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BRILL-NOETHER-PETRI WITHOUT DEGENERATIONS

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Introduction

The purpose of this note is to show that curves generating the Picard group of a $K3$ surface X with $\mathrm{Pic}(X) = \mathbf{Z}$ behave generically from the point of view of Brill-Noether theory. In particular, one gets a quick new proof of Gieseker's theorem [5] concerning the varieties of special divisors on a general algebraic curve.

Let C be a smooth irreducible complex projective curve of genus g. One says that C satisfies *Petri's condition* if the map

$$\mu_0\colon H^0(A) \otimes H^0(\omega_C \otimes A^*) \to H^0(\omega_C)$$

defined by multiplication is injective for every line bundle A on C. Roughly speaking, this condition means that the varieties $W_d^r(C)$ of special divisors on C have the properties one would naively expect. Specifically, it implies that $W_d^r(C)$ is smooth away from $W_d^{r+1}(C)$, and that $W_d^r(C)$ (when nonempty) has the postulated dimension $\rho(r, d, g) =_{\mathrm{def}} g - (r+1)\cdot(g - d + r)$. We refer to [1] for the definition of $W_d^r(C)$, and for a detailed discussion of Petri's condition and its role in Brill-Noether theory. One of the most basic results of this theory is Gieseker's theorem [5] that Petri's condition does in fact hold for the generic curve of genus g.

We prove here the following

Theorem. *Let X be a complex projective $K3$ surface, and let $C_0 \subset X$ be a smooth connected curve. Assume that every divisor in the linear system $|C_0|$ is reduced and irreducible. Then the general curve $C \in |C_0|$ satisfies Petri's condition.*

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The hypothesis is satisfied in particular when Pic(X) is infinite cyclic, generated by the class of C_0. But for any integer $g \geqslant 2$ there exists a $K3$ surface X with Pic(X) $= \mathbf{Z} \cdot [C_0]$ for some curve C_0 of genus g, and thus the theorem implies Gieseker's result [5].

The study of special divisors on a general curve has traditionally centered around degeneration arguments. One of the first results in this area was due to Griffiths and Harris [7], who proved the assertion of Brill and Noether that if C is a general curve of genus g, then $\dim W_d^r(C) = \rho(r, d, g)$ provided that $\rho(r, d, g) \geqslant 0$. Their method was to deduce the theorem from an analogous statement for a rational curve with g nodes, which in turn was proved by a further degeneration. To prove Petri's conjecture, Gieseker [5] combined some rather elaborate combinatorial arguments with a systematic analysis of the limiting linear series on reducible curves arising in a degeneration of g-nodal $\mathbf{P}^1$'s. Eisenbud and Harris [2] subsequently streamlined Gieseker's proof by using a different degeneration, and they have recently extended and given several interesting new applications of these techniques (cf. [4]).

By contrast, the proof of the theorem here does not require any degenerations. Instead the method is simply to exhibit smooth families of g_d^r's. Specifically, we consider triples (C, A, τ) consisting of a nonsingular curve $C \subset X$ in the linear system $|C_0|$, a line bundle $A \in W_d^r(C)$ such that both A and $\omega_C \otimes A^*$ are base-point free, and an isomorphism τ mod scalars of $H^0(A)$ with a fixed vector space of dimension $r + 1$. Such triples are parametrized by a variety P_d^r, and one has an evident map π: $P_d^r \to |C_0|$. The tangent spaces to P_d^r and the derivative of π are computed cohomologically in terms of certain vector bundles $F_{C,A}$ on X which we study in §1. One finds in particular that these bundles have only trivial endomorphisms so long as $|C_0|$ does not contain any reducible curves. Much as in [10] this allows us to show in §2 that P_d^r is nonsingular, and that moreover the morphism π is smooth at (C, A, τ) if and only if the Petri μ_0 map for A is injective. The theorem then follows (§3) from the generic smoothness of π. In as much as it avoids the combinatorics involved in degenerational proofs, the present approach to Brill-Noether-Petri would seem to be simpler than the traditional one. On the other hand, as in [2] the argument only works in characteristic zero, and these techniques do not yield the theorem of Kempf [8] and Kleiman-Laksov [9] that $W_d^r(C)$ is nonempty when $\rho(r, d, g) \geqslant 0$ (which however is elementary nowadays; cf. [1, Chapter VII]).

