the value 1 an *odd* number of times, $j = 1,\cdots,2l+1$. Put

$$F = (f_1,\cdots,f_{2l+1}). \tag{1.19}$$

To obtain M_F^{4l}, glue corresponding boundary components of Σ_1^{4l} and Σ_2^{4l} by the identifications

(1.20)
$$(z_1,\cdots,\hat{z}_j,\cdots,z_{2l+1}) \sim \left(z_1,\cdots,\bar{z}_{i_1(j)},\cdots,\hat{z}_j,\cdots,\bar{z}_{i_{t(j)}(j)},\cdots,z_{2l+1}\right),$$

where $i_1(j),\cdots,i_{t(j)}(j)$ are the integers at which f_j takes the value of 1. The torus action on Σ_1^{4l}, Σ_2^{4l} gives rise to a pure T-structure on M_F^{4l}. To describe the holonomy of the corresponding flat bundle, E, it suffices to consider loops $l_1,\cdots,l_{2l}$, where l_j passes from Σ_1^{4l} to Σ_2^{4l} through ∂B_{2l+1}^{4l} and returns to Σ_j^{4l} through ∂B_j^{4l}. Then using (1.12)–(1.17), it follows that the holonomy around l_j is given by the matrix

$$\begin{pmatrix} (-1)^{\tau_j(1)} & a_j(1) & & \\ \ddots & \vdots & & 0 \\ & (-1)^{\tau_j(j)} & & \\ 0 & \vdots & \ddots & 0 \\ & a_{2l}(1) & & (-1)^{\tau_j(l)} \end{pmatrix}, \tag{1.21}$$

where

$$\tau_j(i) = f_j(k) + f_{2l+1}(k), \tag{1.22}$$

and for $k \neq j$,

$$a_j(k) = \begin{cases} 0, & f_j(k) + f_j(2l+1) \equiv 0 \mod 2, \\ -2, & f_j(k) + f_j(2l+1) \equiv 1 \mod 2, \end{cases} \tag{1.23}$$

Finally, since the identifications on the boundary components are *orientation reversing*, if Σ_1^{4l}, Σ_2^{4l} are *both* given the orientation induced from $\mathbf{C}P(2l)$, then M_F^{4l} also acquires an orientation. Moreover, the signature, $\sigma(M_F^{4l})$, is given by

$$\begin{aligned}(1.24)\qquad \sigma(M_F^{4l}) &= 2\sigma(\Sigma^{4l}) = 2\{\sigma(\mathbf{C}P(2l)) - (2l+1)\sigma(B^{4l})\}\\ &= 2\sigma(\mathbf{C}P(2l)) = 2.\end{aligned}$$

2. Pure polarized collapses with bounded diameter

In this section, we discuss the collapse associated to a pure polarization, $\mathscr{P}$, of an F-structure, $\mathscr{F}$, on a manifold Y_n. Let

$$(2.1)\qquad g = g' + h$$

be a metric which is invariant for $\mathscr{F}$; see Lemma 1.2. Here h vanishes on vectors tangent to the orbits of $\mathscr{P}$, and g' vanishes on vectors normal to these orbits. Put

$$(2.2)\qquad g_\delta = \delta^2 g' + h.$$

Theorem 2.1. *As $\delta \to 0$, the family (Y_n, g_δ) collapses. Moreover,*

$$(2.3)\qquad \operatorname{dist}_{\delta_1}(p,q) \leqslant \operatorname{dist}_{\delta_2}(p,q), \qquad \delta_1 < \delta_2,$$

and for each compact set $\overline{U}$, there is a constant $c(\overline{U})$[1] *such that*

$$(2.4)\qquad \sup_{\overline{U},\delta\geqslant 1} |K_\delta| \leqslant c(\overline{U}).$$

Proof. Observe that (2.3) is obvious and the main point is, of course, (2.4). We claim that when the appropriate coordinates are introduced, (2.4) is obvious as well.

Let $p \in Y$. Take a basis of Killing fields, $\{X_i\}$, tangent to the orbits of $\mathscr{P}$ in a neighborhood of p.

Let N^{n-k} be a local transversal to the orbits $\{\mathscr{O}\}$ with $p \in N^{n-k}$. Choose local coordinates, $(y_1,\cdots,y_{n-k})$ on N^{n-k} with p at the origin. Since $[X_i, X_j] \equiv 0$, there is a unique coordinate system $(x_1,\cdots,x_k)$ on each orbit $\mathscr{O}$, with $\mathscr{O} \cap N^{n-k} = (0,\cdots,0)$ and $X_i = \partial/\partial x_i$. By projecting onto N^{n-k}, i.e. $\mathscr{O} \to \mathscr{O} \cap N^{n-k}$, we obtain coordinates $(x_1,\cdots,x_k,y_1,\cdots,y_{n-k})$ in a neighborhood $B_r(O) \times B_s(O)$ of p with, say, $\Sigma x_i^2 \leqslant r^2$, $\Sigma y_i^2 \leqslant s^2$.

In terms of these coordinates, the matrix $(g(x,y,\delta))$, representing the metric g_δ, can be calculated as follows. Note that translation in the direction of x_i preserves (coordinate fields and) inner products, since $\partial/\partial x_i$ is a Killing field.

[1] The notation $c(\cdot)$ will always mean a constant which depends only on the quantities within the parentheses.

Thus, if we put

$$\frac{\partial}{\partial y_i} = X_i + V_i, \tag{2.5}$$

where X_i is tangent to orbits and V_i is normal to orbits,

$$\langle X_i, V_j \rangle_1 = 0, \tag{2.6}$$

and the following matrices are independent of $x_1, \cdots, x_k$.

$$A(y_1, \cdots, y_{n-k}) = \left(\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_1 \right), \tag{2.7}$$

$$B(y_1, \cdots, y_{n-k}) = \left(\left\langle \frac{\partial}{\partial y_i}, X_j \right\rangle_1 \right), \tag{2.8}$$

$$C(y_1, \cdots, y_{n-k}) = (\langle X_i, X_j \rangle_1), \tag{2.9}$$

$$D(y_1, \cdots, y_{n-k}) = (\langle V_i, V_j \rangle_1). \tag{2.10}$$

Here $\langle \ , \ \rangle_\delta$ denotes the inner product for g_δ. It follows that

$$\begin{aligned} (g(x, y, \delta)) &= \left(\begin{array}{c|c} \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_g & \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_k} \right\rangle_\delta \\ \hline \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_k} \right\rangle_\delta & \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle_\delta \end{array} \right) \\ &= \left(\begin{array}{c|c} \delta^2 A(y) & \delta^2 B(y) \\ \hline \delta^2 B(y) & \delta^2 C(y) + D(y) \end{array} \right). \end{aligned} \tag{2.11}$$

As $\delta \to 0$, the matrix in (2.11) becomes singular. But if we make the change of coordinates

$$u_i = \delta x_i, \quad du_i = \delta dx_i, \quad \frac{\partial}{\partial u_i} = \frac{1}{\delta} \frac{\partial}{\partial x_i}, \tag{2.12}$$

then

$$\begin{aligned} (g(u, y, \delta)) &= \left(\begin{array}{c|c} \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle_\delta & \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial y_k} \right\rangle_\delta \\ \hline \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial y_k} \right\rangle_\delta & \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle_\delta \end{array} \right) \\ &= \left(\begin{array}{c|c} A(y) & \delta B(y) \\ \hline \delta B(y) & \delta^2 C(y) + D(y) \end{array} \right). \end{aligned} \tag{2.13}$$

The family $(g(u, y, \delta))$ can be regarded as being defined on all of $R^k_u \times B_s(0)$. As $\delta \to 0$, it converges smoothly to the *generalized warped product metric*

$$(g(u, y, 0)) = \left(\begin{array}{c|c} A(y) & 0 \\ \hline 0 & D(y) \end{array}\right) \tag{2.14}$$

on $R^k_u \times B_s(0) \subset R^k_u \times R^{n-k}_y$, where for each fixed y, the induced metric on R^k_u is *flat*: compare Example 0.5. Since

$$(g(x, y, \delta)) | B_r(0) \times B_s(0) \subset R^k_x \times R^{n-k}_y \tag{2.15}$$

is isometric to

$$(g(u, y, \delta)) | B_{\delta r}(0) \times B_s(0) \subset R^k_u \times R^{n-k}_y, \tag{2.16}$$

it is clear that $|K_\delta|$ is uniformly bounded independent of δ on compact subsets. This gives (2.4).

To see that (Y_n, g_δ) collapses, consider the closure $\bar{\mathscr{P}}$ of $\mathscr{P}$. The orbits $\{\bar{\mathscr{O}}\}$ of $\bar{\mathscr{P}}$ are compact flat manifolds. Since g_δ restricted to the normal space of any $\bar{\mathscr{O}}$ is independent of δ, it follows easily that the distance between any two such orbits is bounded below independent of δ. If Y_n is not complete, the same holds for $\inf_\delta \operatorname{dist}(\bar{\mathscr{O}}_q, \bar{Y}^n \setminus Y_n) = d_q$ (where $\bar{Y}^n$ is the completion of Y_n). In particular, the closed tubular neighborhood $T_r(\bar{\mathscr{O}}_q)$ is compact, independent of δ, for $r < d_q$. If $B_r(q, g_\delta)$ denotes the ball of radius r about q, with respect to g_δ, clearly

$$B_r(q, g_\delta) \subset T_r(\bar{\mathscr{O}}_q), \tag{2.17}$$

and by (2.2),

$$\lim_{\delta \to 0} \operatorname{Vol}_\delta(B_r(q, g_\delta)) = \lim_{\delta \to 0} \operatorname{Vol}_\delta(T_r(\mathscr{O}_q)) = 0. \tag{2.18}$$

Let $V(c, s)$ denote the volume of the ball of radius s on the sphere of curvature $c = c(T_r(\bar{\mathscr{O}}_q))$, the constant in (2.4). It follows that for any $s < r$, we have

$$i_q(g_\delta) < s, \tag{2.19}$$

if δ is so small that

$$\operatorname{Vol}(B_s(q, g_\delta)) < \operatorname{Vol}(B_r(q, g_\delta)) \leqslant V(c, s). \tag{2.20}$$

Thus, (Y_n, g_δ) collapses.

