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INFIMA OF ENERGY FUNCTIONALS IN HOMOTOPY CLASSES OF MAPPINGS

BRIAN WHITE

0. Introduction

Let M and N be compact connected Riemannian manifolds. The energy of a lipschitz map $f\colon M \to N$ is $\int_M |Df|^2$ (where $|Df(x)|^2 = \sum|\partial f/\partial x_i|^2$ if $x_1,\cdots,x_m$ are normal coordinates for M at x). Mappings for which the first variation of energy vanishes are called harmonic. ([1], [5], and [7] are nice introductions to harmonic maps.) The identity map from M to M is always harmonic, but it may be homotopic to mappings of less energy. For instance the identity map on S^3 is homotopic to mappings of arbitrarily small energy (namely, conformal maps that pull points from the North Pole toward the South Pole). That suggests the question: For which manifolds M is the identity map homotopic to maps of arbitrarily small energy? In this paper we give the simple answer: Those M such that $\pi_1(M)$ and $\pi_2(M)$ are both trivial. More generally we consider energy functionals like $\Phi(f) = \int_M |Df|^p$ and ask: When is the infimum of $\Phi(f)$ in some homotopy class of mappings $f\colon M \to N$ equal to 0?

To answer such questions, it is convenient to regard N as isometrically embedded in a euclidean space $\mathbf{R}^\nu$ and to work with the Sobolev norm

$$\|f\|_{1,p} = \left(\int_M |f|^p\right)^{1/p} + \left(\int_M |Df|^p\right)^{1/p}$$

(where $f\colon M \to \mathbf{R}^\nu$ has distribution derivative Df) and with the associated Sobolev space

$$W^{1,p}(M,N) = \text{the } \|\cdot\|_{1,p} \text{ completion of } \{\text{lipschitz maps } f\colon M \to N\}$$

We say that two continuous maps f, $g\colon M \to N$ are k-homotopic (or have the same k-homotopy type) if their restrictions to the k-dimensional skeleton of some triangulation of M are homotopic. Then the gist of our main theorems (Theorems 1 and 2) can be summarized as follows.

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Theorem 0. *Two lipschitz maps are in the same connected component of* $W^{1,p}(M, N)$ *if and only if they are* $[p]$*-homotopic. Furthermore, the set of lipschitz maps homotopic to a given map* f *is dense (with respect to* $\|\cdot\|_{1,p}$*) in the connected component containing* f*. (Consequently every map in* $W^{1,p}(M, N)$ *has a well-defined* $[p]$*-homotopy type.)*

The theorems have the following consequences for energy functionals. Let Φ be a function that assigns a number $\Phi(x, y, L) \geqslant 0$ to each $x \in M$, $y \in N$, and linear map $L\colon TM_x \to TN_y$. Let Φ also denote the associated functional on lipschitz maps:

$$\Phi(f) = \int_M \Phi(x, f(x), Df(x))\, dx.$$

Corollary 1 (§1). *If* Φ *satisfies* $\Phi(x, y, L) \leqslant c \cdot (1 + |L|^p)$ *(for some* c*), then*

$$\inf\{\Phi(g)\colon g \text{ is homotopic to } f\} = \inf\{\Phi(g)\colon g \text{ is } [p]\text{-homotopic to } f\}.$$

Corollary 2 (§1). *Suppose* Φ *satisfies* $\Phi(x, y, L) \leqslant c \cdot (1 + |L|^p)$ *(for some* c*) and that* $\Phi(\cdot, y, 0) \equiv 0$ *for some* y*. If* f *is* $[p]$*-homotopic to a constant map, then*

$$\inf\{\Phi(g)\colon g \textit{ is homotopic to } f\} = 0.$$

Corollary (§3). *If* Φ *satisfies* $\Phi(x, y, L) \geqslant C \cdot |L|^p$ *(for some* $C > 0$*),* $f\colon M \to N$ *is lipschitz, and if*

$$\inf\{\Phi(g)\colon g \textit{ homotopic to } f\} = 0,$$

then f *is* $[p]$*-homotopic to a constant map (so* $\pi_i(f)$ *is trivial for* $i \leqslant [p]$*).*

Example. $\Phi(x, y, L) = |L|^{5/2} + (\cosh|y|) \cdot |L|$ satisfies the hypotheses of all three corollaries with $p = 5/2$.

Similar results hold if M is a compact manifold with boundary or if N is a fiber bundle over M (§§4, 5).

A key ingredient of the proofs is the homotopy extension theorem, which says that if $A \subset B$ are nice subsets of euclidean space and if

$$H\colon ([0,1] \times A) \cup (\{0\} \times B) \to N$$

is lipschitz, then H may be extended to a lispchitz homotopy $H\colon [0,1] \times B \to N$ on all of B. Here "A is nice" means it is a lipschitz neighborhood retract, i.e. there is a open set U containing A and a lipschitz retraction of U onto A (see [4, p. 13] for a proof).

We also use a construction reminiscent of the Federer-Fleming deformation theorem [2, 4.2.9]. Oddly enough, however, the deformation is applied in the domain rather than in the range (or ambient space).

In another paper we will use similar techniques to give criteria for the infimum to be attained. Results analogous to (but very different from) those of this paper hold for the area functional (see [8]). A special case ($p = 2$, M a simply-connected Lie group) of Theorem 1 is treated (by very different methods) in [6].

I would like to thank Rick Schoen for bringing the problems considered here to my attention, and for many helpful and stimulating conversations. I am also grateful to J. Eells for various suggestions.

