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CURVATURE CHARACTERIZATION OF COMPACT HERMITIAN SYMMETRIC SPACES

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In the study of complex manifolds the following conjecture is a well-known and natural analogue of the elliptic case of the uniformization theorem.

Conjecture I. Suppose X is a compact Kähler manifold of nonnegative holomorphic bisectional curvature and positive Ricci curvature. Then X is biholomorphic to a compact Hermitian symmetric space.

The special case, when X is of positive bisectional curvature and conjectured to be $\mathbf{P}^n$, is the Frankel conjecture, resolved simultaneously and independently by Mori [19] and Siu & Yau [22] in 1979 using very different methods. The general case of Conjecture I is at present still open. A related conjecture in case X is assumed to be Kähler-Einstein is the following.

Conjecture II. Suppose X is a compact Kähler-Einstein manifold of nonnegative holomorphic bisectional curvature and positive Ricci curvature. Then X is *isometric* to a compact Hermitian symmetric space.

The first efforts to resolve Conjecture II were due to Berger [3], who showed in 1966 that a compact Kähler-Einstein manifold of positive sectional curvature is isometric to $\mathbf{P}^n$ and equipped with the Fubini-Study metric (up to a scalar factor). This was reformulated by Goldberg and Kobayashi to the case of positive holomorphic bisectional curvature. Later, Gray [8] proved Conjecture II in 1973 under the stronger assumption of nonnegative Riemannian sectional curvature. He introduced on the unit sphere bundle of X a (degenerate) elliptic operator D and developed a Bochner-Kodaira formula for DR, R denoting the curvature tensor, to prove the vanishing of ∇R on X. The last property is the simplest characterization of locally symmetric spaces in terms of the curvature tensor. Apparently, there are serious difficulties in modifying Gray's argument to the general case of nonnegative holomorphic bisectional

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curvature since D will in general not be (degenerate) elliptic. This has left Conjecture II open for a long time. It was one of the open questions in Kähler geometry raised by Siu [21] in his address in 1983 to the International Congress of Mathematics at Warsaw.

One connection between Conjectures I and II is inspired by the work of Hamilton [10] on deforming Riemannian metrics of positive curvature on a compact 3-manifold to an Einstein metric. It is hoped that such an approach can be applied to compact Kähler manifolds of nonnegative holomorphic bisectional curvature. In this connection we refer the reader to a recent article of Bando [2], who used the evolution equation of Hamilton [10] and results of Siu [20] on characterizing hyperquadrics to obtain an affirmative answer to Conjecture I in the case of dimension 3. (The cases of dimensions 1 and 2 are well known.)

In this article we resolve Conjecture II in the affirmative. Our starting point is the method of Berger [3] on characterizing $\mathbf{P}^n$ with the Fubini-Study metric. He did this by showing that the Kähler manifold X under consideration has constant holomorphic sectional curvature. To do this, he considered a point x_0 on X and a unit tangent vector α of type $(1,0)$ at x_0, where the global maximum of holomorphic sectional curvatures is attained, and applied the maximum principle to $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x_0)$. For Conjecture II, we used the characterization of Hermitian symmetric spaces by the vanishing of ∇R, a property not verifiable by a direct application of Berger's method. In a similar setting as above, assuming X Kähler-Einstein of nonnegative bisectional curvature at $\alpha \in T^{1,0}_{x_0}(X)$, one can show that relative to the Hermitian bilinear form $H_\alpha(\xi, \xi') = R_{\alpha\bar{\alpha}\xi\bar{\xi}}(x_0)$, $T^{1,0}_{x_0}(X)$ decomposes into the orthogonal direct sum of eigenspaces $\mathbf{C}\alpha \oplus \mathscr{H}_\alpha \oplus \mathscr{N}_\alpha$, where $R_{\alpha\bar{\alpha}\xi\bar{\xi}}(x_0) = \frac{1}{2}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x_0)$ for $\xi \in \mathscr{H}_\alpha$, $= 0$ for $\xi \in \mathscr{N}_\alpha$ and moreover $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x_0) = 0$.

Our idea is to prove first of all the invariance of $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ under parallel transport of α along certain curves emanating from x_0. To start with we prove, using the maximum principle, that the global maximum of holomorphic sectional curvatures is attained at every point $x \in X$. Let γ be an integral curve of any vector field of "maximal directions" $\alpha(x)$; we prove the stronger fact that the curvature tensor R is invariant under parallel transport along the curve γ. Using an orthonormal basis $\{e_i\}$ at $x \in \gamma$ consisting of eigenvectors of the Hermitian form $H_\alpha(\xi, \xi') = R_{\alpha\bar{\alpha}\xi\bar{\xi}'}(x)$, $\alpha = \alpha(x)$, we shall actually prove the vanishing of all terms $\nabla_\alpha R_{i\bar{j}k\bar{l}}(x)$. The proof of the vanishing of such covariant derivatives will occupy the bulk of the present article.

Our original aim was to prove that $\nabla^i_\eta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$ at a global maximal direction α for all real tangent vectors η at x and all positive integers i. Since the Kähler-Einstein metric is real-analytic, this would allow us to conclude the invariance of $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ under parallel transport along geodesics. Although this scheme is too involved for higher order radial derivatives, it will be enough to show $\nabla_\eta^i R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$ for $1 \leqslant i \leqslant 7$, which is sufficient to imply the invariance of R under parallel transport along integral curves of maximal directions. The point of departure is the observation that Berger's formula implies $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$ and hence $\nabla_\eta^i R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$ for $1 \leqslant i \leqslant 3$ in view of the global maximality of $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$. It follows that $\nabla_\eta^4 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \geqslant 0$. Define a $(2k)$th order elliptic operator $S^{(2k)}$ on smooth tensors T by taking, at each point y where T is defined, $S^{(2k)}T$ to be the average, suitably normalized, of $\nabla_\eta^{2k}T$ over all $\eta \in T_y(X)$ of unit length. Clearly, $S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \leqslant 0$. On the other hand, we show that $\Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \geqslant 0$. For the Euclidean case, $S^{(4)}$ agrees with Δ^2. However, for Kähler manifolds in general $S^{(4)}$ differs from Δ^2 by some zero-order terms. Such zero-order terms are obtained by a number of commutations. At x we have sufficient knowledge of zero-order terms to conclude that $S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = \Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$, implying both are zero and that $\nabla_\eta^i R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$ for $1 \leqslant i \leqslant 5$.

To proceed further one can consider similarly $S^{(2k)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$ and $S^{(2k-2)}\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$. In general the difference between $S^{(2k)}$ and $S^{(2k-2)}\Delta$ is a differential operator of order $(2k - 4)$. We are able to prove in a way similar to the above that $S^{(6)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = S^{(4)}\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$, implying $\nabla_\eta^i R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$ for $1 \leqslant i \leqslant 7$. This involves proving the vanishing of commutation terms which are second order covariant derivatives of terms of the type $R_{\alpha\bar{\alpha}\alpha\bar{k}}$ or $R_{\alpha\bar{\alpha}k\bar{k}}$. These are obtained from variation equalities or Taylor series expansions of curvature functions along geodesics issuing from x.

In order to prove the vanishing of $\nabla_\alpha R_{i\bar{j}k\bar{l}}(x)$ we make full use of gradient terms arising in formulas $\Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$ and $S^{(4)}\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$. To prove $\nabla_\alpha R_{i\bar{j}k\bar{l}}(x) = 0$ it will actually be necessary also to show $\Delta^3 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$ and to make use of gradient terms arising from $\Delta^3 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$. One surprising thing in this scheme of proof is that, under our special choice of basis at x, $\alpha \in T_x^{1,0}(X)$ a fixed maximal direction, we show that there are only a few types of nonvanishing curvature terms. Such information is also used in the proof of $\nabla_\alpha R_{i\bar{j}k\bar{l}}(x) = 0$.

At each $x \in X$ let V_x be the real linear subspace of $T_x^{1,0}(X)$ generated by the nonempty set of maximal directions $\alpha \in T_x(X)$. We can use the invariance of R under parallel transport along integral curves of vector fields of maximal directions to show that at some point x, the vector subspaces $\operatorname{Re} V_y \subset T_y(X)$ for adjacent points y constitute an integrable distribution. The integral submanifolds are moreover complex, totally geodesic and locally symmetric. Then, we use the theorem of Bonnet-Myers to show that these integral submanifolds extend to complex submanifolds of X for a suitable choice of x, and that they are mutually nonintersecting. We use this to show that the curvature tensor is reducible at each point, that the vector subspaces V_x, $x \in X$, constitute a differentiable vector bundle invariant under parallel transport and that the foliation of X by integral submanifolds of the distribution $x \mapsto \operatorname{Re} V_x$ actually corresponds to a global decomposition of X up to a finite covering. This allows us to prove Conjecture II inductively.

We believe that our analysis of the curvature tensor should also be useful in other problems in Kähler geometry related to locally symmetric Hermitian manifolds.

The main results of the present article, together with a sketch of the methods of proof, has appeared in Mok & Zhong [18].

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0. Statement of results

Our main theorem in this article is the following confirmation of Conjecture II (in the introduction).

Main Theorem. *Let X be a compact Kähler-Einstein manifold of nonnegative holomorphic bisectional curvature and positive Ricci curvature. Then X is* ***isometric*** *to a compact Hermitian symmetric space.*

We remark that the nonnegativity of holomorphic bisectional curvatures is strictly weaker than the nonnegativity of Riemannian sectional curvatures and that the former concept is more natural in the context of complex differential geometry. For a compact Kähler manifold, holomorphic bisectional curvatures are nonnegative if and only if the unit ball bundle of the dual tangent bundle is weakly pseudoconvex.

Our method of proof yields the following generalizations of the Main Theorem:

Corollary 1. *Let X be a compact Kähler manifold of nonnegative holomorphic bisectional curvature and constant scalar curvature. Then X is isometric to a compact Hermitian locally symmetric space.*

Corollary 2. *Let X be a complete Kähler manifold of nonnegative holomorphic bisectional curvature such that the Ricci tensor is parallel. Then X is isometric to a complete Hermitian locally symmetric space.*

From the proof given for the Main Theorem, it is immediate to generalize to the case when X is assumed to be of nonnegative holomorphic bisectional curvature and the Ricci tensor is parallel and positive. The proofs of the corollaries involve essentially a splitting of the flat directions of the Ricci tensor. Corollary 1 follows from Corollary 2 and results of Bishop & Goldberg [4]–[6] which assert that under the hypothesis of Corollary 1, the Ricci tensor is automatically parallel.

1. Background material

(1.1) The curvature tensor on Kähler manifolds and commutation formulas. Let X be a Kähler manifold. Denote by $R = R\langle\,\cdot\,,\cdot,\cdot,\cdot\rangle$ the Riemannian curvature tensor on the underlying Riemannian manifold. By complexification, R acts on $(\mathbf{C}T(X))^4$, $\mathbf{C}T(X)$ denoting the complexified tangent bundle. On X we have a decomposition of $\mathbf{C}T(X)$ into the orthogonal direction sum $T^{1,0}(X) \oplus T^{0,1}(X)$. If we choose a system of holomorphic coordinates $(z_1,\cdots,z_n)$ at $x \in X$, then $\{\partial/\partial z_1,\cdots,\partial/\partial z_n\}$ and $\{\partial/\partial \bar{z}_1,\cdots,\partial/\partial \bar{z}_n\}$ constitute bases of $T_x^{1,0}(X)$ and $T_x^{0,1}(X)$ respectively. In terms of the corresponding decomposition of tensors into (p,q)-types on a Kähler manifold, R is of type $(2,2)$. In terms of the basis $\{\partial/\partial z_1,\cdots,\partial/\partial z_n;\ \partial/\partial \bar{z}_1,\cdots,\partial/\partial \bar{z}_n\}$ of $\mathbf{C}T_{x_0}(X)$ and writing $R_{i\bar{j}k\bar{l}} = R\langle\partial/\partial z_i, \partial/\partial \bar{z}_j, \partial/\partial z_k, \partial/\partial \bar{z}_l\rangle$, etc., the only possible nonzero terms of R_{****} (indices with or without bars) are given by $R_{i\bar{j}k\bar{l}}$ and accompanying terms obtained by permutation of indices. We write $R_{i\bar{j}}$ for the Ricci curvature tensor in terms of coordinates. Our convention on R is such that $R_{1\bar{1}1\bar{1}} > 0$ for the Riemann sphere with the standard Hermitian metric of constant positive curvature.

We say that X is of nonnegative holomorphic bisectional curvature if $R\langle \xi, \bar{\xi}; \zeta, \bar{\zeta}\rangle \geqslant 0$ for all $x \in S$ and $\xi, \zeta \in T_x^{(1,0)}(X)$. In terms of indices this means that $\sum R_{i\bar{j}k\bar{l}} a_i \bar{a}_j b_k \bar{b}_l \geqslant 0$ for all n-tuples $(a_1,\cdots,a_n)$, $(b_1,\cdots,b_n)$ of complex numbers. Every Hermitian (globally) symmetric space carries on invariant Kähler-Einstein metric of nonnegative holomorphic bisectional curvature. In terms of the curvature tensor we have the following characterization of locally symmetric Riemannian manifolds.

Proposition (*cf. Kobayashi & Nomizu* [12]). *A Riemannian manifold X is locally symmetric at $x \in X$ if and only if in a neighborhood of x, $\nabla R \equiv 0$ for the Riemannian curvature tensor R.*

The curvature tensor measures analytically the commutation of covariant differentiations. For example, for a covariant tensor of the type $T_{i\bar{j}k\bar{l}}$, we have, denoting $\nabla_t \nabla_s T_{i\bar{j}k\bar{l}} = T_{i\bar{j}k\bar{l},st}$ etc.,

$$T_{i\bar{j}k\bar{l},st} = T_{i\bar{j}k\bar{l},ts}, \qquad T_{i\bar{j}k\bar{l},\bar{s}\bar{t}} = T_{i\bar{j}k\bar{l},\bar{t}\bar{s}},$$

$$T_{i\bar{j}k\bar{l},s\bar{t}} = T_{i\bar{j}k\bar{l},\bar{t}s} + R_{i\bar{\mu}s\bar{t}}T_{\mu\bar{j}k\bar{l}} - R_{\mu\bar{j}s\bar{t}}T_{i\bar{\mu}k\bar{l}}$$

$$+ R_{k\bar{\mu}s\bar{t}}T_{i\bar{j}\mu\bar{l}} - R_{\mu\bar{l}s\bar{t}}T_{i\bar{j}k\bar{\mu}}.$$

All three equations follow from the definition of R in terms of commutation of covariant differentiations. The first two are consequences of the fact that R is of type $(2,2)$.

In general, for any covariant tensor field $T_{**\cdots**}$, commutation for second-order covariant differentiation occurs only if we commute two indices of opposite type (one barred, one unbarred), in which case there are as many commutation terms as there are indices in T, the sign attached to a commutation term in $T_{**\cdots**,s\bar{t}} - T_{**\cdots,\bar{t}s}$ is positive if it corresponds to a substitution of an unbarred index in $T_{**\cdots**}$, and negative otherwise. This is simply because $-R_{\mu\bar{j}s\bar{t}} = R_{\bar{j}\mu s\bar{t}}$.

Finally we recall that the Bianchi identity implies the equality $R_{i\bar{j}k\bar{l},m} = R_{i\bar{j}m\bar{l},k}$ in the case of Kähler manifolds.

(1.2) Computation of $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$. At $x \in X$ fixed let $(z_1, \cdots, z_n)$ be a system of local holomorphic coordinates. For any smooth tensor T we shall denote by ΔT the operator $\sum_{i,j} g^{i\bar{j}}(\nabla_i \nabla_{\bar{j}} T + \nabla_{\bar{j}} \nabla_i T)$, where $g^{i\bar{j}}$ is the contravariant metric tensor. (See (1.3) for the meaning of ΔT and other averaging differential operators.) We recall here the computation of $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ in Berger [3] for any tangent vector α of type $(1,0)$.

Proposition (1.2). *Let $(z_1, \cdots, z_n)$ be a system of local holomorphic coordinates at $x \in X$ such that $g_{i\bar{j}}(x) = \delta_{ij}$ for the Kähler metric tensor $(g_{i\bar{j}})$. Then, denoting by ρ the Einstein constant, i.e. Ricci form $= \rho$(Kähler form), we have*

$$\frac{1}{2}\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \sum_{i,j} \left| R_{\alpha\bar{i}\alpha\bar{j}} \right|^2 + \rho R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - 2\sum_{i,j} \left| R_{\alpha\bar{\alpha}i\bar{j}} \right|^2.$$

Proof. Obviously the right-hand side is independent of the choice of holomorphic coordinates at x as long as they are unitary at x, and it suffices to prove the proposition for e_α of unit length. We may therefore choose $(z_1,\cdots,z_n)$ so that $\partial/\partial z_1 = \alpha$. Then, from

$$R_{\alpha\bar\alpha\alpha\bar\alpha,i\bar i} = R_{\alpha\bar\alpha\alpha\bar\alpha,\bar i i} + 2\sum_\mu R_{\alpha\bar\mu i\bar i}R_{\mu\bar\alpha\alpha\bar\alpha} - 2\sum_\mu R_{\mu\bar\alpha i\bar i}R_{\alpha\bar\mu\alpha\bar\alpha}$$

we have, using $\sum_i R_{\alpha\bar\mu i\bar i} = \rho\delta_{\alpha\mu}$,

$$\sum_i R_{\alpha\bar\alpha\alpha\bar\alpha,i\bar i} = \sum_i R_{\alpha\bar\alpha\alpha\bar\alpha,\bar i i} + 2\rho R_{\alpha\bar\alpha\alpha\bar\alpha} - 2\rho R_{\alpha\bar\alpha\alpha\bar\alpha} = \sum_i R_{\alpha\bar\alpha\alpha\bar\alpha,\bar i i}.$$

Hence

$$\begin{aligned}\frac{1}{2}\Delta R_{\alpha\bar\alpha\alpha\bar\alpha} &= \sum_i R_{\alpha\bar\alpha\alpha\bar\alpha,i\bar i} = \sum_i R_{\alpha\bar\alpha i\bar\alpha,\alpha\bar i}\\ &= \sum_i R_{\alpha\bar\alpha i\bar\alpha,\bar i\alpha} + \sum_{i,\mu} R_{\alpha\bar\mu\alpha\bar i}R_{\mu\bar\alpha i\bar\alpha} - \sum_{i,\mu} R_{\mu\bar\alpha\alpha\bar i}R_{\alpha\bar\mu i\bar\alpha}\\ &\quad + \sum_{i,\mu} R_{i\bar\mu\alpha\bar i}R_{\alpha\bar\alpha\mu\bar\alpha} - \sum_{i,\mu} R_{\mu\bar\alpha\alpha\bar i}R_{\alpha\bar\alpha i\bar\mu}\\ &= \sum_i R_{\alpha\bar\alpha i\bar i,\bar\alpha\alpha} + \sum_{i,j}\left|R_{\alpha\bar i\alpha\bar j}\right|^2 + \rho R_{\alpha\bar\alpha\alpha\bar\alpha} - 2\sum_{i,j}\left|R_{\alpha\bar\alpha i\bar j}\right|^2\end{aligned}$$

by the Bianchi identity.

Since the metric on X is Kähler-Einstein, we have

$$\sum_i R_{\alpha\bar\alpha i\bar i,\bar\alpha\alpha} = R_{\alpha\bar\alpha,\bar\alpha\alpha} = 0,$$

proving the proposition.