A metric space X is said to be the *Hausdorff limit* (as $\delta \to 0$) of the family of metric spaces X_δ, if for all $\varepsilon_1, \varepsilon_2$ there exists $\delta(\varepsilon_1, \varepsilon_2)$ such that for $\delta < \delta(\varepsilon_1, \varepsilon_2)$ there are ε_1 dense sets $\{p_i(\varepsilon_1, \varepsilon_2, \delta)\}$ in X_δ and $\{p_i(\varepsilon_1, \varepsilon_2)\}$ in X with

$$\begin{aligned} (1 + \varepsilon_2)^{-1} \overline{p_i(\varepsilon_1, \varepsilon_2), p_j(\varepsilon_1, \varepsilon_2)} &\leqslant \overline{p_i(\varepsilon_1, \varepsilon_2, \delta), p_j(\varepsilon_1, \varepsilon_2, \delta)} \\ &\leqslant (1 + \varepsilon_2) \overline{p_i(\varepsilon_1, \varepsilon_2), p_j(\varepsilon_1, \varepsilon_2)}; \end{aligned} \tag{2.21}$$

see [9] for further discussion. Clearly, the Hausdorff limit of the family $g(x, y, \delta)$ on $B_r(0) \times B_s(0) \subset R_x^k \times R_y^{n-k}$ is $B_s(0)$, equipped with the metric corresponding to $(D(y))$. By definition, this is the metric on N^{n-k} for which the length of a vector is the length with respect to g of its projection orthogonal to $\mathcal{O}$. Up to isometry, it is independent of the choice of transversal, and is the unique metric on the *local quotient space* defined by (pieces of) the orbits of $\mathcal{P}$, for which the projection is a riemannian submersion.

If the orbits $\{\mathcal{O}\}$ are not closed, the *global quotient spaces* $X/\mathcal{P}$ is not Hausdorff. But we can still look at the quotient space, $X/\overline{\mathcal{P}}$, for the orbits of $\overline{\mathcal{P}}$. Since $\mathcal{O}$ is dense in $\overline{\mathcal{O}}$ it follows from (2.2) that

$$\lim_{\delta \to 0} \operatorname{diam}_\delta\left(\overline{\mathcal{O}}_q\right) = 0. \tag{2.22}$$

Thus, the Hausdorff limit of a compact subset $\overline{U}$ of Y, which is saturated for $\overline{\mathcal{P}}$, is $\overline{U}/\overline{\mathcal{P}}$, with the obvious quotient metric. Of course, this is not a smooth manifold near exceptional orbits.

Example 1.4 (*continued*). The polarization $\mathcal{P}_\gamma$ defined by the subgroup S_γ is closed if and only if γ is rational. If $1 \neq \gamma = p/q$ is rational, the Hausdorff limit $S^3/\mathcal{P}_\gamma$ is the surface of revolution, obtained by revolving the curve

$$y = \frac{1}{2} \frac{\sin x \cos x}{\left(p^2 \cos^2 x + q^2 \sin^2 x\right)^{1/2}}, \tag{2.23}$$

$0 \leqslant x \leqslant \pi/2$, about the x-axis. Thus it is a topological S^2 with two non-smooth points.

For γ irrational, $S^3/\overline{\mathcal{P}}_\gamma$ is the interval $[0, \pi/2]$.

Example 2.1 (*Tori*). The pure polarizations $\mathcal{P}(E^k)$ of the canonical T-structure on the standard torus, T^n, are parametrized by subspaces, E^k, of R^n which pass through the origin. When T^n is collapsed along E^k, of course

$$\operatorname{Vol}_\delta(T^n) = \delta^k \operatorname{Vol}_1(T^n). \tag{2.24}$$

But if one looks at (T^n, g_δ) up to homothety (i.e. isometry and scaling) it is a classical fact that the family (T^n, g_δ) corresponds to the image in the moduli space $\mathrm{SO}(n, R)\backslash \mathrm{SL}(n, R)/\mathrm{Sl}(n, Z)$ of a geodesic which goes to infinity in $\mathrm{SO}(n, R)\backslash \mathrm{SL}(n, R)$.

For example, if $n = 2$ and $(\mathcal{P}, E_\gamma^1)$ corresponds to a line of slope $\gamma = p/q$, then (T^2, g_δ) goes to infinity in $H^2/\mathrm{SL}(2, Z)$. However, if γ is irrational, the (T^2, g_δ) makes an infinite sequence of excursions which carry it successively further towards infinity, followed by returns to a fixed compact set. The precise behavior is determined by the continued fraction expansion of γ. Thus, for γ rational, up to scaling, (T^2, g_δ) becomes arbitrarily thin as $\delta \to 0$. For γ irrational, there exist δ for which (T^2, g_δ) is arbitrarily thin. But there are also arbitrarily small δ for which it is fat.

Example 2.2 (*Almost flat manifolds; see* [6]). The simplest (but quite typical) almost flat manifolds arise very naturally in our context. Let $S^1 \to N^3 \to T^2$ be a circle bundle with connection over T^2. If this bundle is topologically nontrivial, then N^3 does *not* have the fundamental group of a flat manifold. In fact, N^3 is a quotient of the Heisenberg group, and as such, is a nilmanifold (the fiber S^1 corresponds to the center). If T^2 is given a metric, the connection induces a metric on N^3, for which rotation through the angle θ in the fibers is an isometry. Choose the metric on T^2 to be flat and note that all fibers have the same length. Then by (2.13) and (2.14), for the collapse (N^3, g_δ) along the fibers, g_δ converges locally to a *flat* metric. In fact, for the sequence $(N^3, \delta^2 g_{\delta^{2(1+\varepsilon)}})$ (where $\varepsilon > 0$), both the curvature *and* the diameter approach zero, so that the limit of this collapse is a *point*.

Remark 2.1. It is easy to see that the calculation of Theorem 2.1 can be generalized to the case in which the abelian Lie algebra of Killing fields is replaced by a nilpotent Lie algebra. The latter is collapsed as in Example 2.2, rather than by scaling.

3. Polarized volume collapses

For collapses associated to a polarization for which all orbits are not of the same dimension, we will need a slight generalization of the calculation of Theorem 2.1. Suppose that in (2.2) we replace δ by a function ρ which is constant on orbits;

$$\rho = \rho(y_1, \cdots, y_{n-k}) > 0. \tag{3.1}$$

We fix attention on the origin $(0,0)$ in (x, y)-space and make the change of coordinates,

$$u_i = \rho(0) x_i. \tag{3.2}$$

Now we obtain

$$(g(u, y, \rho)) = \left(\begin{array}{c|c} \dfrac{\rho^2(v)\, A(y)}{\rho^2(0)} & \dfrac{\rho^2(v)\, B(y)}{\rho(0)} \\ \hline \dfrac{\rho^2(v)\, B(y)}{\rho(0)} & \rho^2(y) C(y) + D(y) \end{array}\right). \tag{3.3}$$

It follows that for, say, $|\rho| \leqslant 1$,

$$|K_\rho| \leqslant c\big(A, B, C, D, |\rho'/\rho|, |\rho''/\rho|\big), \tag{3.4}$$

where ρ' and ρ'' denote typical first and second partials of ρ with respect to $y, \cdots, y_{n-k}$

Theorem 3.1. *Let Y^n be compact, let $\mathscr{P}$ be a polarization of an F-structure, $\mathscr{F}$, on Y^n and let g be an invariant metric. Then there exists a family of metrics, g_δ, which are invariant for the F-structure defined by $\bar{\mathscr{P}}$, such that for $\delta \leqslant 1/2$,*

(1) (Y^n, g_δ) is $c\delta$-collapsed,

(2) $\operatorname{diam}(Y^n, g_\delta) \leqslant c|\log \delta|$,

(3) $\operatorname{Vol}(Y^n, g_\delta) \leqslant c\delta^k|\log \delta|^n$,

(4) $|K_\delta| \leqslant c$.

Proof. Let $\{U_\alpha\}$ be as in Definition 1.7 and let f_δ: $U_\alpha \to [1/2, 1]$ be smooth functions such that $f_\alpha \equiv 1$ near U_α and

$$\bigcup_\alpha f^{-1}(1/2) = Y^n. \tag{3.5}$$

As in Lemma 1.4, we can assume that f_α is constant on every orbit $\mathscr{O}$ of $\bar{\mathscr{P}}$. Set

$$\rho_\alpha = \delta^{\log f_\alpha / \log 1/2}. \tag{3.6}$$

The metric g_δ will depend on a choice of ordering, $U_1, U_2, \cdots$ of the $\{U_\alpha\}$ (although its essential properties are independent of this choice). We start with the metric $\log^2\delta \cdot g$ and put

$$\log^2\delta \cdot g = g_1' + h_1 \tag{3.7}$$

on U_1, where the decomposition is as in (2.2) and h_1 vanishes on the orbits of $\mathscr{G}_1$. We then define g_1 by

$$g_1 = \begin{cases} \rho_1^2 g_1' + h_1, & U_1, \\ \log^2\delta g, & Y \setminus U_1, \end{cases} \tag{3.8}$$

where ρ_1 is as in (3.6). Proceeding by induction, we put

$$g_j = g_{j+1}' + h_{j+1}, \tag{3.9}$$

where h_{j+1} vanishes on the orbits of $\mathscr{G}_{j+1}$ and define g_{j+1} by

$$g_{j+1} = \begin{cases} \rho_{j+1}^2 g_{j+1} + h_{j+1}, & U_{j+1}, \\ g_j, & Y \setminus U_{j+1}. \end{cases} \tag{3.10}$$

We claim that

$$g_\delta = g_N \tag{3.11}$$

has the required properties. Note that (2) and (3) are obvious, and that (1) follows as in the proof of Theorem 2.1.