Note added in proof: A. T. Fomenko has informed me that in the case of the ordinary energy ($p = 2$) some results of this paper (in particular Theorem 3 when $M = N$) were proved by A. I. Pluzhnikov [9].

1. The main theorem

Theorem 1. *Let M and N be Riemannian manifolds with M compact. Let f and g be lipschitz maps from M to N that are $[p]$-homotopic. Then for every $\eta > 0$ there is a lipschitz map f' homotopic to f such that*

$$\|f' - g\|_{1,p} < \eta. \tag{1}$$

Indeed, we can choose f' so that

$$\int_{f'(x) \neq g(x)} \left(1 + |Dg(x)|^p + |Df'(x)|^p\right) dx < \eta, \tag{2}$$

where $[p]$ is the greatest integer $\leqslant p$.

Remarks. (1) The proof may be easier to follow if the reader keeps in mind the special case $p = 2$, $\dim M = 3$ (or even $p = 1$, $\dim M = 2$).

(2) N need not be a manifold: any local lipschitz neighborhood retract will do.

Proof. It will be convenient for us to work not with a triangulation of M, but rather with a "cubeulation". That is, we will regard M as the union of m-dimensional cubes, each of which is *isometric* to the standard cube $[-1, 1]^m$ in $\mathbf{R}^m$, and any two which are either disjoint or else intersect along a common lower dimensional face.

(To see that such cubeulation is possible, suppose M has already been triangulated. Now connect the centroid of each triangle to the midpoints of its sides: this divides the triangle into three "squares". Next connect the centroid of each 3-simplex to the centroids of each of its four faces: this divides the 3-simplex into four 3-dimensional cubes, and so on. Equivalently, consider the standard m-dimensional simplex $\Delta = \{\langle x_1, \cdots, x_{m+1}\rangle : \Sigma x_i = 1,\ x_i \geqslant 0\}$ in $\mathbf{R}^{m+1}$. The map $x \to x/\max\{|x_i|\}$ projects Δ out onto those faces of $[0, 1]^{m+1}$ that contain $\langle 1, \cdots, 1\rangle$, thus dividing it into $(m + 1)$ cubes.)

Note that we may choose the metric on M in this way, since changing the metric (in a bilipschitz way) alters the $\|\cdot\|_{1,p}$ norm by a bounded factor.

Let $k = [p]$. We begin by defining a function $\|\cdot\|$: $[-1,1]^m \to \mathbf{R}$ as follows:

$$\|x\| = \min\{\max_{i\in S}|x_i|: S \text{ is a } (k+1)\text{-element subset of } \{1,2,\cdots,m\}\},$$

where $x = \langle x_1, x_2, \cdots, x_m, \rangle$. Thus, for example, if $k = 0$ then $\|x\|$ is the minimum $|x_i|$, and if $k = m - 1$ then $\|x\|$ is the maximum $|x_i|$. To understand $\|\cdot\|$ geometrically, observe that $\{x\colon \|x\| \leqslant \varepsilon\}$ is the set of points x such that $(k+1)$ or more of the coordinates of x lie in $[-\varepsilon, \varepsilon]$. Hence

$$\mathscr{H}^m\{x\colon \|x\| \leqslant \varepsilon\} \leqslant C_1\varepsilon^{k+1}, \tag{3}$$

$$\mathscr{H}^{m-1}\{x\colon \|x\| = \varepsilon\} \leqslant C_1\varepsilon^k \tag{4}$$

for some constant $C_1 = C_1(m)$. (Here $\mathscr{H}^i(S)$ is the i-dimensional measure of S.)

Since M consists of isometric copies of $[-1,1]^m$, we can also regard $\|\cdot\|$ as a function defined on M. Note that $\|\cdot\|$: $M \to R$ is lipschitz. Note also that $\{x \in M\colon \|x\| = 1\}$ is the k-skeleton M^k of M, and that $\{x\colon \|x\| \geqslant \varepsilon\}$ is a "tubular" neighborhood of M^k.

For $0 < \delta \leqslant \varepsilon \leqslant 1$ we will also need a lipschitz map $F_{\varepsilon,\delta}$: $M \to M$ such that:

$$F_{\varepsilon,\delta} \text{ leaves fixed the points in } \{x\colon \|x\| \geqslant \varepsilon\}, \tag{5}$$

$$F_{\varepsilon,\delta} \text{ retracts } \{x\colon \delta \leqslant \|x\| \leqslant \varepsilon\} \text{ onto } \{x\colon \|x\| = \varepsilon\} \tag{6}$$

and such that $F_{\varepsilon,\delta}$ expands $\{x\colon \|x\| \leqslant \delta\}$ to fill the gap. We define $F_{\varepsilon,\delta}$ by specifying it on each of the m-cubes of M:

$$F_{\varepsilon,\delta}\colon [-1,1]^m \to [-1,1]^m,$$
$$F_{\varepsilon,\delta}(x) = F_{\varepsilon,\delta}(x_1, x_2, \cdots, x_m) = (y_1, y_2, \cdots, y_m),$$
$$y_i = x_i \cdot \max\{1, \varepsilon/\max\{\|x\|, |x_i|, \delta\}\}.$$

Observe in addition to (5) and (6) that:

$$|DF_{\varepsilon,\delta}| \leqslant C_2 \cdot \varepsilon/\|x\| \quad \text{for } \delta \leqslant \|x\| \leqslant \varepsilon, \tag{7}$$

$$|DF_{\varepsilon,\delta}| \leqslant C_2 \cdot (\varepsilon/\delta) \quad \text{for } \|x\| \leqslant \delta. \tag{8}$$

Notice also that $F_{\varepsilon,\delta}$ is homotopic to the identity map on M. (Consider $F_{\varepsilon,t}$ with $\delta \leqslant t \leqslant \varepsilon$.)