(1.3) Averaging operators of radial derivatives. Let T be a covariant smooth tensor of order m defined on an open subset U of X. At $x \in U$ let η be a real tangent vector of unit length. Let γ be a geodesic passing through x and within the cut locus of x with η tangent to γ at x. Let $v_1,\cdots,v_k$ be complexified tangent vectors at x and denote by the same symbols the vector fields defined on γ obtained by parallel transport. Then

$$\nabla_\eta^i T\langle v_1,\cdots,v_m\rangle(x) = \left(\nabla_\eta^i T\right)\langle v_1,\cdots,v_m\rangle(x)$$

since $\nabla_\eta v_j = 0$ for $1 \leqslant j \leqslant k$ along γ. Letting k be a positive integer, we define the operator $S^{(2k)}T$ by setting

$$S^{(2k)}T\langle v_1,\cdots,v_m\rangle(x) = c_{2k}\int_\eta \nabla_\eta^{2k}T\langle v_1,\cdots,v_m\rangle(x)$$

at each x where T is defined, where the integration is over the unit sphere of the real tangent space at x, endowed with the unique rotation-invariant metric of unit mass, and where c_{2k} is a constant to be determined. In case $k = 1$ we have

$$S^{(2)}T_{**\cdots**}(x) = c_2\int_\eta \nabla_\eta^2 T_{**\cdots**}(x).$$

Let $(z_1,\cdots,z_n)$ be a local holomorphic coordinate system unitary at x. Then $\eta = \xi + \bar{\xi}$ for some $\xi \in T_x^{1,0}(X)$ of length $1/\sqrt{2}$. Write $\xi = \Sigma_i a_i(\eta)\partial/\partial z_i$,

$$\begin{aligned} S^{(2)}T_{**\cdots**}(x) &= c_2\int_\eta \left(\nabla_{\Sigma_i a_i(\eta)\partial/\partial z_i + \overline{\Sigma_i a_i(\eta)\partial/\partial z_i}}\right)^2 T_{**\cdots**}(x) \\ &= 2\operatorname{Re} c_2\int_\eta \sum_{i,j} a_i(\eta)a_j(\eta)\nabla_i\nabla_j T_{**\cdots**}(x) \\ &\quad + c_2\int \sum_{i,j} a_i(\eta)\overline{a_j(\eta)}\left(\nabla_i\nabla_{\bar{j}} + \nabla_{\bar{j}}\nabla_i\right)T_{**\cdots**}(x). \end{aligned}$$

Denote

$$b_{ij} = \int_\eta a_i(\eta)a_j(\eta), \qquad b_{i\bar{j}} = \int_\eta a_i(\eta)\overline{a_j(\eta)}.$$

Consider on $T_x^{1,0}(X) \approx \mathbf{C}^n$ the transformation

$$(a_1,\cdots,a_i,\cdots,a_j,\cdots,a_n) \to \left(a_1,\cdots,e^{i\theta_i}a_i,\cdots,e^{i\theta_j}a_j,\cdots,a_n\right),$$

where the left-hand side stands for $\Sigma a_i\partial/\partial z_i$. This induces an orthogonal transformation $\eta \mapsto \eta'$ on $T_x(X) = \{\xi + \bar{\xi} \colon \xi \in T_x^{(1,0)}(X)\}$. It follows that

$$b_{ij} = \int_\eta a_i(\eta')a_j(\eta') = e^{i(\theta_i+\theta_j)}b_{ij},$$

$$b_{i\bar{j}} = \int_\eta a_i(\eta')\overline{a_j(\eta')} = e^{i(\theta_i-\theta_j)}b_{i\bar{j}}.$$

Choosing θ_i, θ_j suitably we see that $b_{ij} = 0$ for all i, j and that $b_{i\bar{j}} = 0$ unless $i = j$. By symmetry, clearly $b_{1\bar{1}} = \cdots = b_{n\bar{n}}$. These constants can be computed by taking T to be the function $\Sigma_i|z_i|^2$ and comparing coefficients. In any case $b_{\bar{1}\bar{i}} > 0$ and we choose $c_2 b_{1\bar{1}} = 1$, giving

$$S^{(2)}T_{**\cdots**} = \sum_i \nabla_i\nabla_{\bar{i}}T_{**\cdots**} + \sum_i \nabla_{\bar{i}}\nabla_i T_{**\cdots**},$$

i.e., $S^{(2)}T = \Sigma_i(\nabla_i\nabla_{\bar{i}} + \nabla_{\bar{i}}\nabla_i)T$. We use ΔT to denote $S^{(2)}T$. There are two related 4th order averaging differential operators, namely Δ^2 and $S^{(4)}$. We have

$$\begin{aligned} \Delta^2 T &= \sum_{i,j}(\nabla_i\nabla_{\bar{i}} + \nabla_{\bar{i}}\nabla_i)(\nabla_j\nabla_{\bar{j}} + \nabla_{\bar{j}}\nabla_j)T \\ &= \sum_{i,j} T_{,i\bar{i}j\bar{j}} + T_{,i\bar{i}\bar{j}j} + T_{,\bar{i}ij\bar{j}} + T_{,\bar{i}i\bar{j}j}. \end{aligned}$$

By the same argument as above we have

$$S^{(4)}T = c_4 \sum_i b_{i\bar{i}i\bar{i}} \left(T_{,(i\bar{i}+\bar{i}i)(i\bar{i}+\bar{i}i)} + T_{,ii\bar{i}\bar{i}} + T_{,\bar{i}\bar{i}ii} \right)$$

$$+ c_4 \sum_{i<j} b_{i\bar{i}j\bar{j}} \left(T_{,(i\bar{i}+\bar{i}i)(j\bar{j}+\bar{j}j)} + T_{,(j\bar{j}+\bar{j}j)(i\bar{i}+\bar{i}i)} \right.$$

$$+ T_{,(ij+ji)(\bar{i}\bar{j}+\bar{j}\bar{i})} + T_{,(\bar{i}\bar{j}+\bar{j}\bar{i})(ij+ji)}$$

$$\left. + T_{,(i\bar{j}+\bar{j}i)(j\bar{i}+\bar{i}j)} + T_{,(j\bar{i}+\bar{i}j)(i\bar{j}+\bar{j}i)} \right).$$

Here we are adopting the convention

$$T_{,(i\bar{i}+\bar{i}i)(j\bar{j}+\bar{j}j)} = T_{,i\bar{i}j\bar{j}} + T_{,i\bar{i}\bar{j}j} + T_{,\bar{i}ij\bar{j}} + T_{,\bar{i}i\bar{j}j},$$

etc. The equality above is obtained by noting that the only nonzero terms in b_{****} must come from 2 pairs of conjugates, e.g. $b_{i\bar{i}j\bar{j}}$, $b_{ij\bar{j}\bar{i}}$, etc., which follows from using the transformation

$$(a_1, \cdots, a_i, \cdots, a_j, \cdots, a_k, \cdots, a_l, \cdots, a_n)$$
$$\to \left(a_1, \cdots, e^{i\theta_i}a_i, \cdots, e^{i\theta_j}a_j, \cdots, e^{i\theta_k}a_k, \cdots, e^{i\theta_l}a_l, \cdots, a_n\right).$$

Obviously $b_{i\bar{i}j\bar{j}}$ remains unchanged when indices are permuted, but the corresponding covariant differentiation may differ because of the curvature. We write the expansion for $S^{(4)}T$ in a more uniform manner. Denote by S_4 the permutation group of 4 elements. For any $\sigma \in S_4$ and any 4th order covariant differentiation $T^{\sigma}_{,\alpha\beta\gamma\delta}$ (indices with or without bars), we denote by $T^{\sigma}_{,\alpha\beta\gamma\delta}$ the 4th order covariant derivative obtained by formally permuting the four indices using $\sigma \in S_4$. In this notation we can write

$$S^{(4)}T = c_4 \sum_i \frac{b_{i\bar{i}i\bar{i}}}{4} \sum_{\sigma \in S_4} T^{\sigma}_{,i\bar{i}i\bar{i}} + c_4 \sum_{i<j} b_{i\bar{i}j\bar{j}} \sum_{\sigma \in S_4} T^{\sigma}_{,i\bar{i}j\bar{j}}.$$

Note that in the original expression that there are 6 terms attached to $b_{i\bar{i}i\bar{i}}$ and 24 terms attached to $b_{i\bar{i}j\bar{j}}$, $i<j$. This accounts for the factor of $1/4$ in the first term of the new expression. Our main result in this section is the following:

Proposition (1.3). *For a suitable choice of positive constant c_2, we have the expansion*

$$6S^{(4)}T = \sum_{i,j} \sum_{\sigma \in S_4} T^{\sigma}_{,i\bar{i}j\bar{j}}.$$

Similarly, for any positive integer k

$$\frac{(2k)!}{2^k} S^{(2k)}T = \sum_{i_1, \cdots, i_k} \sum_{\sigma \in S_{2k}} T^{\sigma}_{,i_1\bar{i}_1 \cdots i_k\bar{i}_k}.$$

Proof. We will only prove the special case $k = 2$ since the proof of the general case is exactly the same. Recall that $b_{i\bar{i}j\bar{j}}$ is defined by

$$b_{i\bar{i}j\bar{j}} = \int_\eta a_i(\eta)\,\overline{a_i(\eta)}\,a_j(\eta)\,\overline{a_j(\eta)}\,.$$

Obviously $b_{i\bar{i}i\bar{i}} = b_{i'\bar{i}'i'\bar{i}'}$ for all i and i', $1 \leqslant i, i' \leqslant n$, and $b_{i\bar{i}j\bar{j}} = b_{i'\bar{i}'j'\bar{j}'}$ for $i \neq j$ and $i' \neq j'$. It follows that $S^{(4)}T$ must be of the form

$$(*) \qquad S^{(4)}T = c\sum_i \sum_{\sigma\in S_4} T^\sigma{}_{,i\bar{i}i\bar{i}} + c'\sum_{i\neq j}\sum_{\sigma\in S_4} T^\sigma{}_{,i\bar{i}j\bar{j}}.$$

We claim that in the Euclidean case $S^{(4)}T$ agrees with $\Delta^2 T$ for a suitable choice of c_{2k}. We denote the operators $S^{(4)}$ and Δ^2 in the Euclidean case by $S_0^{(4)}$ and Δ_0^2 respectively. Then, the symbol of the fourth-order operator $S_0^{(4)}$ with constant coefficients is a fourth-order polynomial on $\mathbf{C}^n$ (with coordinates ξ) in $\xi_1,\cdots,\xi_n;\ \bar{\xi}_1,\cdots,\bar{\xi}_n$ invariant under rotations, so that it must be a (positive) multiple of $(\sum_i|\xi_i|^2)^2$, the symbol of Δ_0^2, hence proving the claim. From now on we will choose the constant $c_{2k} > 0$ such that $S_0^{(4)} = \Delta_0^2$. Now from $(*)$ we have

$$c\sum_i \sum_{\sigma\in S_4} T^\sigma{}_{,i\bar{i}i\bar{i}} + c'\sum_{i\neq j}\sum_{\sigma\in S_4} T^\sigma{}_{,i\bar{i}j\bar{j}} = \sum_{i,j} T_{,(i\bar{i}+\bar{i}i)(j\bar{j}+\bar{j}j)}$$

in the Euclidean case. But in this case $T^\sigma{}_{,\alpha\beta\gamma\delta} = T_{,\alpha\beta\gamma\delta}$. By comparing coefficients this yields that $c' = c$, so that in general

$$S^{(4)}T = c\sum_{i,j}\sum_{\sigma\in S_4} T^\sigma{}_{,i\bar{i}j\bar{j}}.$$

The constant c can be obtained by setting the right-hand side equal to $\Delta_0^2 T = 4\sum_l i, j\, T_{,i\bar{i}j\bar{j}}$ in the Euclidean case, yielding the special case of $k = 2$ and in an analogous manner the general case of Proposition (1.3).

(1.4) Conversion of radial derivatives to mixed covariant derivatives. In the argument of showing that certain components of covariant derivatives of the curvature tensor vanish at a given point x, it will be a typical situation first to show radial derivatives of certain orders along geodesics γ through the point x vanishing and then to show similar vanishing phenomena for mixed covariant derivatives. Suppose for some fixed positive integer k we have $\nabla_\eta^k R_{i\bar{j}k\bar{l}}(x) = 0$ for all real tangent vectors η at x. In the Euclidean case this would mean that for $R_{i\bar{j}k\bar{l}}$ all covariant derivatives of degree k at x, symbolically $\nabla^k R_{i\bar{j}k\bar{l}}(x)$, would vanish at x. However, for a general Kähler manifold this is not the case. We have the following proposition in the general case of Riemannian manifolds.

Proposition (1.4). *Let M be an m-dimensional Riemannian manifold and let $x \in M$ such that for the given smooth tensor T, the covariant of order l, $\nabla_\eta^k T_{i_1 i_2 \cdots i_l}(x) = 0$ for some specific indices $i_1, \cdots, i_s$ and for all real tangent vectors η at x. For any $\sigma \in S_k$, we denote by $(\nabla_1^{k_1}\nabla_2^{k_2} \cdots \nabla_m^{k_m})^\sigma T_{i_1 i_2 \cdots i_l}(x)$ the components of covariant derivatives obtained by permuting the order of the $k = k_1 + \cdots + k_m$ derivatives using σ. Then we have*

$$\sum_{\sigma \in S_k} \left(\nabla_1^{k_1}\nabla_2^{k_2} \cdots \nabla_m^{k_m}\right)^\sigma T_{i_1 \cdots i_l}(x) = 0$$

for any set of nonnegative integers $k_1, \cdots, k_m$ such that $k_1 + \cdots + k_m = k$.

Proof. Let $x_1, \cdots, x_m$ be real normal geodesic coordinates at the point $x \in M$. The point x is then the origin in this coordinate system. For any real tangent vector $\eta = a_1 \partial/\partial x_1 + \cdots + a_m \partial/\partial x_m$ of unit length we have

$$T_{i_1 i_2 \cdots i_l}(t\eta) = \sum_{s \geqslant 0} \frac{t^s}{s!} \nabla_\eta^s T_{i_1 i_2 \cdots i_l} = \sum_{s \geqslant 0} \frac{1}{s!} \nabla_{t\eta}^s T_{i_1 i_2 \cdots i_l}.$$

Writing $t\eta = (x_1, x_2, \cdots, x_m)$ we have

$$T_{i_1 i_2 \cdots i_l}(x_1, x_2, \cdots, x_m) = \sum_{s \geqslant 0} \sum_{\substack{s_1 + \cdots + s_m = s \\ s_i \geqslant 0}} \sum_{\sigma \in S_s} \left(\nabla_1^{s_1}\nabla_2^{s_2} \cdots \nabla_m^{s_m}\right)^\sigma T_{i_1 i_2 \cdots i_l}(0) x_1^{s_1} x_2^{s_2} \cdots x_m^{s_m},$$

where S_s is the group of formal permutations of the s indices involved in the covariant differentiation. The proposition follows immediately by setting equal to zero all the coefficients of kth order monomials $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$, $k_1 + k_2 + \cdots + k_m = k$, which must be the case when $\nabla_\eta^k T_{i_1 i_2 \cdots i_l}(0) = 0$ for all real tangent vectors η at x.

Remarks. In the complex case we can rewrite the formula in Proposition (1.4) by allowing the differentiations to be against barred or unbarred indices.

(1.5) Second variation inequalities associated to the curvature tensor. Let X be a Kähler manifold and $x \in X$ be a point where holomorphic bisectional curvatures are nonnegative. There are two important and well-known inequalities associated to $R_{i\bar{j}k\bar{l}}(x)$. They are respectively related to maximal directions of holomorphic sectional curvatures and flat (zero) directions of holomorphic bisectional curvatures. We formulate them in the form of two lemmata.

Lemma (1.5.1). *Let X be a Kähler manifold and $x \in X$ be a point where holomorphic bisectional curvatures are nonnegative. Suppose $\alpha \in T_x^{1,0}(X)$ of unit length is the direction attaining the maximum of all holomorphic sectional curvatures at x. Then, for any $\xi \in T_x^{1,0}(X)$ of unit length which is perpendicular to α, we have*

$$0 \leqslant 2R_{\alpha\bar{\alpha}\xi\bar{\xi}}(x) + \left|R_{\alpha\bar{\xi}\alpha\bar{\xi}}(x)\right| \leqslant R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x).$$

Proof. We shall henceforth call α a maximal direction of holomorphic sectional curvatures at x or simply a maximal direction at x. We have, for any $\varepsilon > 0$ and any real θ,

$$R\left\langle \alpha + \varepsilon e^{i\theta}\xi, \overline{\alpha + \varepsilon e^{i\theta}\xi}\,;\, \alpha + \varepsilon e^{i\theta}\xi, \overline{\alpha + \varepsilon e^{i\theta}\xi} \right\rangle \leqslant (1 + \varepsilon^2)^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}.$$

Since ξ is orthogonal to α, $\|\alpha\| = \|\xi\| = 1$ and α is a maximal direction at x. Here and henceforth in the article we will sometimes drop the reference to the point x when there is no danger of confusion. Comparing the coefficients of ε immediately yields $R_{\alpha\bar{\alpha}\alpha\bar{\xi}} = 0$. Comparing the coefficients of ε^2 then yields

$$4R_{\alpha\bar{\alpha}\xi\bar{\xi}} + 2\,\mathrm{Re}\left(e^{2i\theta}R_{\alpha\bar{\xi}\alpha\bar{\xi}}\right) \leqslant 2R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}.$$

We can always choose the angle θ so that $2\,\mathrm{Re}(e^{2i\theta}R_{\alpha\bar{\xi}\alpha\bar{\xi}}) \geqslant 0$, yielding

$$0 \leqslant 2R_{\alpha\bar{\alpha}\xi\bar{\xi}} + \left|R_{\alpha\bar{\xi}\alpha\bar{\xi}}\right| \leqslant R_{\alpha\bar{\alpha}\alpha\bar{\alpha}},$$

the desired inequality.

Lemma (1.5.2). *Let X be a Kähler manifold and $x \in X$ be a point where holomorphic bisectional curvatures are nonnegative. Suppose $\alpha, \beta \in T_x^{1,0}(X)$ are such that $R_{\alpha\bar{\alpha}\beta\bar{\beta}}(x) = 0$. Let $\xi, \zeta \in T_x^{1,0}(X)$ be arbitrary. Then*

$$\left|R_{\alpha\bar{\xi}\beta\bar{\zeta}}\right|^2 + \left|R_{\alpha\bar{\beta}\xi\bar{\zeta}}\right|^2 \leqslant R_{\alpha\bar{\alpha}\xi\bar{\xi}}R_{\beta\bar{\beta}\zeta\bar{\zeta}}.$$

Proof. Since holomorphic bisectional curvatures are nonnegative at x, we have for all $\delta, \varepsilon > 0$ and θ, ϕ real

$$R\left\langle \alpha + \delta e^{i\theta}\zeta, \overline{\alpha + \delta e^{i\theta}\zeta}\,;\, \beta + \varepsilon e^{i\phi}\xi, \overline{\beta + \varepsilon e^{i\phi}\xi} \right\rangle \geqslant 0.$$

Expanding in terms of δ, z and writing out the coefficients of δ, ε we obtain

$$R_{\alpha\bar{\alpha}\beta\bar{\xi}} = R_{\beta\bar{\beta}\alpha\bar{\zeta}} = 0.$$

From the second-order terms we obtain

$$\varepsilon^2 R_{\alpha\bar{\alpha}\xi\bar{\xi}} + \delta^2 R_{\beta\bar{\beta}\zeta\bar{\zeta}} + 2\,\mathrm{Re}\,\varepsilon\delta\left(e^{-i(\theta+\phi)}R_{\alpha\bar{\xi}\beta\bar{\zeta}}\right) + 2\,\mathrm{Re}\,\varepsilon\delta\left(e^{i(\phi-\theta)}R_{\alpha\bar{\beta}\xi\bar{\zeta}}\right) \geqslant 0$$

for δ, ε sufficiently small and hence for all $\delta, \varepsilon > 0$. By making the transformations $\xi \mapsto e^{i\theta_0}\xi$ and $\zeta \mapsto e^{i\phi_0}\zeta$ so that $R_{\alpha\bar{\xi}\beta\bar{\zeta}}$ is changed to $e^{-i(\theta_0+\phi_0)}R_{\alpha\bar{\xi}\beta\bar{\zeta}}$ and $R_{\alpha\bar{\beta}\xi\bar{\zeta}}$ is changed to $e^{i(\theta_0-\phi_0)}R_{\alpha\bar{\beta}\xi\bar{\zeta}}$ we may without loss of generality assume that to start with both $R_{\alpha\bar{\xi}\beta\bar{\eta}}$ and $R_{\alpha\bar{\beta}\xi\bar{\zeta}}$ are real. Then by choosing $\theta = \phi = 0$, we obtain from the discriminant

$$\left|R_{\alpha\bar{\xi}\beta\bar{\zeta}} + R_{\alpha\bar{\beta}\xi\bar{\zeta}}\right|^2 \leqslant R_{\alpha\bar{\alpha}\xi\bar{\xi}}R_{\beta\bar{\beta}\zeta\bar{\zeta}}.$$

By choosing $\phi = \pi/2$, $\theta = -\pi/2$, we obtain on the other hand

$$\left|R_{\alpha\bar{\xi}\beta\bar{\zeta}} - R_{\alpha\bar{\beta}\xi\bar{\zeta}}\right|^2 \leqslant R_{\alpha\bar{\alpha}\xi\bar{\xi}}R_{\beta\bar{\beta}\zeta\bar{\zeta}}.$$

Summing up the two inequalities and dividing by 2, we obtain the desired inequality

$$|R_{\alpha\bar{\xi}\beta\bar{\zeta}}|^2 + |R_{\alpha\bar{\beta}\xi\bar{\zeta}}|^2 \leqslant R_{\alpha\bar{\alpha}\xi\bar{\xi}} R_{\beta\bar{\beta}\zeta\bar{\zeta}}.$$

2. Zero-order information on curvature terms associated with a global maximal direction

(2.1) By using the computation of (1.2) and Lemma (1.5.1), Berger [3] obtained, in the case of positive sectional curvature, that for a unit vector $\alpha \in T_x^{1,0}(X)$ attaining the global maximum of all holomorphic sectional curvatures, $R_{\alpha\bar{\alpha}\xi\bar{\xi}} = \frac{1}{2} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ for any $\xi \in T_x^{1,0}(X)$ of unit length and orthogonal to α. In the case of nonnegative holomorphic bisectional curvature, his computation yields immediately

Proposition (2.1). *$T_x^{1,0}(X)$ splits into the orthogonal direct sum $\mathbf{C}\alpha \oplus \mathscr{H} \oplus \mathscr{N}$, where $\mathscr{H}$ consists of all $\xi \in T_x^{1,0}(X)$ such that $R_{\alpha\bar{\alpha}\xi\bar{\xi}} = 0$ and $\mathscr{N}$ consists of all $\zeta \in T_x^{1,0}(X)$ such that $R_{\alpha\bar{\alpha}\xi\bar{\xi}} = 0$.*

Proof. Since $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ is a global maximum of all holomorphic sectional curvatures, we have

$$\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = \text{const} \int_\eta \nabla_\eta^2 R\langle \alpha(y), \bar{\alpha}(y), \alpha(y), \bar{\alpha}(y)\rangle(x) \leqslant 0$$

as in (1.3), where η ranges over all real tangent vectors of unit length, $\alpha(y)$ denotes on a neighborhood of x the vector field obtained from $\alpha(x) = \alpha$ by parallel transport along geodesics from x and the integration is with respect to the rotation-symmetric measure of the unit sphere of $T_x(X)$. On the other hand, from (1.2) we have

$$\frac{1}{2}\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \sum_{\substack{i \neq 1 \\ \text{or } j \neq 1}} |R_{\alpha\bar{i}\alpha\bar{j}}|^2 + \sum_{i \neq 1} (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - 2R_{\alpha\bar{\alpha}i\bar{i}}) R_{\alpha\bar{\alpha}i\bar{i}}$$

for an orthonormal basis $\{e_i\}$ of $T_x^{1,0}(X)$ such that $e_1 = \alpha$ and $R_{\alpha\bar{\alpha}i\bar{j}} = 0$ for $i \neq j$. From Lemma (1.5.1) we have $R_{\alpha\bar{\alpha}i\bar{i}} \leqslant \frac{1}{2} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ yielding

$$\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \geqslant 0.$$

Equality holds if and only if $R_{\alpha\bar{\alpha}i\bar{i}} = \frac{1}{2} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ or 0 for $i > 1$ and that $R_{\alpha\bar{i}\alpha\bar{j}} = 0$ for all i, j except $i = j = 1$. In particular we have the orthogonal decomposition of $T_x^{1,0}(X)$ into eigenspaces of the Hermitian bilinear form $H_\alpha(\xi, \xi) = R_{\alpha\bar{\alpha}\xi\bar{\xi}}$.