To see (4), let $p \in Y^n$ and $G_{p,\alpha} = G_p$, the stalk of $\mathscr{P}$. Let $(\underline{x}_1 \cdots \underline{x}_l, \underline{y}_1 \cdots \underline{y}_{n-l})$ $(l \geqslant k = \operatorname{rank} \mathscr{P})$ be coordinates as in (2.11) above, for the metric g such that $p = (0, 0)$. Thus, the $\underline{x}$-coordinates are constant along some transversal to the orbit $\mathcal{O}$ through p ($l = \dim \mathcal{O}$) and the y-coordinates are constant along the orbits of $\mathscr{g}(\tilde{U}_\alpha) = G_p$. We will keep track of the effect at p of the changes of metric corresponding to $j = 1, \cdots, N$ in coordinate systems derived from $(\underline{x}, \underline{y})$.

Observe that for $j = 1, \cdots, N$ the functions ρ_j depend only on $\underline{y}$ (and δ) since they are constant on orbits. Moreover, in the coordinate system

$$(3.12) \qquad x_i = \underline{x}_i \cdot \log \delta, \qquad y_i = \underline{y}_i \cdot \log \delta,$$

the matrix representing the metric $\log^2 \delta \cdot g$ has bounded partial derivatives (of all orders) and the functions $|\rho_j'/\rho|$, $|\rho_j''/\rho|$ ($j = 1, \cdots, N$) are *bounded independent of* δ (as is immediate from (3.7)).

Finally, we need only consider the effect at p of the changes of metric corresponding to those j for which $p \in U_j$. For such j, the orbits of $\mathscr{G}_j$ are contained in those of $\mathscr{G}_\alpha$ on $U_\alpha \cap U_j$.

Let β be the first value of j for which $p \in U_\beta$. By making a linear change of coordinates we can suppose that the orbits of $\mathscr{G}_\beta$ are given by $x_{t+1} = \text{const}, \cdots, x_l = \text{const}$, $y_1 = \text{const}, \cdots, y_{n-l} = \text{const}$, near p. We then introduce new coordinates $(u_1, \cdots, u_t, x_{t+1}, \cdots, x_l, y_1, \cdots, y_{n-l})$ as in (3.3). Since the ρ_j depend only on $y_1 \cdots y_{n-l}$ they have the same expressions as before. Thus, (4) follows by proceeding by induction.

Remark 3.1. The initial step in Theorem 3.1 in which distances are expanded in *all* directions by a factor $|\log \delta|$ is not optional, i.e. $\mathrm{Vol}(Y_n, g_\delta)$ does not always approach zero as rapidly as possible as $\delta \to 0$. This loss of sharpness is not very serious in the present context since at best one could replace $\mathrm{Vol}(Y_n, \delta) \sim \delta^k |\log \delta|^n$ by $\mathrm{Vol}(Y_n, \delta) \sim \delta^k$; compare Example 0.2 (continued). However, in Example 4.2 and in §5 we proceed more carefully.

4. Nonpolarized collapses

(a) Introduction. To be able to collapse when no polarization exists, we must:

(i) Describe a structure (referred to, somewhat informally, as a slice polarization) which replaces that of a polarization and which exists in general.

(ii) Check that there is a collapsing procedure based on this structure.

We begin by illustrating (i) and (ii) in an example which was considered in §1. Then we do the calculation behind (ii). Next we explain how the structure of (i) is constructed in general. Finally, we describe the collapse.

Example 1.7 (*continued*). Let $Z^2 \subset \mathscr{M}^4$ denote the union of the zero sections of the flat bundles $\mathscr{E}_\theta^3$, i.e. the 2-torus on which the dimension of the orbits drops from 2 to 1. The T-structure has nilpotent holonomy (given by the matrix (1.10)) in the direction of the θ-circle in Z^2 (the circle transverse to the orbits). As a consequence, the sub-bundle defined by the isotropy subgroups H_p, $p \in Z^2$, has no complementary flat sub-bundle. Equivalently there is no 1-dimensional polarization near Z^2.

However, for each fixed θ, the restriction of our structure to a 3-dimensional slice, $\mathscr{E}_\theta^3 \subset \mathscr{M}^4$, has no holonomy, and hence admits a 1-dimensional polarization. For example, on each slice we can choose the 1-parameter subgroup of the isometry group of $\mathscr{E}_\theta^3$ induced by parallel translation of $\mathscr{E}_\theta^3$ (see (1.6)). This family of "slice polarizations" varies continuously with θ (the corresponding family of infinitesimal generators gives rise to a vector field V on $\mathscr{M}^4$, which is tangent to $\mathscr{E}_\theta^3$ for each fixed θ, and such that $V|\mathscr{E}_\theta^3$ is a Killing field).

If the metric is collapsed in the direction of V, the curvature does *not* remain bounded, because V deviates from being a Killing field through its dependence on θ. To obtain a collapse with bounded curvature, we must simultaneously expand the metric in the θ direction (at an equal rate). This has the effect of making the above deviation negligible.

(b) Main computation. The essential quantitative features of nonpolarized collapse are captured by the following 3-dimensional situation. To simplify notation, we will only write the computation explicitly in this case.

Let $R^3 = R^2 \times R$, where the third coordinate is denoted by z. Let g be a riemannian metric on R^3 and let V be a nonvanishing vector field such that

(1) V is tangent to the slices, $z = \text{const}$.

(2) The restriction of V to any slice is a Killing field.

(3) There is an abelian Lie algebra, $\mathscr{F}$, of Killing fields such that if $X \in \mathscr{F}$, then X is tangent to every slice, $z = z_0$. Moreover, for each z_0, there exists $X_{z_0} \in \mathscr{F}$, with $X_{z_0}|_{z=z_0} = V|_{z=z_0}$.

It suffices to consider, say, $z_0 = 0$. Choose a local coordinate system (x, y, z) with $\partial/\partial x = X_0$.

To begin with we observe that

$$\left[\frac{\partial}{\partial x}, V\right] \equiv 0, \tag{4.1}$$

where [,] is the Lie bracket. In fact, for each fixed z_0,

$$\left[\frac{\partial}{\partial x}, X_{z_0}\right] \equiv 0, \tag{4.2}$$

since $\partial/\partial x = X_{z_0} \in \mathscr{F}$. But $\partial/\partial x$, V, X_{z_0} are all tangent to the slice $z = z_0$. So for $z = z_0$, the brackets in (4.1) and (4.2) can be computed in this slice, and there $V = X_{z_0}$.

Let

$$\frac{\partial}{\partial z} = U + N \tag{4.3}$$

be the decomposition of $\partial/\partial z$ into components tangent and normal to $z = z_0$ respectively. Since $\partial/\partial x$ is a Killing field tangent to $z = z_0$ we have

$$0 \equiv \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right], \quad 0 \equiv \left[\frac{\partial}{\partial x}, U\right], \quad 0 \equiv \left[\frac{\partial}{\partial x}, N\right]. \tag{4.4}$$

Now let (b_1^α, b_1^β) denote the components of $\partial/\partial x$ with respect to an orthonormal basis adapted to the decomposition $\{V\}$, $\{V\}^\perp \cap \{\partial/\partial x, \partial/\partial y\}$, $\{\partial/\partial x, \partial/\partial y\}^\perp$. Let (b_2^α, b_2^β), $(b_3^\alpha, b_3^\beta, b_3^\gamma)$ be the corresponding component functions for $\partial/\partial y$, $\partial/\partial z$. Notice, that all seven of these functions *do not depend on* x. This follows from (4.1), (4.4) and the fact that $\partial/\partial x$ is a Killing field. For example,

$$\frac{\partial}{\partial x}(b_3^\alpha) = \frac{\partial}{\partial x}\left(\frac{\langle U, V\rangle}{\langle V, V\rangle}\right) = \left\langle \left[\frac{\partial}{\partial x}, U\right], V\right\rangle \frac{1}{\langle V, V\rangle} + \cdots = 0. \tag{4.5}$$

Finally, observe that by (3),

$$b_1^\beta(y, 0) = 0, \tag{4.6}$$

since $\partial/\partial x|_{z=0} = V|_{z=0}$.

Let $g_{ij}(y, z, \delta)$ denote the metric obtained by the following operations (as above the subscripts $i, j = 1, 2, 3$ correspond to the variables x, y, z respectively).

(*) Multiply the metric $g_{ij}(y, z)$ by the factor δ^{-2} in the direction $\{\partial/\partial x, \partial/\partial y\}^\perp$, while leaving it unchanged on $\{\partial/\partial x, \partial/\partial y\}$.

(**) Multiply the metric obtained in (*) by a factor δ^2 in the direction of $\{V\}$, while leaving it unchanged on $\{V\}^\perp$.