Now let $0 < \varepsilon < 1$. Note that M^k is a deformation retract of $\{x\colon \|x\| \geqslant \varepsilon\}$ (by (6), $F_{1,\varepsilon}$ is a retraction). It follows that the restrictions of f and g to $\{x\colon \|x\| \geqslant \varepsilon\}$ are homotopic. Indeed (by the homotopy extension theorem, for example) f is homotopic to a lipschitz map f^* such that

$$f^*(x) = g(x) \quad \text{for } \|x\| \geqslant \varepsilon. \tag{9}$$

Let $0 < \delta < \varepsilon$ and write $f'(x) = f^*(F_{\varepsilon,\delta}(x))$. Then f' is homotopic to f. Note that if $\|x\| \geqslant \varepsilon$, then

$$f'(x) = f^*(F_{\varepsilon,\delta}(x)) = f^*(x) = g(x).$$

Thus

$$\{x\colon f'(x) \neq g(x)\} \subseteq \{x\colon \|x\| < \varepsilon\}.$$

Hence we can estimate the first two terms of (2):

$$\begin{aligned}\int_{f' \neq g} \left(1 + |Dg|^p\right) &\leqslant \left(1 + C_3(\operatorname{lip} g)^p\right) \int_{\|x\| < \varepsilon} dx \\ &\leqslant C_4\left(1 + (\operatorname{lip} g)^p\right)\varepsilon^{k+1} \quad \text{(by (3))}.\end{aligned} \tag{10}$$

Likewise the third term of (2) is

$$\int_{f' \neq g} |Df'|^p = \int_{\|x\| \leqslant \varepsilon} |Df'|^p = \int_{\delta \leqslant \|x\| \leqslant \varepsilon} |Df'|^p + \int_{\|x\| \leqslant \delta} |Df'|^p. \tag{11}$$

We estimate these two terms as follows:

$$\begin{aligned}\int_{\delta \leqslant \|x\| \leqslant \varepsilon} |Df'|^p &= \int_{\delta \leqslant \|x\| \leqslant \varepsilon} \left|D(f^* \circ F_{\varepsilon,\delta})\right|^p \\ &= \int_{\delta \leqslant \|x\| \leqslant \varepsilon} \left|D(g \circ F_{\varepsilon,\delta})\right|^p \leqslant (\operatorname{lip} g)^p \int_{\delta \leqslant \|x\| \leqslant \varepsilon} |DF_{\varepsilon,\delta}|^p \\ &= (\operatorname{lip} g)^p \int_\delta^\varepsilon \int_{\|x\| = r} |DF_{\varepsilon,\delta}|^p \, d\mathscr{H}^{m-1}x \, dr \\ &\leqslant (\operatorname{lip} g)^p \int_\delta^\varepsilon \left[C_2 \cdot (\varepsilon/r)\right]^p \cdot \mathscr{H}^{m-1}\{\|x\| = r\} \, dr \quad \text{(by (7))} \\ &\leqslant C_5(\operatorname{lip} g)^p \int_\delta^\varepsilon (\varepsilon/r)^p r^k \, dr \quad \text{(by (4))} \\ &\leqslant C_5(\operatorname{lip} g)^p \varepsilon^{k+1}/(k - p + 1).\end{aligned} \tag{12}$$

To estimate the second term of (11):

$$\begin{aligned}\int_{\|x\| \leqslant \delta} |Df'|^p = \int_{\|x\| \leqslant \delta} \left|D(f^* \circ F_{\varepsilon,\delta})\right|^p &\leqslant (\operatorname{lip} f^*)^p \int_{\|x\| \leqslant \delta} |DF_{\varepsilon,\delta}|^p \\ &\leqslant C_2(\operatorname{lip} f^*)^p (\varepsilon/\delta)^p \mathscr{H}^m\{\|x\| \leqslant \delta\} \quad \text{(by (8))} \\ &\leqslant C_1 C_2 (\operatorname{lip} f^*)^p \varepsilon^p \delta^{k+1-p} \quad \text{(by (3))}.\end{aligned} \tag{13}$$

Combine (10), (11), (12) and (13) to get

$$(14)\qquad \int_{f'\neq g}\left(1+|Dg|^p+|Df'|^p\right) \leqslant C_6\left(1+(\operatorname{lip} g)^p\right)\cdot\varepsilon^{k+1}+C_7\left(\operatorname{lip} f^*\right)^p\varepsilon^p\delta^{k+1-p}.$$

By choosing ε small, we can make the first term of the right side of (14) arbitrarily small. Now f^* depends on ε but not on δ, so once ε is chosen, f^* is fixed and we can make the second term of (14) arbitrarily small by choosing δ sufficiently small. this proves (2): (1) follows immediately.

