J. DIFFERENTIAL GEOMETRY
23 (1986) 143–174

THURSTON'S RIEMANNIAN METRIC FOR TEICHMÜLLER SPACE

SCOTT A. WOLPERT

0. Introduction

Possibly the most interesting structures on the Teichmüller space are those defined directly from structures on a compact Riemann surface. The quintessential example is due to Riemann himself. The complex structure of the Teichmüller space T_g is uniquely determined by the condition that the period mapping to the Siegel upper half-space be holomorphic. Recall that the period mapping is given by the matrix of periods of abelian differentials, the basic numerical invariant of the complex structure of a compact surface. A second example of such a geometry on the Teichmüller space is the Kobayashi-Teichmüller metric. The Kobayashi metric of a complex manifold M is defined as the solution to an extremal problem for maps of the unit disc into M. On the other hand the Teichmüller metric is defined in terms of singular flat metrics on a compact surface, specifically metrics given as $ds^2 = |\varphi|$, φ a holomorphic quadratic differential. Royden's theorem states that the Kobayashi metric of T_g is the Teichmüller metric and thus the extremal maps from the disc to T_g come from singular flat metrics on a surface [9].

Certainly for surfaces of negative Euler characteristic the fundamental Riemannian structure is the hyperbolic metric. A natural question is to identify the geometry on Teichmüller space coming from the hyperbolic surface geometry. Two elements of the geometry are already known: the geodesic length functions and Thurston's earthquakes. Indeed both are defined in terms of the hyperbolic geometry of a compact surface. In [13], [14] we found that the Weil-Petersson Kähler form is also part of the geometry. Specifically if t_α, l_α on T_g are respectively the earthquake tangent field and the geodesic length function associated to a simple closed geodesic α, then $\omega(t_\alpha, \ ) = -dl_\alpha$, t_α and $-dl_\alpha$ are dual in the Kähler form. An immediate consequence is that the Kähler

Received January 24, 1985. Partially supported by the National Science Foundation; Alfred P. Sloan Fellow.

form is preserved by an earthquake flow. We also found that $\omega(t_\alpha, t_\beta) = t_\alpha l_\beta = \sum_{p \in \alpha\#\beta} \cos\theta_p$, that is $\omega(t_\alpha, t_\beta)$ is simply the sum of cosines of the intersection angles of geodesics α and β. And finally both the Lie bracket $[t_\alpha, t_\beta]$ and the Lie derivatives $t_\alpha t_\beta l_\gamma$ are evaluated by the hyperbolic trigonometry of the geodesics α, β and γ [5], [14].

More recently Thurston has given the description of a Riemann metric for Teichmüller space in terms of hyperbolic trigonometry. His idea is based on Kerckhoff's observation that the geodesic length functions are convex along earthquake paths. Hence a geodesic length function l_α might be thought of as the square of the distance, as measured from the minimum of l_α. Accordingly the second derivative at the minimum would be the metric tensor. Now, given $R \in T_g$, if α, chosen arbitrarily, is replaced by the *generic* R geodesic, then Thurston observed that the corresponding length function indeed has its minimum at R. Specifically consider $\{\beta_j\}$ a sequence of closed geodesics whose images are uniformly distributed in the R unit tangent bundle and given t_α, t_γ earthquake tangent fields Thurston introduces the quantity

$$\langle t_\alpha, t_\gamma \rangle_T = \lim_j \frac{1}{l_{\beta_j}} t_\alpha t_\gamma l_{\beta_j},$$

the *second derivative of the length of a uniformly distributed geodesic*. Positivity follows from Kerckhoff's observation and estimating the growth of $t_\alpha t_\gamma l_{\beta_j}$. A sample of the arguments is given by Thurston's proof that the minimum occurs at R, specifically that

$$\lim_j \frac{1}{l_{\beta_j}} t_\alpha l_{\beta_j} = 0.$$

A simple consequence of the uniform distribution of $\{\beta_j\}$ is that in the limit the distribution of intersection angles θ_p, $p \in \alpha\#\beta_j$, in $(0, \pi)$ is symmetric about $\pi/2$. Applying the formula $t_\alpha l_\beta = \sum_{p \in \alpha\#\beta} \cos\theta_p$, the conclusion follows since $\cos\theta$ is odd relative to the symmetry.

The premier question is to relate Thurston's metric to the classical metrics on T_g. Our main result is quite simple: Thurston's metric is the Weil-Petersson metric (see Corollary 4.3)

$$g_{\mathrm{WP}}(t_\alpha, t_\gamma) = 3\pi(g-1) \lim_j \frac{1}{l_{\beta_j}} t_\alpha t_\gamma l_{\beta_j}. \tag{0.1}$$

We wish to sketch a heuristic argument for this result. The tangent space at $R \in T_g$ is (complex conjugate) isomorphic to $Q(R)$, the space of holomorphic quadratic differentials on R; Thurston's metric gives an inner product on $Q(R)$. By the Riemann Roch Theorem, points $p_j \in R$ and local uniformizers

z_j, $z_j(p_j) = 0$, $j = 1, \cdots, 3g-3$, may be chosen such that the map $\varphi \to (\varphi(z_j(p_j)))_{j=1}^{3g-3}$ of $Q(R)$ to $\mathbf{C}^{3g-3}$ is an isomorphism. Thus $\langle\ ,\ \rangle_T$ can be viewed as a quadratic form on the vector $(\varphi(z_j(p_j)))_{j=1}^{3g-3}$. Now we also recall that point evaluation of holomorphic functions can be given by integration with a continuous kernel. In summary there exist quantities K_1 and K_2 such that Thurston's metric is given as $\langle \varphi, \psi \rangle_T = \int_R \varphi\bar{\psi}K_1 + \varphi\psi K_2$; by invariance considerations K_1 is a $(-1,-1)$ tensor and K_2 a $(-3,1)$ tensor relative to the R complex structure. Now Thurston's metric is constructed naturally relative to the PSL(2; $\mathbf{R}$) geometry on the unit tangent bundle $T_1(R)$. And thus by a general principle (see §3.3) we expect K_1 and K_2 (as functions on $T_1(R)$) to be PSL(2; $\mathbf{R}$) invariant. The only possibility is that K_1 is a multiple of the reciprocal of the hyperbolic area element and that K_2 is trivial. Finally if K is indeed the reciprocal of the hyperbolic area element, then $\langle \varphi, \psi \rangle = \int_R \varphi\bar{\psi}K$ is the Weil-Petersson metric.

In addition to formalizing the above discussion we consider applications. The first is the evaluation of the limit $\lim(1/l_\beta)t_\alpha t_\gamma l_\beta$ as an integral over $T_1(R)$. The result is a formula showing that the limit is completely determined by the sequence of lengths of minimal geodesic arcs connecting α to γ. In particular if α and γ are disjoint, then $g_{\mathrm{WP}}(t_\alpha, t_\gamma)$ is positive. This result can also be applied to the geodesic length functions since the gradient of l_α is simply $-Jt_\alpha$, J the complex structure of T_g.

Teichmüller space considered as a real analytic manifold is isomorphic to $\mathscr{R} = \mathrm{Hom}_{\mathrm{df}}(\pi_1(M), \mathrm{PSL}(2;\mathbf{R}))/\mathrm{PSL}(2;\mathbf{R})$, the space of conjugacy classes of discrete faithful PSL(2; $\mathbf{R}$) representations of the fundamental group of a surface M. An open question has been to describe the complex structure of T_g directly as a structure on the representation space $\mathscr{R}$. We now present a solution to this problem in terms of the earthquake fields t_*. Let J be the T_g complex structure considered as an endomorphism of the tangent bundle. Then we find that at $R \in T_g$ (see formula (5.3))

$$Jt_\alpha = 3\pi(g-1)\lim_j \frac{1}{l_{\beta_j}}\left[t_\alpha, t_{\beta_j}\right], \tag{0.2}$$

where $\{\beta_j\}$ is an R uniformly distributed sequence of closed geodesics. Note that the right-hand side is computed from hyperbolic trigonometry.

The organization of the paper is as follows. §1 is a review of the geometry of the unit tangent bundle $T_1(R)$ of a compact surface R and the description of the canonical map $T_1(R) \to T_1(S)$ associated to the homotopy class $[f]$ of a map f: $R \to S$. A review of Teichmüller theory is given in §2. In §3 we show that an alternate description of Thurston's metric as an integral over $T_1(R)$ is indeed the Weil-Petersson metric. Our proof involves an integral formula for the second variation of the length of a closed geodesic. Unfortunately the formula is in terms of the Hilbert transform and thus manipulation of the integral presents technical difficulties. §4 is devoted to showing that the two characterizations for Thurston's metric, the integral over $T_1(R)$ and the limit over uniformly distributed geodesics, actually coincide. And finally §5 is devoted to applications.

I would like to thank Bill Thurston for suggesting the original question and for his continued encouragement.

1. Geometry of the unit tangent bundle of the hyperbolic plane

1.1. We start with a review of the PSL(2; $\mathbf{R}$) geometry of the unit tangent bundle $T_1(H)$ of the hyperbolic plane. We shall be considering the volume form, the infinitesimal generator of the geodesic flow, the circle at infinity and the natural measure on the Möbius band at infinity. An important observation is that associated to the homotopy class $[f]$ of a map f of compact surfaces R, S with hyperbolic metric is a canonically defined map $F(f)$ of unit tangent bundles $T_1(R)$ and $T_1(S)$. Thurston's metric as we shall see in §3 is a measure of the deviation of $F(f)$: $T_1(R) \to T_1(S)$ from an isomorphism of the PSL(2; $\mathbf{R}$) geometries. We start with a sketch of the construction of the canonical map.

1.2. The unit tangent bundle of a manifold is formally the collection of all unit tangent vectors. If we take the upper half-plane $H \subset \mathbf{R}^2$ with coordinate $z = x + iy$ as a model for the hyperbolic plane, then we may introduce coordinates on $T_1(H)$ as follows. To a unit tangent vector v at $z \in H$ assign the pair (z, θ), where θ is the angle (measured in the ccw sense) formed by v and the positive real axis. An alternate desciption of coordinates may be given relative to the *circle at infinity* $S^1_\infty = \mathbf{R} \cup \{\infty\}$. First recall that the set of oriented complete geodesics in H is naturally parametrized by DMB $= S^1_\infty \times S^1_\infty$-{diagonal}, the double cover of the Mobiüs band at ∞. To a pair $(\alpha, \beta) \in$ DMB, associate the complete geodesic $\widehat{\alpha\beta}$ with initial point $\alpha \in S^1_\infty$ and terminal point $\beta \in S^1_\infty$. Given a third point γ lying in the right-hand component of $S^1_\infty - \{\alpha, \beta\}$, then there is a unique point z on $\widehat{\alpha\beta}$ such that the perpendicular ray (on the right-hand side) converges to $\gamma \in S^1_\infty$. Now if we associate to $(\alpha, \beta, \gamma) \in$ CTB $= (S^1_\infty \times S^1_\infty$-{diagonal}$) \times$ r.h.s.(S^1_∞) the tangent vector v of $\widehat{\alpha\beta}$ at z, then with this construction we have a second coordinate description of $T_1(H)$. In the following paragraphs we shall use the notation CTB for the above coordinate description of the unit tangent bundle.

The group PSL(2; **R**) acts simply transitively on CTB $\approx T_1(H)$ by the rule $h(\alpha,\beta,\gamma) = (h(\alpha), h(\beta), h(\gamma))$, $h \in$ PSL(2; **R**) with PSL(2; **R**) acting by fractional linear transformations on each variable. In fact it is common to make the identification $T_1(H) \approx$ PSL(2; **R**) but we shall not do this. We are interested in the PSL(2; **R**) geometry of $T_1(H)$, especially the geodesic flow. But first to have a heuristic understanding of PSL(2; **R**) invariant tensors we recall a classical transformation law. Consider then $z \in \mathbf{C}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in$ SL(2; **C**) and for $\omega = (az+b)/(cz+d)$ recall that $d\omega = dz/(cz+d)^2$ and let us *formally* write $d\omega^{1/2} = dz^{1/2}/(cz+d)$. The reader may verify that the (formal) tensor $dz^{1/2}d\omega^{1/2}/(z-\omega)$ on $\mathbf{C}\times\mathbf{C}$ is SL(2; **C**) invariant. The tensors we shall encounter can all be written as products of the elementary tensors of the above type.