We have

$$g_{11}(y, z, \delta) = \delta^2 (b_1^\alpha)^2 + (b_1^\beta)^2, \tag{4.7}$$

$$g_{12}(y, z, \delta) = \delta^2 b_1^\alpha b_2^\alpha + (b_1^\beta b_2^\beta), \tag{4.8}$$

$$g_{22}(y, z, \delta) = \delta^2 (b_2^\alpha)^2 + (b_2^\beta)^2, \tag{4.9}$$

$$g_{13}(y, z, \delta) = \delta^2 b_1^\alpha b_3^\alpha + b_1^\beta b_3^\beta, \tag{4.10}$$

$$g_{23}(y, z, \delta) = \delta^2 b_2^\alpha b_3^\alpha + b_2^\beta b_3^\beta, \tag{4.11}$$

$$g_{33}(y, z, \delta) = \delta^2 (b_3^\alpha)^2 + (b_3^\beta)^2 + \frac{1}{\delta^2}(b_3^\gamma)^2. \tag{4.12}$$

Make the change of variables $u = \delta x$, $w = z/\delta$. In terms of these new coordinates

$$g_{11}(y, \delta w, \delta) = (b_1^\alpha)^2 + \left(\frac{1}{\delta} b_1^\beta\right)^2, \tag{4.13}$$

$$g_{12}(y, \delta w, \delta) = \delta b_1^\alpha b_2^\alpha + \left(\frac{1}{\delta} b_1^\beta\right) b_2^\beta, \tag{4.14}$$

$$g_{22}(y, \delta w, \delta) = \delta^2 (b_2^\alpha)^2 + (b_2^\beta)^2, \tag{4.15}$$

$$g_{13}(y, \delta w, \delta) = \delta^2 b_1^\alpha b_3^\alpha + \delta b_1^\beta b_3^\beta, \tag{4.16}$$

$$g_{23}(y, \delta w, \delta) = \delta^3 b_2^\alpha b_3^\alpha + \delta b_2^\beta b_3^\beta, \tag{4.17}$$

$$g_{33}(y, \delta w, \delta) = \delta^4 (b_3^\alpha)^2 + \delta^2 (b_3^\beta)^2 + (b_3^\gamma)^2. \tag{4.18}$$

In view of (4.6),

$$\lim_{\delta \to 0} \frac{1}{\delta} b_1^\beta(y, \delta w) = \frac{\partial}{\partial w} b_1^\beta(y, 0) w. \tag{4.19}$$

Moreover

$$\lim_{\delta \to 0} \left(g_{ij}(y, 0, \delta)\right) = \begin{pmatrix} (b_1^\alpha)^2 & 0 & 0 \\ 0 & (b_2^\beta)^2 & 0 \\ 0 & 0 & (b_3^\gamma)^2 \end{pmatrix} \tag{4.20}$$

which is positive definite. Thus, it is clear that the curvature stays bounded as $\delta \to 0$.

The calculation just given can be generalized. Since the details are straightforward, we will merely state the results.

(A) First of all, instead of the coordinates x, y, z we can as well have several coordinates $x_1, \cdots, x_{n_1}$, $y_1, \cdots, y_{n_2}$, $z_1, \cdots, z_{n_3}$ (where $x_1, \cdots, x_{n_1}$ correspond to $V_1, \cdots, V_{n_1}$). Moreover, we can collapse only, say, $V_1, \cdots, V_{m_1}$ (and make the changes of coordinates $u_1 = \delta x_1 \cdots u_{m_1} = \delta_m x_m$, $w_1 = z_1/\delta \cdots w_{n_3} = z_{n_3}/\delta$). Finally, we can artificially treat a subset of $y_1 \cdots y_{n_2}$ as z-coordinates, even though this is not required in order to keep the curvature bounded.

(B) As in (3.1)–(3.4), δ can be replaced by a function $\rho(y_1, \cdots, y_{n_2}, z_1, \cdots, z_{n_3}, \delta)$. The curvature of the collapsed metric depends on $|\rho'/\rho|$, $|\rho''/\rho|$; compare (3.4).

(c) Construction of slice polarizations. We now explain how the "slice polarizations" which are described in the continuation of Example 1.7 (at the beginning of this section) are obtained in general, starting with the case of a pure structure, $\mathscr{F}$. For this we must consider the orbit stratification associated to $\mathscr{F}$ and construct an invariant metric on Y_n and inner products in the stalks, G_p, which are suitably compatible with this stratification.

Let $\mathscr{F}$ be a pure F-structure of positive rank on Y_n and let g be an invariant metric. There is a natural stratification of Y_n into maximal strata, Σ_i, such that rank $\mathscr{F} = i$ for $p \in \Sigma_i$. Since the groups G_p are abelian, it follows that the identity components, H_p^0, of isotropy groups, H_p, are invariant under parallel translation along any curve in Σ_i. (Recall that the structure sheaf of $\mathscr{F}$ can be regarded as a flat bundle.) Each Σ_i is totally geodesic for the metric g, since locally it can be viewed as the set of common zeros of a collection of Killing fields.

Let $q \in \Sigma_k$ and let $(\Sigma_k)_q$ denote the tangent plane to Σ_k at q. Let $p \in \Sigma_i$ and consider the collection of subspaces of Y_p of the form $\lim_{q \to p}(\Sigma_k)_q = Q_k$ $(k > i)$. Since all groups G_p are abelian, it follows that the Q_k are coordinate hyperplanes relative to some fixed orthogonal basis of Y_p. Moreover, $\{Q_k\}$ is invariant under parallel translation in the normal bundle $\nu(\Sigma_i)$.

Let Σ_{ε_i} denote the set of points of Σ_i at distance $> \varepsilon_i$ from $\partial\Sigma_i$. Let exp be the exponential map of the normal bundle $\nu(\Sigma_{\varepsilon_i})$. For r_i sufficiently small, exp restricted to the subset $S_{\varepsilon_i, r_i} = \{v \in \nu(\Sigma_{\varepsilon_i}) \mid \|v\| < r_i\}$ is a diffeomorphism onto a set $\Sigma_{\varepsilon_i, r_i}$. Let $\pi_i \colon \Sigma_{\varepsilon_i, r_i} \to \Sigma_{\varepsilon_i}$ denote the corresponding projection map.

Lemma 4.1. *The invariant metric g and numbers ε_i, r_i can be chosen such that*

(1) $\bigcup \Sigma_{\varepsilon_i, r_i} = Y$.

(2) *If $i_1 < i_2$, then $\pi_{i_1} = \pi_{i_1}\pi_{i_2}$ on $\Sigma_{\varepsilon_{i_1}, r_{i_1}} \cap \Sigma_{\varepsilon_{i_2}, r_{i_2}}$.*

Proof. Start with any invariant metric g_0. Choose $\varepsilon_1 = 0$ and r_1 so small that $\exp|S_{\varepsilon_1, r_1}$ is a diffeomorphism onto $\Sigma_{\varepsilon_1, r_1}$. There is a *natural metric* on S_{ε_1, r_1} which is *flat on the fibers*, for which the subspace orthogonal to the fibers is given by the connection on $\nu(\Sigma_1)$ and for which projection onto the zero section is a riemannain submersion. Push this metric down to a metric g_1 on $\Sigma_{\varepsilon_1, r_1}$ via exp. Note that g_1 is compatible with $\mathscr{F}|\Sigma_{\varepsilon_1, r_1}$ and hence that $\Sigma_i \cap \Sigma_{\varepsilon_1, r_1}$ is totally geodesic for g_1.

It follows easily from the construction that (2) is satisfied on $\Sigma_{\varepsilon_1, r_1}$. Moreover (using (2)) it follows that near $\Sigma_i \cap \Sigma_{\varepsilon_1, r_1}$, the pullback of g_1 via the exponential map of $\partial(\Sigma_i)$, actually coincides with the natural metric of $\nu(\Sigma_i)$.

Now we can proceed by induction. Extend g_1 to an invariant metric for $\mathscr{F}$ on all of Y_n, choose $r_2 \ll \varepsilon_2 \ll r_1$, and replace the metric $g_1|\Sigma_{\varepsilon_2, r_2}$ with the push down of the natural metric on S_{ε_2, r_2}. Let g_2 be the metric on $\Sigma_{\varepsilon_1, r_1} \cup \Sigma_{\varepsilon_2, r_2}$ so obtained. By what was noted above, g_2 coincides with g_1 on $\Sigma_{\varepsilon_1, r_1}$. By proceeding in this way, we obtain the required metric.

Put $U_i = \Sigma_{\varepsilon_i, r_i}$. Let $q \in U_i$ and let γ be the unique minimal geodesic from q to $\pi_i(q)$. Parallel translation along γ induces an isomorphism $G_q \to G_{\pi_i(q)}$, which we will also denote by π_i.

Note that at each point p, the metric g of Lemma 4.1 induces a natural inner product on the Lie algebra, g_p, of G_p. For this, we identify a Killing field X with $(X(p), \nabla X(p))$ (we assume G_p acts effectively; see Remark 1.2). The resulting inner product is invariant under the local action of G_q, but not under the maps π_i: $G_q \to G_{\pi_i(q)}$.

Lemma 4.2. *There exists an inner product* $\langle\ ,\ \rangle_q$ *on* g_q *which is invariant under the local action of* G_q *and under the projection* π_i *(for* $q \in U_i$*).*

Proof. On Σ_1, define $\langle\ ,\ \rangle_q$ to be the inner product above. Extend $\langle\ ,\ \rangle_q$ to U_1 by making it invariant under π_1. In view of 2) of Lemma 4.1, $\langle\ ,\ \rangle_q$ is invariant under the local action of G_q and under π_i on $U_1 \cup U_i$. Clearly, we can extend $\langle\ ,\ \rangle_q | U_1 \cap \Sigma_{\varepsilon_2}$ to an inner product on g_q, for all $q \in \Sigma_{\varepsilon_2}$, which is invariant under the local action G_q. Then extend to U_2 by composing with π_2. This is consistent with $\langle\ ,\ \rangle_q$ as defined previously on $U_1 \cap U_2$. By proceeding in this way, we construct $\langle\ ,\ \rangle_q$ on all of Y with the desired properties.