Applications. Recall that if Φ is a function that assigns a number $\Phi(x, y, L) \geqslant 0$ to each $x \in M$, $y \in N$, and linear map $L\colon TM_x \to TN_y$ and if $f\colon M \to N$ is lipschitz, we write

$$\Phi(f)=\int_M \Phi(x, f(x), Df(x))\,dx.$$

Corollary 1. *If Φ satisfies $\Phi(x, y, L) \leqslant c\cdot(1+|L|^p)$ for some c, then*

$$\inf\{\Phi(g)\colon g \text{ is homotopic to } f\} = \inf\{\Phi(g)\colon g \text{ is } [p]\text{-homotopic to } f\}.$$

Corollary 2. *Suppose Φ satisfies $\Phi(x, y, L) \leqslant c\cdot(1+|L|^p)$ for some c and that $\Phi(\cdot, y, 0) \equiv 0$ for some y. If f is $[p]$-homotopic to a constant map, then*

$$\inf\{\phi(g)\colon g \text{ is homotopic to } f\} = 0.$$

2. The Poincaré inequality

In this section we prove that the usual Poincaré inequality holds in regular polyhedral complexes. It follows that it also holds (with a worse constant) in regular curvilinear polyhedral complexes (i.e. spaces that are bilipschitz equivalent to regular polyhedral complexes). In the next section we will apply it to the p-dimensional skeleton of a Riemannian manifold: such skeletons are always regular.

Definition. A p-dimensional polyhedral complex X is *regular* if for every connected open subset $U \subset X$, the set $U \setminus X^{(p-2)}$ is also connected (where X^i is the i-skeleton of X).

(For example several triangles meeting along a common edge form a regular complex, but two triangles with only a vertex in common do not.)

Proposition 1 (*global Poincaré inequality*). *Let X be a regular connected p-dimensional polyhedral complex and let $f\colon X \to \mathbf{R}^\nu$ be lipschitz. Then there is a $v \in \mathbf{R}^\nu$ such that*

$$\int_X |f(x)-v|^p \leqslant C\int_X |Df(x)|^p$$

(*where $C = C(X, \nu, p)$ does not depend on f*).

Proof. Let $D_1, \cdots, D_n$ be a list of all the p-cells of X so that each pair D_i, D_{i+1} have a common $(p-1)$-dimensional face. Of course it may be necessary to include a given p-cell several times in the list but such a listing is possible since X is regular. Now make a new complex X' as follows. Let $D_1', D_2', \cdots, D_n'$ be n distinct p-cells such that D_i' is isometric to D_i (for each $i \leqslant n$). Construct X' by joining each pair D_i', D_{i+1}' in the same way that D_i and D_{i+1} are joined. Let π: $X' \to X$ be the obvious covering. Note that π is surjective, locally an isometry, and that each point $x \in X$ has at most n preimages.

Since X' is bilipschitz homeomorphic to the unit p-dimensional ball B^p, we have by the ordinary Poincaré inequality [3, 7.45] for B^p:

$$\int_{X'} |f \circ \pi - v|^p \leqslant c_0 \int_{X'} |D(f \circ \pi)|^p$$

for some $v \in \mathbf{R}^\nu$. But since π is surjective and locally an isometry

$$\int_X |f - v|^p \leqslant \int_{X'} |f \circ \pi - v|^p,$$

and since cardinality $(\pi^{-1}(x)) \leqslant n$

$$\int_{X'} |D(f \circ \pi)|^p \leqslant n \int_X |Df|^p.$$

Hence,

$$\int_X |f - v|^p \leqslant c_0 n \int_X |Df|^p.$$

Proposition 2 (*local Poincaré inequality*). *Let X be a regular p-dimensional complex. Write*

$$\mathscr{B}(x, r) = \{y \in X\colon \operatorname{dist}(y, x) \leqslant r\},$$

where $\operatorname{dist}(y, x)$ *is the geodesic distance from y to x in X. Then there is a $c = c(X, \nu, p)$ such that for any $x \in X$, $r > 0$, and lipschitz function f: $\mathscr{B}(x, 3r) \to \mathbf{R}^\nu$ there exists a $v \in \mathbf{R}^\nu$ so that*

$$r^{-p} \cdot \int_{\mathscr{B}(x,r)} |f - v|^p \leqslant c \cdot \int_{\mathscr{B}(x,3r)} |Df|^p.$$

Proof. First observe that for each $x \in X$ there is an $R(x) > 0$ such that if $0 < r < R(x)$, then $\mathscr{B}(x, r)$ is similar to $\mathscr{B}(x, R(x))$. (I.e. there is a bijection from $\mathscr{B}(x, r)$ to $\mathscr{B}(x, R(x))$ which multiplies all distances by $R(x)/r$.)

If follows that there is an $R > 0$ with the following property. If $0 < r < R$ and $x \in X$, then $\mathscr{B}(x, 3r)$ is similar to $\mathscr{B}(y, 3R)$ for some $y \in X$. To see this, let $3R$ be the Lebesgue number corresponding to the open cover $\{\operatorname{int} \mathscr{B}(z, R(z)/2)\colon z \in X\}$. Then $r < R$ implies that $\mathscr{B}(x, 3r)$ is contained in some $\mathscr{B}(z, R(z)/2)$ and hence is similar to some larger neighborhood.

Now let $\mathscr{B}(x_i, r_i)$ $(1 \leqslant i \leqslant n)$ be a finite collection of neighborhoods such that if $r \geqslant R$ and $x \in M$, then for some i,

$$\mathscr{B}(x, r) \subseteq \mathscr{B}(x_i, r_i), \quad \text{where } r_i < (3/2)r,$$

and hence

$$\mathscr{B}(x, r) \subseteq \mathscr{B}(x_i, r_i) \subseteq \mathscr{B}(x, 3r). \tag{1}$$

Now let $r > 0$.