As the first example consider the infinitesimal generator of the geodesic flow on $T_1(H)$, the vector field

$$\dot{g} = \frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\beta)}\frac{\partial}{\partial\gamma};$$

in particular the flow of a triple (α,β,γ) fixes α, β while γ moves towards β. The element of hyperbolic arc length along a trajectory of geodesic flow is given by the flow invariant 1-form $dl = ((\alpha-\beta)/(\alpha-\gamma)(\beta-\gamma))\,d\gamma$. The vector field and 1-form dl are PSL(2; **R**) invariant; in particular there is a well-defined notion of length along a trajectory of the flow. Specifically the 1-form dl can be integrated to obtain the displacement from $(\alpha,\beta,\gamma) \in$ CTB to $(\alpha,\beta,\gamma) \in$ CTB. The displacement is simply $\log(\alpha,\beta,\gamma,\delta)$, where $(\alpha,\beta,\gamma,\delta) = (\alpha-\delta)(\beta-\gamma)/(\alpha-\gamma)(\beta-\delta)$ is the cross ratio.

There is also a natural flow invariant 2-form $\omega = (d\alpha \wedge d\beta)/(\alpha-\beta)^2$ which assigns a nonzero area to 2-planes of $T_1(H)$ transverse to geodesic flow. And finally there is the flow invariant volume from

$$dV = \frac{d\alpha \wedge d\beta \wedge d\gamma}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)}$$

of $T_1(H)$. The forms ω and dV are PSL(2; **R**) invariant and are related by the following identities

$$dV = \omega \wedge dl, \qquad dV(\dot{g},\,,\,) = \omega.$$

The change of variables $(z,\theta) \to (\alpha,\beta,\gamma)$ carries $-dV$ into the volume form $(dx \wedge dy \wedge d\theta)/2y^2$. We shall be interested in discrete subgroups $\Gamma \subset$ PSL(2; **R**), torsion free, such that H/Γ is a compact Riemann surface with hyperbolic metric. The hyperbolic metric as well as the above tensors on $T_1(H)$

are PSL(2; **R**), hence Γ invariant, and thus project to the quotient $T_1(H)/\Gamma \approx T_1(H/\Gamma)$.

1.3. Now we wish to study triples $(R, S, [f])$, where R and S are compact surfaces with hyperbolic metric and $[f]$ is the homotopy class of a homeomorphism $f\colon R \to S$. The associated canonical map $F(f)\colon T_1(R) \to T_1(S)$ is a homeomorphism which carries the trajectories of the R-geodesic flow monotonically to the trajectories of the S-geodesic flow.

Start by fixing uniformizations $\pi_R\colon H \to R$, $\pi_S\colon H \to S$ and lifting f to $\tilde{f}\colon H \to H$ such that the following diagram is commutative:

$$\begin{array}{ccc} H & \xrightarrow{\tilde{f}} & H \\ \downarrow{\scriptstyle \pi_R} & & \downarrow{\scriptstyle \pi_S} \\ R & \xrightarrow{f} & S \end{array}$$

Now it is a basic fact of hyperbolic geometry that $\tilde{f}$ has an extension $\tilde{\tilde{f}}$ to a homeomorphism of the pair $\tilde{\tilde{f}}\colon (H, S^1_\infty) \to (H, S^1_\infty)$. Classically this corresponds to the result that a quasiconformal homeomorphism of H (or the disc) may be extended to a homeomorphism of the closure. The restriction $\tilde{\tilde{f}}|_{S^1_\infty}$ depends only on the homotopy class $[f]$ of the map f. In particular since R is compact, a homotopy of f is bounded in the hyperbolic metric and thus the extension and restriction to S^1_∞ of its lift will be constant in the homotopy variable. We shall abuse our notation and write $[f]\colon S^1_\infty \to S^1_\infty$ for the homeomorphism $\tilde{\tilde{f}}|_{S^1_\infty}$ of the circles at infinity. Since $[f]$ is orientation preserving there is an induced homeomorphism (same notation) $[f]\colon \mathrm{CTB} \to \mathrm{CTB}$ defined by $[f]$ acting on the components of a triple. If we write $R = H/\Gamma_R$ and $S = H/\Gamma_S$ for the appropriate subgroups of PSL(2; **R**), then for each $g \in \Gamma_R$ there exists an $h(= \tilde{f} \circ g \circ \tilde{f}^{-1})$ in Γ_S with $\tilde{f} \circ g = h \circ \tilde{f}$. It follows readily that the self-map $[f]$ of $T_1(H)$ is $\Gamma_R - \Gamma_S$ equivariant and therefore projects to a homeomorphism (same notation) $[f]\colon T_1(R) \to T_1(S)$. A simple argument will show that $[f]$ is independent of the choice of coverings π_R and π_S. With the above conventions the map is described simply as follows.

$$(\alpha, \beta, \gamma) \bmod \Gamma_R \xrightarrow{[f]} ([f](\alpha), [f](\beta), [f](\gamma)) \bmod \Gamma_S.$$

Finally recall that geodesic flow is given (locally) by the rule: α, β are fixed, γ flows towards β. An immediate consequence is that $[f]$ carries the trajectories of the R-geodesic flow monotonically to the trajectories of the S-geodesic flow. As a reminder to the reader we point out that if $[f]$ preserves distance along the trajectories (or for just the closed trajectories) then the homotopy class $[f]$ contains an isometry. By contrast the general $[f]$ is not absolutely continuous along trajectories [3].

2. A review of Teichmüller theory and the Weil-Petersson metric

2.1. For the sake of clarification we shall give a sketch of the necessary background material. The first item is a description of the construction of a holomorphic coordinate chart for Teichmüller space. In the following section we shall consider the first and second variation relative to a quasiconformal map of the length of a trajectory segment of the geodesic flow. The convergence considerations require some care and are based on the theorem in 2.3. Finally in 2.4 we introduce the Weil-Petersson metric and the harmonic Beltrami differentials.

2.2. The motivation for Teichmüller theory is the study of the variation of invariants of a hyperbolic metric. Suppose $\Gamma(S)$ is an invariant of the hyperbolic geometry of a surface S and furthermore that S_t is a 1-parameter family of hyperbolic metrics. The goal is to develop procedures for calculating the t derivatives of $\Gamma(S_t)$. In order to do this start with the uniformization of S, $S = H/\Gamma$, $\Gamma \subset \mathrm{PSL}(2;\mathbf{R})$, and consider a 1-parameter family f_t of homeomorphisms of H which conjugate Γ into a 1-parameter family of subgroups $\Gamma_t = f_t\Gamma f_t^{-1} \subset \mathrm{PSL}(2;\mathbf{R})$. A 1-parameter family of hyperbolic metrics is given by considering the quotients H/Γ_t. Teichmüller space, itself a complex manifold, can be considered as the space of hyperbolic metrics on a compact surface. Below we recall the definition of the complex structure of Teichmüller space and the description of the local coordinate charts, an essential point for calculations.

Formally we start with $\Gamma \subset \mathrm{PSL}(2;\mathbf{R})$ a torsion free subgroup with H/Γ compact. Denote by B (resp. $B(\Gamma)$) the complex Banach space of (resp. Γ invariant) tensors of type $\partial/\partial z \otimes d\bar{z}$ with measurable coefficients and finite L^∞ norm. Denote by $Q(\Gamma)$ the complex Banach space of Γ invariant holomorphic tensors of type $dz \otimes dz$ with finite L^1 norm. To $\mu \in B(\Gamma)$, a Beltrami differential, and $\varphi \in Q(\Gamma)$, a holomorphic quadratic differential, associate the pairing $(\mu, \varphi) = \int_{H/\Gamma} \mu\varphi$. By Riemann-Roch, $Q(\Gamma)$ has dimension $3g - 3$ where $g \geqslant 2$ is the genus of H/Γ and for $N(\Gamma) = Q(\Gamma)^{\perp} \subset B(\Gamma)$, the $Q(\Gamma)$ null space, $B(\Gamma)/N(\Gamma)$ and $Q(\Gamma)$ are dual. In fact the reader should consult [1] for a formal definition of the Teichmüller space T_g and will find that the diagram

$$\begin{array}{ccc} T^{1,0}T_g \times (T^{1,0})^*T_g & & \\ \updownarrow \qquad\qquad \updownarrow & \searrow & \\ & & C \\ & \nearrow & \\ B(\Gamma)/N(\Gamma) \times Q(\Gamma) & & \end{array} \tag{2.1}$$

characterizes the complex structure of T_g, where $T^{1,0}$ denotes the holomorphic tangent space which is naturally paired with $(T^{1,0})^*$, its dual.

We shall now describe an explicit map Φ: $B(\Gamma) \to T_g$ which is analogous to an exponential map and whose differential $d\Phi|_0$: $B(\Gamma)/N(\Gamma) \approx T^{1,0}T_g$ is the complex isomorphism of (2.1). In brief Φ is given by solving the Beltrami equation for $\mu \in B(\Gamma)$, $\|\mu\|_\infty < 1$,

$$\begin{cases} f\colon H \to H \quad \text{a homeomorphism fixing } 0,1 \text{ and } \infty, \\ f_{\bar z} = \mu f_z. \end{cases} \tag{2.2}$$

Denote the solution of (2.2), a quasiconformal map, by f^μ and for $\mu \in B(\Gamma)$ define

$$\Phi(\mu) = [H/\Gamma^\mu] \in T_g$$

for $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$, where $[H/\Gamma^\mu]$ denotes the point of T_g determined by the map f^μ: $H/\Gamma \to H/\Gamma^\mu$. That $f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ is actually a subgroup of $\mathrm{PSL}(2;\mathbf{R})$ is a basic fact which can be found in the references. The kernel $N(\Gamma) \subset B(\Gamma)$ and hence the quotient $B(\Gamma)/N(\Gamma)$ are complex Banach spaces. By definition of the complex structure on T_g the differential $d\Phi|_0$: $B/N \to T^{1,0}T_g$ is complex linear.

In order to describe local coordinates on T_g we first choose $\mu_1,\cdots,\mu_n \in B(\Gamma)$ ($n = 3g-3$) whose $N(\Gamma)$ cosets form a complex basis for $B(\Gamma)/N(\Gamma)$. And given $\mu \in B(\Gamma)$, $\|\mu\|_\infty < 1$, define

$$L^\mu \nu = \left(\frac{\nu}{1-|\mu|^2}\frac{f_z^\mu}{\overline{f_z^\mu}}\right)\circ (f^\mu)^{-1}. \tag{2.3}$$

Now we shall describe a holomorphic coordinate chart $\tilde{\Phi}$ mapping a neighborhood U of the origin in $\mathbf{C}^n$ to an open set in T_g [1], [2]. First pick U sufficiently small to ensure that for $t = (t_1,\cdots,t_n) \in U$, $\|\mu(t)\|_\infty < 1$ where $\mu(t) = \sum_{j=1}^n t_j\mu_j$. A coordinate mapping $\tilde{\Phi}$: $U \to T_g$ is given simply by $\tilde{\Phi}(t) = [H/\Gamma^{\mu(t)}]$, that is the tuple t is mapped to the equivalence class of $H/\Gamma^{\mu(t)}$. An essential point for calculations is to have a description of the holomorphic coordinate vector fields for the chart $\tilde{\Phi}$. The coordinate fields are given as

$$\left.\frac{\partial}{\partial t_i}\right|_t = \left(L^{\mu(t)}\mu_i\right) \bmod N(\Gamma^{\mu(t)}) \in B(\Gamma^{\mu(t)})/N(\Gamma^{\mu(t)})$$

for $t \in U$. The description of $\tilde{\Phi}$ and the holomorphic coordinate vector fields $\partial/\partial t_i|_t$, $t \in U$, will serve as the basis for our calculations.

2.3. Our considerations will focus on the t dependence of a 1-parameter family $f^{\mu(t)}$ of quasiconformal maps. The discussion of §3 will be divided into two parts: the formal calculation of the appropriate t-derivatives and an analysis of the convergence. The first part of the calculation, since it is formal, can be effected for any model of the hyperbolic plane, in particular the half-plane H can be replaced by the disc D. The advantage of the latter is that the theorem in the t-dependence of $f^{\mu(t)}$ is most readily stated for the disc. The following theorem will suffice for our convergence considerations.

Fix $p > 4$ and consider the Banach space B_p of continuous functions f on D possessing distributional derivatives f_z and $f_{\bar{z}}$ and having finite B_p norm

$$\|f\|_{B_p} \underset{\text{def}}{=} \sup_{z_1, z_2 \in D} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^{1-2/p}} + \|f_z\|_p + \|f_{\bar{z}}\|_p, \tag{2.4}$$

where $\| \ \|_p$ is the standard L^p norm. In the case of the disc D, f^μ is to be interpreted as the unique homeomorphism of D fixing ± 1 and i and satisfying the Beltrami equation $f_{\bar{z}} = \mu f_z$. Recall that a map of an open set $U \subset \mathbf{R}^n$ to a Banach space B is real analytic if it admits at each point of U a B convergent power series expansion.