For $p \in S_{\varepsilon_i, r_i}$, we let K_p^i denote the connected (but not necessarily closed) subgroup whose Lie algebra is the orthogonal complement of that of the isotropy group, H_p. For $q \in U_i$, we put

$$K_q^i = \pi_1^{-1}\left(K_{\pi_i(q)}\right). \tag{4.21}$$

It follows from Lemmas 4.1 and 4.2 that the assignment $q \to K_q^i$ is invariant under the local action of G_q. Moreover, if $q \in U_i \cap U_j$, $i \leqslant j$, then

$$K_q^i \subseteq K_q^j. \tag{4.22}$$

Finally, if $q_1, q_2 \in U_i \cap U_j$, $i \leqslant j$, and $\pi_i(q_1) = \pi_i(q_2)$, then $K_{q_1}^i = K_{q_2}^j$.

(d) Collapse. We can now collapse Y_n by a straightforward variant of the procedure of §3. Choose functions f_i, ρ_i on U_i as in (3.5) and (3.6). Fix q and let $U_{i_1}, \cdots, U_{i_j}$, $i_q < \cdots < i_j$, denote the U_k with $q \in U_k$. Let $Z_{i_1} \subset \cdots \subset Z_{i_j}$ denote the subspaces of Y_q tangent to the orbits of $K_q^{i_1} \cdots K_q^{i_j}$ and $W_{i_j} \subset \cdots \subset W_{i_1}$ the tangent spaces to $\pi_1^{-1}(\mathcal{O}_{\pi_{i_1}}(q)), \cdots, \pi_i^{-1}(\mathcal{O}_{\pi_{i_j}}(q))$. Then

$$Z_{i_1} \subset \cdots \subset Z_{i_j} \subset W_{i_j} \subset \cdots \subset W_{i_1}. \tag{4.23}$$

Let g be as in Lemma 4.1 and put

$$\log^2\delta g = g_1^1 + h_1 + k_1, \tag{4.24}$$

where the decomposition (4.24) corresponds to $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$. Set

$$g_1 = \begin{cases} \rho_1^2 g_1^1 + h_1 + \rho_1^{-2} k_1, & U_1, \\ \log^2\delta g, & Y \setminus U_1. \end{cases} \tag{4.25}$$

Define g_{j+1} by induction:

$$g_{j+1} = \begin{cases} \rho_{j+1}^2 g_{j+1}^1 + h_{j+1} + \rho_{j+1}^{-2} k_{j+1}, & U_{j+1}, \\ g_j, & Y \setminus U_{j+1}, \end{cases} \tag{4.26}$$

where the decomposition corresponds to Z_{j+1}, $Z_{j+1}^{\perp} \cap W_{j+1}$, $W_{j+1}^{\perp}$.

We claim that g_n ($n = \dim Y$) collapses with bounded curvature as $\delta \to 0$ (where ρ_j depends on δ as in (3.6)).

To see that the curvature remains bounded, let $U_{i_1} \cdots U_{i_j}$ be those U_{i_k} with $q \in U_{i_k}$, and choose local coordinates near q as follows. Let

$$m_i = \dim \Sigma_i - i. \tag{4.27}$$

Choose local coordinates functions $s_1, \cdots, s_{m_{i_1}}$ on Σ_{i_1}, which are constant on the orbits. Extend these to $U_{i_1} \cap \cdots \cap U_{i_j}$ by composing with π_{i_1}. Next choose $s_{m_i+1}, \cdots, s_{m_{i_2}}$ on Σ_2 so that $s_1, \cdots, s_{m_{i_2}}$ are coordinates transverse to the orbits on Σ_2. Extend these to $U_{i_1} \cap U_{i_2}$ by composing with π_{i_2} (recall $\pi_{i_1} = \pi_{i_1}\pi_{i_2}$). By proceeding in this way, we obtain $s_1, \cdots, s_{m_{i_j}}$. Extend $s_1, \cdots, s_{m_{i_j}}$ to a complete system of local coordinates transverse to the orbit of $K_q^{i_j}$ through q, by choosing additional functions, $t_1, \cdots, t_{n-i_j-m_{i_j}}$, which are constant on the orbits of $K_p^{i_j}$ (for p near q). Finally, choose $x_1, \cdots, x_{i_k}$, $i = 1, \cdots, m$, such that for fixed $t_1, \cdots, s_{m_{i_j}}$, the fields $\partial/\partial x_1, \cdots, \partial/\partial x_{i_k}$ are Killing fields generated by the action of $K_q^{i_k}$.

$$\begin{aligned} z_1 &= s_1 \\ &\vdots \\ z_{m_{i_1}} &= s_{m_{i_1}} \end{aligned} \tag{4.28}$$

$$\begin{aligned} y_1 &= s_{m_{i_1}+1} \\ &\vdots \\ y_{m_{i_j}-m_{i_1}} &= s_{m_{i_j}} \end{aligned} \tag{4.29}$$

$$\begin{aligned} y_{m_{i_j}-m_{i_1}+1} &= t_1 \\ &\vdots \\ y_{n-i_j-m_{i_1}} &= t_{n-i_j-m_{i_j}}. \end{aligned} \tag{4.30}$$

The effect of the change of metric corresponding to U_i in this coordinate system is to collapse only $x_1, \cdots, x_{i_1}$ while expanding all directions orthogonal to $z_1 = \text{const}, \cdots, z_{m_{i_1}} = \text{const}$. The change corresponding to U_{i_j} collapses all $x_1 \cdots x_{i_j}$ directions while expanding directions normal to $y_1 = \text{const}, \cdots, y_{m_{i_j}-m_{i_1}} = \text{const}$, as well as $z_1 = \text{const}, \cdots, z_{m_{i_1}} = \text{const}$ (compare (A) which follows (4.20) above). The change corresponding to U_{i_k}, $1 < k < j$, has an effect intermediate between the two above. Thus, by successively changing coordinates as in (B) above and observing, as in §3, that $|\rho'_{i_k}/\rho_{i_k}|$, $|\rho''_{i_k}/\rho_{i_k}|$ remain bounded in the new coordinate systems, we see that the curvature of g_n is bounded independent of δ.

To see that g_n collapses as $\delta \to 0$, for each $p \in Y$ choose $r = r(\mathcal{O}_p)$ such that the exponential map of the normal bundle to $\mathcal{O}_p$ is a diffeomorphism when restricted to vectors of length r. Take a finite covering of Y^n by tubular open neighborhoods $T_{r_i/2}(\mathcal{O}_{p_i})$. For $q \in T_{r_i/2(\mathcal{O}_{p_i})}$ it follows that $\operatorname{dist}(q, \partial T_{r_i}(\mathcal{O}_{p_i}) > c(i) > 0$ for some $c(i)$ independent of δ. But through every such q passes a curve of length $c_1(i)\delta$, which is not contractible in $T_{r_i}(\mathcal{O}_{p_i})$. For $c_1(i)\delta < c(i)/2$, this implies that there is a closed noncontractible geodesic loop on q of length $< c_Y(i)\delta$. Hence $i(q) < c_1(i)\delta$.

Finally, we note that if m_i is as in (4.27) and we put

$$\kappa = \min_{U_{i_1} \cap \cdots \cap U_{i_j} \neq \varnothing} \left(i_1 - m_{i_1}\right) + \cdots + \left(i_j - m_{i_j}\right), \tag{4.31}$$

then

$$\operatorname{Vol}(Y^n, g_n(\delta)) \leqslant c\delta^{\kappa} |\log \delta|^n. \tag{4.32}$$

In particular, for this method of collapsing (which we indicate how to sharpen in §5) the volume goes either to infinity or to zero. In fact,

$$\lim_{\delta \to 0} \operatorname{Vol}(Y^n, g_n(\delta)) = 0 \tag{4.33}$$

if and only if for all i,

$$i - m_i > 0. \tag{4.34}$$

or equivalently,

$$i > \tfrac{1}{2} \dim \Sigma_i. \tag{4.35}$$

The procedure just described has a straightforward generalization to structures which are not pure. For this, we choose a regular atlas, $\{U_\alpha\}$, for a substructure of positive rank. Over each U_α we have a pure substructure, G_α (see Remark 1.2), and as in Lemma 1.3 there is a natural partial ordering among the U_α. Note that if $G_{p,\beta} \subset G_{p,\alpha}$, then the orbit stratification for the local action of $G_{p,\alpha}$ refines that for $G_{p,\beta}$. From this we easily obtain the existence of a metric g and inner product $\langle\ ,\ \rangle_q$ on g_q, with properties which generalize in the obvious way those of Lemmas 4.1 and 4.2.

Now on each U_α we collapse as above, except that we modify the cut off functions, ρ_i^α, in such a way that $\rho_i^\alpha \equiv 1$ in a small neighborhood of ∂U_α. By performing these collapses successively, we collapse Y^n with bounded curvature. Thus, we have the following result.

Let Y^n admit an F-structure of positive rank. Let $\{U_\alpha\}$ be a covering as above. Let $m_i^\alpha = \dim \Sigma_i^\alpha$, where Σ_i^α is defined by the action of G_α and put

$$\kappa = \inf_\alpha \kappa_\alpha, \tag{4.36}$$

where κ_α is defined as in (4.31).