Case 1: $r \geqslant R$. Then (1) holds for some i. By the global Poincaré inequality we have

$$\int_{\mathscr{B}(x_i, r_i)} |f - v|^p \leqslant c_i \int_{\mathscr{B}(x_i, r_i)} |Df|^p$$

for some $v \in \mathbf{R}^\nu$. But

$$\int_{\mathscr{B}(x, r)} |f - v|^p \leqslant \int_{\mathscr{B}(x_i, r_i)} |f - v|^p, \quad \int_{\mathscr{B}(x_i, r_i)} |Df|^p \leqslant \int_{\mathscr{B}(x, 3r)} |Df|^p$$

so

$$\int_{\mathscr{B}(x, r)} |f - v|^p \leqslant c_i \int_{\mathscr{B}(x, 3r)} |Df|^p, \tag{2}$$

and since $r \geqslant R$,

$$r^{-p} \int_{\mathscr{B}(x, r)} |f - v|^p \leqslant R^{-p} c_i \int_{\mathscr{B}(x, 3r)} |Df|^p. \tag{3}$$

Case 2: $r < R$. Then there is a $y \in X$ and a similarity ϕ: $\mathscr{B}(y, 3R) \to \mathscr{B}(x, 3r)$. Now apply Case 1 to $f \circ \phi$ to get a $v \in \mathbf{R}^\nu$ such that (by (2))

$$\int_{\mathscr{B}(y, R)} |f \circ \phi - v|^p \leqslant c_i \int_{\mathscr{B}(y, 3R)} |D(f \circ \phi)|^p.$$

But since ϕ is a similarity,

$$R^{-p} \int_{\mathscr{B}(y, R)} |f \circ \phi - v|^p = r^{-p} \int_{\mathscr{B}(x, r)} |f - v|^p,$$

$$\int_{\mathscr{B}(y, 3R)} |D(f \circ \phi)|^p = \int_{\mathscr{B}(x, 3r)} |Df|^p.$$

Hence

$$r^{-p} \int_{\mathscr{B}(x, r)} |f - v|^p \leqslant c_i R^{-p} \int_{\mathscr{B}(x, 3r)} |Df|^p.$$

Thus if we let $c = R^{-p}\max\{c_i\}$ we get the desired conclusion in Case 1 (see (3)) and in Case 2.

3. A converse to the main theorem

Lemma. *Let M be a compact Riemannian manifold. X a k-dimensional Borel subset of M, and F: $M \to [0, \infty]$ a Borel function. Then there is a diffeomorphism ϕ: $M \to M$ homotopic (indeed isotopic) to the identity such that*

$$\int_X F \circ \phi \, d\mathscr{H}^k \leqslant c\mathscr{H}^k(X) \int_M F,$$

where $c = c(M)$ depends only on M.

Proof. By the Nash embedding theorem we may regard M as a submanifold of a euclidean space $\mathbf{R}^\nu$. Let $U = \{x \in \mathbf{R}^\nu: \operatorname{dist}(x, M) \leqslant \varepsilon\}$, where $\varepsilon > 0$ is such that the nearest point retraction π: $U \to M$ is well defined and smooth on U. Note that

$$\int_U F(\pi(x)) \, d\mathscr{L}x \leqslant c_1 \int_M F$$

for some $c_1 = c_1(M, U)$. Now

$$\begin{aligned}\int_{|v|<\varepsilon} \int_{x \in X} F(\pi(x+v)) \, dx \, dv &= \int_{x \in X} \int_{|v|<\varepsilon} F(\pi(x+v)) \, dv \, dx \\ &\leqslant \int_{x \in X} \int_{z \in U} F(\pi(z)) \, dz \, dx \\ &= \mathscr{H}^k(X) \int_U F \circ \pi \\ &\leqslant c_1 \mathscr{H}^k(X) \int_M F,\end{aligned}$$

so for some $v \in \mathbf{R}^\nu$, $|v| < \varepsilon$,

$$\int_{x \in X} F(\pi(x+v)) \, d\mathscr{H}^k x \leqslant c\mathscr{H}^k(X) \int_M F.$$

Thus $\phi(x) = \pi(x+v)$ defines the desired diffeomorphism.

Theorem 2. *Let M be a compact Riemann manifold, N a compact submanifold of $\mathbf{R}^\nu$, and g: $M \to \mathbf{R}^\nu$ a $W^{1,p}$ map such that $g(x) \in N$ for every (sic)$x \in M$. There is an $\varepsilon > 0$ such that if f_1, f_2: $M \to N$ are lipschitz and $\|f_i - g\|_{1,p} < \varepsilon$ $(i = 1, 2)$, then f_1 and f_2 are $[p]$-homotopic.*

Remark. Actually N need not be compact, or even a manifold. The proof requires only that N have a neighborhood U, with $\operatorname{dis}(N, \partial U) > 0$, such that U retracts onto N.

Proof. We may assume that p is an integer since the $\|\cdot\|_{1,[p]}$ norm is dominated by the $\|\cdot\|_{1,p}$ norm. Let f_i: $M \to N$ be a sequence of lipschitz maps such that

$$(1) \qquad \|f_i - g\|_{1,p} < 2^{-i}.$$

It suffices to show that there is an I such that if i, $j \geqslant I$, then f_i and f_j are $[p]$-homotopic.