Theorem 2.1 [4]. *There exists an $\varepsilon = \varepsilon(p)$, $\varepsilon > 0$, such that if $\mu(t)$ varies real analytically in L^∞ with $\|\mu(t)\|_\infty < \varepsilon$, then $f^{\mu(t)}$ varies real analytically in B_p.*

2.4. One method for considering the quotient space $B(\Gamma)/N(\Gamma)$ is to choose a representative from each coset. An example of such a choice is given by the harmonic Beltrami differentials. If $HB(\Gamma)$ is the subspace of $B(\Gamma)$ of harmonic Beltrami differentials, then the natural map i: $HB(\Gamma) \to B(\Gamma)/N(\Gamma)$ induced by the inclusion $HB(\Gamma) \to B(\Gamma)$ is a complex linear isomorphism. Consequently $HB(\Gamma)$ provides an alternate model for the tangent space at $[H/\Gamma]$ of T_g. The Weil-Petersson metric is easily described on $HB(\Gamma)$ and furthermore there is a natural projection operator P: $B(\Gamma) \to HB(\Gamma)$ which induces the inverse of the isomorphism i: $HB(\Gamma) \to B(\Gamma)/N(\Gamma)$. We start by discussing the harmonic Beltrami differentials and the projection P.

A Beltrami differential $\mu \in B(\Gamma)$ given on H is harmonic provided there exists a $\varphi \in Q(\Gamma)$ such that $\mu = (z - \bar{z})^2 \overline{\varphi(z)}$. The essential properties of the harmonic Beltrami differentials are simple consequences of the existence of the projection operator P: $B(\Gamma) \to HB(\Gamma)$ defined by

$$P[\mu] = \frac{12(\operatorname{Im} z)^2}{\pi} \int_H \frac{\mu(\zeta)}{(\zeta - \bar{z})^4} \, d\sigma(\zeta)$$

for $\mu \in B(\Gamma)$ and $d\sigma$ the Euclidean area element [2]. With the hypothesis that H/Γ is compact the projection property $P^2 = P$ provides that $N(\Gamma) = \operatorname{Ker} P$ and from this it follows that the inclusion $HB(\Gamma) \to B(\Gamma)$ induces an isomorphism i: $HB(\Gamma) \to B(\Gamma)/N(\Gamma)$.

The Weil-Petersson Hermitian pairing on $HB(\Gamma)$ is given simply as

$$\langle \mu, \nu \rangle = \int_{H/\Gamma} \mu \bar{\nu}\, dA$$

for $\mu, \nu \in HB(\Gamma)$ and dA the area element of the hyperbolic metric [1], [2]. The metric can be given in a more general form by substituting the projection P into the above integral: for μ, ν (arbitrary) representatives of cosets $B(\Gamma)/N(\Gamma)$, $\langle \mu, \nu \rangle = \int_{H/\Gamma} \mu \overline{P[\nu]}\, dA$. In fact P is selfadjoint with respect to the inner product and thus a consequence of $N(\Gamma) = \operatorname{Ker} P$ is that the pairing does not depend on the choice of coset representatives. The Weil-Petersson Riemannian metric is given by the symmetric 2-tensor $g_{\mathrm{WP}}(\mu, \nu) = 2 \operatorname{Re}\langle \mu, \nu \rangle$, $\mu, \nu \in HB(\Gamma)$. For the sake of background we recall that the metric is not complete [12], has negative sectional curvature [16] and that T_g is Weil-Petersson convex in the sense that any two points are joined by a unique geodesic [17].

3. The R-average of the S-length of a geodesic

3.1. An alternate approach to Teichmüller space is to consider equivalence classes of triples $(R, S, [f])$, where $[f]$ is the homotopy class of a homeomorphism f: $R \to S$ of Riemann surfaces and triples $(R, S_1, [f_1])$, $(R, S_2, [f_2])$ are equivalent if there exists a conformal homeomorphism h such that $h \circ f_1$ is homotopic to f_2. The description of Teichmüller space by triples shall be used in the following sections. An example is given by the discussion in §1 of the canonical homeomorphism $[f]$: $T_1(R) \to T_1(S)$ of unit tangent bundles. A second example is given by the following definition of $A([R],[S])$, the R-average of the S-length of a geodesic.

First recall the definition of the PSL(2; **R**) invariant 2-form ω and 1-form dl on $T_1(H)$; we denote with subscripts R or S the projections of either form to $T_1(R)$ or $T_1(S)$. Thurston's first construction of the Riemannian metric for T_g starts with the following integral:

$$A([R],[S]) = \int_{T_1(R)} \omega_R \wedge [f]^* dl_S \tag{3.1}$$

(the present definition of A is strictly heuristic; a formal definition will be given below). $A([R],[R]) = 4\pi^2(g-1)$ is simply the volume of $T_1(R)$. For $[R]$ fixed, $A([R],[S])$ as a function of $[S]$ is defined on T_g. Thurston observed that $\dot{A}([R],[R]) = 0$ where $\dot{}$ indicates the first derivative of $A([R],[S])$ if $[S]$ varies in a 1-parameter family with initial value $[R]$. Now recall that if the differential of a function vanishes at a point, then the second derivative defines a symmetric 2-tensor at that point. Using earthquakes and the convexity of the geodesic length functions Thurston shows that $\ddot{A}([R],[R])$ is a positive definite 2-tensor. This is Thurston's metric tensor at $[R] \in T_g$.

We shall now apply the techniques of §2 to compute the variations $\dot{A}$ and $\ddot{A}$ directly. Our current approach is independent of the connection between A and the earthquake deformation and is an application of the general techniques for computing the variation of a hyperbolic structure. We start with the result $\dot{A}([R],[R]) = 0$ and then calculate that $\ddot{A}([R],[R]) = 2\pi g_{\mathrm{WP}}/3$, where g_{WP} is the Weil-Petersson metric tensor.

3.2. The first matter is the definition of A. The canonical map $[f]$: $T_1(R) \to T_1(S)$ is typically not absolutely continuous and thus the pullback $[f]^*dl_S$ cannot be defined in the usual way. An important property of $[f]$ is that it carries the trajectories of the R-geodesic flow monotonically to trajectories of the S-geodesic flow. Thus $[f]^*dl_S$ when restricted to a trajectory of the R-geodesic flow can be interpreted as the derivative in the sense of measures of a monotone function. This interpretation is compatible with the exterior operations required for $\omega_R \wedge [f]^*dl_S$. To define the integral of $\omega_R \wedge [f]^*dl_S$ we use the monotonicity of $[f]$ and the invariance of ω. First integrate along the trajectories of the geodesic flow and then over the space of trajectories. Specifically represent $T_1(R)$ (resp. $T_1(S)$) as a quotient $T_1(H)/\Gamma_R$ (resp. $T_1(H)/\Gamma_S$) and then an open set $U \subset T_1(R)$ can be lifted to $\tilde{U} \subset T_1(H) \approx$ CTB. Letting p be the projection p: $T_1(H) \to$ DMB onto the first two factors we define

$$\int_U \omega_R \wedge [f]^*dl_S = \int_{x \in p(\tilde{U})} \tilde{\omega}(x) L(x),$$

where for $x \in p(\tilde{U})$, $L(x)$ is the dl_S length of the image by $[f]$: $T_1(R) \to T_1(S)$ of the trajectory segment $p^{-1}(x) \in \tilde{U}$ and $\tilde{\omega}$ is simply the 2-form $(d\alpha \wedge d\beta)/(\alpha - \beta)^2$ on DMB. The reader may check that the Γ invariance of $\tilde{\omega}$ and dl_S taken together with the Γ_R, Γ_s equivariance of $[f]$ guarantee that the integral is indeed independent of the choice of universal coverings of R and S and of the lift of U. Furthermore the properties of $\tilde{\omega}$ and L guarantee that the integral of $\tilde{\omega}L$ is independent of the choice of fundamental domain.

3.3. Now we shall review the formula for the first variation of a quasiconformal map. This formula will serve as the basis for our calculations, in particular the first application is the formula for the first variation of the cross ratio of points on the circle at infinity.

We start with a Beltrami differential $\mu \in B$, $\|\mu\|_\infty < 1$, and recall f^μ the normalized solution of the Beltrami equation. By theorem 2.1 $f^{\varepsilon\mu}$ admits an expansion in ε for ε small. The standard result is that if $f^{\varepsilon\mu}(z) = z + \varepsilon\dot{f}[\mu] + O(\varepsilon^2)$, $z \in \mathbf{C}$ (Theorem 2.1 describes the convergence), then

$$\dot{f}[\mu](z) = -\frac{1}{\pi}\int_H \mu(\zeta)P(\zeta, z) + \overline{\mu(\zeta)}\,P(\bar{\zeta}, z)\,d\sigma(\zeta), \tag{3.2}$$

where $P(\zeta, z) = 1/(\zeta - z) + (z - 1)/\zeta - z/(\zeta - 1)$ and $d\sigma(\zeta)$ is the Euclidean area form [1], [2]. An application is given by considering the first variation of the cross ratio $(\alpha, \beta, \gamma, \delta) = (\alpha - \delta)(\alpha - \gamma)/(\alpha - \gamma)(\beta - \delta)$, $\alpha, \beta, \gamma, \delta \in S^1_\infty$. It is shown in [14] that for $\mu \in B$,

$$\begin{aligned}&\frac{d}{d\varepsilon}(f^{\varepsilon\mu}(\alpha), f^{\varepsilon\mu}(\beta), f^{\varepsilon\mu}(\gamma), f^{\varepsilon\mu}(\delta))\\ &\qquad = -\frac{2}{\pi}(\alpha, \beta, \gamma, \delta)\int_H \mu(\zeta)K(\zeta, \alpha, \beta, \gamma, \delta),\end{aligned} \tag{3.3}$$

where $K = (\alpha - \beta)(\delta - \gamma)/(\zeta - \alpha)(\zeta - \beta)(\zeta - \gamma)(\zeta - \delta)$. In order to clarify future transformation calculations it is important that we consider K to be a tensor of type $d\zeta \otimes d\zeta$, i.e. a quadratic differential. In particular, by the remark of §1.2, K satisfies the following transformation law:

$$K(h(\zeta), h(\alpha), h(\beta), h(\gamma), h(\delta))h'(\zeta)^2 = K(\zeta, \alpha, \beta, \gamma, \delta)$$

for $h \in \mathrm{PSL}(2; \mathbf{R})$. In fact by a general principle, which we shall now sketch, the first variation of a $\mathrm{PSL}(2; \mathbf{R})$ invariant tensor is given by a $\mathrm{PSL}(2; \mathbf{R})$ invariant tensor. First we recall that the normalized solution f^μ, $\mu \in B$, $\|\mu\|_\infty < 1$, of the Beltrami equation is unique. For $h \in \mathrm{PSL}(2; \mathbf{R})$ and $\mu \in B$ we write $h^*\mu$ for the expression $\mu(h)\overline{h'}/h'$ (h' is the complex derivative).

Lemma 3.1 [4]. *Given* $\mu \in B$, $\|\mu\|_\infty < 1$ *and* $h \in \mathrm{PSL}(2; \mathbf{R})$, *then there exists* $\hat{h} \in \mathrm{PSL}(2; \mathbf{R})$ *such that* $f^{h^*\mu} = \hat{h} \circ f^\mu \circ h$.

Proof. The homeomorphism $f^{h^*\mu}$ and $f^\mu \circ h$ satisfy the same Beltrami equation and hence only differ in their normalization. The transformation $\hat{h}$ is uniquely determined by the requirement that $\hat{h} \circ f^\mu \circ h$ fix 0, 1 and ∞.

Now consider L a tensor depending on the variables $\alpha_1, \cdots, \alpha_n$ and invariant under the diagonal $\mathrm{PSL}(2; \mathbf{R})$ action. By the above lemma for $\nu = h^*\mu$, $L(f^\nu(\alpha_j)) = L(f^\mu(h(\alpha_j)))$ for $h \in \mathrm{PSL}(2; \mathbf{R})$ arbitrary. Hence applying h to μ is equivalent to applying h to $(\alpha_1, \cdots, \alpha_n)$. Now proceeding formally the first variation in μ of L is a linear functional of μ depending on the vector $(\alpha_1, \cdots, \alpha_n)$. By the above, $\mathcal{L}(h^*\mu, \alpha_j) = \mathcal{L}(\mu, h(\alpha_j))$ and if we express $\mathcal{L}$ by an absolutely convergent integral $\mathcal{L} = \int_S H\mu\hat{\mathcal{L}}$, the PSL(2; **R**) invariance of $\hat{\mathcal{L}}$ is a simple consequence.