Special divisors on a curve C on a $K3$ surface X appear to have been first considered by Reid [13], who showed that under suitable numerical hypotheses a special pencil on C is the restriction of one on X. A beautiful conjecture of Mumford, Harris and Green (see [6, §5]) asserts that all curves in a given linear series on X have the same Clifford index. This conjecture—which would generalize the well-known fact that if $C_0 \subset X$ is hyperelliptic, then so too is any other smooth curve in $|C_0|$—has been verified in special cases by Donagi and Morrison, and by Green and the author. Serrano-Garcia [14] has extended some of Reid's results to surfaces other than $K3$'s.

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1. The vector bundles $F_{C,A}$

This section is devoted to the study of certain vector bundles that play an important role in the argument. But first some notation. Throughout the paper X denotes a complex projective $K3$ surface, and $C_0 \subset X$ is a smooth irreducible curve of genus g. Given a curve C, and integers d and r, we define

$$V_d^r(C) \subset \operatorname{Pic}^d(C)$$

to be the open subset of $W_d^r(C)$ consisting of line bundles A on C such that:

(i) $h^0(A) = r + 1$, $\deg(A) = d$; and

(ii) both A and $\omega_C \otimes A^*$ are generated by their global sections.

Fix now a smooth curve $C \subset X$ in the linear series $|C_0|$, and a line bundle $A \in V_d^r(C)$. We associate to the pair (C, A) a vector bundle $F_{C,A}$ on X, of rank $r + 1$, as follows. Thinking of A as a sheaf on X, there is a canonical surjective evaluation map

$$e_{C,A}\colon H^0(A) \otimes_{\mathbf{C}} \mathcal{O}_X \twoheadrightarrow A$$

of $\mathcal{O}_X$-modules. Take

$$F_{C,A} \underset{\text{def}}{=} \ker e_{C,A}$$

to be its kernel. [Note that A, being locally isomorphic to $\mathcal{O}_C$, has homological dimension 1 over $\mathcal{O}_X$. Hence $F_{C,A}$ is indeed a vector bundle.]

The basic properties of these bundles are easily determined. Specifically, setting $F = F_{C,A}$ one has by construction the exact sequence

$$0 \to F \to H^0(A) \otimes_{\mathbf{C}} \mathcal{O}_X \to A \to 0 \tag{1.1}$$

of sheaves on X. Since $\mathcal{O}_X = \mathcal{O}_X$, dualizing (1.1) gives:

$$0 \to H^0(A)^* \otimes_{\mathbf{C}} \mathcal{O}_X \to F^* \to \omega_C \otimes A^* \to 0, \tag{1.2}$$

and from (1.1) and (1.2) one sees that:

(i) $c_1(F) = -[C_0]$, $c_2(F) = \deg(A) = d$;

(ii) F^* is generated by its global sections [recall: $h^1(\mathcal{O}_X) = 0$];

(iii) $H^0(F) = H^2(F^*) = 0$,
$H^1(F) = H^1(F^*) = 0$,
$h^0(F^*) = h^0(A) + h^1(A)$.

Furthermore, one has:

(iv) $\chi(F \otimes F^*) = 2 \cdot h^0(F \otimes F^*) - h^1(F \otimes F^*) = 2 - 2 \cdot \rho(A)$,
where $\rho(A) = g(C) - h^0(A) \cdot h^1(A)$.

Proof. The first equality follows from Serre duality. If E is a vector bundle of rank e on X, Riemann-Roch gives $\chi(E \otimes E^*) = (e-1) \cdot c_1(E)^2 - 2e \cdot c_2(E) + 2e^2$. Now compute

The presence or absence of reducible curves in $|C_0|$ comes into play via

Lemma 1.3. *Fix a smooth curve C in $|C_0|$ and a line bundle $A \in V_d^r(C)$, and let $F = F_{C,A}$. If F has nontrivial endomorphisms, i.e. if $h^0(F \otimes F^*) \geqslant 2$, then the linear system $|C_0|$ contains a reducible (or multiple) curve.*