From now on we shall fix an α and call $\mathscr{H}_\alpha = \mathscr{H}$ the half-space and $\mathscr{N}_\alpha = \mathscr{N}$ the null-space associated to α.

Also from here on we shall fix an $\alpha \in T^{1,0}(X)$ of unit length such that $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \sup_{\xi \in T^{1,0}(X), \|\xi\|=1} R_{\xi\bar{\xi}\xi\bar{\xi}}$, where $\alpha \in T_x^{1,0}(X)$ shall be termed a global maximal direction (of holomorphic sectional curvatures). We shall also fix an orthonormal basis of $T_x^{1,0}(X)$, according to the orthogonal decomposition $T_x^{1,0}(X) = \mathbf{C}\alpha \oplus \mathscr{H} \oplus \mathscr{N}$, consisting of $\{e_1, \cdots, e_n\}$ such that $e_1 = \alpha$, $e_p \in \mathscr{H}$ for $2 \leqslant p \leqslant m$ and $e_q \in \mathscr{N}$ for $m + 1 \leqslant q \leqslant n$. Write $H = \{2, \cdots, m\}$ and $N = \{m + 1, \cdots, n\}$. For $\alpha \in T_x^{1,0}(X)$ fixed, we shall typically use the indices given by the above choice of bases. Since the choice of $\{e_p\colon\ p \in H\}$ and $\{e_q\colon\ q \in N\}$ is arbitrary within $\mathscr{H}$ and $\mathscr{N}$ as long as they form orthonormal bases of $\mathscr{H}$ and $\mathscr{N}$ respectively, we shall also use the notations e_p and e_q for general elements of $\mathscr{H}$ and $\mathscr{N}$ of unit length. Any orthonormal basis $\{e_1\} \cup \{e_p\colon\ p \in H\} \cup \{e_q\colon\ q \in N\}$ associated to $\alpha \in T_{x_0}^{1,0}(X)$ will be called a privileged orthonormal basis associated to α.

To systematize once and for all the choice of notations, we use, as has been the case, ξ, ζ to denote the complexified tangent vectors of type $(1,0)$ and η to denote the general real tangent vectors. The new indices arising from substitution in commutation formulas will be denoted by μ, ν.

(2.2) Equations satisfied by curvature terms associated to a global maximal direction. We collect here the necessary information on $R_{i\bar{i}j\bar{k}}$ associated to a global maximal direction of holomorphic sectional curvatures $\alpha = e_1 \in T_{x_0}^{1,0}(X)$. Some of these equations are already contained in Proposition (2.1) and its proof.

Proposition (2.2.1). *Let $\alpha \in T_{x_0}^{1,0}(X)$ of unit length be a global maximal direction of holomorphic sectional curvatures and let $\{e_1\} \cup \{e_p\colon\ p \in H\} \cup \{e_q\colon\ q \in N\}$ be a privileged orthonormal basis of $T_{x_0}^{1,0}(X)$ associated to α. In terms of this basis, we have*

(a) $R_{1\bar{1}p\bar{p}} = \frac{1}{2}R_{1\bar{1}1\bar{1}} > 0$ *for* $p \in H$.

(b) $R_{1\bar{i}1\bar{j}} = 0$ *for* $i \neq 1$ *or* $j \neq 1$.

(c) $R_{1\bar{q}i\bar{j}} = 0$ *for* $q \in N$ *and* $1 \leqslant i, j \leqslant n$. (*In particular* $R_{1\bar{1}q\bar{q}} = R_{1\bar{1}p\bar{q}} = 0$ *for* $p \in H$ *and* $q \in N$.)

(d) $R_{1\bar{p}p\bar{p}} = 0$ *for* $p \in H$.

(e) $\Sigma_{r \in H}|R_{1\bar{\xi}q\bar{r}}|^2 = R_{1\bar{1}\xi\bar{\xi}}R_{\xi\bar{\xi}q\bar{q}}$ *for* $\xi \in \mathscr{H}$ *and* $q \in N$.

(e)′ $R_{p\bar{p}'q\bar{q}'} = (2/R_{1\bar{1}1\bar{1}})\Sigma_{r \in H} R_{1\bar{p}'q\bar{r}}R_{p\bar{1}r\bar{q}'}$.

Before proving Proposition (2.2.1) we give a few remarks on our formulation of the equations. Since the equations are stated for arbitrary choices of privileged orthonormal basis associated to α, the equations are satisfied for e_p an arbitrary unit vector of the half-space $\mathscr{H}$ and e_q an arbitrary unit vector of the null-space $\mathscr{N}$. For example, the equation $R_{1\bar{1}q\bar{q}} = 0$ implies by polarization

$R_{1\bar{1}q\bar{q}'} = 0$ for $q, q' \in N$, and the equation $R_{1\bar{p}p\bar{p}} = 0$ for $p \in H$ implies by polarization $R_{1\bar{p}p'\bar{p}''} = 0$ for $p, p', p'' \in H$. Equation (e) implies by polarization the following representation of curvature terms $R_{p\bar{p}'q\bar{q}'}$ in terms of $R_{1\bar{i}j\bar{k}}$.

Proof of Proposition (2.2.1). (a), (b) and the special case of (c) in parentheses are included either in the statement or the proof of Proposition (2.1). We first prove (d) by a third-order variation equality at $\alpha = e_1$. Consider the function $F(\varepsilon)$ in the real variable ε defined by

$$F(\varepsilon) = \frac{1}{(1+\varepsilon^2)^2} R\left\langle e_1 + \varepsilon e_p, \overline{e_1 + \varepsilon e_p}, e_1 + \varepsilon e_p, \overline{e_1 + \varepsilon e_p} \right\rangle.$$

Proposition (2.2) implies the vanishing of both the first and the second variation of $F(\varepsilon)$ at $\varepsilon = 0$. In fact

$$F(\varepsilon) = \frac{1}{1 + 2\varepsilon^2 + \varepsilon^4}\left(R_{1\bar{1}1\bar{1}} + 4\varepsilon^2 R_{1\bar{1}p\bar{p}} + 4\varepsilon^3 \operatorname{Re} R_{1\bar{p}p\bar{p}} + \varepsilon^4 R_{p\bar{p}p\bar{p}}\right)$$

since $R_{1\bar{1}1\bar{p}} = R_{1\bar{p}1\bar{p}} = 0$.

Since $R_{1\bar{1}p\bar{p}} = \frac{1}{2} R_{1\bar{1}1\bar{1}}$ we obtain

$$\begin{aligned} F(\varepsilon) &= R\left\langle \frac{e_1 + \varepsilon e_p}{\sqrt{1+\varepsilon^2}}, \frac{\overline{e_1 + \varepsilon e_p}}{\sqrt{1+\varepsilon^2}}, \frac{e_1 + \varepsilon e_p}{\sqrt{1+\varepsilon^2}}, \frac{\overline{e_1 + \varepsilon e_p}}{\sqrt{1+\varepsilon^2}} \right\rangle \\ &= \frac{1}{(1 + 2\varepsilon^2 + \varepsilon^4)}\left(R_{1\bar{1}1\bar{1}} + 2\varepsilon^2 R_{1\bar{1}1\bar{1}} + 4\varepsilon^3 \operatorname{Re} R_{1\bar{p}p\bar{p}} + \varepsilon^4 R_{p\bar{p}p\bar{p}}\right). \end{aligned}$$

Since $F(0) = R_{1\bar{1}1\bar{1}}$, by comparing coefficients of Taylor expansions of the denominator and the numerator, we have immediately $dF(0)/d\varepsilon = d^2F(0)/d\varepsilon^2 = 0$. Since $F(\varepsilon) \leqslant R_{1\bar{1}1\bar{1}}$ the third-order variation equality yields

$$\frac{d^3F}{d\varepsilon^3}(0) = 4 \operatorname{Re} R_{1\bar{p}p\bar{p}} = 0.$$

Since the same equality holds with e_p replaced by $e^{i\theta}e_p$, we conclude the equation (c)

$$R_{1\bar{p}p\bar{p}} = 0 \quad \text{for all } p \in H.$$

(Recall that $e_p \in \mathscr{H}$, $\|e_p\| = 1$, is arbitrary so that by polarization $R_{1\bar{p}p'\bar{p}''} = 0$ for $p, p', p'' \in H$.)

To finish the proof of Proposition (2.2.1) we shall need the following analysis at the zero directions $R_{1\bar{1}q\bar{q}}(x_0)$ of holomorphic bisectional curvatures.

Proposition (2.2.2). *Let $q \in N$ and $\alpha = e_1 \in T^{1,0}_{x_0}(X)$ be a global maximal direction of holomorphic sectional curvatures. Then, $\Delta R_{1\bar{1}q\bar{q}}(x_0) = 0$. Hence $\nabla^i_\eta R_{1\bar{1}q\bar{q}}(x_0) = 0$ for $0 \leqslant i \leqslant 3$.*

Proof. First, we compute $\Delta R_{1\bar{1}q\bar{q}}$. It is immediate that $R_{1\bar{1}q\bar{q},i\bar{i}} = R_{1\bar{1}q\bar{q},\bar{i}i}$ by a commutation formula. Hence

$$\frac{1}{2}R_{1\bar{1}q\bar{q}} = \sum_i R_{1\bar{1}q\bar{q},i\bar{i}} \quad \text{(summations over } i \text{ with unspecified ranges will henceforth mean } 1 \leqslant i \leqslant n)$$

$$= \sum_i R_{1\bar{1}i\bar{q},q\bar{i}}$$

$$= \sum_i R_{1\bar{1}i\bar{q},\bar{i}q} + \sum_{i,\mu}\big(R_{1\bar{\mu}q\bar{i}}R_{\mu\bar{1}i\bar{q}} - R_{\mu\bar{1}q\bar{i}}R_{1\bar{\mu}i\bar{q}} + R_{i\bar{\mu}q\bar{i}}R_{1\bar{1}\mu\bar{q}} - R_{\mu\bar{q}q\bar{i}}R_{1\bar{1}i\bar{\mu}}\big).$$

Since

$$\sum_i R_{1\bar{1}i\bar{q},\bar{i}q} = \sum_i R_{1\bar{1}i\bar{i},\bar{q}q} = R_{1\bar{1},\bar{q}q} = 0,$$

we obtain at x_0

$$\frac{1}{2}\Delta R_{1\bar{1}q\bar{q}} = \sum_{i,\mu}\left|R_{1\bar{\mu}q\bar{i}}\right|^2 + R_{1\bar{1}q\bar{q}} - \sum_{i,\mu}\left|R_{1\bar{q}i\bar{\mu}}\right|^2 - \sum_{i,\mu}R_{1\bar{1}i\bar{\mu}}R_{\mu\bar{i}q\bar{q}}.$$

From $R_{1\bar{1}q\bar{q}} = 0$ and first variation equalities, noting that bisectional curvatures on X are nonnegative, we obtain immediately

$$R_{1\bar{i}q\bar{q}} = R_{1\bar{1}q\bar{i}} = 0 \quad \text{for } 1 \leqslant i \leqslant n,\ e_q \in \mathscr{N},\ \|e_q\| = 1 \text{ arbitrary.}$$

This yields at x_0

$$\frac{1}{2}\Delta R_{1\bar{1}q\bar{q}} = \sum_{p,r\in H}\left|R_{q\bar{p}q\bar{r}}\right|^2 - \sum_{i,j}\left|R_{1\bar{q}i\bar{j}}\right|^2 - \sum_{p\in H}R_{1\bar{1}p\bar{p}}R_{p\bar{p}q\bar{q}}.$$

We claim that from the second-variation inequality Lemma (1.5.2)

$$\Delta R_{1\bar{1}q\bar{q}}(x_0) \leqslant 0.$$

Since X carries nonnegative holomorphic bisectional curvatures

$$\Delta R_{1\bar{1}q\bar{q}}(x_0) \geqslant 0,$$

yielding Proposition (2.2.2). To prove the inequality $\Delta R_{1\bar{1}q\bar{q}}(x_0) \leqslant 0$ we give two different approaches. First, we recall the following lemma in linear algebra.

Lemma (2.2.3). *Let $S(z;\ z')$ be a complex symmetric bilinear form on a complex vector space $\mathbf{C}^n$ represented by the matrix S with respect to the canonical coordinates of $\mathbf{C}^n$. Then there exists a unitary transformation U of $\mathbf{C}^n$ (relative to the Euclidean Hermitian structure) such that U^tSU is a diagonal matrix.*

Using Lemma (2.2.3), we diagonalize the complex symmetric bilinear form $S(\xi;\ \xi) = R_{1\bar{\xi}q\bar{\xi}}$ on $\mathscr{H}$ yielding $R_{1\bar{p}q\bar{p}'} = 0$ for $p \neq p$ for some choice of orthonormal basis $\{e_p\}$ of $\mathscr{H}$. In this coordinate system we have

$$\frac{1}{2}\Delta R_{1\bar{1}q\bar{q}} = \sum_{p\in H} |R_{1\bar{p}q\bar{p}}|^2 - \sum_{i,j} |R_{1\bar{q}i\bar{j}}|^2 - \sum_{p\in H} R_{1\bar{1}p\bar{p}} R_{p\bar{p}q\bar{q}}.$$

Note that $\sum_{p\in H}|R_{1\bar{p}q\bar{r}}|^2$ is invariant under the unitary transformations on $\mathscr{H}$ since $\operatorname{tr}(U^t S\overline{S}\overline{U}) = \operatorname{tr}(U^t S\overline{S}\overline{U}) = \operatorname{tr}(S\overline{S})$. From the second-variation inequality (1.5.2) we have

$$|R_{1\bar{p}q\bar{p}}|^2 + |R_{1\bar{q}p\bar{p}}|^2 \leqslant R_{1\bar{1}p\bar{p}} R_{p\bar{p}q\bar{q}}.$$

This yields

$$\frac{1}{2}\Delta R_{1\bar{1}q\bar{q}} \leqslant -\sum_{i,j} |R_{i\bar{q}i\bar{j}}|^2 \leqslant 0.$$

This yields a proof of Proposition (2.2.2) and with it also the equation (d)

$$R_{i\bar{q}i\bar{j}} = 0 \quad \text{for all } i, j, 1 \leqslant i, j \leqslant n.$$

Since Lemma (2.2.3) is proved entirely by algebraic means, it would be desirable to give a geometric proof of

$$(\#) \qquad \sum_{p,r} |R_{1\bar{p}q\bar{r}}|^2 \leqslant \sum_{p} R_{1\bar{1}p\bar{p}} R_{p\bar{p}q\bar{q}}$$

in our situation without a special choice of coordinates. We claim that for any $\xi \in \mathscr{H}$ and any orthonormal basis $\{e_r\}$ of $\mathscr{H}$,

$$(*) \qquad \sum_{r\in H} |R_{1\bar{\xi}q\bar{r}}|^2 + \sum_{r\in H} |R_{1\bar{q}\xi\bar{r}}|^2 \leqslant \frac{1}{2} R_{1\bar{1}1\bar{1}} R_{\xi\bar{\xi}q\bar{q}}.$$

Then, integrating $(*)$ over $\xi \in \mathscr{H}$ of unit length using a rotation-symmetric metric on the unit sphere yields immediately $(\#)$ and hence another proof of Proposition (2.2.2).

To prove $(*)$ observe first of all that Lemma (1.5.2) yields only

$$|R_{1\bar{p}q\bar{r}}|^2 \leqslant \tfrac{1}{2} R_{1\bar{1}1\bar{1}} R_{\xi\bar{\xi}q\bar{q}}.$$

We shall now prove $(*)$ by making a better use of the argument of the second-variation used in Lemma (1.5.2). Consider the Taylor expansion of

$$G(\varepsilon) = R\left(e_1 + \varepsilon\xi,\ \overline{e_1 + \varepsilon\xi},\ e_q + \varepsilon \sum_{r\in H} C_r e_r,\ \overline{e_q + \varepsilon \sum_{r\in H} C_r e_r}\right).$$

We then have

$$G(\varepsilon) = \varepsilon^2\left(R_{\xi\bar{\xi}q\bar{q}} + \sum_{r\in H} |C_r|^2 R_{1\bar{1}r\bar{r}} + \sum_{r\in H} 2\operatorname{Re}\overline{C}_r R_{1\bar{\xi}q\bar{r}} + \sum_{r\in H} 2\operatorname{Re} C_r R_{1\bar{q}r\bar{\xi}}\right).$$

Since X carries nonnegative holomorphic bisectional curvatures, $\partial^2 G(0)/\partial\varepsilon^2$ is always nonnegative for any choice of complex numbers C_r. It follows that the quadratic form Q in $(z_1, z_2, \cdots, z_m)$, defined by

$$Q((z_1, \cdots, z_m); (z_1, \cdots, z_m)) = |z_1|^2 R_{\xi\bar{\xi}q\bar{q}} + \sum_{2 \leqslant r \leqslant m} |z_r|^2 \frac{R_{1\bar{1}1\bar{1}}}{2} + \sum_{2 \leqslant r \leqslant m} 2\,\mathrm{Re}\big(z_1 \bar{z}_r R_{1\bar{\xi}q\bar{r}} + \bar{z}_1 z_r R_{1\bar{q}r\bar{\xi}}\big),$$

is positive semidefinite. Now take z_r to be of the form $x_r e^{i\theta_r}$, x_r, θ_r real, for $2 \leqslant r \leqslant m$ and take $z_1 = x_1$ real and positive. Choose θ_r and replace e_q by $e^{i\alpha} e_q$ for an appropriate real α so that $e^{-i\theta_r} R_{1\bar{\xi}q\bar{r}}$ is real and $\geqslant 0$ while $e^{i\theta_r} R_{1\bar{q}\xi\bar{r}}$ is $\leqslant 0$. By computing the determinant of the symmetric matrix representing the real symmetric bilinear form Q_θ given by

$$\begin{aligned} &Q_\theta((x_1, \cdots, x_m); (x_1, \cdots, x_m)) \\ &\quad = Q\big(\big(x_1, x^2 e^{i\theta_2}, \cdots, x_m e^{i\theta_m}\big); \big(x_1, x e^{i\theta_2}, \cdots, x e^{i\theta_m}\big)\big), \end{aligned}$$

we conclude immediately that

$$\sum_{r \in H} \Big|\, |R_{1\bar{\xi}q\bar{r}}| - |R_{1\bar{q}\xi\bar{r}}| \,\Big|^2 \leqslant \frac{R_{1\bar{1}1\bar{1}}}{2} R_{\xi\bar{\xi}q\bar{q}}.$$

By a similar argument we have

$$\sum_{r \in H} \Big|\, |R_{1\bar{\xi}q\bar{r}}| + |R_{1\bar{q}\xi\bar{r}}| \,\Big|^2 \leqslant \frac{R_{1\bar{1}1\bar{1}}}{2} R_{\xi\bar{\xi}q\bar{q}}.$$

Adding the two equations and dividing by 2, we obtain immediately the statement $(*)$.

End of proof of Proposition (2.2.1). The second proof of Proposition (2.2.2) yields immediately from the equality $\Delta R_{1\bar{1}q\bar{q}}(x_0) = 0$ the equality

$$\text{(e)} \qquad \sum_{r \in H} |R_{1\bar{\xi}q\bar{r}}|^2 = \frac{R_{1111}}{2} R_{\xi\bar{\xi}q\bar{q}}.$$

(e)′ is obtained easily from (e) by polarization. To see this define a tensor $T_{p\bar{p}'q\bar{q}'}$ of type (2,2) for $p, p' \in H$ and $q, q' \in N$ by

$$T_{p\bar{p}'q\bar{q}'} = \frac{2}{R_{1\bar{1}1\bar{1}}} \sum_{r \in H} R_{1\bar{p}'q\bar{r}} R_{p\bar{1}r\bar{q}'}.$$

Clearly $T_{\xi\bar{\xi}\zeta\bar{\zeta}} = R_{\xi\bar{\xi}\zeta\bar{\zeta}}$ for $\xi \in \mathscr{H}$ and $\zeta \in \mathscr{N}$. It suffices therefore to show that from $S_{\xi\bar{\xi}\zeta\bar{\zeta}} = T_{\xi\bar{\xi}\zeta\bar{\zeta}} - R_{\xi\bar{\xi}\zeta\bar{\zeta}} = 0$, one can prove $S_{p\bar{p}'q\bar{q}'} = 0$, proving $T_{p\bar{p}'q\bar{q}'} = R_{p\bar{p}'q\bar{q}'}$, i.e. (e)′. But now for ε, δ real and θ, ϕ real angles

$$S\Big(e_p + \varepsilon e^{i\theta} e'_p,\ \overline{e_p + \varepsilon e^{i\theta} e'_p};\ e_q + \delta e^{i\phi} e'_q,\ \overline{e_q + \delta e^{i\phi} e'_q}\Big) \equiv 0$$

giving by computing the coefficient of $\varepsilon\delta$

$$2\,\mathrm{Re}\, e^{-i(\theta+\phi)} R_{p\bar{p}'q\bar{q}'} + 2\,\mathrm{Re}\, e^{i(\theta-\phi)} R_{p'\bar{p}q\bar{q}'} = 0$$

for all real θ and ϕ. This implies $R_{p\bar{p}'q\bar{q}'} = R_{p'\bar{p}q\bar{q}'} = 0$.