3.4. Now we are ready to consider the first variation of $A([R],[S])$. Fix $\mu \in B(\Gamma)$, $\Gamma = \Gamma_R$, $\tilde{\Delta}$ a compact fundamental domain for Γ acting on $T_1(H)$ and Δ a compact fundamental domain for Γ acting on H. By Theorem 2.1 the cross ratio $(f^{\varepsilon\mu}(\alpha), f^{\varepsilon\mu}(\beta), f^{\varepsilon\mu}(\gamma), f^{\varepsilon\mu}(\delta))$ is real analytic in ε, ε small, and Hölder continuous in $\alpha, \beta, \gamma, \delta \in S^1_\infty$. An easy argument shows that we may differentiate under the integral to obtain

$$\dot{A}([R],[R]) = -\frac{2}{\pi}\operatorname{Re}\int_{\tilde{\Delta}} \tilde{\omega} \int_H \mu(\zeta) K(\zeta, \alpha, \beta, \gamma, \delta),$$

where indicates the derivative of $[S]$ in the μ *direction*. The kernel K considered as a function of ζ lies in $L^1(H)$ and varies continuously in $L^1(H)$ as a function of $\alpha, \beta, \gamma, \delta \in S^1_\infty$ (all distinct). Consequently $\int_H |\mu K|$ is a majorant continuous in $\alpha, \beta, \gamma, \delta$ for the first integral. This is sufficient to justify the following formal manipulations (a standard argument): start by writing $H = \bigcup_{h \in \Gamma} h(\Delta)$ and thus

$$\int_{\tilde{\Delta}} \tilde{\omega} \int_H \mu K = \int_{\tilde{\Delta}} \tilde{\omega} \sum_{h \in \Gamma} \int_{h(\Delta)} \mu K = \int_{\tilde{\Delta}} \tilde{\omega} \sum_{h \in \Gamma} \int_{\Delta} h^*\mu K(h(\zeta), \alpha, \beta, \gamma, \delta) h'(\zeta)^2;$$

now note that μ is Γ invariant, K is invariant under the diagonal PSL(2; **R**), hence Γ action as is $\tilde{\omega}$:

$$\sum_{\substack{k \in \Gamma \\ k = h^{-1}}} \int_{\tilde{\Delta}} k^*\tilde{\omega} \int_{\Delta} \mu K(\zeta, k(\alpha), k(\beta), k(\gamma), k(\delta)) = \sum_{k \in \Gamma} \int_{k(\tilde{\Delta})} \tilde{\omega} \int_{\Delta} \mu K$$

$$= \int_{T_1(H)} \tilde{\omega} \int_{\Delta} \mu K.$$

The argument also provides that the last integral is absolutely convergent. Interchanging order of integration we are left to consider $\int_{T_1(H)} \tilde{\omega} K$. The integral along the trajectories of the geodesic flow is given simply by forming the limit

$$\lim_{\substack{\gamma \to \alpha \\ \delta \to \beta}} K(\zeta, \alpha, \beta, \gamma, \delta) = \lim_{\substack{\gamma \to \alpha \\ \delta \to \beta}} \frac{(\alpha - \beta)(\delta - \gamma)}{(\zeta - \alpha)(\zeta - \beta)(\zeta - \gamma)(\zeta - \delta)}$$

$$= \frac{-(\alpha - \beta)^2}{(\zeta - \alpha)^2(\zeta - \beta)^2}.$$

Finally we are left to consider the integral of

$$\omega_1 = (-d\alpha \wedge d\beta)/(\zeta - \alpha)^2(\zeta - \beta)^2$$

over DMB. This last integral will be determined by formal considerations. The quantity ω_1 is a tensor of type $d\alpha \wedge d\beta \otimes d\zeta^2$ and as such is invariant under the diagonal action of PSL(2; **R**). It follows that the integral of ω_1 over DMB is a PSL(2; **R**) invariant tensor of type $d\zeta^2$. To determine this tensor consider ζ_0 an arbitrary point of H and $\mathrm{Rot}(\zeta_0) \subset \mathrm{PSL}(2; \mathbf{R})$ the group of rotations with fixed point ζ_0. By considering the action of $\mathrm{Rot}(\zeta_0)$ in the tangent space at ζ_0 we see that a tensor of type $d\zeta^2$ invariant under $\mathrm{Rot}(\zeta_0)$ necessarily vanishes at ζ_0. Consequently the integral of ω_1 over DMB is trivial.

3.5. Now we proceed and compute the second variation in $[S]$ of $A([R],[S])$. The argument is similar to the above although in this case convergence is a more delicate matter. If $[S]$ close to $[R]$ is given in the form $[S] = [f^{\varepsilon\mu}(R)]$, $\mu \in B(\Gamma)$, ε small, then by §2.2 and the first part of §3.4 the ε derivative of A is given as

$$\dot{A}([R],[f^{\varepsilon\mu}(R)]) = -\frac{2}{\pi}\mathrm{Re}\int_{\tilde{\Delta}} \tilde{\omega} \int_H L^{\varepsilon\mu}\mu K, \tag{3.4}$$

where $L^{\varepsilon\mu}\mu$ defined by (2.3) represents the tangent vector at $\varepsilon\mu$ of the 1-parameter family $[f^{\varepsilon\mu}(R)]$. We start by writing out the first integral and making the change of variables $\zeta = f^{\varepsilon\mu}(s)$:

$$\begin{aligned}\int_H & L^{\varepsilon\mu}\mu K \\ &= \int_H \left(\frac{\mu}{1-|\varepsilon\mu|^2}\frac{f^{\varepsilon\mu}{}_z}{\overline{f^{\varepsilon\mu}{}_z}}\right)\circ f^{\varepsilon\mu}(\zeta)^{-1} K(\zeta, f^{\varepsilon\mu}(\alpha), f^{\varepsilon\mu}(\beta), f^{\varepsilon\mu}(\gamma), f^{\varepsilon\mu}(\delta)) \\ &= \int_H \mu(s) K(f^{\varepsilon\mu}(s), f^{\varepsilon\mu}(\alpha), f^{\varepsilon\mu}(\beta), f^{\varepsilon\mu}(\gamma), f^{\varepsilon\mu}(\delta))(f_s^{\varepsilon\mu}(s))^2.\end{aligned}$$

The first term is to verify that we may differentiate under the integrals in (3.4). For this and the remaining convergence questions we shall transform to the unit disc (without further mention). A conformal isomorphism of H to D is an element of PSL(2; **C**) and thus it follows by the remark of §1.2 that $\tilde{\omega}$ and K on D are given by the same formula (note that $\alpha, \beta, \gamma, \delta$ are now complex variables restricted to the unit circle). Composing the tensor K with the map $f^{\varepsilon\mu}$ with the obvious notation gives

$$\begin{aligned}(f^{\varepsilon\mu})^* K & \\ = & \frac{(f^{\varepsilon\mu}(\alpha) - f^{\varepsilon\mu}(\beta))(f^{\varepsilon\mu}(\delta) - f^{\varepsilon\mu}(\gamma))(f_s^{\varepsilon\mu}(s))^2}{(f^{\varepsilon\mu}(s) - f^{\varepsilon\mu}(\alpha))(f^{\varepsilon\mu}(s) - f^{\varepsilon\mu}(\beta))(f^{\varepsilon\mu}(s) - f^{\varepsilon\mu}(\gamma))(f^{\varepsilon\mu}(s) - f^{\varepsilon\mu}(\delta))}.\end{aligned}$$

The factors of $(f^{\varepsilon\mu})^*K$ are of three types as illustrated by: $(f^{\varepsilon\mu}(\alpha) - f^{\varepsilon\mu}(\beta))$, $(f^{\varepsilon\mu}(s) - f^{\varepsilon\mu}(\alpha))$ and $(f_s^{\varepsilon\mu}(s))^2$. Now referring to Theorem 2.1 the ε derivative of $f^{\varepsilon\mu}$ converges in B_p, $p > 4$, in particular the ε derivative of $(f^{\varepsilon\mu}(\alpha) - f^{\varepsilon\mu}(\beta))$ converges in L^∞ and the ε derivative of $(f^{\varepsilon\mu}(s))^2$ converges in $L^{p/2}$, $p/2 > 2$. For the remaining denominator type term write $f^{\varepsilon\mu}(z) = z + \varepsilon f_1(\varepsilon, z)$ and

$$\frac{1}{\varepsilon}\left(\frac{1}{(f^{\varepsilon\mu}(s) - f^{\varepsilon\mu}(\alpha))} - \frac{1}{(s-\alpha)}\right) = -\frac{(f_1(\varepsilon, s) - f_1(\varepsilon, \alpha))}{(s-\alpha)^2}.$$

By definition of the B_p norm, $(f_1(\varepsilon, s) - f_1(\varepsilon, \alpha))/|s - \alpha|^{1-2/p}$ converges uniformly in s and α. Recalling that a standard technique for bounding the integral $\int_\Delta f(s)/(s-\alpha)\, d\sigma$, $f \in L^q$, $q > 2$, is to apply the Hölder inequality; the partial quotients for the ε-derivative are now bounded by combining the above estimates with the Hölder inequality.

Now in order to derive the formula for $d(f^{\varepsilon\mu})^*K/d\varepsilon$ we recall the following formulas:

$$(3.5)\qquad \begin{aligned} K(s,\alpha,\beta,\gamma,\delta) &= \frac{(\alpha-\beta)(\delta-\gamma)}{(s-\alpha)(s-\beta)(s-\gamma)(s-\delta)},\\ \frac{df^{\varepsilon\mu}}{d\varepsilon}(s)\bigg|_{\varepsilon=0} &= -\frac{1}{\pi}\int_H \mu(\zeta)P(\zeta, z) + \overline{\mu(\zeta)}\,P(\bar\zeta, z)\, d\sigma(\zeta), \end{aligned}$$

where $P(\zeta, s) = 1/(\zeta - s) + (s-1)/\zeta - s/(\zeta - 1)$ and the standard result [4] that the s-derivative of $\dot f[\mu]$ is given by differentiating under the integral provided the resulting integral is interpreted as a Cauchy principal value. Now using a $\dot{}$ to denote the first variation with respect to $f^{\varepsilon\mu}$ at $\varepsilon = 0$ we have

$$\dot{(f^{\varepsilon\mu})^*K} = K(s,\alpha,\beta,\gamma,\delta)\left(\frac{(\alpha \dot{-} \beta)}{(\alpha-\beta)} + \frac{(\gamma \dot{-} \delta)}{(\gamma-\delta)} - \frac{(s \dot{-} \alpha)}{(s-\alpha)} - \frac{(s \dot{-} \beta)}{(s-\beta)} - \frac{(s \dot{-} \gamma)}{(s-\gamma)} - \frac{(s \dot{-} \delta)}{(s-\delta)} + 2\left(\dot{f}_s^{\varepsilon}\right)\right),$$

then substituting the formula (3.5) to evaluate the quantity in the brackets,

$$\dot{(f^{\varepsilon\mu})^*K} = -\frac{1}{\pi}K(s,\alpha,\beta,\gamma,\delta)\int_H \mu(\zeta)K_1(\zeta, s,\alpha,\beta,\gamma,\delta) + \overline{\mu(\zeta)}\,K_1(\bar\zeta, s,\alpha,\beta,\gamma,\delta)\, d\sigma(\zeta),$$

where

$$K_1 = \left(\frac{1}{(\zeta-\alpha)(\zeta-\beta)} + \frac{1}{(\zeta-\gamma)(\zeta-\delta)} - \frac{1}{(\zeta-s)(\zeta-\alpha)} \right.$$
$$\left. - \frac{1}{(\zeta-s)(\zeta-\beta)} - \frac{1}{(\zeta-s)(\zeta-\gamma)} - \frac{1}{(\zeta-s)(\zeta-\delta)} + \frac{2}{(\zeta-s)^2} \right)$$

represents a quadratic differential in the variable ζ and the integral is a Cauchy principal limit. Furthermore

$$K_1 = \frac{(s-\alpha)(s-\beta)(\zeta-\gamma)(\zeta-\delta) + (\zeta-\alpha)(\zeta-\beta)(s-\gamma)(s-\delta)}{(\zeta-\alpha)(\zeta-\beta)(\zeta-\gamma)(\zeta-\delta)(\zeta-s)^2}$$

is PSL(2; **R**) invariant under the diagonal action. Combining the above considerations we obtain the formula for the second variation of A

$$\ddot{A}([R],[R]) = \frac{2}{\pi^2}\operatorname{Re}\int_{\tilde{\Delta}} \tilde{\omega} \int_H \mu(s)K(s,\alpha,\beta,\gamma,\delta) \int_H \mu(\zeta)K_1(\zeta,s,\alpha,\beta,\gamma,\delta)$$
$$+ \overline{\mu(\zeta)}\, K_1(\bar{\zeta},s,\alpha,\beta,\gamma,\delta)\, d\sigma(\zeta)\, d\sigma(s).$$

3.6. The technique for evaluating the integral is similar to that of §3.4. First we must analyze the convergence of the integral involving K_1 as well as the absolute convergence of the triple integral over $\tilde{\Delta} \times H \times H$. Given this we apply the same basic argument as before to show that the invariance allows us to replace the $\tilde{\Delta} \times H \times H$ integral with an $T_1(H) \times H \times \Delta$ integral. By the absolute convergence we may reverse the order of integration. Then the $T_1(H)$ integral merely simplifies the integrand, the H integral produces the projection operator P: $B(\Gamma) \to BH(\Gamma)$ of §2.4 and the Δ integral gives the Weil-Petersson pairing.