Proof. Set $E = F^*$. Since $h^0(E \otimes E^*) \geqslant 2$, there exists by a standard argument a nonzero endomorphism v: $E \to E$ which drops rank everywhere on X. [Take any endomorphism w of E, $w \neq (\text{const}) \cdot 1$, and set $v = w - \lambda \cdot 1$, where λ is an eigenvalue of $w(x)$ for some $x \in X$. Then

$$\det(v) \in H^0\big(\det(E^*) \otimes \det(E)\big) = H^0(\mathcal{O}_X)$$

vanishes at x, and hence is identically zero.] Let

$$N = \operatorname{im} v, \qquad M_0 = \operatorname{coker} v,$$

and put

$$M = M_0/T(M_0),$$

where $T(M_0)$ is the torsion subsheaf of M_0. Thus

$$[C_0] = c_1(E) = c_1(N) + c_1(M) + c_1\big(T(M_0)\big)$$

in the Chow group $A_1(X) = \operatorname{Pic}(X)$. Now $c_1(T(M_0))$ is represented by a nonnegative linear combination of the codimension one irreducible components (if any) of $\operatorname{supp}(T(M_0))$. So it is enough to show that $c_1(N)$ and $c_1(M)$ are represented by nonzero effective curves. But N and M are torsion-free sheaves of positive rank, and—being quotients of E—are generated by their global sections. Furthermore, since $H^0(E^*) = 0$ neither of these can be trivial vector bundles. So the lemma follows from the elementary fact:

> Let U be a torsion-free sheaf on a smooth projective surface. If U is generated by its global sections, then $c_1(U)$ is represented by an effective (or zero) divisor. Moreover $c_1(U) = 0$ $\Leftrightarrow$ U is a trivial vector bundle.

Indeed, the double dual U^{**} of U is locally free, and the canonical inclusion $U \to U^{**}$ is an isomorphism outside of a finite set (cf. [12, II.1.1]). Thus $c_1(U) = c_1(U^{**})$, and U^{**} is generated by its sections away from finitely many points. Therefore $H^0(\det(U^{**})) \neq 0$, and (by Porteous) $c_1(U^{**}) = 0$ if and only if U^{**}—and hence also U—is a trivial bundle. q.e.d.

It is amusing to note that the lemma already yields a special case of the Brill-Noether theorem [7], namely that a general curve C of genus g does not carry any line bundle A with $\rho(A)\,[= g(C) - h^0(A) \cdot h^1(A)] < 0$. In fact:

Corollary 1.4. *Assume that every member of the linear series $|C_0|$ is reduced and irreducible. Then for every smooth curve $C \in |C_0|$ and every line bundle A on C one has $\rho(A) \geqslant 0$.*

When $h^0(A) = 2$ the corollary was proved by Donagi and Morrison (unpublished) using very different methods of Reid [13], and independently by Reid himself (private communication). Compare also [3].

Proof of Corollary 1.4. Observe that if B is a base-point free special line bundle on C, and if Δ is the divisor of base-points of $\omega_C \otimes B^*$, then $B(\Delta)$ is again base-point free. Hence we can assume in (1.4) that both A and $\omega_C \otimes A^*$ are generated by their global sections, and then the assertion follows from (iv) and (1.3).

2. Infinitesimal calculations

Keeping notation as in §1, we now fix positive integers r and d, and a vector space H of dimension $r + 1$.

Definition 2.1. *Let P_d^r denote the quasi-projective scheme (constructed below) parametrizing the set of all triples (C, A, λ), where:*

(i) *$C \subset X$ is a smooth curve in the linear system $|C_0|$;*

(ii) *$A \in V_d^r(C)$; and*

(iii) *λ is a surjective homomorphism of $\mathcal{O}_X$-modules:*

$$\lambda\colon H \otimes_{\mathbf{C}} \mathcal{O}_X \to A \to 0$$

inducing an isomorphism $H \simeq H^0(A)$, two such homomorphisms being identified if they differ only by multiplication by a nonzero scalar.

Construction of P_d^r: P_d^r is an open subset of a Hilbert scheme classifying curves in $X \times \mathbf{P}(H)$. Specifically, given a triple (C, A, λ) as above, the quotient $\lambda|C\colon H \otimes_{\mathbf{C}} \mathcal{O}_C \to A$ determines an embedding

$$C \subset \mathbf{P}(H \otimes_{\mathbf{C}} \mathcal{O}_X) = X \times \mathbf{P}(H),$$

and distinct triples give rise to distinct subvarieties of $X \times \mathbf{P}(H)$. The subschemes of $X \times \mathbf{P}(H)$ arising in this manner are parametrized by a Zariski-open subset of the Hilbert scheme of curves in $X \times \mathbf{P}(H)$ (with appropriate Hilbert polynomial defined with respect to some ample divisor on $X \times \mathbf{P}(H)$). We take this open set to be P^r_d.