3. Structure of the space of maximal directions on $T^{1,0}(X)$

(3.1) Everywhere existence of global maxima of holomorphic sectional curvature. From now on X will stand for a compact Kähler-Einstein manifold of nonnegative holomorphic bisectional curvatures. We denote by $S_x^{1,0}(X)$ the unit sphere of the hermitian vector space $T_x^{1,0}(X)$ of complexified tangent vectors of type $(1,0)$. $S^{1,0}(X)$ will denote the sphere bundle thus obtained. Define the function f on X by $f(x) = \sup_{\xi \in S_x^{1,0}(X)} R_{\xi\bar{\xi}\xi\bar{\xi}}$. If $\alpha \in S_x^{1,0}(X)$ and $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = f(x)$ we shall call α a maximal direction (of holomorphic sectional curvature at x). Clearly f is a continuous function. Our main result here is the following proposition:

Proposition (3.1). *The function f is constant on X. In other words, the global maximum $\sup_{\xi \in S^{1,0}(X)} R_{\xi\bar{\xi}\xi\bar{\xi}}$ of all holomorphic sectional curvatures is attained at every single point of X.*

Proof. We prove the proposition by using the maximum principle. It suffices to show that f is subharmonic in the generalized sense. The starting point is the following consequence of Berger's computation.

(*) Let $\alpha \in S_x^{1,0}(X)$ be a maximal direction of holomorphic sectional curvatures at x. Then $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \geqslant 0$.

The proof was given in Proposition (2.2.1). There, for the verification $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \geqslant 0$, it suffices to assume that $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = \sup_{\xi \in S_x^{1,0}(X)} R_{\xi\bar{\xi}\xi\bar{\xi}}$. We note that since $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ is the Laplacian of the tensor $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ evaluated at $\langle \alpha, \bar{\alpha};\, \alpha, \bar{\alpha} \rangle$ we cannot apply the maximum principle directly. Instead we claim that at each $x \in X$, and for any $\alpha \in S_x^{1,0}(X)$ such that $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = f(x)$,

$$(**) \qquad \Delta f(x) \geqslant \Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \geqslant 0$$

in the generalized sense. To prove (**) we construct local barrier functions for f, denoted by g_x, as follows. Fix $x \in X$ and $\alpha \in S_x^{1,0}(X)$ with $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = f(x) = \sup_{\xi \in S_x^{1,0}(X)} R_{\xi\bar{\xi}\xi\bar{\xi}}$. In an open neighborhood U_x of x within the cut-locus of x we shall denote by $\alpha(y)$ the complexified tangent vector at y of type $(1,0)$ obtained by parallel transport of $\alpha = \alpha(x)$ along the unique geodesic joining x to y within the cut-locus of x. Define $g_x(y) = R\langle \alpha(y), \overline{\alpha(y)}, \alpha(y), \overline{\alpha(y)} \rangle$ for $y \in U_x$. From the discussion in (1.3) of averaging operators of radial derivatives we know that

$$\Delta g_x(x) = \Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x).$$

We know that on U_x, $g_x \leqslant f$ and that $g_x(x) = f(x)$. For the Laplacian of continuous functions, we have the generalized definition (following Oka)

$$\Delta f(x) = c_{2n} \lim_{r \to 0} \frac{1}{\gamma^2}\left(\frac{\int_{B(x;r)} f}{\int_{B(x;r)} 1} - f(x)\right).$$

With this definition f is subharmonic on X if and only if $\Delta f(x) \geqslant 0$ at each point $x \in X$. Obviously $\Delta f(x) \geqslant \Delta g_x(x)$ since $g_x(x) = f(x)$ and $g_x \leqslant f$ on $B(x;\ r)$. It follows that

$$\Delta f(x) \geqslant \Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \geqslant 0$$

in the generalized sense. Thus, f is subharmonic and hence constant on X.

(3.2) Structure of the bundle of "maximal subspaces". On the unit sphere $S_x^{1,0}(X)$ of $T_x^{1,0}(X)$, we shall denote by $\mathscr{M}_x$ the set of all $\alpha \in S_x^{1,0}(X)$ attaining the global maximum of holomorphic sectional curvatures. By (3.1) $\mathscr{M}_x$ is nonempty for any $x \in X$. We denote by V_x the complex linear span of $\mathscr{M}_x$ and call it the "maximal subspace" at x. We call $V = \bigcup_{x \in X} V_x \subset T^{1,0}(X)$ the bundle of maximal subspaces. Note that we do not know at this point that V is a differentiable vector subbundle of $T^{1,0}(X)$. Denoting by π: $T^{1,0}(X) \to X$ the canonical projection, we shall write $V|_U = V \cap \pi^{-1}(U)$ for the restriction of V to the open set U. We claim that

Proposition (3.2). *There exists a point $x \in X$ such that in some open neighborhood U_x of x, $V|_{U_x}$ is a differentiable complex vector subbundle of $T^{1,0}(U_x)$.*

Proof. Denote by $\mathscr{M} = \bigcup_{x \in X} \mathscr{M}_x$ the bundle of maximal direction $\mathscr{M} \subset S^{1,0}(X)$, $\mathscr{M}$ is defined by the real-analytic equation $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \sup_{\xi \in S^{1,0}(X)} R_{\xi\bar{\xi}\xi\bar{\xi}}$, so that it is a real-analytic subvariety of the compact real-analytic manifold $S^{1,0}(X)$. Let $\mathscr{M} = \mathscr{M}_1 \cup \cdots \cup \mathscr{M}_l$ be the decomposition of $\mathscr{M}$ into irreducible components. Since the global maximum of holomorphic sectional curvatures is attained at each $x \in X$, we have $\bigcup_{1 \leqslant i \leqslant l} \pi(\mathscr{M}_i) = X$. We arrange the $\mathscr{M}_i$ such that π is a submersion at some regular point of $\mathscr{M}_i$ if and only if $1 \leqslant i \leqslant k$, and denote by $\mathscr{M}'$ the union $\bigcup_{1 \leqslant i \leqslant k} \mathscr{M}_i$. Denote by U the nonempty open set $X - \pi(\bigcup_{i > k} \mathscr{M}_i)$. Then for each $x \in U$, $\mathscr{M}_x \subset \mathscr{M}'$. Let E be the union of singular points of $\mathscr{M}'$ and regular points of $\mathscr{M}'$ at which π: $\mathrm{Reg}(\mathscr{M}') \to X$ fails to be a submersion. We claim:

Lemma. *E is a real-analytic subvariety of $\mathscr{M}'$.*

Proof. The problem being local, it suffices to prove the following

(*) Let W be an irreducible real-analytic subvariety of some open subset G of $\mathbf{R}^N$ such that the projection $\rho_w(x_1,\cdots,x_N) = (x_1,\cdots,x_N)$ onto the first n coordinates is of rank n at some point of W. Let S be the union of the singular points of W and the regular points of W at which ρ fails to be a submersion. Then S is a real-analytic subvariety of W.

To prove (*) assume $0 \in W$ and let $f_1,\cdots,f_k$ be generators of the reduced ideal sheaf $\mathscr{F}_W \cap G'$ for some open neighborhood G' of 0, $G' \Subset G$. We can regard $\mathbf{R}^N$ as the real part of $\mathbf{C}^N$ and extend (cf. Gunning & Rossi [9]) $f_1,\cdots,f_k$ to holomorphic functions $F_1,\cdots,F_k$ on a Stein neighborhood D of 0 in $\mathbf{C}^N$ such that $D \cap \mathbf{R}^N = G'$. Then the common zero set of $F_1,\cdots,F_k$ is a complex-analtyic subvariety C of D such that $C \cap G' = W \cap G'$. We can assume, by shrinking D if necessary, that C is an irreducible complex-analytic variety such that the complex dimension of C equals the real dimension of W. Moreover, C is smooth at smooth points of $W \cap G'$. Now $\rho_W\colon W \to \mathbf{R}^n$ is a submersion at $x \in W$ if and only if the real n form $\rho_W^*(dx_1 \wedge \cdots \wedge dx_n)$ vanishes at x. Consider the projection $\rho_C\colon C \to \mathbf{C}^n$ defined by $\rho_C(z_1,\cdots,z_N) = (z_1,\cdots,z_N)$ extending $\rho_W|_{W \cap G'}$. Then clearly $\rho_W^*(dx_1 \wedge \cdots \wedge dx_n) = 0$ if and only if $\rho_C^*(dz_1 \wedge \cdots \wedge dz_n)(x) = 0$. Let H be the union of the singular set of C and regular points of C at which ρ_C fails to be a submersion. Then, $S \cap G' = (H \cap G') \cup \operatorname{Sing}(W \cap G')$ which clearly yields (*) if we know that H is a complex-analytic subvariety of C.

To show that H is a complex-analytic subvariety of C, We resort to the Coherence Theorem of Oka and Theorem A of Cartan-Oka (cf. Gunning & Rossi [9]). Consider the sheaf mapping $\phi\colon \mathcal{O}_C^N \to \mathcal{O}_C^k$, $\mathcal{O}_C$ denoting the reduced structure sheaf of C, defined by

$$\phi(g_1,\cdots,g_N) = \left(\left\langle dF_1, g_1\frac{\partial}{\partial z_1} + \cdots + g_N\frac{\partial}{\partial z_N}\right\rangle,\right.$$
$$\left.\cdots,\left\langle dF_k, g_1\frac{\partial}{\partial z_1} + \cdots + g_N\frac{\partial}{\partial z_N}\right\rangle\right),$$

where the pairing between 1-forms and vector fields is defined by

$$\left\langle \omega_1\frac{\partial}{\partial z_1} + \cdots + \omega_N\frac{\partial}{\partial z_n}, g_1\frac{\partial}{\partial z_1} + \cdots + g_N\frac{\partial}{\partial z_n}\right\rangle = \omega_1 g_1 + \cdots + \omega_N g_N.$$

Clearly ϕ is a morphism between coherent sheaves. Denote the kernel by $\mathscr{F}$. $\mathscr{F}$ can be regarded as a coherent sheaf of restrictions of local holomorphic vector fields on D to C which are tangent to regular points of C. Since $\mathscr{F}$ is a coherent subsheaf of $\mathcal{O}_C^N$ we can define the coherent subsheaf $\Lambda^n\mathcal{F}$ of $\Lambda^n\mathcal{O}_C^N$. At a regular point z of C, ρ_C: $C \to \mathbf{C}^n$ fails to be a submersion if and only if, under the natural pairing of n-forms and n-vector fields, we have

$$\langle dz_1 \wedge \cdots \wedge dz_n, v_1 \wedge \cdots \wedge v_n \rangle(z) = 0$$

for all holomorphic tangent vectors $v_1, \cdots, v_n$ of C at z. By Theorem A of Cartan-Oka, $(\Lambda^n\mathcal{F})_z$ is generated by $\Gamma(D, \Lambda^n\mathcal{F})$ since D is Stein. It follows that H is the set of common zeros on C of $\langle dz_1 \cdots dz_n, h\rangle$, where h runs over all holomorphic sections of $\Lambda^n\mathcal{F}$ on D. Hence, H is a complex analytic subvariety of C. This finishes the proof of the lemma.

Continuation of proof of Proposition (3.2). Recall that U is an open subset of X, $\pi^{-1}(U) \cap \mathcal{M} = \mathcal{M}' = \bigcup_{1 \leqslant i \leqslant k} \mathcal{M}_i$, $\pi|_{\mathcal{M}_i}$ is a submersion at some regular point, and E is the union of singular points of $\mathcal{M}'$ and regular points of $\mathcal{M}'$ at which π fails to be a submersion. By the preceding lemma, E is a real-analytic subvariety which obviously does not contain any component of $\mathcal{M}'$. Recall also that the bundle $V = \bigcup_{x \in X} V_x$ is obtained by taking V_x = $\mathbf{C}$-linear span of $\mathcal{M}_x$. We assert

(#) There exists a nonempty open subset U' of U and a finite number of subsets $S_1, \cdots, S_m$ of $\mathcal{M}' \cap \pi^{-1}(U')$ such that

(i) each S_i is a locally closed real-analytic submanifold (possibly disconnected) of $\pi^{-1}(U')$,

(ii) $\pi|_{S_i}$ is everywhere a submersion,

(iii) $\mathcal{M}' \cap \pi^{-1}(U') = S_1 \cup \cdots \cup S_m$.

We now set forth to prove (#). Let $E = \bigcup_{1 \leqslant i \leqslant l} E_i$ be a decomposition of E into irreducible components and assume $E' = \bigcup_{1 \leqslant i \leqslant k} E_i$ is the union of irreducible components containing the branches of $E \cap \operatorname{Reg}(\mathcal{M}')$. By Sard's Theorem $\pi|_{E_i \cap \operatorname{Reg}(\mathcal{M}')}$ is not a submersion at any point. Hence $\pi(E_i)$ is a closed semianalytic subset (in the sense of Łojasiewicz [17]) of U of measure zero. Define $U_1 = U - \pi(E')$. Then $\pi|_{\operatorname{Reg}(\mathcal{M}') \cap \pi^{-1}(U_1)}$ is everywhere a submersion. We shall choose some $U_1' \subset U_1$, to be determined later, and define S_1 by $S_1 = \operatorname{Reg}(\mathcal{M}') \cap \pi^{-1}(U')$. On U_1, let $\operatorname{Sing}(\mathcal{M}') \cap \pi^{-1}(U_1) = \bigcup_{1 \leqslant i \leqslant p} T_i$ be a decomposition of $\operatorname{Sing}(\mathcal{M}') \cap \pi^{-1}(U_1)$ into irreducible components. For each T_i either $\pi(T_i)$ is a closed semianalytic subset of U_1 or π is a submersion at some regular point of T_i. We arrange T_i such that π is a submersion at some regular point of T_i if and only if $1 \leqslant i \leqslant q$. Now let $1 \leqslant \leqslant q$. $\pi|_{T_i}$: $T_i \to U_1$ is not necessarily surjective. Let T_i' be the union of the singular set of T_i and regular points of T_i at which $\pi|_{T_i}$ fails to be a submersion. $\pi(T_i)$ is a closed semianalytic subset of U_1 (because of properness) and $\pi(T_i - T_i')$ is an open subset of U_1 dense in $\pi(T_i)$. Define $F_i = \pi(T_i) - \pi(T_i - T_i')$. Then, on each connected component Ω of $U_1 - F_i$, either $\pi|_{T_i \cap \pi^{-1}(\Omega)}$ maps $T_i \cap \pi^{-1}(\Omega)$ properly and surjectively onto Ω or $T_i \cap \pi^{-1}(\Omega) = \varnothing$. Applying Sard's Theorem to the mapping $\pi|_{\mathrm{Reg}(T_i)}$: $\mathrm{Reg}(T_i) \to U_1$ we obtain then a closed subset $\tilde{F}_i$ of measure zero of $U_1 - F_i$ such that $\pi|_{\mathrm{Reg}(T_i) \cap \pi^{-1}(U_1 - F_i - \tilde{F}_i)}$ is everywhere a submersion. Since the boundary of each Ω in U_1 is contained in F_i, clearly $F_i \cup \tilde{F}_i$ is a closed subset (of measure zero) of U_1. We now define $U_2 = U_1 - \bigcup_{1 \leqslant i \leqslant q}(F_i \cup \tilde{F}_i) - \bigcup_{q \leqslant i \leqslant p} \pi(T_i)$. For $U' \subset U_2$ to be determined we define $S_2 = \mathrm{Reg}(T_1) \cap \pi^{-1}(U')$, etc. It is now clear that one can go on by removing step-by-step the singular set of irreducible components of the preceding singular set in order to obtain open sets $U, U_1, U_2, \cdots, U_s$ all derived from the preceding set by removing a closed (semianalytic) subset of measure zero until we obtain the last open set $U' = U_s$ and the closed real-analytic submanifolds S_i, $1 \leqslant i \leqslant m$, of $\pi^{-1}(U')$ on which $\pi|_{S_i}$ is everywhere a submersion. Obviously $\mathcal{M}' \cap \pi^{-1}(U') = S_1 \cup \cdots \cup S_m$.

Propositions (3.2) will now be proved by picking some point $x \in U'$ and some open neighborhood U_x of x contained in U'. Let $x \in U'$ be a point such that V_x is of maximum dimension among points on U'. Suppose $\{v_1, \cdots, v_s\}$ is a basis of V_s with $v_i \in \mathcal{M}'_x$. Each v_i is contained in one of the pieces S_j, $1 \leqslant j \leqslant m$. Since $\pi|_{S_j}$: $S_j \to U'$ is a submersion it follows that there exist vector fields $v_i(y)$ defined for y sufficiently close to x such that $v_i(x) = v_i$, $1 \leqslant i \leqslant s$. For y sufficiently close to x, say $y \in U_x$, $\{v_1(y), \cdots, v_s(y)\}$ are linearly independent. But since $\dim_{\mathbf{C}} V_y \leqslant \dim V_x$ it follows that $V|_{U_x}$ is an s-dimensional complex vector bundle generated at each point by $\{v_1(y), \cdots, v_s(y)\}$.

4. The maximum principle for fourth-order radial derivatives

(4.1) The equality $\Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \geqslant 0$ at maximal directions α. The main objective of §4 is to prove the vanishing of fourth-order radial derivatives $\nabla_\eta^4 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ for any maximal direction α of holomorphic sectional curvatures and any real tangent vector η at $x = \pi(\alpha)$. As was explained in the introduction, we know $\nabla_\eta^i R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ for $1 \leqslant i \leqslant 3$. We will first prove $\Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \geqslant 0$ and then compute the difference between $\Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ and $S^{(4)} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$, the averaging operator of fourth-order radial derivatives introduced in §1, (1.3). Then we will conclude $S^{(4)} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \geqslant 0$, implying $\nabla_\eta^4 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0$.

Proposition (4.1). *Let α be a maximal direction of holomorphic sectional curvatures, $\pi(\alpha) = x$. Then $\Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \geqslant 0$.*

Proof. (I) Without loss of generality we may assume that α is a unit vector. Let $\{e_1, \cdots, e_n\}$ be a privileged basis of $T_x^{1,0}(X)$, $x = \pi(\alpha)$, associated to the unit maximal direction α. From §1, Proposition (1.2), we have

$$\frac{1}{2}\Delta R_{1\bar{1}1\bar{1}} = \sum_{i,j}\left|R_{1\bar{i}1\bar{j}}\right|^2 + \rho R_{1\bar{1}1\bar{1}} - 2\sum_{i,j}\left|R_{1\bar{1}i\bar{j}}\right|^2,$$

where ρ is the Einstein constant and the equality holds in a neighborhood of x when $R_{i\bar{j}k\bar{l}}(y)$, for y sufficiently close to x, is interpreted as $R\left\langle e_i(y), \overline{e_j(y)}, e_k(y), \overline{e_l(y)}\right\rangle$ with $e_i(y)$ obtained from $e_i = e_i(x)$ by parallel transport along geodesics emanating from x. Letting η be a real tangent vector at x, we have, at x

$$\frac{1}{2}\Delta R_{1\bar{1}1\bar{1},\eta\eta} = 2\sum_{i,j}\left|R_{1\bar{i}1\bar{j},\eta}\right|^2 - 4\sum_{p\in H} R_{1\bar{1}p\bar{p}}R_{1\bar{1}p\bar{p},\eta\eta} - 4\sum_{i,j}\left|R_{1\bar{1}i\bar{j},\eta}\right|^2.$$

Here we have used the equalities of §2, Proposition (2.2.1), at x and the fact that $R_{1\bar{1}1\bar{1},\eta\eta}(x) = 0$. It follows that at x,

$$\begin{aligned}\frac{1}{8}\Delta R_{1\bar{1}1\bar{1},\eta\eta} &\geqslant -\sum_{p\in H} R_{1\bar{1}p\bar{p}}R_{1\bar{1}p\bar{p},\eta\eta} - \sum_{i,j}\left|R_{1\bar{1}i\bar{j},\eta}\right|^2 \\ &= -\frac{R_{1\bar{1}1\bar{1}}}{2}\sum_{p\in H} R_{1\bar{1}p\bar{p},\eta\eta} - \sum_{i,j}\left|R_{1\bar{1}i\bar{j},\eta}\right|^2.\end{aligned}$$

Since X is Kähler-Einstein, we have $\sum_i R_{1\bar{1}i\bar{i},\eta\eta} = 0$, giving

$$\sum_{p\in H} R_{1\bar{1}p\bar{p},\eta\eta} = -R_{1\bar{1}1\bar{1},\eta\eta} - \sum_{q\in N} R_{1\bar{1}q\bar{q},\eta\eta} = -\sum_{q\in N} R_{1\bar{1}q\bar{q},\eta\eta} \quad \text{at } x.$$

Hence, we obtain the inequality

$$(*)_0 \qquad \frac{1}{8}\Delta R_{1\bar{1}q\bar{q},\eta\eta} \geqslant \frac{R_{1\bar{1}1\bar{1}}}{2}\sum_{q\in N} R_{1\bar{1}q\bar{q},\eta\eta} - \sum_{i,j}\left|R_{1\bar{1}i\bar{j},\eta}\right|^2 \quad \text{at } x.$$

(II) We claim that the only possible nonzero terms in the summation $\sum_{i,j}|R_{1\bar{1}i\bar{j},\eta}|^2$ are of the type $|R_{1\bar{1}p\bar{q},\eta}|^2$. In other words, we have

Lemma. *For any real tangent vector η at x, we have*

(i) $R_{1\bar{1}1\bar{j},\eta} = 0$ *for* $1 \leqslant j \leqslant n$,

(ii) $R_{1\bar{1}p\bar{p},\eta} = 0$ *for* $p \in H$,

(iii) $R_{1\bar{1}q\bar{q},\eta} = 0$ *for* $q \in N$.