Define a function F: $M \to [0, \infty]$ by

$$(2) \qquad F(x) = |f_1(x)| + \sum |f_{i+1}(x) - f_i(x)| + |Df_1(x)| + \sum |Df_{i+1}(x) - Df_i(x)|.$$

By (1) (and the Minkowski inequality), $\int_M F^p < \infty$. Note also that for all i,

$$(3) \qquad |f_i(x)| + |Df_i(x)| \leqslant F(x).$$

Let X be the p-skeleton of some triangulation of M. By the lemma there is a diffeomorphism ϕ: $M \to M$ such that

$$(4) \qquad \int_X (F \circ \phi)^p \, d\mathscr{H}^p < \infty.$$

For simplicity, let us assume that ϕ is the identity (otherwise replace F, f_i, and g by $F \circ \phi$, $f_j \circ \phi$, and $g \circ \phi$ in the following argument).

Let $\varepsilon > 0$. Then there is an $R > 0$ such that for every $x \in X$

$$\int_{\mathscr{B}(x,3R)} F(y)^p \, d\mathscr{H}^p y < \varepsilon,$$

where $\mathscr{B}(x, r)$ is the geodesic ball of radius r in X. Hence by (3)

$$(5) \qquad \int_{\mathscr{B}(x,3R)} |Df_j|^p < \varepsilon,$$

where here and in the following, integration is with respect to $d\mathscr{H}^p$.

For $r > 0$ and $x \in X$, let $H_i(r, x)$ be the vector $v \in \mathbf{R}^\nu$ that minimizes $\int_{\mathscr{B}(x,r)} |f_i - v|^p$. (If $p > 1$, this v is unique since the L^p norm is strictly convex. For $p = 1$, let $H_i(r, x)$ be the average of f_i over $\mathscr{B}(x, r)$.) Note that $\lim H_i(r, x) = f_i(x)$ as $r \to 0$, so let $H_i(0, x) = f_i(x)$.

By (5) and the local Poincaré inequality,

$$(6) \qquad r^{-p} \int_{\mathscr{B}(x,r)} |f_i(y) - H_i(r, x)|^p \, d\mathscr{H}^p y \leqslant c\varepsilon$$

whenever $r \leqslant R$. Consequently

(7) $$\operatorname{dist}\big(H_i(r,x),N\big) \leqslant (c'\varepsilon)^{1/p} \quad \text{if } r \leqslant R,$$

where $c' = c \cdot \max\{r^p/\mathscr{H}^p(\mathscr{B}(y,r))\colon y \in X, r \leqslant R\}$. Also

$$\begin{aligned}
&\big|H_i(R,x) - H_j(R,x)\big| \cdot \mathscr{H}^p(\mathscr{B}(x,R))^{1/p} \\
&\qquad = \left(\int_{\mathscr{B}(x,R)} \big|H_i(R,x) - H_j(R,x)\big|^p \, d\mathscr{H}^p y\right)^{1/p} \\
&\qquad \leqslant \left(\int_{\mathscr{B}(x,R)} \big|H_i(R,x) - f_i(y)\big|^p \, d\mathscr{H}^p y\right)^{1/p} \\
&\qquad\quad + \left(\int_{\mathscr{B}(x,R)} |f_i - f_j|^p\right)^{1/p} \\
&\qquad\quad + \left(\int_{\mathscr{B}(x,R)} \big|f_j(y) - H_j(R,x)\big|^p \, d\mathscr{H}^p y\right)^{1/p}.
\end{aligned}$$

Hence by (6)

(8) $$\big|H_i(R,x) - H_j(R,x)\big| \leqslant 2R(c\varepsilon)^{1/p} + \left(\int_X |f_j - f_i|^p\right)^{1/p}.$$

By (2) and (4) we see that $\{f_j \mid X\}$ is a Cauchy sequence in $L^p(X,\mathbf{R}^\nu)$. Thus (8) implies that there is an $I = I(\varepsilon)$ such that

(9) $$\big|H_i(R,x) - H_j(R,x)\big| \leqslant 3R(c\varepsilon)^{1/p} \quad \text{if } i,\ j \geqslant I.$$

Now let U be a neighborhood of N in $\mathbf{R}^\nu$ that retracts onto N, let the distance from N to ∂U be 3δ, and choose $\varepsilon > 0$ so that the right-hand sides of (7) and (9) are less than δ. Consider the homotopy $H_i\colon [0,R] \times X \to \mathbf{R}^\nu$ from $f_i| X$ to $g_i(\cdot) = H_i(R,\cdot)$. By (7), the image of H_i is contained in U, so $f_i \mid X$ is homotopic to g_i in U. By (7) and (9),

$$\operatorname{dist}\big(g_i(x),N\big) \leqslant \delta, \qquad \big|g_i(x) - g_j(x)\big| \leqslant \delta$$

for $i,\ j \geqslant I$. Hence the line segment joining $g_i(x)$ to $g_j(x)$ lies in U. Thus g_i and g_j (and therefore also $f_i \mid X$ and $f_j \mid X$) are homotopic in U. But U retracts onto N, so $f_i \mid X$ and $f_j \mid X$ are actually homotopic in N.

Corollary. *Suppose $\Phi(x,y,L) \geqslant C \cdot |L|^p$ for every $x \in M$, $y \in N$, and $L\colon TM_x \to TN_y$. If $f_0\colon M \to N$ is lipschitz, and*

$$\inf\left\{\int_M \Phi(x,f(x),Df(x))\,dx\colon f \text{ homotopic to } f_0\right\} = 0,$$

then f_0 is $[p]$-homotopic to a constant map.