Theorem 4.1. *There exists a family of invariant metrics, g_δ, on Y^n such that for $\delta \leqslant 1/2$.*

(1) (Y^n, g_δ) *is $c\delta$-collapsed.*

(2) $\operatorname{diam}(Y^n, g_\delta) \leqslant c|\log \delta|$.

(3) $\operatorname{Vol}(Y^n, g_\delta) \leqslant c\delta^\kappa |\log \delta|^n$.

(4) $|K_\delta| \leqslant c$.

Example 4.1 (*Nonpolarized volume collapse*). Let $R^2 \to \mathscr{M}^4 \xrightarrow{\pi} S^1 \times S^1_\theta$ be the space of flat bundles considered in Example 1.7, where S^1 denotes the zero section of $\mathscr{E}^3(\theta)$ and S^1_θ the circle which parametrizes the $\eta^3(\theta)$. If $\mathscr{M}_1, \mathscr{M}_2$ are 2-copies of $\mathscr{M}$, we can form

$$R^4 \to \mathscr{M}_1 \times \mathscr{M}_2 \xrightarrow{\pi_1 \times \pi_2} S^1 \times S^1_{\theta_1} \times S^1 \times S^1_{\theta_2}.$$

Let

$$T^3 = \{(x_1, \theta, x_2, \theta)\} \subset S^1 \times S^1_{\theta_1} \times S^1 \times S^1_{\theta_2} \tag{4.37}$$

and let $\mathscr{N}^7 = (\pi_1 \times \pi_2)^{-1}(T^3)$. The T-structure on $\mathscr{M}$ gives rise to an obvious nonpolarizable T-structure on $\mathscr{N}^7$ with orbits of dimension 2, 3, 4. The corresponding strata satisfy $\dim \Sigma_2 = 3$, $\dim \Sigma_3 = 5$, $\dim \Sigma_4 = 7$. Thus, (4.35) holds. By regarding $R^4 \to \mathscr{N}^7 \to T^3$ and letting Y^7 denote the double of the corresponding disc bundle, we obtain a specific example of a compact manifold which can be volume collapsed by means of a nonpolarizable T-structure.

Remark 4.1. The above Y^7 actually does admit a polarized T-structure. But probably there exist manifolds which can be volume collapsed although they admit no such structure.

The following example indicates how the construction of Theorem 4.1 can be sharpened.

Example 4.2 (*Collapsing M_F^4 with bounded volume*). The manifolds M_F^4 of Example 1.9 have F-structures which are of rank 2, except on Σ_1, which is the union of three connected codimension 2 submanifolds. These are either tori or Klein bottles depending on the particular choice of F. It will suffice to collapse tubular neighborhoods of the components of Σ_1, such that the volume stays finite and near the boundary the collapse agrees with the standard collapse of a pure rank 2 structure.

Let M^2 be a component of Σ_1. We start with a metric on the normal bundle, $\nu(M^2)$, which is cylindrical on the fibers. That is, on each fiber, R^2, it is of the form $dr^2 + f^2(r)d\theta^2$, where $f(r) \equiv 1$ for $r \geqslant 1$.

Given $\delta > 0$, we can construct a δ-collapsed metric on the disk-bundle, $0 \leqslant r \leqslant 1$, by means of a slice polarization. Thus, we multiply the metric by δ^2 on the subspace, X, tangent to the 1-dimensional orbits of the slice polarization and multiply the metric by δ^{-2} on the subspace W, orthogonal to the slices.

Let V be the orthogonal complement of X in the tangent space to the orbit. We extend the collapse to the annular region $1 \leqslant r \leqslant |\log \delta|$ by multiplying the metric in the direction of V by a factor $\rho^2(r)$, where $|\rho'/\rho|$, $|\rho''/\rho|$ are bounded, $\rho \equiv 1$ near $r = 1$ and $\rho \equiv \delta$ near $r = |\log \delta|$. Observe that the volume of this region is bounded independent of δ. Moreover, near $r = |\log \delta|$ we have the standard collapse of a pure structure. However, the metric is still expanded by a factor $\rho^2(r - |\log \delta| + 1)$ on the subspace W and for different components M_1^2, M_2^2 the subspaces W_1, W_2 do not correspond. Thus, we extend the collapse to the region $|\log \delta| \leqslant r \leqslant 2|\log \delta| - 1$ by multiplying the metric by a factor $\delta^2(r - |\log \delta| + 1)$ on the subspace W. It is easy to see that the curvature remains bounded independent of δ as does the volume.

By gluing the metrics just constructed onto the standard δ-collapsed metric for the rank 2 polarization on the remaining piece of M_F^4, we obtain the required δ-collapsed metric on M_F^4, with curvature and volume bounded independent of δ.

5. *F*-structures and complete metrics on open manifolds

(a) Introduction. In this section, we consider an open manifold, Y^n, which carries an F-structure, $\mathscr{F}$, or polarization, $\mathscr{P}$, on the complement of some compact subset. We treat in detail the case of a polarization, showing that Y^n admits a complete metric, g_∞, such that $|K_{g_\infty}| \leqslant 1$, $\mathrm{Vol}(Y^n, g_\infty) < \infty$. The analogous result for F-structures is the existence of a complete metric, g_∞, such that $|K_{g_\infty}| \leqslant 1$ and the injectivity radius goes uniformly to zero as $p \to \infty$; i.e. the family $Y^n \setminus B_R(q)$ collapses as $R \to \infty$. The proof of this latter result will be omitted since the ingredients which are required (beyond those of §4) will be presented in proving the existence of metrics of finite volume.

It is necessary to refine the constructions of the previous sections at two points.

Point 1. The invariant metric constructed in Lemma 1.3 is not guaranteed to have additional nice properties such as completeness, bounded curvature, etc. in case the manifold is open. Thus, we must begin by showing the existence of such an invariant metric, g_0, for a subpolarization $\mathscr{P}' \subset \mathscr{P}$. Moreover, this g_0 can be chosen such that $\mathscr{P}'$, when measured with respect to g_0, has essentially the same kind of uniform local behavior as in the compact case. This is achieved by making the metric grow sufficiently fast at infinity, and there is no attempt to control the volume at this stage.

Point 2. Rather than passing from g_0 to $\log^2\delta \cdot g_0$ (as in (3.27) and (4.24)) we will make a sequence of changes which expand g_0 by a factor $\log^2\delta_\alpha$ in a single (radial) direction near the boundary of each U_α'. The construction is such that the numbers $|\log\delta_\alpha|$ can be selected independently. If $c_\alpha = \mathrm{Vol}(U_\alpha, g_0)$, we choose $\{|\log\delta_\alpha|\}$ so small that

$$(5.1) \qquad \sum_{\alpha=1}^{\infty} c_\alpha\delta_\alpha \cdot |\log\delta_\alpha| < \infty.$$

Then, by proceeding as in §3, we obtain

$$(5.2) \qquad \mathrm{Vol}(Y^n, g_\infty) < c\sum_{\alpha=1}^{\infty} c_\alpha\delta_\alpha \cdot |\log\delta_\alpha|,$$

where g_∞ is the required metric.

(b) Construction of a complete metric g_0. We can assume that the polarization $\mathscr{P}$ is regular and that the boundaries $\{\partial U_\alpha\}$ are smooth and interest transversally. Moreover, after modifying the invariant metric, we can assume that g is such that

(1) The exponential map on the normal bundle, $\nu(\partial U_\alpha)$, is a diffeomorphism when restricted to vectors of length $\leqslant 2\varepsilon_\alpha$.

(2) Let $\underline{r}_\alpha$ denote the distance function from ∂U_α. Then on $T_{2\varepsilon_\alpha}(\partial U_\alpha) \cap T_{2\varepsilon_\beta}(\partial U_\beta)$

$$(5.3) \qquad \left\langle \mathrm{grad}_g \underline{r}_\alpha, \mathrm{grad}_g \underline{r}\right\rangle_g \equiv 0.$$

(3) The sets $\{U_\alpha \setminus \overline{T_{2\varepsilon}(\partial U_\alpha)}\}$ cover a neighborhood of infinity.

Let $\mathscr{P}' \subset \mathscr{P}$ denote the polarization defined by $\{U_\alpha'\}$, where $U_\alpha' = U_\alpha \setminus T_{\varepsilon_\alpha}(\partial U_\alpha)$.

The construction of the metric g_0 is based on the following lemma which is essentially a restatement of Lemma 5.4 and Theorem 5.5 of [3]. Unfortunately, the presentation of these results in [3] was somewhat garbled due to a confusion between the functions $\bar{k}$ and $1/\bar{k}$ below. For this reason, we will repeat some of the details here.

Lemma 5.1. *Let $k(p)$ be a locally bounded nonnegative function on a riemannian manifold, Y^n, with possibly incomplete metric g. Then there exists a smooth function, k^*, such that*

(1) $k \leqslant 3k^*$.

(2) *If $g_0 = (k^*)^2 g = e^{2\log k^*} g$, then g_0 is complete with curvature $|K(g_0)| \leqslant 1$ and injectivity radius $i_p(g_0) \geqslant 1$ for all p. Moreover,*

$$\left\| \operatorname{grad}_{g_0} \log k^* \right\|_{g_0} \leqslant c(n), \tag{5.4}$$

$$\left\| \operatorname{Hess}_{g_0} \log k^* \right\|_{g_0} \leqslant c(n). \tag{5.5}$$

(3) *If Y^n carries regular polarizations $\mathscr{P}' \subset \mathscr{P}$ and the metric g as above, then g_0 can be chosen invariant for $\mathscr{P}'$.*

Proof. By increasing the function k if necessary, we can assume

$$k(p) > \sup_{\tau \in \Lambda^2(TY_p)} |K_g(\tau)|^{1/2}, \tag{5.6}$$

$$k(p) \geqslant 1/i_p(g). \tag{5.7}$$

$$k(p) > 1/\overline{p, \infty}, \tag{5.8}$$

$$k \not\equiv 0, \tag{5.9}$$

where $\overline{p, \infty}$ denotes the supremum of the radii of open metric balls at p whose closure is compact.