Observe that there is a natural morphism

$$\pi\colon P^r_d \to |C_0|$$

sending a triple (C, A, λ) to the point $\{C\}$. Note also that for every $(C, A, \lambda) \in P^r_d$, the sheaf $\ker\lambda$ is isomorphic to the bundle $F_{C,A}$ introduced in §1. Consequently the discussion of §1 applies to these kernels.

The basic fact for us is that one has good infinitesimal control over P^r_d and π:

Proposition 2.2. *Fix any point* $(C, A, \lambda) \in P^r_d$, *and let* $F = \ker\lambda$. *Assume that* $h^0(F \otimes F^*) = 1$. *Then*:

(i) P^r_d *is smooth at* (C, A, λ), *of dimension* $\rho(A) + g + \{h^0(A)^2 - 1\}$; *and*

(ii) *The map* π *is smooth at* (C, A, λ), *i.e.* $d\pi_{(C,A,\lambda)}$ *is surjective, if and only if the Petri homomorphism*

$$\mu_0\colon H^0(A) \otimes H^0(\omega_C \otimes A^*) \to H^0(\omega_C)$$

is injective.

Remark. Observe that there is no assumption on the integers r and d. However it may well be that P^r_d is empty [cf. Corollary 1.4].

Proof of Proposition 2.2. Consider the embedding $C \subset X \times \mathbf{P}(H)$ determined by λ. Denoting by $\Phi\colon C \to \mathbf{P}(H)$ the projection of C to $\mathbf{P}(H)$, one has a canonical exact sequence of tangent and normal bundles:

$$(*) \qquad 0 \to \Phi^*(\Theta_{\mathbf{P}(H)}) \to N_{C/X\times\mathbf{P}(H)} \to N_{C/X} \to 0,$$

and $d\pi_{(C,A,\lambda)}$ is identified with the resulting homomorphism

$$T_{(C,A,\lambda)}P^r_d = H^0(N_{C/X\times\mathbf{P}(H)}) \to H^0(N_{C/X}) = T_{(C)}|C_0|.$$

Grant for the time being the following

Claim. If $h^0(F \otimes F^*) = 1$, *then the map*

$$(**) \qquad H^1(N_{C/X\times\mathbf{P}(H)}) \to H^1(N_{C/X})$$

determined by $(*)$ *is bijective.*

Then first of all one gets an isomorphism $\operatorname{coker} d\pi_{(C,A,\lambda)} \simeq H^1(\Phi^*(\Theta_{\mathbf{P}(H)}))$. But $\Phi = \Phi_A$ is the morphism determined by the complete linear system associated to A, and hence $H^1(\Phi^*(\Theta_{\mathbf{P}(H)}))$ is Serre dual to $\ker\mu_0$. This proves (ii).

For (i) we argue much as in [10] that the obstructions to the smoothness of the Hilbert scheme of $X \times \mathbf{P}(H)$ at (C, A, λ) vanish. Specifically, let R be a local artinian $\mathbf{C}$-algebra, let $I \subset R$ be a one-dimensional square-zero ideal, and set $S = R/I$. Consider an infinitesimal deformation

$$(+) \qquad \underline{C} \subset X \times \mathbf{P}(H) \times \operatorname{Spec}(S)$$

of C in $X \times \mathbf{P}(H)$ over Spec(S). The obstruction to extending $(+)$ to a deformation over Spec(R) is given by an element $o_{(+)} \in H^1(N_{C/X\times \mathbf{P}(H)})$. On the other hand, $(+)$ determines by projection an infinitesimal deformation

$$(\#) \qquad \underline{C} \subset X \times \operatorname{Spec}(S)$$

of C in X, and one has a corresponding obstruction class $o_{(\#)} \in H^1(N_{C/X})$. Furthermore, $o_{(+)}$ maps to $o_{(\#)}$ under the homomorphism $(**)$; this can be checked, e.g., using the explicit description of the obstruction classes in [11, Lecture 23] by observing that the local equation of $\underline{C}$ in $X \times \operatorname{Spec}(S)$ can be taken as one of the equations locally cutting out $\underline{C}$ in $X \times \mathbf{P}(H) \times \operatorname{Spec}(S)$. But the Hilbert scheme $|C_0|$ of C in X is smooth, and hence $o_{(\#)} = 0$. Therefore $o_{(+)} = 0$ thanks to the claim, and this proves that P_d^r is smooth at (C, A, λ). (One could also deduce (i) from Theorem (0.1) of [10].)