Proof. Apply the theorem with $g =$ a constant map. (Use the Poincaré inequality [3, 7.45] on M to obtain the g.)

4. Manifolds with boundary: the Dirichlet problem

Now suppose M is a compact Riemannian manifold with boundary. In this section we show that Theorems 1 and 2 and their corollaries remain true provided we replace "homotopic" and "$[p]$-homotopic" by "homotopic $(\operatorname{rel}\partial M)$" and "$[p]$-homotopic $(\operatorname{rel}\partial M)$". First we recall the definition of relative homotopy.

Definition. The maps f, g: $M \to N$ are homotopic $(\operatorname{rel}\partial M)$ if there is a homotopy H: $[0,1] \times M \to N$ from f to g such that

$$H(t,x) = f(x) = g(x) \quad \text{for } x \in \partial M.$$

We say that f and g are k-homotopic $(\operatorname{rel}\partial M)$ provided there is a homotopy

$$H\colon [0,1] \times (\partial M \cup M^k) \to N$$

such that $H(0,\cdot) = f(\cdot)$, $H(1,\cdot) = g(\cdot)$, and $H(t,x) = f(x) = g(x)$ for $x \in \partial M$, where M^k is the k-skeleton of M. By the homotopy extension theorem this is equivalent to the existence of a homotopy H: $[0,1] \times M \to N$ such that

$$H(0,x) = f(x) \quad \text{for } x \in M,$$
$$H(1,x) = g(x) \quad \text{for } x \in M^k,$$
$$H(t,x) = f(x) = g(x) \quad \text{for } x \in \partial M.$$

In the proof of Theorem 1, only the definition of the function $\|\cdot\|$ needs to be modified. First note that M can be cubeulated so that each m-dimensional cube of M is either disjoint from ∂M or else intersects ∂M in one of its $(m-1)$-dimensional faces. (To see this, first cubeulate M in any way. Then $([0,1] \times \partial M) \cup (\{1\} \times M)$ is homeomorphic to M and clearly has a cubeulation with the desired property.)

On each m-cube of M disjoint from ∂M, define $\|\cdot\|$ exactly as before. If on the other hand a cube $[-1,1]^m$ intersects ∂M along the face $\{1\} \times [-1,1]^{m-1}$, then we define $\|x\|$ to be the maximum of x_1 and the previously defined value of $\|x\|$. Note that $\{x \in M\colon \|x\| = 1\}$ is now $M^k \cup \partial M$.

The modification in the proof of Theorem 2 is slightly more complicated. Let M' be M with a collar attached:

$$M' = ([0,1] \times \partial M) \cup (\{1\} \times M),$$

and let M'' be the double of M':

$$M'' = ([-1,1] \times \partial M) \cup (\{-1,1\} \times M).$$

Extend f_i, g, and F to M'' in the obvious way, e.g. $g(\langle t, x\rangle) = g(x)$. Apply the lemma of §3 to M'' to get a diffeomorphism ϕ of M'' so that

$$\int_{X''} (F \circ \phi)^p \, d\mathscr{H}^p < \infty,$$

where X'' is the p-skeleton of M''. As before, we may assume that ϕ is the identity (otherwise replace f_i, g, F, and M' by $f_i \circ \phi$, $g \circ \phi$, $F \circ \phi$, and $\phi^{-1}(M')$). Now let $X' = \phi(X'') \cap M'$ and let $Y = X' \cup \partial M'$. Define H_i: $[0, R] \times Y \to \mathbf{R}^\nu$ as follows. For $r \leqslant \operatorname{dist}(x, \partial M')$, let $H_i(r, x)$ be the $v \in \mathbf{R}^\nu$ which minimizes

$$\int_{\mathscr{B}'(r,x)} |f_i - v|^p \, d\mathscr{H}^p,$$

where $\mathscr{B}'(x, r)$ is the geodesic ball in X'. If $r > \operatorname{dist}(x, \partial M')$, let $H_i(r, x)$ be $H_i(\operatorname{dist}(x, \partial M'))$.

Essentially the same argument as before shows that for i, $j \geqslant$ some I, f_i and f_j are p-homotopic ($\operatorname{rel} \partial M'$) as maps from M' to N. It follows that they are p-homotopic ($\operatorname{rel} \partial M$) as maps from M to N.

5. Sections of nontrivial bundles

A map from M to N may be regarded as a section of the trivial N-bundle over M. One can also consider energy functionals on sections of more general bundles. Specifically, suppose for each $x \in M$ there is assigned a compact smooth submanifold N_x of $\mathbf{R}^\nu$ in such a way that N_x depends smoothly on x. A map f: $M \to \mathbf{R}^\nu$ is said to be a *section* (of the bundle) if $f(x) \in N_x$ for every $x \in M$.

Two sections f and g are said to be *homotopic through sections* if there is a homotopy H: $[0,1] \times M \to \mathbf{R}^\nu$ such that $H(t, x) \in N_x$ for all x and t (i.e. such that each $H(t, \cdot)$ is a section).

We claim that Theorems 1 and 2 remain true provided we replace "map" by "section", "homotopic" by "homotopic through sections", and "$[k]$-homotopic" by "$[k]$-homotopic through sections". We now indicate the necessary modifications of the proofs.