Put

$$\bar{k}(p) = \inf\left\{ \frac{1}{R} \,\middle|\, \sup_{q \in B_R(p)} k(q) \leqslant 1/R \right\}. \tag{5.10}$$

It follows directly from (5.10) that if $\lambda \geqslant 0$ and $\overline{p, q} \leqslant \lambda / \bar{k}(p)$, then

$$\frac{1}{1 + \lambda} \bar{k}(p) \leqslant \bar{k}(q). \tag{5.11}$$

Moreover, if $\lambda < 1$,

$$\bar{k}(q) \leqslant \frac{1}{1 - \lambda} \bar{k}(p) \tag{5.12}$$

((5.11) and (5.12) replace Lemma 5.4 of [3]). The construction of k^* now proceeds as in the proof of Theorem 5.5 of [3], but with the following proviso: $\bar{k}$ is to be replaced by $1/\bar{k}$, except in the expression $\bar{k}^2 g$. This gives (1) and (2).

(3) If $\mathscr{P}' \subset \mathscr{P}$ as above, the function $\bar{k}(p)$ need not be invariant for $\mathscr{P}$ at points $p \in U_\alpha$ with

$$\bar{k}(p) \leqslant 1/\operatorname{dist}_g(p, \partial U_\alpha). \tag{5.13}$$

However, if we put

$$\eta(p) = \sup_{p \in U'_\alpha} \varepsilon_\alpha \tag{5.14}$$

(where $U'_\alpha = U_\alpha \setminus \overline{T_{\varepsilon_\alpha}(\partial U_\alpha)}$) and require in addition to (5.9)–(5.12) that, say,

$$k(p) \geqslant 10\eta(p), \tag{5.15}$$

then $\bar{k}(p)$ is invariant for $\mathscr{P}'$. If we now combine the argument of [3] with a standard averaging argument, (3) follows.

Let $p \in Y^n$ and let $(\underline{z}_1, \cdots, \underline{z}_n)$ be a local coordinate system with p at the origin. Suppose that on the $\underline{z}$-coordinate ball, $B_\varepsilon(p)$, the matrix $(g_{ij}(z))$ for the metric g satisfies, say

$$\tfrac{1}{2} \leqslant \det g_{ij}(\underline{z}) \leqslant 2, \tag{5.16}$$

$$\left| g'_{ij}(\underline{z}) \right| \leqslant \Omega, \tag{5.17}$$

$$\left| g''_{ij}(\underline{z}) \right| \leqslant \Omega^2. \tag{5.18}$$

We choose $k^* \geqslant \max(1/\varepsilon, \Omega)$ and make the change of variables

$$z_i = k^*(0)\underline{z}_i. \tag{5.19}$$

Then the metric $g_0 = (k^*)^2 g$ satisfies

$$\tfrac{1}{9} \leqslant \det(g_0)_{ij}(z) \leqslant 12. \tag{5.20}$$

$$\left| (g_0)'_{ij}(z) \right| \leqslant c(n), \tag{5.21}$$

$$\left| (g_0)''_{ij}(z) \right| \leqslant c(n) \tag{5.22}$$

on the z-coordinate ball $B_1(0)$ (see (5.4), (5.5), (5.11), (5.12) and (1) above).

Remark 5.1. By making the function k grow sufficiently rapidly we can find at each point a coordinate system satisfyiing (5.20)–(5.22) in which the basic computation, (2.11)–(2.14), will apply.

(c) Expansion of g_0. Let $\mathscr{P}'$ be as above and put $I_\alpha = U'_\alpha \cap \overline{T_\varepsilon(\partial U'_\alpha)}$, where as in (5.3), the tubular neighborhood is with respect to the metric g. If $p \in I_1 \cap \cdots \cap I_l$ we can introduce a local coordinate system, $(\underline{x}, \underline{y}, \underline{r})$, near p, as follows. As usual, the fields $\partial/\partial\underline{x}_1, \cdots, \partial/\partial\underline{x}_k$ are Killing fields spanning the orbit $\mathscr{O}_p$ (with respect to $\mathscr{P}'$). The functions $\underline{r}_1, \cdots, \underline{r}_l$ are as in (5.3). In view of (5.3) we can find additional coordinates $\underline{y}_1, \cdots, \underline{y}_t$, such that the matrix of g for these coordinates is of the form

$$\begin{pmatrix} A(\underline{y}, r) & 0 \\ 0 & I \end{pmatrix}, \tag{5.23}$$

where I is the identity matrix and $A(x, y, \underline{r})$ represents the inner product on the subspace spanned by $\{\partial/\partial\underline{x}_1, \cdots, \partial/\partial\underline{y}_t\}$. As in Remark 5.1, we can assume that for the coordinate system

$$(5.24) \qquad x_i = k^*(p)\underline{x}_i, \quad y_i = k^*(p)\underline{y}_i, \quad r_i = k^*(p)\underline{r}_i,$$

the matrix

$$(5.25) \qquad (g_0(x, y, r)) = \frac{(k^*(y, r))^2}{(k^*(p))^2}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

satisfies (5.20)–(5.22). Here

$$(5.26) \qquad A = A(y/k^*(p), r/k^*(p))$$

and the r_i take values which include the interval $(0, 1)$.

In order to expand g_0, we choose functions h_d: $[0, 1] \to [1, \infty)$, each $d \in R^+$, such that

(i) $h_d \equiv 1$ on fixed intervals $[0, \varepsilon]$, $[1 - \varepsilon, 1]$.

(ii) $\int_0^1 h_d = d$.

(iii) The derivatives of the function $1/h_d$ are uniformly bounded independent of d.

Now on each subset I_α, multiply the metric in the direction of grad r_α by the function $h_{d_\alpha}^2(r_\alpha)$, while leaving it unchanged in the orthogonal directions. Call the new metric g_{ex}. The constant d_α will be specified below.

To see the effect of this change of metric on $I_1 \cap \cdots \cap I_l$, we make the change of variables

$$(5.27) \qquad \int_0^{r_i} h_{d_i}(v)\, dv = s_i, \qquad h_{d_i}(r_i)\, dr_i = ds_i,$$

$$(5.28) \qquad \frac{1}{h_{d_i}(r_i)} \frac{\partial}{\partial r_i} = \frac{\partial}{\partial s_i}.$$

Then using (iii) and (5.28) it follows that

$$(5.29) \qquad (g_0(x, y, s)) = \frac{(k^*(y, r))^2}{(k^*(p))^2}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

satisfies (5.20)–(5.22) (for some $c(n)$ independent of $d_1, \cdots, d_l$). Moreover, the functions, s_i, take values which include the interval $(0, d_i)$.

(d) Collapse of the expanded metric. Now choose nonincreasing functions ρ_d: $[0, d] \to [0, 1]$ such that $\rho_d \equiv 1$ on $[0, \varepsilon_1]$ and $\rho_d \equiv e^{-d}$ near d. These can be chosen such that $|\rho_d'/\rho_d|$, $|\rho_d''/\rho_d|$ are bounded independent of d (compare

(3.4)). Finally, by choosing $\varepsilon_1 < \varepsilon$ (where ε is as in (i)) we can arrange that

$$|h_d(r(x)) \cdot \rho_d(s)| \leqslant 1, \tag{5.30}$$

where r and s are related as in (5.27).

The function $\rho_{d_\alpha}(s_\alpha)$, defined near $\partial U'_\alpha$, has an obvious extension to all of Y^n. Start with the metric g_{ex} and successively multiply the inner product on the subspace tangent to the orbits of each $\mathscr{G}'_\alpha$ by the function $\rho^2_{d_\alpha}$. Call the resulting metric g_∞. In view of (5.30), the volume form of g_∞ is pointwise smaller than that of g_0. Now choose $d_\alpha = |\log \delta_\alpha|$, where δ_α is as in (5.1). Then it follows from (3) of the subsection above that $\mathrm{Vol}(Y^n, g_\infty) < \infty$.

Finally, since $\mathscr{P}'$ is regular, each point is contained in at most n different sets U'_α. So by Remark 5.1 and the bounds for (5.29) it is clear that $|K_{g_\infty}|$ is uniformly bounded. Thus, we have

Theorem 5.2. (1) *If Y^n admits a polarization $\mathscr{P}$ on the complement of a compact subset, C, then Y^n admits a complete metric, g_∞, invariant for some $\mathscr{P}' \subset \mathscr{P}$, with $|K_{g_\infty}| \leqslant 1$ and $\mathrm{Vol}(Y^n, g_\infty) < \infty$.*

(2) *If C is empty, Y^n admits a family, $g_{\infty,\delta}$, of such complete metrics, with $|K_{g_{\infty,\delta}}| \leqslant 1$ and $\lim_{\delta\to 0}\mathrm{Vol}(Y^n, g_{\infty,\delta}) = 0$.*

In the same way we have

Theorem 5.3. (1) *If Y^n admits an F-structure, $\mathscr{F}$, of positive rank of the complement of a compact subset, C, then Y^n admits a complete metric g_∞, invariant for some $\mathscr{F}' \subset \mathscr{F}$, such that $|K_{g_\infty}| \leqslant 1$ and $i_p \to 0$ uniformly as $p \to \infty$.*

(2) *If C is empty, Y^n admits a family, $g_{\infty,\delta}$, of such metrics, such that $|K_{g_{\infty,\delta}}| \leqslant 1$ and $(Y^n, g_{\infty,\delta})$ collapses.*

Remark 5.2. Clearly, a sharper statement of Theorem 5.2 is possible; see Theorem 4.1 and Example 4.2.