Proof of Lemma. To prove (i) and (ii) we consider the Taylor series expansion of $R_{1\bar{1}1\bar{1}}$ along geodesics issuing from x. Let $\gamma(t)$, $-\delta < t < \delta$, be a geodesic parametrized by arc length such that $\gamma(0) = x$, $\dot{\gamma}(0) = \eta$. We know that

$$R_{1\bar{1}1\bar{1}}(\gamma(t)) = R_{1\bar{1}1\bar{1}}(x) + \frac{1}{4!}R_{1\bar{1}1\bar{1},\eta\eta\eta\eta}(x)t^4 + \cdots \quad \text{with } R_{1\bar{1}1\bar{1},\eta\eta\eta\eta} \leqslant 0$$

by the maximality of $R_{1\bar{1}1\bar{1}}(x)$. Recall that $\nabla_\eta^i R_{1\bar{1}1\bar{1}}(x) = 0$ for $1 \leqslant i \leqslant 3$. Consider the expansion

$$(*) \qquad \begin{aligned} &R\left\langle e_1 + \varepsilon e_j, \overline{e_1 + \varepsilon e_j}; e_1 + \varepsilon e_j, \overline{e_1 + \varepsilon e_j}\right\rangle(\gamma(t)) \\ &\quad = R_{1111}(\gamma(t)) + 2\varepsilon \operatorname{Re} R_{1\bar{1}1\bar{j}}(\gamma(t)) \\ &\quad\quad + \varepsilon^2\left(4R_{1\bar{1}j\bar{j}} + 2\operatorname{Re} R_{1\bar{j}1\bar{j}}\right)(\gamma(t)) + \cdots . \end{aligned}$$

Now choose $\varepsilon = t^2$ and change e_j to $e^{i\theta}e_j$ for some real θ so that $R_{1\bar{1}1\bar{j},\eta}(x)$ is real and $\geqslant 0$. We have

$$\begin{aligned} &\frac{1}{(1+\varepsilon)^2} R\left\langle e_1 + \varepsilon e_j, \overline{e_1 + \varepsilon e_j}; e_1 + \varepsilon e_j, \overline{e_1 + \varepsilon e_j}\right\rangle(\gamma(t)) \\ &\quad = \frac{1}{(1+t^4)^2}\left(R_{1\bar{1}1\bar{1}}(x) + O(t^4) + \left(2R_{1\bar{1}1\bar{j},\eta}\right)(t^3) + O(t^4)\right). \end{aligned}$$

From the fact that $R_{1\bar{1}1\bar{1}}(x) = \sup_{\xi \in T^{1,0}(X)} R_{\xi\bar{\xi}\xi\bar{\xi}}$ and comparing the Taylor expansions of the denominator and the numerator, we obtain immediately

$$R_{1\bar{1}1\bar{j},\eta}(x) = 0 \quad \text{for } j \geqslant 1 \qquad \text{(i)}.$$

To prove (iii), $R_{1\bar{1}p\bar{p},\eta} = 0$, we also use $(*)$. Choosing $j = p \in H$ in the expansion $(*)$ and setting $\varepsilon = t$ for $t > 0$ we define

$$\begin{aligned} F(t) &= R\left\langle e_1 + t^\sigma e_p, e_1 + t^\sigma e_p; e_1 + t^\sigma e_p, \overline{e_1 + t^\sigma e_p}\right\rangle(\gamma(t)) \\ &= R_{1\bar{1}1\bar{1}}(\gamma(t)) + 2\operatorname{Re} R_{1\bar{1}1\bar{p}}(\gamma(t))t^\sigma \\ &\quad + \left(4R_{1\bar{1}p\bar{p}} + 2\operatorname{Re} R_{1\bar{p}1\bar{p}}\right)(\gamma(t)) \cdot t^{2\sigma} + 4\operatorname{Re} R_{1\bar{p}p\bar{p}}(\gamma(t)) \cdot t^{3\sigma} \\ &\quad + R_{p\bar{p}p\bar{p}}(\gamma(t)) \cdot t^{4\sigma}. \end{aligned}$$

We have $R_{1\bar{1}1\bar{p},\eta}(x) = 0$ and $R_{1\bar{p}1\bar{p}}(x) = R_{1\bar{p}p\bar{p}}(x) = 0$ (Proposition (2.2.1)), so that

$$\begin{aligned} F(t) &= R_{1\bar{1}1\bar{1}}(x) + O(t^4) + O(t^{\sigma+2}) + 2R_{1\bar{1}1\bar{1}}(x) \cdot t^{2\sigma} \\ &\quad + \left(4R_{1\bar{1}p\bar{p},\eta}(x) + 2\operatorname{Re} R_{1\bar{p}1\bar{p},\eta}(x)\right)t^{2\sigma+1} + O(t^{3\sigma+1}) + O(t^{4\sigma}). \end{aligned}$$

Now choose $\sigma = 0.9$. We get

$$F(t) = R_{1\bar{1}1\bar{1}}(x)(1 + 2t^{1.8}) + \left(4R_{1\bar{1}p\bar{p},\eta}(x) + 2\operatorname{Re} R_{1\bar{p}1\bar{p},\eta}(x)\right)t^{2.8} + O(t^{2.9}).$$

By comparing the Taylor expansion of $(1+\varepsilon^2)^2$, $\varepsilon = t^{0.9}$, and that of $x(t)$, we obtain from $F(t)/(1+t^{1.8})^2 \leqslant R_{1\bar{1}1\bar{1}}(x)$ the inequality

$$4R_{1\bar{1}p\bar{p},\eta}(x) + 2\operatorname{Re} R_{1\bar{p}1\bar{p},\eta}(x) \leqslant 0.$$

Without loss of generality we may assume $\operatorname{Re} R_{1\bar{p}1\bar{p},\eta}(x) \geqslant 0$ (by some change $e_p \mapsto e^{i\theta}e_p$) so that

$$R_{11pp,\eta}(x) \leqslant 0.$$

Since the same inequality applies to the geodesic γ with orientation reversed we have also $R_{1\bar{1}p\bar{p},-\eta}(x) \leqslant 0$, giving

$$R_{1\bar{1}p\bar{p},\eta}(x) = 0 \quad \text{(ii)}.$$

Finally,

$$R_{1\bar{1}q\bar{q},\eta}(x) = 0 \quad \text{(iii)}$$

follows immediately from $R_{1\bar{1}q\bar{q}}(x) = 0$ and the fact that X carries nonnegative holomorphic bisectional curvature, so that $R_{1\bar{1}q\bar{q}}(x)$ is a minimum $R_{\xi\bar{\xi}\xi'\bar{\xi}'}$, $\xi, \xi' \in T^{1,0}(X)$.

(III) The equations $R_{1\bar{1}1\bar{1},\eta}(x) = 0$, (i) $R_{1\bar{1}1\bar{j},\eta}(x) = 0$ for $j > 1$, (ii) $R_{1\bar{1}p\bar{p},\eta}(x) = 0$ and (iii) $R_{1\bar{1}q\bar{q},\eta}(x) = 0$ can now be used to yield the estimate from $(*)_0$

$$(*) \qquad \frac{1}{8}\Delta R_{1\bar{1}1\bar{1},\eta\eta} \geqslant \frac{R_{1\bar{1}1\bar{1}}}{2} \sum_{q \in N} R_{1\bar{1}q\bar{q},\eta\eta} - \sum_{\substack{p \in N \\ q \in N}} \left|R_{1\bar{1}p\bar{q},\eta}\right|^2 \quad \text{at } x.$$

In order to finish the proof of Proposition (4.1) it suffices to prove the inequality, for each $q \in N$,

$$(\#) \qquad \frac{R_{1\bar{1}1\bar{1}}}{2} R_{1\bar{1}q\bar{q},\eta\eta} \geqslant \sum_{p \in H} \left|R_{1\bar{1}p\bar{q},\eta}\right|^2 \quad \text{at } x.$$

In fact, from the discussion of §1, (1.3), $\Delta^2 R_{1\bar{1}1\bar{1}}(x)$ is the average of $\Delta R_{1\bar{1}1\bar{1},\eta\eta}$ over the unit sphere $S_x^{1,0}(X)$ of $T_x^{1,0}(X)$, up to a multiplicative constant.

(IV) To prove $(\#)$ we apply the Schwarz inequality. Let $\gamma(t)$, $-\delta < t < \delta$, denote the same geodesic as above. Then

$$R_{1\bar{1}q\bar{q}}(\gamma(t)) = \tfrac{1}{2} R_{1\bar{1}q\bar{q},\eta\eta}(x) t^2 + \cdots.$$

On the other hand, for any $p \in H$

$$R_{1\bar{1}p\bar{p}}(\gamma(t)) = \frac{R_{1\bar{1}1\bar{1}}}{2}(x) + \frac{1}{2} R_{1\bar{1}p\bar{p},\eta\eta}(x) t^2 + \cdots.$$

By the Schwarz inequality applied to the semidefinite Hermitian form $H_t(\xi, \zeta) = R_{1\bar{1}\xi\bar{\xi}}(\gamma(t))$ at $\gamma(t)$

$$\left|R_{1\bar{1}p\bar{q}}(\gamma(t))\right|^2 \leqslant R_{1\bar{1}p\bar{p}}(\gamma(t)) R_{1\bar{1}q\bar{q}}(\gamma(t)),$$

yielding

$$\left|R_{1\bar{1}p\bar{q}}(\gamma(t))\right|^2 \leqslant \frac{R_{1\bar{1}1\bar{1}}}{2}(x) R_{1\bar{1}q\bar{q},\eta\eta}(x) t^2.$$

But $R_{1\bar{1}p\bar{q}}(x) = 0$ and $R_{1\bar{1}p\bar{q}}(\gamma(t)) = R_{1\bar{1}p\bar{q},\eta}(x) t + \cdots$, so that by comparing the Taylor expansion we have immediately

$$(\#)_0 \qquad \left|R_{1\bar{1}p\bar{q},\eta}(x)\right|^2 \leqslant \frac{R_{1\bar{1}1\bar{1}}}{4}(x) R_{1\bar{1}q\bar{q},\eta\eta}(x).$$

(V) At first glance $(\#)_0$ is not strong enough to yield $(\#)$ unless $\mathscr{H} = \mathscr{H}_\alpha$ is at most 2-dimensional. However, the estimate $(\#)_0$, for $e_q \in \mathscr{N}_\alpha$ fixed, $\|e_q\| = 1$, is true for any $e_p \in \mathscr{H}_\alpha$ of unit length. This means that, if we fix one privileged orthonormal basis of $T_x^{1,0}(X)$ associated to α, $(\#)_0$ can be applied to $\sum_{p \in H} a_p e_p$ in place of e_p for any $(a_1, \cdots, a_p)$ such that $\sum_{p \in H} |a_p|^2 = 1$. In general,

$$\left| \sum_{p \in H} a_p R_{1\bar{1}p\bar{q},\eta}(x) \right|^2 \leqslant \left(\frac{R_{1\bar{1}1\bar{1}}}{4}(x) R_{1\bar{1}q\bar{q},\eta\eta}(x) \right) \left(\sum_{p \in H} |a_p|^2 \right).$$

In particular, if we choose $a_p = \overline{R_{1\bar{1}p\bar{q},\eta}}$, then

$$\left(\sum_{p \in H} \left| R_{1\bar{1}p\bar{q},\eta}(x) \right|^2 \right)^2 \leqslant \left(\frac{R_{1\bar{1}1\bar{1}}}{4}(x) R_{1\bar{1}q\bar{q},\eta\eta}(x) \right) \left(\sum_{p \in H} \left| R_{1\bar{1}p\bar{q},\eta}(x) \right|^2 \right),$$

yielding

$$\sum_{p \in H} \left| R_{1\bar{1}p\bar{q},\eta}(x) \right|^2 \leqslant \frac{R_{1\bar{1}1\bar{1}}}{4}(x) R_{1\bar{1}q\bar{q},\eta\eta}(x),$$

an equality even sharper than the required inequality $(\#)$, proving Proposition (4.1).

(4.2) Comparing $S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ and $\Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$. Recall that from §1, (1.3), $S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$ was defined by

$$S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = c_4 \int_\eta \nabla_\eta^4 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x),$$

where c_4 is a positive constant, η a real tangent vector of unit length, and the integral is over the unit tangent sphere with the canonical metric.

Recall that from Proposition (1.3) we have the formula

$$(*)_0 \qquad 6S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \sum_{i,j} \sum_{\sigma \in S_4} R^\sigma_{\alpha\bar{\alpha}\alpha\bar{\alpha},i\bar{i}j\bar{j}},$$

where S_4 denotes the symmetry group of order 4, and $R^\sigma_{\alpha\bar{\alpha}\alpha\bar{\alpha},i\bar{i}j\bar{j}}$ is the fourth-order covariant derivative of $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ obtained by formally permuting the last four elements using σ. Our main result in this section is the following proposition.

Proposition (4.2). *Let α be a maximal direction of holomorphic sectional curvatures, $\pi(\alpha) = x$. Then $S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = \Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$. Hence $\nabla_\eta^i R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$ for $1 \leqslant i \leqslant 5$.*

Proof. (I) To prove Proposition (4.2) it suffices to show that

$$S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = \Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x).$$

In fact, since $\nabla^i_\eta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$, $1 \leqslant i \leqslant 3$, and is a global maximal direction of holomorphic sectional curvatures, we have

$$\nabla^4_\eta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \leqslant 0.$$

Integrating over η of unit length, we have

$$S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \leqslant 0.$$

From Proposition (4.1) the equality $S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = \Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$ would imply $S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$ and hence $\nabla^i_\eta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) = 0$ for $1 \leqslant i \leqslant 5$ and for all $\eta \in T_x(X)$.

(II) From the equality $(*)$ we have

$$(*) \qquad 6S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 4\operatorname{Re} \sum_{i,j} R_{\alpha\bar{\alpha}\alpha\bar{\alpha},i\bar{i}j\bar{j}+i\bar{i}\bar{j}j+ij\bar{i}\bar{j}+ij\bar{j}\bar{i}+i\bar{j}\bar{i}j+i\bar{j}j\bar{i}}.$$

To see this, there are 24 terms on the right-hand side of $(*)_0$ of the form $i***$, $\bar{i}***$, $j***$ or $\bar{j}***$ in the order of differentiation. By interchanging the roles of i and j in the same terms of the expansion of $(*)_0$, we obtain the expansion $(*)$. Furthermore we have the equalities

$$R_{\alpha\bar{\alpha}\alpha\bar{\alpha},ij\bar{i}\bar{j}} = R_{\alpha\bar{\alpha}\alpha\bar{\alpha},ij\bar{j}\bar{i}}, \qquad R_{\alpha\bar{\alpha}\alpha\bar{\alpha},i\bar{i}\bar{j}j} = R_{\alpha\bar{\alpha}\alpha\bar{\alpha},i\bar{j}\bar{i}j}.$$

Recall that by our definition of Δ^2 we have

$$\Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \sum_{i,j} R_{\alpha\bar{\alpha}\alpha\bar{\alpha},i\bar{i}j\bar{j}+i\bar{i}\bar{j}j+\bar{i}ij\bar{j}+\bar{i}i\bar{j}j}.$$

Our approach of computing $S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ at x is by converting all terms to $\sum_{i,j} R_{\alpha\bar{\alpha}\alpha\bar{\alpha},i\bar{i}j\bar{j}}$. To start with we fix at x a privileged system of orthonormal basis of $T^{1,0}_x(X)$ associated to the maximal direction α. Then, at x,

$$R_{1\bar{1}1\bar{1},i\bar{i}\bar{j}j} = R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}} + 2\sum_\mu R_{\mu\bar{1}1\bar{1},i\bar{i}}R_{1\bar{\mu}j\bar{j}} - 2\sum_\mu R_{1\bar{\mu}1\bar{1},i\bar{i}}R_{\mu\bar{1}j\bar{j}}$$
$$+ \sum_\mu R_{1\bar{1}1\bar{1},\mu\bar{i}}R_{i\bar{\mu}j\bar{j}} - \sum_\mu R_{1\bar{1}1\bar{1},i\bar{\mu}}R_{\mu\bar{i}j\bar{j}}.$$

Summing up over j we immediately have, using the Einstein condition,

$$\sum_j R_{1\bar{1}1\bar{1},i\bar{i}\bar{j}j} = \sum_j R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}}.$$

In particular, we have

$$\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}\bar{j}j} = \sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}}, \qquad \sum_{i,j} R_{1\bar{1}1\bar{1},\bar{i}ij\bar{j}} = \sum_{i,j} R_{1\bar{1}1\bar{1},\bar{i}i\bar{j}j},$$

where the second equation is obtained from the first by conjugation. Furthermore,

$$R_{1\bar11\bar1,\bar i ij\bar j} = \Big(R_{1\bar11\bar1,\bar i i} + 2\sum_\mu R_{\mu\bar11\bar1}R_{1\bar\mu \bar i i} - 2\sum_\mu R_{1\bar\mu1\bar1}R_{\mu\bar1\bar i i}\Big)_{j\bar j}$$

$$= R_{1\bar11\bar1,i\bar ij\bar j} + 2\Big(\sum_\mu R_{\mu\bar11\bar1}R_{1\bar\mu i\bar i}\Big)_{j\bar j} - 2\Big(\sum_\mu R_{1\bar\mu1\bar1}R_{\mu\bar1i\bar i}\Big)_{j\bar j}.$$

Summing up over i, we have, using the Einstein condition,

$$\sum_i R_{1\bar11\bar1,\bar iij\bar j} = \sum_i R_{1\bar11\bar1,i\bar ij\bar j}.$$

In particular, combined with equalities above

$$\sum_{i,j} R_{1\bar11\bar1,i\bar ij\bar j} = \sum_{i,j} R_{1\bar11\bar1,i\bar i\bar jj} = \sum_{i,j} R_{1\bar11\bar1,\bar ii\bar jj} = \sum_{i,j} R_{1\bar11\bar1,\bar iij\bar j}$$

so that

$$\Delta^2 R_{1\bar11\bar1} = 4\sum_{i,j} R_{1\bar11\bar1,i\bar ij\bar j}.$$

(Hence the last term is real.)

(III) From the expansion (*) for $S^{(4)}R_{\alpha\bar\alpha\alpha\bar\alpha} = S^{(4)}R_{1\bar11\bar1}$, we now have

$$6S^{(4)}R_{1\bar11\bar1} = 4\Big(3\sum_{i,j} R_{1\bar11\bar1,i\bar ij\bar j} + 2\,\mathrm{Re}\sum_{i,j} R_{1\bar11\bar1,ij\bar i\bar j} + \mathrm{Re}\sum_{i,j} R_{1\bar11\bar1,i\bar jj\bar i}\Big).$$

Now we convert the last two terms to $\sum_{i,j} R_{1\bar11\bar1,i\bar ij\bar j}$:

$$R_{1\bar11\bar1,ij\bar i\bar j} = \Big(R_{1\bar11\bar1,i\bar ij} + 2\sum_\mu R_{\mu\bar11\bar1,i}R_{1\bar\mu j\bar i} - 2\sum_\mu R_{1\bar\mu1\bar1,i}R_{\mu\bar1j\bar i} + \sum_\mu R_{1\bar11\bar1,\mu}R_{i\bar\mu j\bar i}\Big)_{\bar j}$$

$$= R_{1\bar11\bar1,i\bar ij\bar j} + 2\sum_\mu R_{\mu\bar\omega1\bar1,i\bar j}R_{1\bar\mu j\bar i} + 2\sum_\mu R_{\mu\bar11\bar1,i}R_{1\bar\mu j\bar i,\bar j}$$

$$-2\sum_\mu R_{1\bar\mu1\bar1,i\bar j}R_{\mu\bar1j\bar i} - 2\sum_\mu R_{1\bar\mu1\bar1,i}R_{\mu\bar1j\bar i,\bar j}$$

$$+\sum_\mu R_{1\bar11\bar1,\mu\bar j}R_{i\bar\mu j\bar i} + \sum_\mu R_{1\bar11\bar1,\mu}R_{i\bar\mu j\bar i,\bar j}.$$

Summing up over i, j, and applying the Bianchi identity and the Einstein condition, we obtain

$$\sum_{i,j} R_{1\bar11\bar1,ij\bar i\bar j} = \sum_{i,j} R_{1\bar11\bar1,i\bar ij\bar j} + 2\sum_{i,j,\mu} R_{\mu\bar11\bar1,i\bar j}R_{1\bar\mu j\bar i}$$

$$-2\sum_{i,j,\mu} R_{1\bar\mu1\bar1,i\bar j}R_{\mu\bar1j\bar i} + \Delta R_{1\bar11\bar1}.$$

From previous information we know the first term is real and equal to $\frac{1}{4}\Delta^2 R_{1\bar{1}1\bar{1}}$, so that at x

$$2\operatorname{Re}\sum_{i,j} R_{1\bar{1}1\bar{1},\bar{i}j\bar{j}i} = 2\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}} + 4\operatorname{Re}\sum_{i,j,\mu} R_{\mu\bar{1}1\bar{1},\bar{i}j} R_{1\bar{\mu}j\bar{i}}$$
$$-4\operatorname{Re}\sum_{i,j,\mu} R_{1\bar{\mu}1\bar{1},\bar{i}j} R_{\mu\bar{1}j\bar{i}}.$$

Regrouping the terms we have, at x,

$$2\operatorname{Re}\sum_{i,j} R_{1\bar{1}1\bar{1},\bar{i}j\bar{j}i} = 2\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}} + 2\sum_{i,j,\mu}\left(R_{\mu\bar{1}1\bar{1},\bar{i}j} - R_{\mu\bar{1}1\bar{1},j\bar{i}}\right) R_{1\bar{\mu}j\bar{i}}$$
$$+2\sum_{i,j,\mu}\left(R_{1\bar{\mu}1\bar{1},j\bar{i}} - R_{1\bar{\mu}1\bar{1},\bar{i}j}\right) R_{\mu\bar{1}j\bar{i}}.$$

We compute the commutation terms inside the parentheses to obtain

$$R_{\mu\bar{1}1\bar{1},\bar{i}j} - R_{\mu\bar{1}1\bar{1},j\bar{i}} = \sum_{\nu} R_{\nu\bar{1}1\bar{1}} R_{\mu\bar{\nu}\bar{i}j} - 2\sum_{\nu} R_{\mu\bar{\nu}1\bar{1}} R_{\nu\bar{1}\bar{i}j} + \sum_{\nu} R_{\mu\bar{1}\nu\bar{1}} R_{1\bar{\nu}\bar{i}j}.$$

The other commutation can be computed by conjugation, yielding at x

$$2\operatorname{Re}\sum_{i,j} R_{1\bar{1}1\bar{1},\bar{i}j\bar{j}i} = 2\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}} + 4\operatorname{Re}\sum_{i,j,\mu} R_{1\bar{1}1\bar{1}}\left|R_{\mu\bar{1}i\bar{j}}\right|^2$$
$$-8\operatorname{Re}\sum_{i,j,\mu} R_{1\bar{1}\mu\bar{\mu}}\left|R_{\mu\bar{1}i\bar{j}}\right|^2 + 4\operatorname{Re}\sum_{i,j} R_{1\bar{1}1\bar{1}}\left|R_{1\bar{1}i\bar{j}}\right|^2.$$