The first issue is to show that the limit for the Cauchy principal value of (3.6) converges uniformly.

Lemma 3.2. *Assume $\nu \in B(\Gamma)$ is smooth. Then*

$$\lim_{\varepsilon \to 0} \int_{\rho(\zeta,s) \geq \varepsilon} \nu(\zeta) K_1(\zeta,s,\alpha,\beta,\gamma,\delta)\, d\sigma(\zeta)$$

converges uniformly to the Cauchy principal value, where ρ is hyperbolic distance.

Proof. We consider the integral on the unit disc. The first matter is to show that it suffices to consider $s = 0$. By Lemma 3.1, given $h \in \operatorname{Aut}(D) \subset$ PSL(2; **C**), there exists $\hat{h} \in \operatorname{Aut}(D)$ satisfying $f^{\varepsilon\lambda} = \hat{h} \circ f^{\varepsilon\nu} \circ h$, $\lambda = h^*\nu$, hence in the above notation

$$(f^{\varepsilon\lambda})^* K(s,\alpha,\beta,\gamma,\delta) = (\hat{h} \circ f^{\varepsilon\nu} \circ h)^* K(s,\alpha,\beta,\gamma,\delta),$$

where by invariance of K the last term is

$$(f^{\varepsilon\nu})^*K(h(s),h(\alpha),h(\beta),h(\gamma),h(\delta)).$$

Thus by an appropriate choice of h (i.e. $h(s)=0$) we may assume that $s=0$. Now for the case $s=0$ the difference of the integral over annuli $\varepsilon_1 \leqslant |\zeta| \leqslant 1$ and $\varepsilon_2 \leqslant |\zeta| \leqslant 1$ is the integral over $\varepsilon_1 \leqslant |\zeta| \leqslant \varepsilon_2$. Expand the integrand as

$$\nu K = \frac{a_0 + a_1\zeta + a_2\bar{\zeta} + E(\zeta)}{\zeta^2}, \qquad \zeta \in D,$$

and integrate ζ in polar coordinates over the annulus, performing the angle integral first. We are left with the integral

$$\int_{\varepsilon_1 \leqslant |\zeta| \leqslant \varepsilon_2} \frac{E(\zeta)}{\zeta^2}\, d\sigma(\zeta), \quad \text{where } E \text{ is } O(|\zeta|^2),$$

with the constant determined by the second derivatives of $(\zeta^2\nu(\zeta)K_1(\zeta))$ at 0. An inspection shows that the bound is uniform in α, β, γ, and δ, the desired result.

Now the observation that $K(\zeta,\alpha,\beta,\gamma,\delta)$ lies in $L^1(H)$ and varies continuously in $L^1(H)$ as a function of $\alpha,\beta,\gamma,\delta$ is sufficient to bound the s integral. In particular given this and that $(\alpha,\beta,\gamma,\delta)$ varies in a compact set it follows from the above Lemma 3.2 that the limit for the Cauchy principle value can be interchanged with the s integral and the $(\alpha,\beta,\gamma,\delta)$ integral. Accordingly fix $\varepsilon > 0$ and assume that the integrals are over $(s,\zeta) \in H \times H$, $\rho(s,\zeta) \geqslant \varepsilon$. The next matter is the absolute convergence of the ζ integral. The technique is illustrated if we bound $\int \nu(\zeta)K_1(\zeta)\,d\sigma(\zeta)$.

Lemma 3.3. *Given $\nu \in B(\Gamma)$ and K_1 as above*

$$\int_{\rho(s,\zeta)\geqslant\varepsilon} |\nu(\zeta)K_1(\zeta,s,\alpha,\beta,\gamma,\delta)|\,d\sigma(\zeta)$$
$$\leqslant C(\varepsilon)\|\nu\|_\infty \log(4(\alpha,\bar{s},\beta,s)(\gamma,\bar{s},\delta,s)),$$

where $C(\varepsilon)$ depends only on ε.

Proof. As above we may assume $s=0$ and consider

$$\int_{\varepsilon\leqslant|\zeta|\leqslant 1} |\nu(\zeta)K_1(\zeta)|\,d\sigma(\zeta),$$

where

$$K_1(\zeta) = \frac{\alpha\beta}{(\zeta-\alpha)(\zeta-\beta)\zeta^2} + \frac{\gamma\delta}{(\zeta-\gamma)(\zeta-\delta)\zeta^2}.$$

By a standard argument [4] the integral is bounded by

$$C(\varepsilon)\|\nu\|_\infty(\log(2/|\alpha-\beta|)+\log(2/|\gamma-\delta|)).$$

Now, by definition of the cross ratio, $1/|\alpha-\beta| = (\alpha,\infty,\beta,0)$, where 0 and ∞ are inverses in the unit circle; the invariant expression is (α, s^*, β, s), where s^* is the inverse of s in the circle. This is the desired result.

In order to show that the integral (3.6) for $\ddot{A}$ is absolutely bounded it suffices by the above estimate to consider the following

$$\int_{\tilde{\Delta}}\int_D |K(s,\alpha,\beta,\gamma,\delta)|\big(c+\log|(\alpha,\bar{s},\beta,s)(\gamma,\bar{s},\delta,s)|\big)\,d\sigma(s).$$

The discussion of [4] covers a similar integral. It is shown that

$$\int_\Delta\left|\frac{(a-b)f(\zeta)}{(\zeta-a)(\zeta-b)}\right|d\sigma(\zeta)\leqslant c_p|a-b|^{1-2/p}\|f\|_p.$$

A similar estimate can be applied to the above and we find that the integral is bounded provided (γ,δ) is bounded away from (α,β), a condition satisfied by the compact fundamental domain $\tilde{\Delta}\subset T_1(H)$.

At this time we have all the necessary convergence estimates and so we proceed to the formal steps for evaluating the integral. The total integrand $I = I(\zeta,\alpha,\beta,\gamma,\delta)$ is invariant under the diagonal Γ action and since the integral converges absolutely we may proceed as in §3.4

$$\begin{aligned}\int_{\tilde{\Delta}\times H\times H} I\,d\sigma &= \sum_{k\in\Gamma}\int_{\tilde{\Delta}\times H\times h^{-1}(\Delta)} I\,d\sigma\\ &= \sum_{h\in\Gamma}\int_{\tilde{\Delta}\times H\times\Delta} I(h^{-1}(\zeta),s,\alpha,\beta,\gamma,\delta)\,d\sigma\\ &= \sum_{h\in\Gamma}\int_{\tilde{\Delta}\times H\times\Delta} I(\zeta,h(s),h(\alpha),h(\beta),h(\gamma),h(\delta))\,d\sigma\end{aligned}$$

and since h is a bijection of H by a change of variables we may replace $h(s)$ by s,

$$=\sum_{h\in\Gamma}\int_{h(\tilde{\Delta})\times H\times\Delta} I\,d\sigma=\int_{T_1(H)\times H\times\Delta} I\,d\sigma.$$

Furthermore by virtue of the absolute convergence we may interchange the order of integration and begin with the $T_1(H)$ integral. As before the integral along the trajectories of the geodesic flow is given by forming a limit:

$$\begin{aligned}&\lim_{\substack{\gamma\to\alpha\\ \delta\to\beta}}\omega K\big(K_1(\zeta)+K_1(\bar{\zeta})\big)\\ &=\frac{-2}{(s-\alpha)(s-\beta)}\left(\frac{1}{(\zeta-\alpha)(\zeta-\beta)(\zeta-s)^2}+\frac{1}{(\bar{\zeta}-\alpha)(\bar{\zeta}-\beta)(\bar{\zeta}-s)^2}\right).\end{aligned}$$

Next we integrate over the space of trajectories $(\alpha, \beta) \in S^1_\infty \times S^1_\infty$-{diagonal} and obtain $8\pi^2/(s - \bar{\zeta})^2$, in particular substituting back gives

$$\ddot{A} = 16\,\mathrm{Re} \int_\Delta \mu(s) \int_H \frac{\overline{\mu(\zeta)}}{(\bar{s} - \zeta)^4}\, d\sigma(\zeta)\, d\sigma(s).$$

Substituting $d\sigma(s) = (\mathrm{Im}\, s)^2\, dA(s)$, where dA is the hyperbolic area form we recognize the first integral as the projection P: $B(\Gamma) \to HB(\Gamma)$ and have finally $\ddot{A} = (4\pi/3) Re \langle \mu, P[\mu] \rangle$. Thurston's metric is the Weil-Petersson metric.

Theorem 3.4. *Let $A([R],[S])$ be the R-average of the S-length of a geodesic. Given $\mu \in B(\Gamma)$ let $\partial/\partial x(\mu)$ be the corresponding real tangent vector and $\partial/\partial t(\mu)$ the corresponding complex tangent vector. Then for $\mu, \nu \in B(\Gamma)$*

$$\text{(i)} \quad \frac{\partial}{\partial x(\mu)} \frac{\partial}{\partial x(\nu)} A = \frac{2\pi}{3} g_{\mathrm{WP}}(\mu, \nu),$$

$$\text{(ii)} \quad \frac{\partial}{\partial t(\mu)} \frac{\partial}{\partial \overline{t(\nu)}} A = \frac{2\pi}{3} \langle \mu, P[\nu] \rangle,$$

$$\text{(iii)} \quad \frac{\partial}{\partial t(\mu)} \frac{\partial}{\partial t(\nu)} A = \frac{\partial}{\partial \overline{t(\mu)}} \frac{\partial}{\partial \overline{t(\nu)}} A = 0.$$

Proof. The formulas are an immediate consequence of polarization.

Corollary 3.5. *For $[R], [S] \in T_g$ with $[S]$ close to $[R]$, then*

$$A([R],[S]) = 4\pi^2(g-1) + \frac{\pi}{3} d_{\mathrm{WP}}([R],[S])^2 + O\big(d_{\mathrm{WP}}([R],[S])^3\big).$$

4. Thurston's approach via earthquakes

4.1. Thurston's construction of a Riemannain metric is based on the observation by Kerckhoff that the geodesic length functions are convex along earthquake paths [6]. Heuristically a geodesic length function l_α is a candidate for the square of the distance measured from a base point, say its minimum on T_g; the second derivative of l_α is a candidate for the metric tensor. In order to avoid the arbitrary choice of a geodesic α it would be natural to instead consider the *generic* geodesic. In principle the *generic* geodesic traverses the surface in a uniform manner. This concept is quantified with the definition of a uniformly distributed sequence of closed geodesics. Now recall that the first derivative of l_α with respect to an infinitesimal Fenchel-Nielsen twist t_β about β (a simple earthquake) is the sum of the cosines of the intersection angles of α

and β (the cosine formula) [6], [14]. Now if $\{\kappa_j\}$ is a uniformly distributed sequence of geodesics, then $\lim_j(1/l_{\kappa_j})t_\beta l_{\kappa_j}$ vanishes by the cosine formula and the uniform distribution of the intersection angles in the interval $(0,\pi)$. It follows that $\langle t_\alpha, t_\beta\rangle = \lim_j(1/l_{\kappa_j})t_\alpha t_\beta l_{\kappa_j}$ defines a symmetric 2-tensor; the positive definiteness follows from Kerckhoff's convexity result.

Given a continuous function F on $T_1(R)$, a basic property of a uniformly distributed sequence is the limit formula

$$\int_{T_1(R)} F\,dV = V(T_1(R))\lim_j \frac{1}{l_{\kappa_j}}\int_{\kappa_j} F\,dl.$$

Thurston's idea is to use this formula to show that

$$\ddot{A}(t_\alpha, t_\alpha) = V(T_1(R))\lim_j \frac{1}{l_{\kappa_j}} t_\alpha t_\alpha l_{\kappa_j};$$

thus connecting the two constructions of a Riemannian metric. An immediate difficulty is that the quantity on $T_1(R)$ which interpolates the derivative $(1/l_{\kappa_j})t_\alpha t_\beta l_{\kappa_j}$, defined for the closed orbits, is not a continuous function but a measure. On the other hand the calculation is invariant under the geodesic flow and so we may convolve the measure with geodesic flow to obtain a continuous function. Our discussion starts with this matter. §4.3 is a brief review of the symplectic geometry of twist vector fields and geodesic length functions. Then in §4.4 we give Thurston's argument for the vanishing of the limit $\lim_j(1/l_{\kappa_j})t_\alpha l_{\kappa_j}$. The discussion of $\lim_j(1/l_{\kappa_j})t_\alpha t_\beta l_{\kappa_j}$ is postponed until the next section, where a formula is given for the limit.