It remains to verify the claim. Denoting by p and q the projections of $X \times \mathbf{P}(H)$ onto X and $\mathbf{P}(H)$ respectively, note first that C is defined in $X \times \mathbf{P}(H)$ as the zero-locus of the evident section of $p^*(F^*) \otimes q^*(\mathcal{O}_{\mathbf{P}(H)}(1))$. Therefore

$$N_{C/X\times \mathbf{P}(H)} = F^*|C \otimes A.$$

We next compute $h^1(C, F^*|C \otimes A) = h^1(X, F^* \otimes A)$. To this end, observe that since F^* is locally free, λ determines an exact sequence

$$0 \to F \otimes F^* \to H \otimes_{\mathbf{C}} F^* \to A \otimes F^* \to 0$$

of sheaves on X. Using the computations of $H^i(F^*)$ in §1 one sees that $H^1(X, A \otimes F^*) = H^2(X, F \otimes F^*)$, and so by duality plus the hypothesis on $F \otimes F^*$ one finds that $h^1(N_{C/X\times \mathbf{P}(H)}) = 1$. Since also $h^1(N_{C/X}) = h^1(\omega_C) = 1$, the claim follows. Finally, using facts (iii) and (iv) from §1, one gets the stated value for $h^0(X, F^* \otimes A) = \dim_{(C,A,\lambda)} P_d^r$.

Remark. Suppose that the linear system $|C_0|$ does not contain any reducible members. Then it follows from the proposition and Lemma 1.3 that P_d^r (if nonempty) has pure dimension $g + \rho(d, r, g) + \{(r+1)^2 - 1\}$. Observing that the fiber of π over a point $\{C\} \in |C_0|$ is a PGL$(r+1)$-bundle over $V_d^r(C)$, one can use this to give a proof of the Brill-Noether theorem of Griffiths and Harris [7]. But at this point it is quicker for us to get dimensionality via Petri.

3. Proof of the Theorem

We assume that the linear system $|C_0|$ does not contain any reducible or multiple members, and we wish to show that almost every curve in $|C_0|$ satisfies Petri's condition.

To begin with fix arbitrary positive integers r and d. We claim that there is a nonempty Zariski-open set $U_d^r \subset |C_0|$ of smooth curves such that for all $C \in U_d^r$:

$$\mu_0\colon H^0(A) \otimes H^0(\omega_C \otimes A^*) \to H^0(\omega_C) \text{ is injective}$$
$$\text{for every line bundle } A \in V_d^r(C).$$

Indeed, it follows from Lemma 1.3 and the assumption on $|C_0|$ that for any point $(C, A, \lambda) \in P_d^r$, the bundle $F = \ker \lambda$ satisfies $h^0(F \otimes F^*) = 1$. Thus by Proposition 2.2 the variety P_d^r is nonsingular (or empty). As we are in characteristic zero the theorem on generic smoothness applies, and there exists a nonempty open set $U_d^r \subset |C_0|$ over which the map $\pi\colon P_d^r \to |C_0|$ is smooth. Invoking the proposition again, it follows that U_d^r has the stated property.

We assert next that there is a nonempty open set $U \subset |C_0|$ of smooth curves such that for any $C \in U$:

> μ_0 is injective for every line bundle A on C such that both A and $\omega_C \otimes A^*$ are generated by their global sections.

In fact, for a fixed genus g the injectivity of μ_0 for A is nontrivial for only finitely many values of $d = \deg(A)$ and $r = r(A)$ [e.g., $0 \leqslant 2r \leqslant d \leqslant 2g - 2$]. It suffices to take U to be the intersection of the corresponding U_d^r's.

Using the remark at the beginning of the proof of Corollary 1.4, the theorem now follows from the observation that if D is any effective divisor on C, and if Δ is the divisor of base-points of $|D|$, then the injectivity of μ_0 for $\mathcal{O}_C(D - \Delta)$ implies the injectivity of μ_0 for $\mathcal{O}_C(D)$.

Remark. It is not generally the case that Petri's condition holds for *all* smooth curves in $|C_0|$. Furthermore, one cannot avoid the hypothesis on $|C_0|$: e.g. for $n \geqslant 2$ the general member of $|n \cdot C_0|$ does not satisfy Petri. Similarly one can not expect to weaken too greatly the hypothesis that X be a $K3$, since for instance the theorem already fails for the general surface of degree $\geqslant 5$ in $\mathbf{P}^3$.

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