Here we have used equation (b) of Proposition (2.2.1), i.e., $R_{1\bar{i}i\bar{j}} = 0$ unless $i = j = 1$. We can furthermore regroup the commutation terms according to whether $\mu = 1$, $\mu \in H$ or $\mu \in N$, yielding

$$2\operatorname{Re}\sum_{i,j} R_{1\bar{1}1\bar{1},\bar{i}j\bar{j}i} - 2\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}}$$
$$= \sum_{i,j}\sum_{p\in H}\left(4R_{1\bar{1}1\bar{1}} - 8R_{1\bar{1}p\bar{p}}\right)\left|R_{p\bar{1}i\bar{j}}\right|^2 + 4R_{1\bar{1}1\bar{1}}\sum_{q\in N}\sum_{i,j}\left|R_{q\bar{1}i\bar{j}}\right|^2.$$

Since $R_{1\bar{1}p\bar{p}} = \frac{1}{2}R_{1\bar{1}1\bar{1}}$ and $R_{q\bar{1}i\bar{j}} = 0$ for all i, j, $1 \leqslant i, j \leqslant n$, we have obtained

$$2\operatorname{Re}\sum_{i,j} R_{1\bar{1}1\bar{1},\bar{i}j\bar{j}i} = 2\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}}.$$

(IV) Similarly, we compute

$$\operatorname{Re}\sum_{i,j} R_{1\bar{1}1\bar{1},\bar{i}\bar{j}ji} - \sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}}$$

by commutation at x,

$$R_{1\bar{1}1\bar{1},\bar{i}\bar{j}ji} = R_{1\bar{1}1\bar{1},\bar{i}j\bar{j}i} + 2\sum_{\mu} R_{\mu\bar{1}1\bar{1},\bar{i}j} R_{1\bar{\mu}j\bar{i}} - 2\sum_{\mu} R_{1\bar{\mu}1\bar{1},\bar{i}j} R_{\mu\bar{1}j\bar{i}}$$
$$+\sum_{\mu} R_{1\bar{1}1\bar{1},\mu j} R_{i\bar{\mu}j\bar{i}} - \sum_{\mu} R_{1\bar{1}1\bar{1},i\bar{\mu}} R_{\mu\bar{j}j\bar{i}}.$$

Summing over i, j and using the Einstein condition we have

$$2\,\mathrm{Re}\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{j}j\bar{i}} - 2\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}} = 4\,\mathrm{Re}\sum_{i,j,\mu} R_{\mu\bar{1}1\bar{1},i\bar{j}} R_{1\bar{\mu}j\bar{i}}$$

$$-4\,\mathrm{Re}\sum_{i,j,\mu} R_{1\bar{\mu}1\bar{1},i\bar{j}} R_{\mu\bar{1}j\bar{i}}.$$

The same computation as in (III) yields the equality

$$\mathrm{Re}\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{j}j\bar{i}} = \sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}} = \sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}}.$$

From this and equalities in (II) we obtain at x

$$6S^{(4)}R_{1\bar{1}1\bar{1}} = 24\sum_{i,j} R_{1\bar{1}1\bar{1},i\bar{i}j\bar{j}} = 6\Delta^2 R_{1\bar{1}1\bar{1}},$$

proving Proposition (4.2).

5. The maximum principle for sixth-order radial derivatives and computation of $\Delta^3 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$

(5.1) The major objective of this section is to extract further zero-order information on the curvature tensor with respect to a privileged orthonormal basis relative to any maximal direction α at any point $x \in X$. (Results of this section will be used in §6 to prove the crucial fact $\nabla_\alpha R = 0$ for most $\alpha \in \mathscr{M}$.) In order to do this it will be necessary to make use of gradient terms arising in the expressions of $S^{(4)}\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$ and $\Delta^3 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$. Since the computation of these two quantities resemble the computation of $S^{(4)}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$ and $\Delta^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x)$, which were carried out in the last section, we will be contented with sketching the steps of such computation, and indicating only the necessary modifications and new methods of applying the computation.

Keeping notations as before, we will fix some $x \in X$, some maximal direction α at x and use a fixed privileged orthonormal basis of $T_x^{1,0}(X)$ adapted to α. Recall that $T_x^{1,0}(x) = \mathbf{C}\alpha \oplus \mathscr{H}_\alpha \oplus \mathscr{N}_\alpha$; the index set of the basis for $\mathscr{H} = \mathscr{H}_\alpha$ is denoted by H and that $\mathscr{N} = \mathscr{N}_\alpha$ is denoted by N. The indices will be denoted respectively by $p, p', \cdots$ and $q, q', \cdots$. For the sake of simplicity we shall say that a curvature from $R_{1\bar{p}p'\bar{p}''}$, for example, is of type $R_{1\bar{p}p\bar{p}}$, etc., meaning that the indices p, q appearing in terms of type R_{****} can take arbitrary values in H and N respectively. We can therefore group the curvature terms into those of types $R_{1\bar{1}1\bar{1}}$, $R_{1\bar{1}1\bar{p}}$, $R_{1\bar{1}p\bar{p}}, \cdots$, etc. We shall say that a curvature term $R_{i\bar{j}k\bar{l}}$ is of type R_{****} up to conjugation and permutation of $R_{i\bar{j}k\bar{l}}$ can be obtained from R_{****} by conjugation, the allowable permutations of indices due to symmetry and by substituting any p and q indices by arbitrary indices in H and N respectively. Our major objective here is the following result on the structure of R.

Proposition (5.1). *With $x \in X$ and $\alpha \in \mathscr{M}_x$ fixed as above, the only possible nonvanishing terms of $R_{i\bar{j}k\bar{l}}$ are those of the following types up to conjugation and permutation*:

$$R_{1\bar{1}1\bar{1}},\ R_{1\bar{1}p\bar{p}},\ R_{p\bar{p}q\bar{q}},\ R_{p\bar{p}p\bar{p}},\ R_{q\bar{q}q\bar{q}} \text{ and } R_{1\bar{p}q\bar{p}}.$$

Proposition (5.1) says that almost all nonvanishing curvature terms are of bisectional type $R_{k\bar{k}l\bar{l}}$, $k, l = 1, p, q$, with the possible exception of $R_{1\bar{p}q\bar{p}}$, are in actual fact nonzero in many cases. It contains, in addition to results of Proposition (2.2.1) the fact that all curvature terms of types $R_{p\bar{p}p\bar{q}}$, $R_{p\bar{q}q\bar{q}}$ and $R_{p\bar{q}p\bar{q}}$ are zero. It is somewhat surprising that the vanishing of such terms can be derived from computations related to $R_{1\bar{1}1\bar{1}}$ since all the information in Proposition (2.2.1) obtained from variational inequalities are on terms associated to the maximal direction $e_1 = \alpha$. For the derivation of Proposition (5.1) we need the following lemma.

Lemma. *Suppose second order covariant derivatives of $R_{1\bar{1}p\bar{q}}$ vanish at x. Then, all curvature terms of types $R_{p\bar{p}p\bar{q}}$, $R_{p\bar{q}q\bar{q}}$ and $R_{p\bar{q}p\bar{q}}$ vanish.*

Proof. By polarization it suffices to prove the vanishing of the given terms, i.e., the indices p and q can be assumed to carry the same meaning. Under the hypothesis of the lemma, we have, at x,

(i) $R_{1\bar{1}p\bar{q},p\bar{p}} - R_{1\bar{1}p\bar{q},\bar{p}p} = 0$,
(ii) $R_{1\bar{1}p\bar{q},q\bar{q}} - R_{1\bar{1}p\bar{q},\bar{q}q} = 0$,
(iii) $R_{1\bar{1}p\bar{q},p\bar{q}} - R_{1\bar{1}p\bar{q},\bar{q}p} = 0$.

We compute these differences by commutation separately.

(i)

$$\begin{aligned} R_{1\bar{1}p\bar{q},p\bar{p}} - R_{1\bar{1}p\bar{q},\bar{p}p} &= \sum_\mu R_{\mu\bar{1}p\bar{q}} R_{1\bar{\mu}p\bar{p}} - \sum_\mu R_{1\bar{\mu}p\bar{q}} R_{\mu\bar{1}p\bar{p}} \\ &\quad + \sum_\mu R_{1\bar{1}\mu\bar{q}} R_{p\bar{\mu}p\bar{p}} - \sum_\mu R_{1\bar{1}p\bar{\mu}} R_{\mu\bar{q}p\bar{p}}. \end{aligned}$$

From Proposition (2.2.1), statements (c) and (d), we have the vanishing of curvature terms of types $R_{1\bar{q}i\bar{j}}$ and $R_{1\bar{p}p\bar{p}}$, so that

$$0 = R_{1\bar{1}p\bar{q},p\bar{p}} - R_{1\bar{1}p\bar{q},\bar{p}p} = -R_{1\bar{1}p\bar{p}} R_{p\bar{p}p\bar{q}}.$$

Since $R_{1\bar{1}p\bar{p}} = \frac{1}{2} R_{1\bar{1}1\bar{1}} \neq 0$, we obtain immediately $R_{p\bar{p}p\bar{q}} = 0$. Similarly we have

(ii) $R_{1\bar{1}p\bar{q},q\bar{q}} - R_{1\bar{1}p\bar{q},\bar{q}q} = -R_{1\bar{1}p\bar{p}} R_{p\bar{q}q\bar{q}}$,
(iii) $R_{1\bar{1}p\bar{q},p\bar{q}} - R_{1\bar{1}p\bar{q},\bar{q}p} = -R_{1\bar{1}p\bar{p}} R_{p\bar{q}p\bar{q}}$.

It follows therefore, under the hypothesis of the lemma,

$$R_{p\bar{p}p\bar{q}} = R_{p\bar{q}q\bar{q}} = R_{p\bar{q}p\bar{q}} = 0.$$

Our next step is therefore to prove the vanishing of second order covariant derivatives of $R_{1\bar{1}p\bar{q}}$. Recall that from the computation of $S^{(4)}R_{1\bar{1}1\bar{1}} = \Delta^2 R_{1\bar{1}1\bar{1}} = 0$ we obtain at the same time the vanishing of $R_{1\bar{1}p\bar{q},\eta}$ for any real tangent vector η at x. The term $R_{1\bar{1}p\bar{q},\eta}$ appears in the expression

$$\Delta R_{1\bar{1}1\bar{1},\eta\eta} = \sum_{i,j} |R_{1i1j,\eta}|^2 + R_{1\bar{1}1\bar{1},\eta\eta}$$
$$-2\sum_{i,j} |R_{1\bar{1}i\bar{j},\eta}|^2 - 2\,\mathrm{Re}\sum_{i,j} R_{1\bar{1}i\bar{j}}\,\overline{R_{1\bar{1}i\bar{j},\,\eta\eta}}.$$

Similarly

$$\Delta R_{1\bar{1}1\bar{1},\eta\eta\eta\eta} = \sum_{i,j} |R_{1i1j,\eta\eta}|^2 + R_{1\bar{1}1\bar{1},\eta\eta\eta\eta}$$
$$-2\sum_{i,j} |R_{1\bar{1}i\bar{j},\eta\eta}|^2 - 2\,\mathrm{Re}\sum_{i,j} R_{1\bar{1}i\bar{j}}\overline{R_{1\bar{1}i\bar{j},\,\eta\eta\eta\eta}}.$$

Here we have already used the facts $R_{1i1j,\eta} = R_{1\bar{1}i\bar{j},\eta} = 0$ derived together with the vanishing of $\Delta R_{1\bar{1}1\bar{1},\eta\eta}$. It is plausible from the preceding expression that the vanishing of $\Delta R_{1\bar{1}1\bar{1},\eta\eta\eta\eta}$ can be used to derive the vanishing of second order *radial* derivatives of $R_{1\bar{1}p\bar{q}}$. This is in fact the case. For this purpose we are going to compute the sixth order term $S^{(4)}\Delta R_{1\bar{1}1\bar{1}}$ in the same spirit as in §4 for $S^{(2)}\Delta R_{1\bar{1}1\bar{1}} = \Delta^2 R_{1\bar{1}1\bar{1}}$. Notice that the equation $R_{1\bar{1}p\bar{q},\eta\eta} = 0$ does not imply the vanishing of second order covariant derivatives. However, if instead we compute the expression

$$\Delta^3 R_{1\bar{1}1\bar{1}} = \sum_{\alpha,\beta} \Delta R_{1\bar{1}1\bar{1},(\alpha\bar{\alpha}+\bar{\alpha}\alpha)(\beta\bar{\beta}+\bar{\beta}\beta)},$$

then the gradient terms attached to $R_{1\bar{1}i\bar{j}}$ will be of the form $|R_{1\bar{1}i\bar{j},\alpha\beta}|^2$, $|R_{1\bar{1}i\bar{j},\alpha\bar{\beta}}|^2$, etc. Therefore we will further compute the commutation from $S^{(4)}\Delta R_{1\bar{1}1\bar{1}}$ to $\Delta^3 R_{1\bar{1}1\bar{1}} = S^{(2)}\Delta^2 R_{1\bar{1}1\bar{1}}$.

We will now collect the computational results into the following two propositions.

Proposition (5.2). *At an arbitrary $x \in X$ and for any $\alpha \in \mathcal{M}_x$, we have, in terms of notations used before, $S^{(4)}\Delta R_{1\bar{1}1\bar{1}} = S^{(6)}R_{1\bar{1}1\bar{1}} = 0$, so that $\nabla^i_\eta R_{1\bar{1}1\bar{1}} = 0$ for $1 \leqslant i \leqslant 7$.*

Proposition (5.3). *At an arbitrary $x \in X$ and for any $\alpha \in \mathcal{M}_x$, in terms of notations used before,*

$$\Delta^3 R_{1\bar{1}1\bar{1}} = S^{(4)}\Delta R_{1\bar{1}1\bar{1}} = 0.$$

This implies in particular that for any real tangent vector η at x

$$R_{1\bar{1}p\bar{q},\alpha\beta} = R_{1\bar{1}p\bar{q},\alpha\bar{\beta}} = R_{1\bar{1}p\bar{q},\bar{\alpha}\beta} = R_{1\bar{1}p\bar{q},\bar{\alpha}\bar{\beta}} = 0$$

for $p \in H$, $q \in N$ and $1 \leqslant \alpha, \beta \leqslant n$.

(5.2) Sketch of the proof of Proposition (5.2)—$S^{(4)}\Delta R_{1\bar{1}1\bar{1}} = S^{(6)}R_{1\bar{1}1\bar{1}} = 0$. By the same argument as step (I) of Proposition (4.2) the equality $S^{(6)}R_{1\bar{1}1\bar{1}} = 0$ will imply $\nabla_\eta^i R_{1\bar{1}1\bar{1}} = 0$ for $1 \leqslant i \leqslant 7$. The proof of Proposition (5.2) therefore reduces to the following two statements.

Proposition (5.2), Part I. *Notations as above, for any real tangent vector* η *at* x *we have* $\Delta R_{1\bar{1}1\bar{1},\eta\eta\eta\eta} \geqslant 0$ *so that in particular* $S^{(4)}\Delta R_{1\bar{1}1\bar{1}} \geqslant 0$.

Proposition (5.2), Part II. *Notations as above, we have at* x $S^{(6)}R_{1\bar{1}1\bar{1}} \geqslant S^{(4)}\Delta R_{1\bar{1}1\bar{1}} \geqslant 0$.

For the derivation of the first inequality we need the following lemma for which we include a proof.

Lemma. *At* x, *for any* i, j, $1 \leqslant i, j \leqslant n$, *for any* $p \in H$, $q \in N$ *and* η *any real tangent vector at* x, *we have*

(i) $R_{1\bar{i}1\bar{j},\eta} = R_{1\bar{1}i\bar{j},\eta} = 0$,

(ii) $R_{1\bar{1}1\bar{j},\eta\eta} = 0$,

(iii) $R_{1\bar{1}q\bar{q},\eta\eta} = 0$

(iv) $R_{1\bar{1}p\bar{p},\eta\eta} = R_{1\bar{p}1\bar{p},\eta\eta} = 0$.

Proof. From the proof of Proposition (4.1) (see formula (*), step (III) where we dropped the first term and obtained an inequality) we have

$$(\#)\quad \frac{1}{8}\Delta R_{1\bar{1}1\bar{1},\eta\eta} = \frac{1}{2}\sum_{i,j}\left|R_{1\bar{i}1\bar{j},\eta}\right|^2 + \frac{R_{1\bar{1}1\bar{1}}}{2}\sum_{q\in N} R_{1\bar{1}q\bar{q},\eta\eta} - \sum_{\substack{p\in H\\ q\in N}}\left|R_{1\bar{1}p\bar{q},\eta\eta}\right|^2.$$

From the last formula of step (V) we have at x

$$\frac{R_{1111}}{4}R_{11qq,\eta\eta} \geqslant \sum_{p\in H}\left|R_{1\bar{1}p\bar{q},\eta}\right|^2.$$

From these we derive

$$\frac{1}{8}R_{1\bar{1}1\bar{1},\eta\eta} \geqslant \frac{1}{2}\sum_{i,j}\left|R_{1\bar{i}1\bar{j},\eta}\right|^2 + \sum_{\substack{p\in H\\ q\in N}}\left|R_{1\bar{1}p\bar{q},\eta}\right|^2.$$

Recalling formulas (i), (ii), (iii) of step (II) of Proposition (4.1), we have

$$R_{1\bar{1}1\bar{j},\eta} = R_{1\bar{1}p\bar{p},\eta} = R_{1\bar{1}q\bar{q},\eta} = 0 \quad \text{for } 1 \leqslant i, j \leqslant n.$$

Since $\Delta R_{1\bar{1}1\bar{1},\eta\eta} = 0$ by Proposition (4.2), we thus obtain

$$R_{1\bar{i}1\bar{j},\eta} = R_{1\bar{1}i\bar{j},\eta} = 0,$$

proving (i) of the lemma.

To prove (ii) we need only to consider the case $j > 1$. Let $\gamma(t)$, $-\delta < t < \delta$, be a geodesic parametrized by arc length such that $\gamma(0) = x$ and $\gamma(0) = \eta$. Since $\nabla_\eta^i R_{1\bar{1}1\bar{1}} = 0$ at x for $1 \leqslant i \leqslant 5$, we have

$$R_{1111}(\gamma(t)) = R_{1111}(x) + \frac{1}{6!}\nabla^6 R_{1111}(x)t^6 + \cdots = R_{1111}(x) + O(t^6).$$

On the other hand, we have

$$
\begin{aligned}
&R\left\langle e_1 + \varepsilon e_j, \overline{e_1 + \varepsilon e_j}; e_1 + \varepsilon e_j, \overline{e_1 + \varepsilon e_j}\right\rangle(\gamma(t)) \\
(*)\qquad &= R_{1\bar{1}1\bar{1}}(\gamma(t)) + 2\varepsilon \operatorname{Re} R_{1\bar{1}1\bar{j}}(\gamma(t)) + \varepsilon^2\left(4R_{1\bar{1}j\bar{j}} + 2\operatorname{Re} R_{1\bar{j}1\bar{j}}\right)(\gamma(t)) \\
&\quad + \varepsilon^3\left(4\operatorname{Re} R_{1\bar{j}j\bar{j}}(\gamma(t))\right) + \varepsilon^4 R_{j\bar{j}j\bar{j}}(\gamma(t)).
\end{aligned}
$$

Substituting $\varepsilon = t^3$ and changing e_j to $e^{i\theta}e_j$ for some real θ so that $R_{1\bar{1}1\bar{j},\eta\eta}(x)$ is real and $\geqslant 0$ (recall that $R_{1\bar{1}1\bar{j},\eta} = 0$ at x for $j > 1$), we have, for $j > 1$,

$$
\begin{aligned}
&\frac{1}{(1+\varepsilon^2)^2} R\left\langle e_1 + \varepsilon e_j, \overline{e_1 + \varepsilon e_j}; e_1 + \varepsilon e_j, \overline{e_1 + \varepsilon e_j}\right\rangle(\gamma(t)) \\
&= \frac{1}{(1+t^6)^2}\left(R_{1\bar{1}1\bar{1}}(x) + O(t^6) + 2R_{1\bar{1}1\bar{j},\eta\eta}(x)t^5 + O(t^6)\right).
\end{aligned}
$$

But since the holomorphic sectional curvature of $(e_1 + \varepsilon e_j)/\sqrt{1+\varepsilon^2}$ at $\gamma(t)$ is smaller than $R_{1\bar{1}1\bar{1}}(x)$, we see immediately that $R_{1\bar{1}1\bar{j},\eta\eta}(x) = 0$.

Equation (iii), $R_{1\bar{1}q\bar{q},\eta\eta} = 0$ for all $q \in N$, has already been proved in Proposition (2.2.2). To prove (iv) we use the same expansion $(*)$ above. We choose $j = p \in H$ and set $\varepsilon = t^\sigma$ for $t > 0$. By taking $\sigma = 1.5$ and comparing expansions in terms of t we obtain

$$
4R_{1\bar{1}p\bar{p},\eta\eta}(x) + 2\operatorname{Re} R_{1\bar{p}1\bar{p},\eta\eta}(x) \leqslant 0.
$$

By replacing e_p by $e^{i\theta}e_p$ for a suitable real θ we may assume that $\operatorname{Re} R_{i\bar{p}i\bar{p},\eta\eta}(x) \geqslant 0$, so that

$$
R_{1\bar{1}p\bar{p},\eta\eta}(x) \leqslant 0.
$$

Since $R_{1\bar{1}1\bar{1},\eta\eta}(x) = R_{1\bar{1}q\bar{q},\eta\eta}(x) = 0$ for all $q \in N$, from the Einstein condition we obtain $\sum_{p\in H} R_{1\bar{1}p\bar{p},\eta\eta}(x) = 0$, giving

$$
R_{1\bar{1}p\bar{p},\eta\eta}(x) = R_{1\bar{p}1\bar{p},\eta\eta}(x) = 0
$$

and completing the proof of the lemma.