4.2. As a review recall the definition of the integral $A([R],[S])$ given in §3.2, $A = \int_{p(\tilde{\Delta})}\tilde{\omega}L$. In particular p: $T_1(H)\to$ DMB is the projection onto the first factor, $\tilde{\Delta}\subset T_1(H)$ is a Γ_R fundamental domain and L is the S-length of $p^{-1}(x)\cap\tilde{\Delta}$, $x\in p(\tilde{\Delta})$. In brief the function L is integrated over $p(\tilde{\Delta})$. Now for simplicity we can choose $\tilde{\Delta}$ relatively compact such that $I_x = p^{-1}(x)\cap\tilde{\Delta}$ is an interval. Furthermore if g_t is the time t geodesic flow on $T_1(H)$, then recall that: $p\circ g_t = p$, ω is g_t invariant and $g_t(\tilde{\Delta})$ is also a Γ_R fundamental domain. Finally the fibration p: $T_1(H)\to$ DMB is trivial and so we may choose a smooth section s and use this to define a coordinate $l(\alpha)=\int_s^\alpha dl$ along each trajectory of g_t.

The first matter is the variation of L. If I_x is the trajectory interval $[\alpha_x,\beta_x]$, then, by additivity, $L(x)=L(I_x)$ is the difference of the S-length of $[f]$ of $[s(x),\alpha_x]$ and $[s(x),\beta_x]$ (s is the section). Now if we consider a 1-parameter family $(R,S^\varepsilon,[f^\varepsilon])$ of deformations, then the quantity $L^\varepsilon([s(x),\gamma])$ is by Theorem 2.1 real analytic in ε and continuous in x and γ; by the estimates of §3 we may differentiate under the integral. In particular if $\mathscr{L}_1(\gamma)$ (resp. $\mathscr{L}_2(\gamma)$)

is the first (resp. second) variation of $L^\varepsilon([s(x),\gamma])$ evaluated at $\varepsilon = 0$, then

$$\dot{A} = \int_{p(\tilde{\Delta})} \tilde{\omega}(\mathscr{L}_1(\beta_x) - \mathscr{L}_1(\alpha_x)) \quad \text{and} \quad \ddot{A} = \int_{p(\tilde{\Delta})} \tilde{\omega}(\mathscr{L}_2(\beta_x) - \mathscr{L}_2(\alpha_x)).$$

The following convolution argument for $\mathscr{L}$ a variation (first or second) of L will be used to replace the integrals over $p(\tilde{\Delta})$ by the integrals of continuous functions over $\tilde{\Delta}$.

The argument is based on a simple result.

Lemma 4.1. *Let $F(x)$ be a continuous function of $x \in \mathbf{R}$. Then*

$$\int_0^1 F(\beta + t) - F(\alpha + t)\, dt = \int_\alpha^\beta F(x+1) - F(x)\, dx.$$

Proof. For $G(x) = \int_0^1 F(x+t)\,dt$, then G is differentiable with derivative $G'(x) = F(x+1) - F(x)$. The conclusion is the Fundamental Theorem of Calculus applied to G.

Now we start the convolution argument with the observation that $g_t(\tilde{\Delta}) \subset T_1(H)$ is a fundamental domain and thus

$$\begin{aligned}\ddot{A} &= \int_0^1 \int_{g_t(\tilde{\Delta})} \tilde{\omega}(\mathscr{L}(\beta_x) - \mathscr{L}(\alpha_x))\, dt \\ &= \int_{p(\tilde{\Delta})} \int_0^1 \tilde{\omega}(\mathscr{L}(g_t(\beta_x)) - \mathscr{L}(g_t(\alpha_x)))\, dt,\end{aligned}$$

where the interchange of integrals is valid since the integrand is continuous and the domain is relatively compact; the remaining manipulations are consequences of the g_t invariance. By the above lemma the last integral is

$$\int_{p(\tilde{\Delta})} \int_{[\alpha_x, \beta_x]} \omega(\mathscr{L}(l+1) - \mathscr{L}(l))\, dl,$$

where l is the natural coordinate along a g_t orbit. In particular the above is merely

$$\int_{\tilde{\Delta}} (\mathscr{L}(l+1) - \mathscr{L}(l))\omega \wedge dl, \tag{4.1}$$

the integral of the continuous function $(\mathscr{L}(l+1) - \mathscr{L}(l))$ over $T_1(R)$.

Now recall that the integral of a continuous function on $T_1(R)$ can be computed as the limit of the integrals on a sequence $\{\kappa_j\}$ of uniformly distributed geodesics. The basic observation is that for $U \subset T_1(R)$, an open set with measure 0 boundary, the ratio of the length of the intersection $U \cap \kappa_j$ and the length of κ_j converges to the ratio $V(U)/V(T_1(R))$. In preparation for applying this to the above situation consider $\tilde{\kappa}$ a lift to $T_1(H)$ of a closed geodesic κ of R. Represent $\tilde{\kappa}$ as a segment $[\alpha, \beta]$ of a g_t orbit and apply

Lemma 4.1 a second time to obtain

$$\int_{[\alpha,\beta]} (\mathscr{L}(l+1) - \mathscr{L}(l))\,dl = \int_0^1 \mathscr{L}(\beta+t) - \mathscr{L}(\alpha+t)\,dt.$$

Now since κ is closed, $[\alpha+t, \beta+t]$ is also a lift of κ; in particular, independent of t, $\mathscr{L}(\beta+1) - \mathscr{L}(\alpha+t)$ is simply the variation of the length of κ.

Before stating the theorem we review the definition of the geodesic length function l_κ on the Teichmüller space. Given κ a geodesic on R define $l_\kappa([S])$ to be the hyperbolic length of the unique geodesic on S freely homotopic to $[f(\kappa)]$. A discussion of the basic properties of the geodesic length functions can be found in [6], [8], [13], [14], [15]. Let $\{\kappa_j\}$ be an R-uniformly distributed sequence of closed geodesics. Furthermore, let ε be the parameter of a family $(R, S^\varepsilon, [f^\varepsilon])$, of deformations. A summary of the above is given in the following results.

Theorem 4.2. *With the above notation*

$$\dot{A}([R],[R]) = V \lim_j \frac{1}{l_{\kappa_j}} \frac{dl_{\kappa_j}(S^\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0},$$

$$\ddot{A}([R],[R]) = V \lim_j \frac{1}{l_{\kappa_j}} \frac{d^2 l_{\kappa_j}(S^\varepsilon)}{d\varepsilon^2}\bigg|_{\varepsilon=0},$$

where $V = V(T_1(R))$ is the volume of the unit tangent bundle.

Corollary 4.3. *With the above notation,*

$$g_{\mathrm{WP}}(t_\alpha, t_\beta) = 3\pi(g-1) \lim_j \frac{1}{l_{\kappa_j}} t_\alpha t_\beta l_{\kappa_j}.$$

4.3. An introduction to the theory of earthquakes can be found in [6]. An earthquake is a generalization of the classical Fenchel-Nielsen deformation. For the sake of exposition we shall concentrate on the classical deformation. Start with a surface R with hyperbolic metric and a simple closed geodesic α. A one-parameter family of deformations is defined as follows. Cut the surface along α, rotate one *side* of the cut relative to the other, and then attach the *sides* in their new position. A geodesic intersecting the cut is deformed to a broken geodesic. The hyperbolic structure in the complement of the cut extends naturally to a hyperbolic structure on the new surface. By varying the amount of rotation, a one-parameter family of deformations is defined. In fact since R is arbitrary a flow is defined on the Teichmüller space. Let t_α, a Fenchel-Nielsen vector field, be the tangent field of this flow.

If ω_{WP} is the Weil-Petersson Kähler form and l_β is a geodesic length function, the basic formulas are [13], [14]

$$\omega_{\mathrm{WP}}(t_\alpha, \ ) = -dl_\alpha, \tag{4.2}$$

$$\omega_{\mathrm{WP}}(t_\alpha, t_\beta) = t_\alpha l_\beta = \sum_{p \in \alpha \# \beta} \cos\theta_p, \tag{4.3}$$

$$\begin{aligned} t_\alpha t_\beta l_\gamma = &\sum_{(p,q) \in \alpha\#\gamma x \beta\#\gamma} \frac{e^{l_1} + e^{l_2}}{2(e^{l_\gamma} - 1)} \sin\theta_p \sin\theta_q \\ &- \sum_{(r,s) \in \alpha\#\beta x \beta\#\gamma} \frac{e^{m_1} + e^{m_2}}{2(e^{l_\beta} - 1)} \sin\theta_r \sin\theta_s. \end{aligned} \tag{4.4}$$

In particular the r.h.s. of (4.3) is the sum of the cosines of the intersection angles (measured ccw from α to β) of α and β. Similarly the r.h.s. of (4.4) is a sum of trigonometric invariants for pairs of intersections; l_1 and l_2 are the lengths of the segments on γ defined by p, q and, likewise, for m_1 and m_2 relative to β. The reader should see [5] for the generalizations of these formulas to other representation spaces.

We point out that an earthquake vector field on Teichmüller space is the limit in the C^1 compact-open topology of the Fenchel-Nielsen vector-fields; the C^1 compact-open topology is compatible with the topology of measured geodesic laminations [6]. Thurston showed that the Fenchel-Nielsen fields are dense in the earthquake fields and thus the formulas for earthquakes are obtained from the above on passing to the limit.

4.4. The discussion starts with the geometry of a tubular neighborhood $N \subset R$ of a closed geodesic α and the lift $\tilde{N} \subset T_1(R)$. The first observation is that for ε sufficiently small an ε tubular neighborhood N is an annulus with a standard geometry dependent only on l_α and ε. A vector in $\tilde{N}$ is tangent to a complete geodesic segment g in N intersecting α at most once with angle θ (measured ccw from α to g). $\tilde{N}$ is stratified by the intersection angle. Consider a strata $\tilde{N}_\theta$, the subset corresponding to segments g with a given intersection angle θ, and also the open subset $\tilde{N}_+ = \bigcup_{0<\theta<\pi} \tilde{N}$ corresponding to a non-empty intersection. An important function of θ will be the length $\lambda(\theta)$ of a segment g with intersection angle θ. An explicit description of $\tilde{N}$ can be given as the product $N \times S^1$, where $(n, 0)$ corresponds to the initial tangent of the shortest unit speed geodesic from n to α. Using the description we see that the reflection ρ of $\tilde{N}$ is given by the rule $\rho(n, \phi) = (n, -\phi)$; clearly $\rho(\tilde{N}_\theta) = \tilde{N}_{\pi-\theta}$, $\lambda \circ \rho = \lambda$, $\cos\theta \circ \rho = -\cos\theta$ and $\rho^* dV = -dV$, dV the $T_1(H)$ volume form.

Lemma 4.4. *Let t_α be a Fenchel-Nielsen vector field. The associated first variation of $A([R], [S])$ vanishes.*

Proof. By Theorem 4.2 the first variation is simply $V \lim_j (1/l_{\kappa_j})(t_\alpha l_{\kappa_j})$ for a uniformly distributed sequence of closed geodesics. Now fix a tubular neighborhood N of α and observe that there is a 1-1 correspondence between the intersection points $\alpha \# \kappa_j$ and the components $\kappa_j \cap \tilde{N}_+$. Now let I, an interval, be a small neighborhood of an angle θ_0 and $\tilde{N}_I = \bigcup_{\theta \in I} \tilde{N}_\theta$. Then approximately we have that: each component of $\kappa_j \cap \tilde{N}_I$ has length $\lambda(\theta_0)$, the number of components is the length $l(\kappa_j \cap \tilde{N}_I)$ divided by $\lambda(\theta_0)$, the ratio $V(\tilde{N}_I)/V(T_1(R))$ is $l(\kappa_j \cap \tilde{N}_I)/l_{\kappa_j}$ and finally each intersection contributes $\cos\theta_0$ to the sum for $t_\alpha l_{\kappa_j}$. It follows that $V(t_\alpha l_{\kappa_j})/l_{\kappa_j}$ converges to the integral $\int_{\tilde{N}_+} \lambda^{-1}(\theta)\cos\theta \, dV$. Now to evaluate the integral we simply observe that the integrand is odd relative to the reflection ρ of $\tilde{N}$; the integral is identically zero.