For each $x \in M$, let $\mathscr{R}(x, \cdot)$ be a retraction of a neighborhood of N_x onto N_x, and let $\delta_0 > 0$ be such that $\mathscr{R}(x, y)$ is defined and lipschitz in both arguments whenever $\operatorname{dist}(y, N_x) < 3\delta_0$. (Actually we do not need to assume N_x is smooth; we only need the existence of $\mathscr{R}$ and δ_0.) Without loss of generality assume that

$$\operatorname{dist}(z, N_y) \leqslant |x - y| \quad \text{if } z \in N_x \text{ and } x, y \in M.$$

To begin with, we need a homotopy extension theorem for sections: that is, if $A \subset B$ and $H\colon (\{0\} \times B) \cup ([0,1] \times A) \to N$ is a lipschitz map that satisfies

$$(*) \qquad H(t,x) \in N_x,$$

then H may be extended to a lipschitz map defined on all of $[0,1] \times B$ and still satisfying $(*)$. Note we only use this theorem when B is a polyhedral complex and A is a union of closed cells (of various dimensions) of B. By subdivision we may assume each cell of B has diameter $\leqslant \delta_0$. By induction, it suffices to prove the result when B is a k-cell and A is its boundary. But in that case if

$$\tilde{H}\colon [0,1] \times B \to \bigcup_{x \in B} N_x$$

is any extension of H, then $(x,t) \mapsto \mathscr{R}(x, \tilde{H}(t,x))$ will be an extension satisfying $(*)$.

Now the proof of Theorem 1 is essentially as before, except that we replace f' by the map $x \mapsto \mathscr{R}(x, f'(x))$. (If $\varepsilon < \delta_0$ this is well defined.)

As for Theorem 2, the proof is the same up to line (7). Note that

$$\operatorname{dist}(f_i(y), N_x) \leqslant \operatorname{dist}(y,x) \leqslant R \quad \text{if } y \in \mathscr{B}(x,R),$$

and therefore that

$$r^{-p}\int_{\mathscr{B}(x,r)} \operatorname{dist}(f_i(y), N_x)^p \leqslant cR^p \qquad (r \leqslant R).$$

Consequently by (6) (in the proof of Theorem 2),

$$(7)' \qquad \begin{aligned} r^{-p}\int_{\mathscr{B}(x,r)} \operatorname{dist}(H_i(x,t), N_x)^p &< (cR^p + c\varepsilon), \\ \operatorname{dist}(H_i(x,r), N_x) &< c''(R^p + \varepsilon)^{1/p} < \delta_0, \end{aligned}$$

provided R and ε are sufficiently small (independent of i). Then the proof as before (with δ_0 replacing δ) gives a homotopy L from $f_i \mid X$ to $f_j \mid X$ such that $\operatorname{dist}(L(t,x), N_x) \leqslant 2\delta$. Now $(t,x) \mapsto \mathscr{R}(x, L(t,x))$ is the desired homotopy through sections.

6. An example

The author is grateful to J. Eells for pointing out the following consequence of the preceding theorems.

Theorem 3. *Let M and N be compact connected Riemannian manifolds. If*

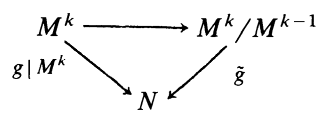
(1) $\pi_1(M) = \pi_2(M) = 0$, *or*

(2) $\pi_1(M) = \pi_2(N) = 0$, *or*

(3) $\pi_1(N) = \pi_2(N) = 0$,
then every lipschitz map f: $M \to N$ is homotopy to maps g with $\int_M |Dg|^2$ arbitrarily small.

Lemma. *If g: $M \to N$ is $(k-1)$-homotopic to a constant map, and $\pi_k(N) = 0$, then g is k-homotopic to a constant map.*

Proof of Lemma. By hypothesis, we may assume that g maps the $(k-1)$-skeleton M^{k-1} to a point $x_0 \in N$. Thus we can find a map $\tilde{g}$: $M^k/M^{k-1} \to N$ so that the following diagram commutes:



Now M^k/M^{k-1} is a collection of k-spheres joined at a point so, since $\pi_k(N) = 0$, $\tilde{g}$ is homotopic to a constant map. Therefore $g\,|\,M^k$ is also.

Proof of Theorem 3. We must show that f is homotopic to a map f' such that $f'(M^k)$ is a point. (See Corollary 2 (§1).)

Case (1). By the Lemma, there is a map ϕ: $M \to M$ homotopic to the identity map such that $\phi\,|\,M^2$ is constant. Then $f \circ \phi$ is homotopic to f and $f \circ \phi\,|\,M^2$ is constant.

Case (2). By the Lemma, there is a map ϕ: $M \to M$ homotopic to the identity map such that $\phi\,|\,M^1$ is constant. Now apply the Lemma to $f \circ \phi$ to get the desired map g.

Case (3). Apply the Lemma twice to f. q.e.d.

Note that the conclusion does not follow from assuming $\pi_2(M) = \pi_1(N) = 0$. For example, let M and N be the two-dimensional torus and sphere, respectively, and let f be any degree 1 map. Then $\pi_2(M) = \pi_1(N) = 0$, but f is not 2-homotopic to a constant map.

The proof easily generalizes to give:

Theorem 3′. *Suppose that $\Phi(x, y, L) \leqslant c \cdot (1 + |L|^p)$ for some c and that $\Phi(\cdot, y, 0) \equiv 0$ for some y. If for some j*

$$\pi_i(M) = 0 \quad (i \leqslant j), \qquad \pi_i(N) = 0 \quad (j < i \leqslant [p]),$$

then every map f: $M \to N$ is homotopic to maps g with $\Phi(g)$ arbitrarily small.

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