Appendix: Pure polarized structures on essential manifolds

Let X^n be a closed oriented manifold and let $f\colon X \to K(\pi, 1)$ be the classifying map, where $\pi \simeq \pi_1(X^n)$. We call X^n *essential* if the fundamental class, $[X^n] \in H_n(X^n, R)$, satisfies $f_*([X^n]) \neq 0$ (compare [6], [7]).

Theorem A.1. *Let $\mathscr{F}$ be a pure F-structure on an essential manifold X^n, such that the group which acts locally is isomorphic to a k-torus, T^k. Then $\dim \mathscr{O}_p p = k$ for all $p \in X^n$. Moreover, there exists a free normal abelian subgroup, $A^k \subset \pi_1(V)$ of rank k, whose action on the higher homotopy groups $\pi_i(X^n)$ ($i \geqslant 2$) is trivial.*

Corollary A.2. *The connected sum $X^n \# M^n$, where M^n is an arbitrary n-dimensional manifold which is not a homology sphere, admits no pure F-structure of rank* $\geqslant 1$.

Example A.1. If $n = 2l + 1$ is odd, and M^n admits a (possibly nonpure) T-structure which is of rank l on some open set, then $T^n \# M^n$ also admits a (nonpure) T-structure. In fact, let $p \in T^n$, $q \in M^n$ lie on principal orbits $\mathcal{O}_p$, $\mathcal{O}_q$ of rank l. Let $T_\varepsilon(\mathcal{O}_p)$, $T_\varepsilon(\mathcal{O}_q)$ denote the small (saturated) tubular neighborhoods of p, q. If we form $T^n \# M^n$ by removing balls of radius $\varepsilon/2$ about p and q, we can regard $T^n \setminus T_\varepsilon(\mathcal{O}_p)$ and $M^n \setminus T_\varepsilon(\mathcal{O}_q)$ as contained in $T^n \# M^n$. On these sets, the T-structure on $T^n \# M^n$ can be taken to coincide with the restrictions of the given structures on T^n and M^n (compare [12]).

Proof of Theorem A.1. By Proposition 1.1, $\pi^*(\tilde{X}^E) \cong T^k$ acts on $\tilde{X}^E$, the holonomy covering. Let $\tilde{x} \in \tilde{X}^E$ and consider the orbit map $T^k \times \tilde{x} \to \tilde{\mathcal{O}}_x$. We claim that it suffices to show that the induced homomorphism

$$\mathbf{Z}^k = \pi_1(T^k) \xrightarrow{i_*} \pi_1(\tilde{X}^E, \tilde{x}) \subset \pi_1(X, x) \tag{A.1}$$

is injective. To see this, note that if $\dim \mathcal{O}_x < k$ for some x, then $\ker i_*$ contains the image in $\pi_1(T^k)$ of $\pi_1(H_x)$, where H_x denotes the isotropy group of x. For the second assertion, we observe that it is well known and easy to see that $i_*(\pi_1(T^k)) \subset \pi_1(X, x)$ is central and acts trivially on $\pi_i(X, x)$, $i \geqslant 2$.

Let $T^l_* \subset T^k$ denote the unique sub-torus commensurable with $\ker i_*$. Then T^l_* defines a substructure, $\mathcal{F}^*$, with the following property. for each orbit, $\mathcal{O}^*_p$, of $\mathcal{F}_1^*$ there is a finite covering $\tilde{\mathcal{O}}^*_x \to \mathcal{O}^*_x$ such that the induced map $\pi_1(\tilde{\mathcal{O}}^*_x) \to \pi_1(X)$ is the zero map (see below for further details).

Suppose first that $\tilde{\mathcal{O}}^*_x = \mathcal{O}^*_x$ for all x, and so $\pi_1(\mathcal{O}^*_x) \to \pi_1(X)$ is the zero map. Then if $X/\mathcal{F}^*$ denotes the orbit space, it follows that the induced map $p_*\colon \pi_1(X) \to \pi_1(X/\mathcal{F}^*)$ is an isomorphism. In fact, since the inverse image, ω^*_x, of each point in $X/\mathcal{F}^*$ is connected, p_* is surjective. Moreover, $\ker p_*$ is spanned by the normal subgroup generated by $\bigcup_x [Im(\pi_1(\mathcal{O}^*_x)) \subset \pi_1(X)] = 0$.

Since homotopy classes of maps $f\colon X \to K(\pi_1)$ are in 1-1 correspondence with (conjugacy classes of) homomorphisms, $f_*\colon \pi_1(X) \to \pi_1(K(\pi_1)) = \pi$, it follows that f is homotopic to $\tilde{f} \circ p$ for some $\tilde{f}\colon X/\mathcal{F}^* \to K(\pi, 1)$. Since X is essential, we can assume that $(\tilde{f} \circ p)_*$ is not the zero map. But then $H_n(X^n/\mathcal{F}^*) \neq 0$, which is possible only if $\dim T^l_* = 0$. Thus $\ker i_* = 0$ and the theorem follows in this case.

If $\tilde{\mathcal{O}}^*_x \neq \tilde{\mathcal{O}}_x$ for some x, the idea is similar but requires some further technical elaboration. Let $U(x)$ be a small equivariant tubular neighborhood of x. By passing to a finite covering, $\tilde{U}'_x \to U_x$, we can assume that the lifted orbit, $\tilde{\mathcal{O}}^{*\prime}_x$, is induced by the action of T^l_* (see Definition 1.2). Note that T^l_* is only commensurable with $\ker i_*$ and that $\tilde{\mathcal{O}}_x^{*\prime}$ might be a multiple orbit in its stratum. Thus, $\pi_1(\tilde{\mathcal{O}}_x^{*\prime}) \to \pi_1(\mathcal{O}_x^*)$ need not be the zero map. But after passing to a finite covering, $\tilde{\mathcal{O}}_x^* \to \tilde{\mathcal{O}}_x^{*\prime}$, we can assume that is the case for $\pi_1(\tilde{\mathcal{O}}_x^*) \to \pi_1(\mathcal{O}_x^*)$. Moreover, if $\tilde{U}_x \to U_x$ denotes the corresponding covering of U_x, then the same holds for any $\tilde{\mathcal{O}}_y^* \subset \tilde{U}_x$, since the inclusion, $\tilde{\mathcal{O}}_x^* \to \tilde{U}_x$, is a homotopy equivalence.

In order to make use of the covering spaces, $\tilde{U}_x \to U_x$, we need the following lemma.

Lemma A.3. *Let Y^n be a closed manifold which is the union of open submanifolds $U_1^n \cdots U_m^n$, whose (smooth) boundaries $\{\partial U_\gamma^n\}$ intersect transversally. Let π_j: $\tilde{U}_j \to U_j$ be finite coverings. Then there exists an n-dimensional polyhedron $\overline{Y}^n$ and a continuous map g: $\overline{Y}^n \to Y^n$, such that*

(1) *g_*: $H_*(\overline{Y}^n, Q) \to H_*(Y^n, Q)$ is surjective.*

(2) *If $C_j \subset U_j^n$ is closed, the map $g|g^{-1}(C_j) \subset Y^n$ factors through a map h: $g^{-1}(C_j) \to \tilde{U}_j$.*

Proof. Let Z be an arbitrary topological space, $U \subset Z$ an open subset and π_1: $\tilde{U} \to U$ a finite covering map. Denote by $\overline{Z} = X \vdash \tilde{U}$ the set $(Z \setminus U) \cup \tilde{U}$, and by $\bar{\pi}$: $\overline{Z} \to Z$ the obvious map. Define the topology in $\overline{W}$ by the condition that $A \subset \overline{Z}$ is closed if and only if $\bar{\pi}(A) \subset Z$ is closed. It is easy to see that $\bar{\pi}$ is surjective on rational homology and that the covering π factors through a unique map $\tilde{U} \to Z$.

Now specialize to the case of a closed manifold $Z = Y = \bigcup_1^m U_j$ as above. Put $Y' = Y \vdash \tilde{U}_1$, $U_j' = \pi_1^{-1}(U_j)$, $j = 2, \cdots, m$, and let π_j': $\tilde{U}_j' \to U_j'$ be the covering maps induced by $\bar{\pi}_1$ from κ_j. Then take $Y'' = Y' \vdash \tilde{U}_2'$, $Y''' = Y'' \vdash \tilde{U}_2''$, etc. After m steps, we reach the required polyhedron $Y^{(m)} = \overline{Y}$.

To complete the proof of Theorem A.1, we consider some sufficiently fine open covering, $\{U_j\}$, of Y by *saturated* open subsets with smooth boundary such that $\{\partial U_j\}$ intersect transversally; compare Lemma 1.4. The local actions on $\tilde{U}_i$ induce corresponding actions on $\overline{Y}$. As in the special case considered above, we want to show that if f: $X \to K(\pi, 1)$, then up to homotopy, $f \circ g$: $\overline{X} - K(\pi, 1)$ lifts to $\tilde{f} \circ p \circ g$. But it follows as above, that if $(p \circ g)_*$: $\pi_1(\overline{X}) \to \pi_1(X/\mathscr{F}^*)$ and $(f \circ g)_*$: $\pi_1(\overline{X}) \to \pi_1(K(\pi, 1))$, then $\ker(f \circ g)_* \subset \ker(p \circ g)_*$. This implies that the desired lift exists and suffices to complete the proof.

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