Actually the preceding argument has two additional consequences. Recall that the Fenchel-Nielsen vector fields everywhere span the tangent space of Teichmüller space [13]. An immediate consequence is that $\dot{A}$ vanishes for all tangent vectors. A second observation is that the limit $\lim_j (1/l_{\kappa_j}) t_{\kappa_j}$ exists and in fact is zero. Immediate consequences of (4.2) are the skew symmetry $t_\alpha l_\kappa = -t_\kappa l_\alpha$ and that the length differentials span the cotangent space. Hence the vanishing

$$\lim_j \frac{1}{l_{\kappa_j}} t_{\kappa_j} l_\alpha = 0$$

for all α implies that $\lim_j (1/l_{\kappa_j}) t_{\kappa_j}$ is the trivial tangent vector. The vanishing of the limit can be contrasted with the work of Kerckhoff and Thurston on geodesic laminations [6], [10]. In brief, if $\{\alpha_j\}$ is a sequence of *simple* closed geodesics, then a subsequence of $\{(1/l_{\alpha_j}) t_{\alpha_j}\}$ converges to a nonzero vector. A sequence of simple geodesics is never uniformly distributed.

By an argument similar to that for the above lemma we could establish the following but instead we shall actually evaluate the integral in the next section.

Theorem 4.5. *Let t_α be a Fenchel-Nielsen vector field. The associated second variation of $A([R],[S])$ is positive.*

5. The Hermitian geometry of the Fenchel-Nielsen vector fields

5.1. The symplectic geometry of the Weil-Petersson Kähler form, geodesic length functions and Fenchel-Nielsen vector fields was considered in [13], [14], [15]. Now we shall examine the Hermitian geometry of the Weil-Petersson metric and the complex structure of T_g for the Fenchel-Nielsen fields and the geodesic length functions. We start in §5.3 with a formula for the Weil-Petersson pairing of twist fields t_α and t_β. The pairing is completely determined by the sequence of lengths of minimal geodesic arcs connecting α to β. In particular if α and β are disjoint, then the pairing $g_{\mathrm{WP}}(t_\alpha, t_\beta)$ is positive and an application to geodesic length functions is made. As a second matter we consider the complex structure J of Teichmüller space and obtain the following formula:

$$Jt_\beta = 3\pi(g-1)\lim_{\{\kappa\}} \frac{1}{l_\kappa}\left[t_\beta, t_\kappa\right],$$

where $\{\kappa\}$ is a uniformly distributed sequence of closed geodesics.

5.2. To a pair of geodesics in H we shall consider an integral, the visibility integral, which is a measure of their relative displacement. The integral occurs naturally as the limit over κ, uniformly distributed, of $(1/l_\kappa)t_\alpha^2 l_\kappa$. In particular the integral is defined in terms of hyperbolic geometry and therefore is PSL(2; **R**) invariant. Specifically given α, β geodesics in H there are three cases to consider, α and β are either: disjoint, coincide or intersect.

Consider first the case of $\tilde{\alpha}$ and $\tilde{\beta}$ fixed disjoint geodesics. A point of DMB $= S^1_\infty \times S^1_\infty$-{diagonal} defines a geodesic $\tilde{\gamma} \subset H$. We wish to consider the following invariant of $\tilde{\gamma}$ (of the triple $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$):

$$i(\tilde{\gamma}) = \tfrac{1}{2} l(p,q)^{-1} e^{-l(p,q)} \sin\theta_p \sin\theta_q,$$

which by definition is either 0 or the indicated quantity if $\tilde{\gamma}$ intersects $\tilde{\alpha}$ and $\tilde{\beta}$ say at p and q with intersection angles θ_p and θ_q, and $l(p,q)$ the distance between p and q. It is immedaite that $i(\tilde{\gamma})$ is a continuous function on DMB which can be lifted to $T_1(H)$ (same notation). As a related matter we define the subset $\Omega(\tilde{\alpha}, \tilde{\beta}, x) \subset T_1(H)$ of tangents of geodesic segments g, where g connects $\tilde{\alpha}$ to $\tilde{\beta}$ and has length at most x; set $\Omega(\tilde{\alpha}, \tilde{\beta}) = \Omega(\tilde{\alpha}, \tilde{\beta}, \infty)$. The first integral to consider,

$$I_1(\tilde{\alpha}, \tilde{\beta}, x) = \int_{\Omega(\tilde{\alpha}, \tilde{\beta}, x)} i\,dV = \tfrac{1}{2}\int_{\Omega(\tilde{\alpha}, \tilde{\beta}, x)} l(p,q)^{-1} e^{-l(p,q)} \sin\theta_p \sin\theta_q\, dV,$$

$$I_1(\tilde{\alpha}, \tilde{\beta}) = I_1(\tilde{\alpha}, \tilde{\beta}, \infty),$$

is an invariant of the relative geometry of the pair $(\tilde{\alpha}, \tilde{\beta})$. The convergence follows after an initial integration along the orbits of the geodesic flow on $T_1(H)$. The initial integration yields the following integral over the projection $p(\Omega) \subset$ DMB:

$$I_1 = \tfrac{1}{2}\int_{p(\Omega)} e^{-l(p,q)} \sin\theta_p \sin\theta_q\, \tilde{\omega}, \tag{5.1}$$

where $\tilde{\omega}$ is the 2-form on DMB and this integral converges since $p(\Omega)$ has finite area. The pair $(\tilde{\alpha}, \tilde{\beta})$ is determined, modulo translation by PSL(2; **R**), by a single invariant, the cross ratio of its endpoints, or equivalently the minimal distance $\rho(\tilde{\alpha}, \tilde{\beta})$ from $\tilde{\alpha}$ to $\tilde{\beta}$; the integral $I_1(\tilde{\alpha}, \tilde{\beta}) = I_1(\rho(\tilde{\alpha}, \tilde{\beta}))$ is a function of $\rho(\tilde{\alpha}, \tilde{\beta})$. This last observation is needed for applications.

For the second case of $\tilde{\alpha} = \tilde{\beta}$ we consider a closed geodesic α on R and choose a tubular neighborhood $\tilde{N} \subset T_1(R)$. In terms of the intersection angle θ and the length λ of a segment in $\tilde{N}$ (see §4.4) the second integral is simply

$$I_2(\alpha, x) = \tfrac{1}{2}\int_{\tilde{N}(x)} \lambda^{-1} \sin^2\theta \, dV,$$
$$I_2(\alpha) = I_2(\alpha, \infty),$$

where $\tilde{N}(x) = \{\tilde{n} \in \tilde{N} | \lambda(\theta(\tilde{n})) \leqslant x\}$. The convergence of the I_2 integral is verified by the analogue of the I_1 argument. The description of $\tilde{N}$ as $N \times S^1$ can be used to introduce coordinates for the integration. In particular displacement along α and then normal displacement from α give coordinates on N. Performing the integral relative to these coordinates with the last integration along α, we see that $I_2(\alpha)$ is a constant times the length of α: $I_2(\alpha) = C_2 l_\alpha$. Integral I_2 can also be written as an integral over the projection $p(\tilde{N}) \subset$ DMB; it follows that C_2 is independent of the width of N.

For the third case, an intersection, consider α and β closed geodesics on R intersecting at a point p. We allow the possibility that α and β intersect elsewhere. Before defining the integral recall from §4.4 the description of geodesic segments in N, a tubular neighborhood of β. Fix a point r on β; then given a geodesic segment $g \subset \tilde{N}$ intersecting β. Define $l(\tilde{n})$, $\tilde{n}$ a point of g, to be the minimal distance along β from r to the intersection $\beta \cap g$. In terms of the quantities θ, $\lambda(\theta)$ and $l(\tilde{n})$ the third integral is

$$I_3(\beta, x) = \frac{1}{2}\int_{\tilde{N}(x)} \lambda^{-1}(\theta) \frac{e^{l(\tilde{n})} + e^{l_\beta - l(\tilde{n})}}{e^{l_\beta} - 1} \sin\theta \, dV,$$
$$I_3(\beta) = I_3(\beta, \infty).$$

The convergence of the I_3 integral is verified by the analogue of the I_1 argument. Introducing coordinates on $\tilde{N}$ as for the case two integral we find that this integral is a constant: $I_3(p) = C_3$. Integral I_3 is also independent of the width of N by the analogue of the I_2 argument.

5.3. Now we shall present the formula for the Weil-Petersson pairing of twist vectors t_α and t_β, α and β are simple closed geodesics. First we introduce $\mathrm{CN}(\alpha, \beta)$, the set of free homotopy classes, relative to α and β, of arcs connecting α and β (nonsimple arcs are allowed and if $\alpha = \beta$ or $\alpha \# \beta \neq \varnothing$ we omit the trivial class). Arcs γ and δ are in the same CN class if there exists a homotopy with the endpoints sliding along α and β. We could even allow α and β to be nonsimple provided that the homotopy defining CN equivalence is interpreted to mean that points on α and β slide straight through selfintersection points. Each equivalence class in $\mathrm{CN}(\alpha,\beta)$ contains a unique minimal length geodesic: denote its length as $|\gamma|$ for γ a homotopy class in $\mathrm{CN}(\alpha,\beta)$. The main result is that the pairing of t_α and t_β is completely determined by the sequence of lengths $|\gamma|$, $\gamma\in\mathrm{CN}(\alpha,\beta)$.

Theorem 5.1. *With the above notation*,

$$g_{\mathrm{WP}}(t_\alpha,t_\alpha)=V(T_1(R))\Big(C_2 l_\alpha+\sum_{\gamma\in\mathrm{CN}(\alpha,\alpha)}I_1(|\gamma|)\Big),$$

$$g_{\mathrm{WP}}(t_\alpha,t_\beta)=V(T_1(R))\Big(\sum_{\gamma\in\mathrm{CN}(\alpha,\beta)}I_1(|\gamma|)+C_3\sum_{p\in\alpha\#\beta}\sin\theta_p\Big).$$

The plan for the proof is to identify the integral which $(1/l_\kappa)t_\alpha^2 l_\kappa$ converges to. Recall that the Lie derivative $t_\alpha^2 l_\kappa$ is given by a sum over the intersection points $(p,q)\in(\alpha\#\kappa)^2$. The first step is to group the terms of the sum according to the homotopy class of the segment $\widehat{pq}$ of κ. Then by the uniform distribution of the geodesics $\{\kappa\}$ the sum for each group of terms converges to the corresponding visibility integral and the proof is complete. Before starting the proof we consider three preliminary matters. First a priori the sum $\sum_\gamma I_1(|\gamma|)$ may not converge. Accordingly we introduce the following convention: for $\sum_m a_m$ a sum of positive terms and $\varepsilon>0$, the partial sum $\sum_M=\sum_{m=1}^M a_m$ is *within* ε *of its limit* if provided $\sum_m a_m$ is convergent to A, then $|\sum_M - A|<\varepsilon$ or if $\sum_m a_m$ is divergent, then $\sum_M>1/\varepsilon$. The second matter is to associate to p,q (intersection) points on κ the length l_{pq} of the shortest segment κ_{pq} of κ connecting p to q.

The third item is an estimate for the number of intersections. In particular for α fixed there exists a constant c such that for each $m\in\mathbf{Z}^+$ the number of pairs $(p,q)\in(\alpha\#\kappa)^2$ with $m<l_{pq}\leqslant m+1$ is at most cl_κ. To see this fix a tubular neighborhood N of α and recall that a complete geodesic segment g in N intersects α at most once; consequently the intersection points of α and κ are separated along κ by at least the width of N. It follows immediately that for $p\in(\alpha\#\kappa)$ fixed there is an absolute bound on the number of q's with $x\leqslant l_{pq}\leqslant x+1$; the conclusion follows since the cardinality of $\alpha\#\kappa$ is at most l_κ divided by the width of N.

Proof of Theorem 5.1. We shall only consider the first formula; the second is an exercise for the reader. The first step is to choose a truncation of the right-hand side. Given $\varepsilon>0$ pick a finite number of terms $T\subset\mathrm{CN}$ such that the sum $\Sigma_{\gamma \in T} I_1(|\gamma|)$ is within ε of its limit. Now the integrals I_1 and I_2 are convergent; hence we may pick M_0 such that the finite sum $I_2(\alpha, M_0) + \Sigma_{\gamma \in T} I_1(|\gamma|, M_0)$ is within 2ε of its limit. Furthermore since the integrand of an I integral is positive, the sum is only increased (thus made closer to its limit) if T is replaced by $T(M) = \{\gamma \in \mathrm{CN} \mid |\gamma| \leqslant M\}$, $M = \sup_{\gamma \in T}\{M_0, |\gamma|\}$ and M is substituted for M_0 as the truncation limit for the integrals. In summary, $\Sigma_M = I_2(\alpha, M) + \Sigma_{\gamma \in T(M)} I_1(|\gamma|, M)$ is within 2ε of its limit. An important point is that this last integral is over all geodesic segments (connecting α to α) of length at most M.