Using the lemma one obtains immediately from the formula of Berger

$$
\begin{aligned}
&\frac{1}{8}\Delta R_{1\bar{1}1\bar{1},\eta\eta\eta\eta} = \frac{3}{2}\sum_{i,j}\left|R_{1\bar{i}1\bar{j},\eta\eta}\right|^2 + 3\sum_{\substack{p\in H\\ q\in N}}\left|R_{1\bar{1}p\bar{q},\eta\eta}\right|^2 \\
(\#\#)\qquad & \cdot \sum_{q\in N}\left(\frac{R_{1\bar{1}1\bar{1}}}{2}R_{1\bar{1}q\bar{q},\eta\eta\eta\eta} - 3\sum_{p\in H}\left|R_{1\bar{1}p\bar{q},\eta\eta}\right|^2\right).
\end{aligned}
$$

By the same application of the Schwarz inequality as in Proposition (4.2) we can show that the term inside the bracket is always nonnegative, proving $\frac{1}{8}\Delta R_{1\bar{1}1\bar{1},\eta\eta\eta\eta} \geqslant 0$ and hence the integrated form $S^{(4)}\Delta R_{1\bar{1}1\bar{1}} \geqslant 0$, proving Part I of Proposition (5.2).

To prove Part II we need only to show $S^{(6)}R_{1\bar{1}1\bar{1}} \geqslant S^{(4)}\Delta R_{1\bar{1}1\bar{1}}$. This is done by the same commutation technique as in Proposition (4.2). Since covariant derivatives with both barred and unbarred indices are involved, we will need a conversion of our knowledge of radical derivatives into that of general covariant derivatives. We shall only indicate the procedure by an example. It will be necessary, for example, to use the fact that all second order covariant derivatives of $R_{1\bar{1}p\bar{p}}$, $p \in H$, vanish. By the preceding lemma we know that all second-order radial derivatives of $R_{1\bar{1}p\bar{p}}$, $p \in H$, vanish at z, i.e., $R_{1\bar{1}p\bar{p},\eta\eta} = 0$ for all $\eta \in T_x(X)$. By polarization (Proposition (1.4)) we obtain, for $1 \leqslant \alpha, \beta \leqslant n$,

$$R_{11p\bar{p},\alpha\beta} = R_{1\bar{1}p\bar{p},\bar{\alpha}\bar{\beta}} = R_{1\bar{1}p\bar{p},\alpha\bar{\beta}} + R_{1\bar{1}p\bar{p},\bar{\beta}\alpha} = 0.$$

To prove $\nabla^2 R_{1\bar{1}p\bar{p}} = 0$ it suffices therefore to show $R_{1\bar{1}p\bar{p},\alpha\bar{\beta}} - R_{1\bar{1}p\bar{p},\bar{\beta}\alpha} = 0$, which can be obtained by the formula for commutation and our knowledge of zero order information on the curvature tensor.

The rest of the proof of $S^{(6)}R_{1\bar{1}1\bar{1}} \geqslant S^{(4)}\Delta R_{1\bar{1}1\bar{1}} \geqslant 0$ follows the same line of thought as in Proposition (4.2) and will be omitted.

(5.3) Sketch of proof of Proposition (5.3)—$\Delta^3 R_{1\bar{1}1\bar{1}} = S^{(4)}\Delta R_{1\bar{1}1\bar{1}} = 0$. From Proposition (5.2) one can derive the vanishing of a number of second-order radial derivatives of the curvature tensor. In addition to the list given in the lemma, we obtain from the actual expression (# #) of $\Delta R_{1\bar{1}1\bar{1},\eta\eta\eta\eta}$ (recall $\Delta R_{1\bar{1}1\bar{1},\eta\eta\eta\eta} \geqslant 0$ for $\eta \in T_x(X)$) the vanishing of $R_{1\bar{i}1\bar{j},\eta\eta}$ and $R_{1\bar{1}p\bar{q},\eta\eta}$ for $1 \leqslant i, j \leqslant n$, $p \in H$ and $q \in N$. Recall that for the derivation we needed the vanishing of $\nabla^2 R_{1\bar{1}p\bar{q}}$. For this purpose we need Proposition (5.3). First we write down the following simplified formula of $\Delta^3 R_{1\bar{1}1\bar{1}}$ using our knowledge of certain vanishing covariant derivatives as indicated at the end of (5.2):

$$(\#\,\#)' \quad \frac{1}{2}\Delta^3 R_{1\bar{1}1\bar{1}} = 2R_{1\bar{1}1\bar{1}} \sum_{q\in N} \Delta^2 R_{1\bar{1}q\bar{q}} - \sum_{k,l} \sum_{\substack{p\in H\\ q\in N}} \left(\left|R_{1\bar{1}p\bar{q},kl}\right|^2 + \left|R_{1\bar{1}p\bar{q},\bar{k}l}\right|^2\right).$$

The derivation of this formula is very much the same as the formula for $\Delta R_{1\bar{1}1\bar{1},\eta\eta\eta\eta}$ in (5.2). Here the covariant derivatives associated to $R_{1\bar{i}1\bar{j}}$ are discarded because one can derive the equality $\nabla^2 R_{1\bar{i}1\bar{j}} = 0$ from $R_{1\bar{i}1\bar{j},\eta\eta} = 0$ for all $\eta \in T_x(X)$, by the argument of the last paragraph of (5.2). To prove Proposition (5.3) it suffices therefore to show $\Delta^3 R_{1\bar{1}1\bar{1}} = 0$ and $\Delta^2 R_{1\bar{1}q\bar{q}} \leqslant 0$. The derivation of $\Delta^3 R_{1\bar{1}1\bar{1}} = 0$ follows the same pattern as the derivation of the inequality $S^{(6)}R_{1\bar{1}1\bar{1}} \geqslant S^{(4)}\Delta R_{1\bar{1}1\bar{1}} \geqslant 0$. Namely, we compare $\Delta^3 R_{1\bar{1}1\bar{1}} = S^{(2)}\Delta^2 R_{1\bar{1}1\bar{1}}$ against $S^{(4)}\Delta R_{1\bar{1}1\bar{1}}$ by the formula for commutation. (See the proof of Proposition (4.2).) In the present situation we actually obtain $\Delta^3 R_{1\bar{1}1\bar{1}} = S^{(4)}\Delta R_{1\bar{1}1\bar{1}} = 0$ directly. The derivation of $\Delta^2 R_{1\bar{1}q\bar{q}} \leqslant 0$ is more involved conceptionally. It suffices to show $\Delta R_{1\bar{1}q\bar{q},\eta\eta} \leqslant 0$ for all $\eta \in T_x(X)$. We derive from Proposition (2.2.2) (which gives $\Delta R_{1\bar{1}q\bar{q}} = 0$) the expression

$$\Delta R_{1\bar{1}q\bar{q},\eta\eta} = \sum_{p,r\in H} \left|R_{1\bar{p}p\bar{r}}\right|^2_{,\eta\eta} - \frac{R_{1\bar{1}1\bar{1}}}{2} \sum_{p\in H} R_{p\bar{p}q\bar{q},\eta\eta}$$

$$= \int_{\substack{\|\xi\|=1\\ \xi\in H}} \left[\sum_{r\in H} \left|R_{1\bar{\xi}q\bar{r}}\right|^2 - R_{1\bar{1}\xi\bar{\xi}} R_{\xi\bar{\xi}q\bar{q}} \right]_{\eta\eta}.$$

Note that the term inside the bracket in the integrand vanishes at x by formula (e) of Proposition (2.2.1). Call this expression $\delta_q(\xi)$ in a neighborhood of x. (As usual the vectors e_q and ξ in a neighborhood of x are understood to be obtained by parallel transport from x of $e_q(x)$ and $\xi(x)$ along geodesics.) Recall that in Proposiiton (2.2.2) $\delta_q(\xi)(x)$ was interpreted as the discriminant of some quadratic polynomial associated with $R_{1\bar{1}q\bar{q}}$. In fact, we defined at x

$$G(\varepsilon) = R\left(e_1 + \varepsilon\xi, \overline{e_1 + \varepsilon\xi}, e_q + \sum_{r\in H} C_r e_r, \overline{e_q + \varepsilon \sum_{r\in H} C_r e_r}\right),$$

and $\delta_q(\xi)$ is the discriminant of the coefficient of ε^2 in the Taylor expansion of $G(\varepsilon)$ in ε, regarded as a quadratic polynomial in the variables C_r, $r \in H$. This quadratic polynomial is positive definite (since $R_{1\bar{1}q\bar{q}} = 0$ and $G(\varepsilon) \geqslant 0$ because X carried semipositive bisectional curvature). The vanishing of the discriminant then implies the existence of a nonzero set of coefficients $(C_r)_{r\in H}$ such that the coefficient of ε^2 in $G(\varepsilon)$ vanishes. In fact, this is given by the formula $C_r = -(R_{1\bar{\xi}q\bar{r}}/R_{1\bar{1}r\bar{r}})(x)$. Fix a geodesic $\gamma(t)$ passing through x with $\gamma(0) = x$ and $\dot{\gamma}(0) = \eta$ and define now $C_r(t) = -(R_{1\bar{\xi}q\bar{r}}/R_{1\bar{1}r\bar{r}})(\gamma(t))$ obtained by parallel transport. Consider the function for $t \geqslant 0$

$$F_\sigma(t) = R\Big(e + t^\sigma\xi, \overline{e_1 + t^\sigma\xi}, e_q + \sum_{r\in H} C_r(t) e_r, e_q$$
$$+ t^\sigma \sum_{r\in H} C_r(t) e_r\Big)(\gamma(t)) \geqslant 0.$$

Writing $R_{1\bar{\xi}q\bar{r}(t)}$ for $R_{1\bar{\xi}q\bar{r}}(\gamma(t))$ etc, the coefficient of $t^{2\sigma}$ in the expansion of $F_\sigma(t)$ in t is given by

$$K(t) = R_{\xi\bar{\xi}q\bar{q}}(t) - \sum_{r\in H} \frac{\left|R_{1\bar{\xi}q\bar{r}}\right|^2}{R_{1\bar{1}r\bar{r}}}(t).$$

We note that $K(0) = -4/R_{1\bar{1}1\bar{1}}(0)$, $\delta_q(\xi)\ (0) = 0$ and that $K''(0) = (-4/R_{1\bar{1}1\bar{1}}(0))\nabla_\eta^2\delta_q(\xi)(0)$. To finish the proof of $\Delta^2 R_{1\bar{1}q\bar{q}}(x) \leqslant 0$ it suffices to show $K''(0) \geqslant 0$. The proof of this follows the same line of argument as in the lemma of (5.2). Namely, by choosing appropriate σ, we conclude successively the vanishing of certain coefficients of powers of t. The starting point of this algorithm is the estimate $R_{1\bar{1}q\bar{q}}(t) = O(t^6)$. To see this from the expansion $(\#\#)$ of $\Delta R_{1\bar{1}1\bar{1},\eta\eta\eta\eta}$ and its vanishing at x we have $\nabla_\eta^i R_{1\bar{1}q\bar{q}}(x) = 0$ for $0 \leqslant i \leqslant 4$, yielding immediately $R_{1\bar{1}q\bar{q}}(t) = O(t^6)$ since $R_{1\bar{1}q\bar{q}}(x) = 0$ is a minimum of bisectional curvatures. The rest of the argument is routine and will be omitted.

The vanishing of $\Delta^3 R_{1\bar{1}1\bar{1}}$ and $\Delta^2 R_{1\bar{1}q\bar{q}} \leqslant 0$ imply the vanishing of $\nabla^2 R_{1\bar{1}p\bar{q}}$ by the expansion $(\#\#)'$ of $\Delta^3 R_{1\bar{1}1\bar{1}}$, which in turn implies the main result Proposition (5.1) of this section, as was indicated in (5.1).

6. Invariance of R along integral curves of vector fields of maximal directions

(6.1) We will make use of our preceding knowledge of the curvature tensor and first-order covariant derivatives to show that there is a nonempty open set U such that for any $x \in U$ and any $\alpha \in \mathscr{M}_x$, $\nabla_\alpha R(x) = 0$. It follows immediately that if $\gamma(t)$ is a curve in U, $\gamma(t)$ is a multiple of some $\alpha \in \mathscr{M}_{\gamma(t)}$, then the curvature tensor is invariant under parallel transport along γ. In order to prove $\nabla_\alpha R(x) = 0$ we first collect all information about first-order covariant derivatives at x. As usual we will fix $x \in X$, $\alpha \in \mathscr{M}_x$ and use a privileged orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_x^{1,0}(X)$ adapted to $\alpha = e_1$.

Lemma 1. *At x, $\nabla R_{1\bar{i}1\bar{j}} = \nabla R_{1\bar{1}i\bar{j}} = \nabla R_{1\bar{q}i\bar{j}} = \nabla R_{1\bar{p}p\bar{p}} = 0$ for all $p \in H$, $q \in N$ and for $1 \leqslant i, j \leqslant n$.*

Proof. The only thing that was not already contained in Lemma 3 of the Appendix, Step VII, is the equation $\nabla R_{1\bar{p}p\bar{p}} = 0$. To prove this consider the expansion along any geodesic $\gamma(t)$, $-\delta < t < \delta$, passing through x with $\gamma(0) = x$ and $\dot{\gamma}(0) = \eta$. Writing $R_{1\bar{1}1\bar{1}}(\gamma(t)) = R_{1\bar{1}1\bar{1}}(t)$, etc., we define for $\sigma > 0$, $0 \leqslant t < \delta$

$$
\begin{aligned}
F_\sigma(t) &= R\left(e_1 + t^\sigma e_p, \overline{e_1 + t^\sigma e_p}, e_1 + t^\sigma e_p, \overline{e_1 + t^\sigma e_p}\right)(\gamma(t)) \\
&= R_{1\bar{1}1\bar{1}}(t) + 4t^\sigma \operatorname{Re} R_{1\bar{1}1\bar{p}}(t) + t^{2\sigma}\left(4R_{1\bar{1}p\bar{p}}(t) + 2\operatorname{Re} R_{1\bar{p}1\bar{p}}(t)\right) \\
&\quad + 4t^{3\sigma}\operatorname{Re} R_{1\bar{p}p\bar{p}}(t) + t^{4\sigma} R_{p\bar{p}p\bar{p}}(t).
\end{aligned}
$$

Recall that from Proposition (5.2) and Lemma 3 of the Appendidx we have

$$
\begin{aligned}
&R_{1\bar{1}1\bar{1}}(t) = R_{1\bar{1}1\bar{1}}(0) + O(t^8), \qquad R_{1\bar{1}1\bar{p}}(t) = O(t^3), \\
&R_{1\bar{p}1\bar{p}}(t) = O(t^3), \qquad R_{1\bar{1}p\bar{p}}(t) = \tfrac{1}{2} R_{1\bar{1}1\bar{1}}(0) + O(t^3).
\end{aligned}
$$

From the maximality of $R_{1\bar{1}1\bar{1}}(0)$ we have

$$F_\sigma(t) \leqslant R_{1111}(0) \cdot (1 + t^{2\sigma})^2.$$

Take any $\sigma > 0$ and comparing the coefficients on both sides of the inequality, we have

$$1 + O(t^8) + O(t^{\sigma+3}) + 2t^{2\sigma} + O(t^{2\sigma+3}) + 4t^{3\sigma+1}R_{1\bar{p}p\bar{p},\eta} + O(t^{4\sigma}) \leqslant 1 + 2t^{2\sigma} + t^{4\sigma},$$

noting that $R_{1\bar{p}p\bar{p}} = 0$ by Proposition (2.2.2). Substituting $\sigma = 0.5$ we immediately obtain $R_{1\bar{p}p\bar{p},\eta} \leqslant 0$. Applying the inequality to the geodesic $\gamma^-(t) = \gamma(-t)$ we obtain $-R_{1\bar{p}p\bar{p},\eta} \leqslant 0$ and hence $R_{1\bar{p}p\bar{p},\eta} = 0$, proving the lemma.

The main result of §6 is the following proposition:

Proposition (6.1). *In the notation of Lemma* 1, *there exists a nonemty dense open set U such that we have, at any point $x \in U$ and for any $e_1 = \alpha \in \mathscr{M}_x$,*

$$\nabla_1 R_{i\bar{j}k\bar{l}} \quad \textit{for } 1 \leqslant i, j, k, l \leqslant n.$$

Proof. By means of polarization it suffices to prove $\nabla_1 R_{i\bar{j}k\bar{l}}$ for $i, j, k, l = 1$, p or q, where p and q represent typical elements of H and N respectively. We will first prove this for all types with one exception by using Lemma 1 and the Bianchi identity. First, we can classify curvature terms into groups of types up to conjugation and permutation of indices:

(i) $R_{1\bar{i}j\bar{k}}$ for $1 \leqslant i, j, k \leqslant n$;
(ii) $R_{p\bar{p}p\bar{p}}$ for $p \in H$;
(iii) $R_{q\bar{q}q\bar{q}}$ for $q \in N$;
(iv) $R_{p\bar{p}q\bar{q}}$ for $p \in H$, $q \in N$;
(v) $R_{p\bar{p}p\bar{q}}$ for $p \in H$, $q \in N$;
(vi) $R_{p\bar{q}q\bar{q}}$ for $p \in H$, $q \in N$;
(vii) $R_{p\bar{q}p\bar{q}}$ for $p \in H$, $q \in N$.

Since this division is up to conjugation (and permutation of indices), it is necessary to prove $R_{i\bar{j}k\bar{l},1} = R_{i\bar{j}k\bar{l},\bar{1}} = 0$ for all terms given in the list. We have

(i) $R_{1\bar{i}j\bar{k},1} = R_{1\bar{i}1\bar{k},j} = 0$, $R_{1\bar{i}j\bar{k},\bar{1}} = R_{1\bar{1}j\bar{k},i} = 0$;
(ii) $\overline{R_{p\bar{p}p\bar{p},\bar{1}}} = R_{p\bar{p}p\bar{p},1} = R_{1\bar{p}p\bar{p},p} = 0$;
(iii) $\overline{R_{q\bar{q}q\bar{q},\bar{1}}} = R_{q\bar{q}q\bar{q},1} = R_{1\bar{q}q\bar{q},q} = 0$;
(iv) $\overline{R_{p\bar{p}q\bar{q},\bar{1}}} = R_{p\bar{p}q\bar{q},1} = R_{1\bar{q}p\bar{p},q} = 0$;
(v) $R_{p\bar{p}p\bar{q},1} = R_{1\bar{q}p\bar{p},p} = 0$, $R_{p\bar{p}p\bar{q},\bar{1}} = R_{p\bar{p}p\bar{1},\bar{q}} = 0$;
(vi) $R_{p\bar{q}q\bar{q},1} = R_{1\bar{q}p\bar{q},q} = 0$, $R_{p\bar{q}q\bar{q},\bar{1}} = R_{q\bar{1}p\bar{q},\bar{q}} = 0$;
(vii) $R_{p\bar{q}p\bar{q},1} = R_{1\bar{q}p\bar{q},p} = 0$, $R_{p\bar{q}p\bar{q},\bar{1}} = ?$.

Everything is proved except for $R_{p\bar{q}p\bar{q},\bar{1}}$, because the only possible application of Bianchi identity $R_{p\bar{q}p\bar{q},\bar{1}} = R_{p\bar{1}p\bar{q},\bar{q}}$ does not yield a curvature term for which one can apply Lemma 1. To complete the proof of Proposition (6.1) it suffices therefore to prove

Lemma 2. *There exists a dense open set U of X such that for any $\alpha \in \mathscr{M}_x$ and for any $\eta \in T_x(X)$, we have, in terms of a privileged basis at x adapted to α,* $R_{p\bar{q}p\bar{q},\eta} = 0$.