We use the same quantity M to divide the terms of the sum for $t_\alpha^2 l_\kappa$ into two classes: (p, q) with $l_{pq} > M$ and (p, q) with $l_{pq} \leqslant M$. We start with the estimate for the remainder terms. Further subdivide the terms (p, q), $l_{pq} > M$, by the rule $m < l_{pq} \leqslant m + 1$, $m \in \mathbf{Z}^+$. By the preliminary remarks for each m there are at most cl_κ such terms. The (p, q) summand is

$$\frac{e^{l_{pq}} + e^{l_\kappa - l_{pq}}}{2(e^{l_\kappa} - 1)} \sin\theta_p \sin\theta_q$$

which is bounded by $e^{l_\kappa - m}/(e^{l_\kappa} - 1)$. In summary the contribution to $(1/l_\kappa)t_\alpha^2 l_\kappa$ by the terms (p, q), with $m < l_{pq} \leqslant m + 1$, is bounded by $ce^{-m}/(1 - e^{-l_\kappa})$ ($(1/l_\kappa) \times$ number of terms $\times$ bound on terms). Consequently the total contribution of terms (p, q) with $l_{pq} > M$ is at most

$$\frac{ce^{-M}}{1 - e^{-l_\kappa}} \sum_{m=0}^{\infty} e^{-m}.$$

Now recall that if M is increased, the quantity Σ_M (discussed above) increases and thus is still within 2ε of its limit; choose M such that the contribution to $(1/l_\kappa)t_\alpha^2 l_\kappa$ by the terms (p, q) with $l_{pq} > M$ is at most ε.

We are left to consider the contribution to $(1/l_\kappa)t_\alpha^2 l_\kappa$ by terms (p, q) with $l_{pq} \leqslant M$. First we group these terms according to the equivalence class of κ_{pq} in CN. The proof will be complete if we verify that the limit over κ, $\{\kappa_j\}$ uniformly distributed, of a group of terms converges to the corresponding visibility integral. Specifically let γ be a particular class in $\mathrm{CN}(\alpha, \alpha)$ and consider the sum

$$\sum_{\substack{(p,q) \\ \kappa_{pq} \in \gamma}} \frac{e^{l_{pq}} + e^{l_\kappa - l_{pq}}}{2(e^{l_\kappa} - 1)} \sin\theta_p \sin\theta_q.$$

We are to approximate the integral over $\Omega(\tilde{\alpha}_0, \tilde{\alpha}_1, M)$, where γ determines a pair $\tilde{\alpha}_0, \tilde{\alpha}_1$ of lifts of α. Consider all segments κ_{rs} in a neighborhood U of κ_{pq} defined by the condition r near p and s near q; the κ_{rs} summand is approximately that of κ_{pq} and the length of κ_{rs} is approximately l_{pq}. Consequently the contribution to the sum for segments $\kappa_{rs} \subset U$ is approximately the number of such terms times

$$\frac{e^{l_{pq}} + e^{l_\kappa - l_{pq}}}{2(e^{l_\kappa} - 1)} \sin\theta_p \sin\theta_q.$$

Now by the uniform distribution of the $\{\kappa_j\}$ the number of such terms is approximately $V(U)/l_{pq}V(T_1(R))$. In conclusion the total contribution of terms κ_{rs}, $\kappa_{rs} \subset U$, is approximately

$$\frac{e^{l_{pq}} + e^{l_\kappa - l_{pq}}}{2l_{pq}(e^{l_\kappa} - 1)} \sin\theta_p \sin\theta_q \frac{V(U)}{V(T_1(R))}.$$

Certainly for l_κ sufficiently large this is approximately

$$\frac{e^{-l_{pq}}}{2l_{pq}} \sin\theta_p \sin\theta_q \frac{V(U)}{V(T_1(R))}.$$

Finally observe that the integrand is invariant under geodesic flow and that $V(U)/l_{pq}$ approximates the area of the projection $p(U) \subset$ DMB. The conclusion now follows from the description of I_1 as an integral over $p(\Omega) \subset$ DMB. The proof is complete.

5.4. We shall now discuss a simple application of the above to the geodesic length functions and then use the above formulas to study the complex structure of Teichmüller space.

Lemma 5.2. *If α and β are simple disjoint geodesics, then $g_{\mathrm{WP}}(t_\alpha, t_\beta) > 0$. In particular the twist tangents for geodesics disjoint from α lie in a common cone.*

Proof. Provided α and β do not intersect, each term of the infinite sum for g_{WP} is positive.

Lemma 5.3. *If $\alpha_1, \cdots, \alpha_n$ are disjoint geodesics on R, then there exists a tangent vector v to T_g at R such that vl_{α_j}, $j = 1, \cdots, n$, is positive.*

Proof. If J is the complex structure of T_g, then formula (4.2) can be written as $g_{\mathrm{WP}}(Jt_\alpha, \ ) = -dl_\alpha$. The vector $-Jt_\alpha$ is the Weil-Petersson gradient of l_α. Now by the above the tangents $t_{\alpha_1}, \cdots, t_{\alpha_n}$ are contained in a common cone and since J is an orthogonal transformation, it follows that the gradients are contained in a common cone, the desired conclusion.

An alternate proof of the above appears in [11] where the result is used in the search for a mapping class group invariant Morse function on the Teichmüller space.

As the final topic we give two descriptions of the complex structure of Teichmüller space in terms of the hyperbolic trigonometry (lengths and intersection angles) of closed geodesics. Let R, a compact surface represent a point of Teichmüller space. Geodesics $\alpha_1,\cdots,\alpha_n$, $n = 6g-6$, can be chosen such that the length functions $l_1,\cdots,l_n$ ($l_j = l_{\alpha_j}$) give local coordinates at $R \in T_g$. By formula (4.2) the twist vectors $t_1,\cdots,t_n$ ($t_j = t_{\alpha_j}$) span the tangent space at $R \in T_g$. Now if J is the complex structure, then by formula (4.3) and Theorem 5.1 the pairings

$$g_{ij} = g_{\mathrm{WP}}(t_i, t_j), \qquad g_{\mathrm{WP}}(Jt_i, t_j) = \omega_{\mathrm{WP}}(t_i, t_j)$$

are evaluated by hyperbolic trigonometry. By a simple linear algebra computation we obtain the first description

$$Jt_i = \sum_{j,k} \omega(t_i, t_k) g^{kj} t_j,$$

where (g^{kj}) is the inverse matrix.

The second description requires two formulas for the Fenchel-Nielsen fields. An alternative normalization for a twist field is $T_\alpha = (4\sinh l_\alpha/2)t_\alpha$. The fields T_α, α a closed geodesic, span a Lie algebra over $\mathbf{Z}$, in particular

$$[T_\alpha, T_\beta] = \sum_{p\in\alpha\#\beta} T_{\alpha_p\beta^+} - T_{\alpha_p\beta^-}, \tag{5.2}$$

where $\alpha_p\beta^\pm$ is the homotopy class of a loop obtained from α and β by a surgery at p [5], [14]. The second formula is the identity [14]

$$t_\alpha t_\beta l_\kappa + t_\kappa t_\alpha l_\beta + t_\beta t_\kappa l_\alpha = 0.$$

By the skew symmetry $t_\alpha l_\beta = -t_\beta l_\alpha$ we have

$$t_\alpha t_\beta l_\kappa + [t_\beta, t_\kappa] l_\alpha = 0,$$

and if we divide by l_κ and take the limit over $\{\kappa_j\}$, uniformly distributed, then by Corollary 4.3 and (4.2) we obtain

$$g_{\mathrm{WP}}(t_\alpha, t_\beta) + 3\pi(g-1)\omega_{\mathrm{WP}}\left(\lim \frac{1}{l_\kappa}[t_\beta, t_\kappa], t_\alpha\right) = 0.$$

Certainly if J is the complex structure, then $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ and since t_α is arbitrary

$$t_\beta + 3\pi(g-1)J\lim\frac{1}{l_\kappa}[t_\beta, t_\kappa] = 0.$$

The final formula is

$$Jt_\beta = 3\pi(g-1)\lim_j \frac{1}{l_{\kappa_j}}[t_\beta, t_{\kappa_j}]. \tag{5.3}$$

Observe that by formula (5.2) the Lie bracket of twist fields and thus J are evaluated in terms of hyperbolic trigonometry. The second remark is that the right-hand side of the above indeed represents a tensor at $R \in T_g$. It suffices to check that for V a vector field on T_g vanishing at R, $\lim(1/l_\kappa)[V, t_\kappa]$ vanishes. This is an immediate consequence of the observation in §4.4 that $\lim(1/l_\kappa)t_\kappa$ vanishes. The final remark is of a more general nature. The standard description of the complex structure of T_g is given in terms of the holomorphic quadratic differentials on a compact surface. Apparently the solving of a partial differential equation (the Cauchy Riemann equation) is replaced in the above by forming the limit over a uniformly distributed sequence of closed geodesics.

An alternative understanding of the formulas for a uniformly distributed sequence $\{\kappa_j\}$ of geodesics is given by introducing local coordinates $z = (z_1, \cdots, z_n)$ for a neighborhood U of $R \in T_g$. We normalize the coordinates by assuming $p \in U$ represents R, $z(p) = 0$, and that $\{\partial/\partial z_k\}_{k=1}^n$ is a unitary basis at p, i.e., $g_{\mathrm{WP}}(\partial/\partial z_j, \partial/\partial z_k) = \delta_{jk}$. Furthermore we introduce the vector field

$$\mathscr{J} = \sum_{k=1}^{n} x_k \frac{\partial}{\partial y_k} - y_k \frac{\partial}{\partial x_k},$$

where $z_k = x_k + iy_k$ on U and let $\mathscr{I}(U)$ (resp. $\hat{\mathscr{I}}(U)$) be the module of vector fields (resp. 1-forms) on U vanishing to second order at p. Now we assert the preceding formulas are equivalent to the single observation

$$(5.4) \qquad 3\pi(g-1) \lim_j \frac{t_{\kappa_j}}{l_{\kappa_j}} \equiv \mathscr{J} \mod \mathscr{I}(U).$$

In fact since $\{\partial/\partial z_k\}$ is a unitary basis at p it is immediate that

$$3\pi(g-1) \lim_j \frac{dl_\kappa}{l_\kappa} \equiv -\omega_{\mathrm{WP}}(\mathscr{J}, \ ) \equiv \sum_k x_k\, dx_k + y_k\, dy_k \equiv \frac{1}{2} d \sum_k |z_k|^2 \mod \hat{\mathscr{I}}(U).$$

The reader will check that Corollary 4.3 and formula (5.3), as well as the identity $[\mathscr{J}, [\mathscr{J}, V]]_p = -V_p$ for V an arbitrary vector field, indeed all follow from (5.4). Conversely if we write

$$\mathscr{L} = \sum_k e_k \frac{\partial}{\partial x_k} + f_k \frac{\partial}{\partial y_k}$$

for the possible limit of $3\pi(g-1)t_\kappa/l_\kappa$, then the previous observation $\lim_j(t_\kappa/l_\kappa) = 0$ and Corollary 4.3 show the limit exists $\mathrm{mod}\, \mathscr{I}(U)$ and allow us to solve for the coefficients e_k and f_k. The limit is $\mathscr{J}$.

References

[1] L. V. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. of Math. **74** (1961) 171–191.

[2] ______, *Curvature properties of Teichmüller's space*, J. Analyse Math. **9** (1961) 161–176.

[3] ______, *Lectures on quasiconformal mappings*, Van Nostrand, New York, 1966.

[4] L. V. Ahlfors & L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. **72** (1960) 385–404.

[5] W. H. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. **54** (1984), 200–225.

[6] S. P. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. **117** (1983) 235–265.

[7] ______, *Earthquakes are analytic*, Comm. Math. Helv. **60** (1985) 17–30.

[8] ______, *Lines of minima in Teichmüller space*, preprint.

[9] H. L. Royden, *Automorphisms and isometries of Teichmüller space*, Ann. of Math. Studies 66, Princeton University Press, Princeton, NJ, 1971, 369–383.

[10] W. P. Thurston, *The geometry and topology of* 3-*manifolds*, notes.

[11] ______, *A spine for Teichmüller space*, preprint.

[12] S. A. Wolpert, *Noncompleteness of the Weil-Petersson metric for Teichmüller space*, Pacific J. Math. **61** (1975) 573–577.

[13] ______, *The Fenchel-Nielsen deformation*, Ann. of Math. **115** (1982) 501–528.

[14] ______, *On the symplectic geometry of deformations of a hyperbolic surface*, Ann. of Math. **117** (1983) 207–234.

[15] ______, *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math. **107** (1985) 969–997.

[16] ______, *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. Math. to appear.

[17] ______, *Geodesic length functions and the Nielsen problem*, preprint.

UNIVERSITY OF MARYLAND