Proof of Lemma 2. In Proposition (5.1) we used the vanishing of $\nabla^2 R_{1\bar{1}p\bar{q}}$ to conclude that $R_{p\bar{q}p\bar{q}} = R_{p\bar{p}p\bar{q}} = R_{p\bar{q}q\bar{q}} = 0$ at x. By the same argument it would be possible to deduce $\nabla R_{p\bar{q}p\bar{q}} = 0$, etc., at x if we know $\nabla^3 R_{1\bar{1}p\bar{q}} = 0$. But this would necessitate the computation of $\Delta^4 R_{1\bar{1}1\bar{1}}$. Instead we will show that our knowledge of the structure of the curvature tensor (Proposition (5.1)) and additional knowledge on covariant derivatives is sufficient for proving $\nabla R_{p\bar{q}p\bar{q}} = 0$ at x wherever $\alpha \in \mathscr{M}'$, where $\mathscr{M}'$ as defined in (3.1) is the union of components $\mathscr{M}_i$, $1 \leqslant i \leqslant i$, such that $\pi|_{\mathscr{M}_i}$ is a submersion at some smooth point. This contains in particular Lemma 2. To do this it suffices by continuity to prove $\nabla R_{p\bar{q}p\bar{q}} = 0$ at x for $\alpha \in \operatorname{Reg}\mathscr{M}_i - \{\alpha \in \mathscr{M}_i$ where π fails to be a submersion at $\alpha\}$ for $1 \leqslant i \leqslant k$. For any such $\alpha \in \mathscr{M}_x$ there exists a smooth vector field $\tilde{\alpha}(y)$ defined on a neighborhood W of x such that $\tilde{\alpha}(x) = \alpha$ and $\tilde{\alpha}(y) \in \mathscr{M}_y$ for each $y \in W$. At $y \in W$ we have the orthogonal decomposition

$$T_y^{1,0} = \mathbf{C}\tilde{\alpha}(y) \oplus \mathscr{H}_{\tilde{\alpha}(y)} \oplus \mathscr{N}_{\tilde{\alpha}(y)}.$$

Since the dimension of $\mathscr{H}_{\tilde{\alpha}(y)}$ and $\mathscr{N}_{\tilde{\alpha}(y)}$ are both independent of y, as a consequence of the Einstein condition or simply of the fact that the Ricci tensor is continuous, the splitting given above for each $y \in W$ actually yields an orthogonal splitting of the smooth vector bundle $T^{1,0}(W)$ as

$$T^{1,0}(W) = \mathscr{A}(W) \oplus \mathscr{H}(W) \oplus \mathscr{N}(W),$$

where by definition $\mathscr{A}(W) = \bigcup_{y \in W} \mathbf{C}\tilde{\alpha}(y)$, etc. Fix a geodesic $\gamma(t)$, $-\delta < t < \delta$, $\gamma(0) = x$ through x lying in W and denote by $\alpha(y) \in T_y^{1,0}(X)$ obtained by parallel transport of $\alpha(x) = \alpha$ along γ. Denote by η the tangent vector $\gamma'(0)$ at x. Fix some $e_q \in \mathscr{N}_\alpha$, $\|e_q\| = 1$, and denote by $e_q(y) \in T_y^{1,0}(X)$, $y \in \gamma$, the corresponding vector similarly obtained by parallel transport. We write $\alpha(t)$ for $\alpha(y)$, etc. for $y = \gamma(t)$. We have the orthogonal decomposition

$$\alpha(t) = a(t)\tilde{\alpha}(t) + \xi(t) + \zeta(t), \quad \text{with } \xi(t) \in \mathscr{H}_{\tilde{\alpha}(t)},\ \zeta(t) \in \mathscr{N}_{\tilde{\alpha}(t)},$$

$$e_q(t) = b(t)\tilde{\alpha}(t) + \xi'(t) + \zeta'(t), \quad \text{with } \xi'(t) \in \mathscr{H}_{\tilde{\alpha}(t)},\ \zeta'(t) \in \mathscr{N}_{\tilde{\alpha}(t)}.$$

Here obviously $\xi(0) = \zeta(0) = \xi'(0) = b(0)\tilde{\alpha}(0) = 0$. We assert that

$$(*) \qquad b(t) = O(t^2).$$

The estimate $(*)$ will be used to study the behavior of $R_{p\bar{q}p\bar{q}}(t)$ for $p \in H$ and $q \in N$ in order to conclude $R_{p\bar{q}p\bar{q},\eta}(0) = 0$. To prove $(*)$ recall that we have

$$R_{1\bar{1}1\bar{q}}(0) = R_{1\bar{1}1\bar{q},\eta} = 0$$

which means that

$$R\left(\alpha(t), \overline{\alpha(t)}, \alpha(t), \overline{e_q(t)}\right) = O(t^2).$$

Substituting the decompositions of $\alpha(t)$ and $e_q(t)$ into the preceding equation and using the fact that

$$\begin{aligned}
&R\big(\tilde\alpha(t), \overline{\tilde\alpha(t)}, \tilde\alpha(t), \overline{\xi'(t)+\zeta'(t)}\big) \quad (\text{type } R_{1\bar 11\bar j} \text{ for } j>1)\\
&\quad = R\big(\tilde\alpha(t), \overline{\tilde\alpha(t)}, \alpha(t), \overline{\zeta'(t)}\big) \quad (\text{type } R_{1\bar q i\bar j})\\
&\quad = R\big(\tilde\alpha(t), \overline{\xi(t)+\zeta(t)}, \tilde\alpha(t), \overline{\zeta'(t)+\xi'(t)}\big) \quad (\text{type } R_{1\bar i 1\bar j},\ i, j \neq 1)\\
&\quad = 0,
\end{aligned}$$

as could be read off from Proposition (2.2.1) on the structure of R_{1***}, we see immediately that

$$\begin{aligned}
&R\big(\alpha(t), \overline{\alpha(t)}, \alpha(t), \overline{e_q(t)}\big)\\
&\quad = a^3(t)b(t)R(\tilde\alpha(t), \tilde\alpha(t), \tilde\alpha(t), \tilde\alpha(t)) + O(t^2).
\end{aligned}$$

Since $R(\tilde\alpha(t), \overline{\tilde\alpha(t)}, \tilde\alpha(t), \overline{\tilde\alpha(t)}) = R_{\alpha\bar\alpha\alpha\bar\alpha} > 0$ for all t, $-\delta < t < \delta$, we have established the estimate $(*)$.

To make use of $(*)$, let e_p be a fixed unit vector in $\mathscr{H}$ and consider the decomposition

$$e_p(t) = c(t)\tilde\alpha(t) + \xi''(t) + \zeta''(t), \quad \text{with } \xi''(t) \in \mathscr{H}_{\tilde\alpha(t)},\ \zeta''(t) \in \mathscr{N}_{\tilde\alpha(t)}.$$

Clearly, $c(0)\tilde\alpha(0) = \zeta''(0) = 0$. Then,

$$\begin{aligned}
&R\big(e_p(t), \overline{e_q(t)}, e_p(t), \overline{e_q(t)}\big)\\
&\quad = R\big(\xi''(t), \overline{\zeta'(t)}, \xi''(t), \overline{\zeta'(t)}\big)\\
&\qquad +2R\big(c(t)\tilde\alpha(t)+\zeta''(t), \overline{\zeta'(t)}, \xi''(t), \overline{\zeta'(t)}\big)\\
&\qquad +2R\big(\xi''(t), \overline{b(t)\tilde\alpha(t)+\xi'(t)}, \xi''(t), \overline{\zeta'(t)}\big) + O(t^2).
\end{aligned}$$

From Proposition (5.1) on the structure of the curvature tensor we obtain

$$\begin{aligned}
&R\big(\xi''(t), \overline{\zeta'(t)}, \xi''(t), \overline{\zeta'(t)}\big) \quad (\text{type } R_{p\bar q p\bar q})\\
&\quad = R\big(\tilde\alpha(t), \overline{\zeta'(t)}, \xi''(t), \overline{\zeta''(t)}\big) \quad (\text{type } R_{1\bar q i\bar j})\\
&\quad = R\big(\zeta''(t), \overline{\zeta'(t)}, \xi''(t), \overline{\zeta'(t)}\big) \quad (\text{type } R_{p\bar q q\bar q})\\
&\quad = R\big(\xi''(t), \overline{\xi'(t)}, \xi''(t), \overline{\zeta'(t)}\big) \quad (\text{type } R_{p\bar p p\bar q})\\
&\quad = 0
\end{aligned}$$

so that

$$\begin{aligned}
&R\big(e_p(t), \overline{e_q(t)}, e_p(t), \overline{e_q(t)}\big)\\
&\quad = 2b(t)R\big(\xi''(t), \overline{\tilde\alpha(t)}, \xi''(t), \overline{\zeta'(t)}\big) + O(t^2).
\end{aligned}$$

Notice that curvature terms of type $R_{p\bar{l}p\bar{q}}$ may be nonzero. However, by (*) we have $b(t) = O(t^2)$, yielding

$$R\left(e_p(t), \overline{e_q(t)}, e_p(t), \overline{e_q(t)}\right) = O(t^2),$$

hence $R_{p\bar{q}p\bar{q},\eta}(0) = 0$, proving Lemma 2 and thus establishing Proposition (6.1).

7. Totally geodesic Hermitian symmetric integral submanifolds and isometric decomposition of X

(7.1) Recall that by Proposition (3.2) there exists a nonempty open subset U of X such that the bundle of maximal subspaces $V = \bigcup_{x \in X} V_x$, $V_x =$ **C**-linear span of $\mathscr{M}_x$, is a differentiable vector bundle on U. By Proposition (6.1), by shrinking U if necessary, we can assume that for any $\alpha \in \mathscr{M}_x$, $x \in U$, we have $\nabla_\alpha R_{i\bar{j}k\bar{l}} = 0$ for $1 \leqslant i, j, k, l \leqslant n$, so that $\nabla_\xi R_{i\bar{j}k\bar{l}}$ for all $\xi \in V_x$. On U we now consider the distribution $\operatorname{Re} V|_U = \{\xi + \bar{\xi}\colon \xi \in V|_U\}$ of vector subspaces of the tangent spaces. Our main result in this section is the following

Proposition (7.1). *The distribution* $\operatorname{Re} V|_U$ *of vector subspaces of* $T_x(X)$, $x \in U$, *is integrable. Moreover, the integral submanifolds are complex, totally geodesic and locally symmetric.*

Proof. By the theorem of Frobenius, to prove $\operatorname{Re} V|_U$ is integrable, all we need to show is that it is closed under taking Lie brackets. Since the metric on X is Riemannian, for any smooth tangent vector fields Y, Z on any open set,

$$\nabla_Y Z - \nabla_Y Z - [Y, Z] = 0.$$

It suffices for the proof of the integrability of $\operatorname{Re} V|_U$ to show that $\nabla_\eta \eta'$ takes values in $\operatorname{Re} V|_U$ for any tangent vector fields η, η' on an open subset of U with values in $\operatorname{Re} V|_U$. Fix $x \in U$ and let $\alpha_1, \cdots, \alpha_m$ be a basis of V_x consisting of maximal directions. We may further assume, as in Proposition (6.1), that there exist smooth vector fields $\alpha_1(y), \cdots, \alpha_m(y)$ in a neighborhood of x such that $\alpha_i(y) \in \mathscr{M}_y$ and $\alpha_i(x) = \alpha_i$. Let $\gamma = \gamma(t)$, $-\delta < t < \delta$, $\gamma(0) = x$, be any integral curve of $\operatorname{Re} V|_U$, i.e. $\dot{\gamma}(t) \in \operatorname{Re} V_{\gamma(t)}$ for each t. Then the curvature tensor is invariant under parallel transport along γ. In particular, if $\beta_i(t)$ is the parallel transport of α_i along γ to $\gamma(t)$, then

$$\frac{d}{dt} R\left(\beta(t), \overline{\beta(t)}, \beta(t), \overline{\beta(t)}\right) \equiv 0$$

since $\nabla_{\dot{\gamma}(t)} R \equiv 0$ and $\nabla_{\dot{\gamma}(t)} \beta(t) = 0$. It follows that $\beta(t)$ is also a maximal direction. In particular, $\beta(t) \in V_{\gamma(t)}$. Write

$$\beta_i(t) = \sum_j a_{ij}(t) \alpha_j(t), \qquad \alpha_j(t) = \alpha_j(\gamma(t)).$$

Write $\eta = \dot\gamma(0)$. Then, at x, $\nabla_\eta \beta_i(t) = 0$, i.e.

$$\sum a_{ij}(0)\nabla_\eta \alpha_j(0) + \sum_j a'_{ij}(0)\alpha_j(0) = 0.$$

From the definition of β_i it is clear that $a_{ij}(0) = \delta_{ij}$, so that

$$\nabla_\eta \alpha_i(0) = -\sum_j a'_{ij}(0)\alpha_j(0),$$

proving that $\nabla_\eta \alpha_i(0) \in V_x$. Since η is real, we obviously have

$$\nabla_\eta(\operatorname{Re}\alpha_i)(0) \in \operatorname{Re} V_x.$$

But this applies to any $\eta = \dot\gamma(0)$ with γ an integral curve of $\operatorname{Re} V|_U$. It follows therefore that for any open $U' \subset U$ and any real tangent vector fields η, η' on U' such that $\eta(x), \eta'(x) \in \operatorname{Re} V_x$, we have

$$(*) \qquad \nabla_\eta \eta'(x) \in \operatorname{Re} V_x \quad \text{for all } x \in U',$$

proving in particular the integrability of $\operatorname{Re} V|_U$. Obviously the integral submanifolds are complex because $\eta \in V_x$ implies $J\eta \in V_x$ for the J-operator on the complex manifold X. Finally $(*)$ implies that $\nabla'_\eta \eta'(x) = \nabla_\eta \eta'(x)$ for the Riemannian connection ∇' on Z, from which it follows that Z is totally geodesic, proving Proposition (7.1).

(7.2) The local foliation on U by locally symmetric complex totally geodesic submanifolds Z_x is a strong indication that X is itself Hermitian symmetric. In this subsection our contention is that each Z_x is contained in a compact Hermitian symmetric submanifold $\tilde Z_x$. To be precise, we have

Proposition (7.2). *For each $x \in U$ there exists a totally geodesic compact Hermitian symmetric submanifold $\tilde Z_x$ containing x such that $\tilde Z_x \cap U = Z_x$ is the integral submanifold of $\operatorname{Re} V|_U$ passing through x.*

Proof. We will prove Proposition (7.2) using the theorem of Bonnet-Meyers, which asserts that every complete Riemannian manifold of Ricci curvature bounded from below by a positive constant is necessarily compact. Let $r > 0$ be less than the injectivity radius of X so that for any $y \in X$, the exponential map at y is a diffeomorphism on the Euclidean ball $\overline{B(r)} = \overline{B(0;\, r)}$ on the tangent space $T_y(X)$, equipped with the obvious Euclidean metric. Without loss of generality we may let U be the open geodesic ball $B(x;\, r)$ so that Z_x is nothing other than $\exp_x(B(r) \cap \operatorname{Re} V_x)$. We can step-by-step enlarge the piece $Z = Z_x$ as follows. Define $Z_0 = Z$. We will define Z_i in general as a locally closed extendable submanifold of X, in the sense that there exists some locally closed submanifold Z'_i of X such that $Z_i \subset\subset Z'_i$. Suppose Z_i is defined. Fix $\varepsilon > 0$ such that $r + \varepsilon <$ injectivity radius. Choose a finite subset S_i of Z_i such that for each $y_0 \in Z_i$ there exists $y \in Z_i$ such that $d(y_0, y) < \varepsilon$. This can be done because Z_i is extendible. Define

$$Z_{i+1} = \bigcup_{y \in S_i} A_{i+1}(y),$$

where

$$A_{i+1}(y) = \exp_y\bigl(B(r) \cap T_y(Z_i)\bigr).$$

We have chosen S_i so that $Z_i \subset\subset Z_{i+1}$. We claim that Z_{i+1} is a locally closed extendible submanifold. By definition Z_{i+1} is locally closed. To show that Z_{i+1} is a submanifold locally, it suffices to show that for $y, y' \in S_i$, either $A_{i+1}(y) \cap A_{i+1}(y') = \varnothing$ or $A_{i+1}(y) \cup A_{i+1}(y')$ is a locally closed connected submanifold extending both $A_{i+1}(y)$ and $A_{i+1}(y')$. To prove this one has to rule out the possibility that they intersect each other in a subset of smaller dimension. If $A_{i+1}(y) \cup A_{i+1}(y')$ is not smooth, we would have either

(i) $A_{i+1}(y)$ intersects $A_{i+1}(y')$ tangentially at some y'', or

(ii) there exists $y'' \in A_{i+1}(y) \cap A_{i+1}(y')$ such that $T_{y''}(A_{i+1}(y)) \cup T_{y''}(A_{i+1}(y'))$ span a real linear subspace of $T_{y''}(X)$ of dimension larger than $2\dim_{\mathbf{C}} V_x =$ real dimension of Z_x.

Possibility (i) cannot happen because both $A_{i+1}(y)$ and $A_{i+1}(y')$ must be totally geodesic at y'' (by the identity theorem for real analytic functions), so that they are determined by their tangent planes at y''. To rule out possibility (ii) observe that both $T_{y''}(A_{i+1}(y))$ and $T_{y''}(A_{i+1}(y'))$ are generated by real parts of maximal directions at y'' (obtained by parallel transport from y and y' respectively). Then translating them back from y'' to the point x along broken geodesics on $Z_i \cup A_{i+1}(y)$ will yield more than $\dim_{\mathbf{C}} V_x$ $\mathbf{C}$-linearly independent maximal directions at x, contradicting with the definition of V_x. This establishes our claim that $Z_{i+1} = \bigcup_{y \in S_i} A_{i+1}(y)$ is a locally closed submanifold. That Z_{i+1} is extendible follows easily by taking

$$Z'_{i+1} = \bigcup_{y \in S_i} \exp_y\bigl(B(r+\varepsilon) \cap T_y(Z_i)\bigr) \quad \text{for some } \varepsilon > 0 \text{ sufficiently small.}$$

For $r + \varepsilon <$ injectivity radius of X, clearly Z'_{i+1} is also a locally closed submanifold such that $Z_{i+1} \subset\subset Z'_{i+1}$.

We now have a sequence of real-analytic manifolds Z_i such that

$$Z_1 \subset\subset Z_2 \subset\subset \cdots \subset\subset Z_k \subset\subset Z_{k+1} \subset\subset \cdots,$$

where Z_k, equipped with the restriction of the Kähler metric on X, is necessarily locally symmetric by the identity theorem of real-analytic functions. Moreover, if we define $\tilde{Z}$ to be the union $\bigcup_{k \geqslant 1} Z_k$ equipped with the induced metric, $\tilde{Z}$ is necessarily a complete Kähler manifold. In fact, at each $z \in Z_k$ there exists some $y \in S_k$, $d(y, z) < \varepsilon$ so that Z_{k+1} contains $B(z;\, r - \varepsilon)$. This implies that for each $z \in Z$ we have $B(z;\, r-\varepsilon) \subset\subset Z$, which in turn implies the completeness of $\tilde{Z}$. Recall that at each $z \in \tilde{Z}$, $T_z(\tilde{Z})$ is generated by the real parts of maximal directions at z. Because of local symmetry, $\tilde{Z}$ with the induced metric splits locally into products of Hermitian symmetric spaces and flat tori. If there is a flat torus as a local factor, it would not be possible for $T_z(\tilde{Z})$ to be generated by real parts of maximal directions. Hence $\tilde{Z}$ is a complete Kähler manifold with positive Ricci curvature bounded from below by some $c > 0$. By the theorem of Bonnet-Myers, $\tilde{Z}$ must be compact, proving Proposition (7.2).

(7.3) Pointwise reducibility of bisectional curvatures. Let $x \in U$. Denote by ξ a typical element of $T_x^{1,0}(Z_x) = V_x$ and by ζ a typical element of $V_x^{\perp}$, the orthogonal complement of V_x in $T_x^{1,0}(X)$. To prove that X is Hermitian symmetric it suffices to show that

$$\nabla_\xi R_{i\bar{j}k\bar{l}} = \nabla_\zeta R_{i\bar{j}k\bar{l}} = 0 \quad \text{for } 1 \leqslant i, j, k, l \leqslant n.$$

What remains to be proved is the vanishing of $\nabla_\zeta R_{\zeta\bar{\zeta}\zeta\bar{\zeta}}$. In fact, any terms of the form $\nabla_\zeta R_{\xi***}$ or $\nabla_\zeta R_{*\bar{\xi}**}$ would also be zero because of Proposition (6.1) and the Bianchi identity. We may assume without loss of generality tht $V|_U \neq T^{1,0}(U)$. In order to prove $\nabla_\zeta R_{\zeta\bar{\zeta}\zeta\bar{\zeta}} = 0$ we will first show that $\operatorname{Re} V^{\perp}|_U$ is an integrable distribution, where $V_x^{\perp}$ denotes the orthogonal complement of V_x in $T_x^{1,0}(X)$. From this and the arguments of Proposition (7.2) we will be able to obtain integral submanifolds $Z^{\perp}$ of $\operatorname{Re} V^{\perp}|_U$ which extend to totally geodesic compact complex submanifolds $\tilde{Z}^{\perp}$ of X. Moreover, in the process of proof we will also show that $R_{\xi\bar{\xi}\zeta\bar{\zeta}} = 0$ for all $\xi \in V_x$ and $\zeta \in V_x^{\perp}$, $x \in U$. This allows us to conclude that each such $\tilde{Z}^{\perp}$ is Kähler-Einstein. Moreover, holomorphic bisectional curvatures are nonnegative on $\tilde{Z}^{\perp}$ (because $\tilde{Z}^{\perp}$ is totally geodesic). To prove the Main Theorem, by induction on dimension we can assume that $\tilde{Z}^{\perp}$ is isometric to a Hermitian symmetric space, so that $\nabla_\zeta R_{\zeta\bar{\zeta}\zeta\bar{\zeta}} = 0$ for all $\zeta \in V_x^{\perp}$, proving $\nabla R \equiv 0$ on X by the identity theorem for real-analytic functions, thus establishing the Main Theorem.

In order to show that $\operatorname{Re} V^{\perp}|_U$ is integrable we will first of all show that $V|_U$ is invariant under parallel transport along all curves on U. For the proof of this we will need the reducibility of bisectional curvatures as stated above:

Proposition (7.3). *For each $x \in U$ and for all $\xi \in V_x$, $\zeta \in V_x^{\perp}$, we have $R_{\xi\bar{\xi}\zeta\bar{\zeta}} = 0$.*

Proof. For each $x \in U$ there exists a totally geodesic compact complex submanifold $\tilde{Z}_x$ of X such that $Z_x = \tilde{Z}_x \cap U$ is an integral submanifold of the distribution $\operatorname{Re} V|_U$. Suppose $y \in U$, $y \notin Z_x$; we assert that $\tilde{Z}_x \cap \tilde{Z}_y = \varnothing$. In fact, the proof of this is exactly as in Proposition (7.2), where it was shown that the $\tilde{Z}_x$ cannot have self-intersections. Since adjacent extended integral submanifolds are mutually nonintersecting, the normal bundle $N_{\mathbf{R}}$ of the real manifold $\tilde{Z}_x$ in X must be trivial as a differentiable vector bundle. As a differentiable bundle, $N_{\mathbf{R}}$ is simply isomorphic to the bundle $\operatorname{Re} V^{\perp}|_{\tilde{Z}_x}$, where obviously we can assume that the open set U contains $\tilde{Z}_x$. Sincce $N_{\mathbf{R}}$ is differentiably trivial, the complex bundle $N_{\mathbf{C}} = N_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ is a differentiably trivial complex vector bundle. We have the decomposition

$$N_{\mathbf{C}} = N^{1,0} \oplus N^{0,1},$$

where $N_x^{1,0}$ is the eigenspace of J on $N_{\mathbf{C},x} = N_{\mathbf{R},x} \otimes_{\mathbf{R}} \mathbf{C}$ corresponding to the eigenvalue i, and $N_x^{0,1}$ is that corresponding to the eigenvalue $-i$. Note that this decomposition is possible because $V_x^{\perp}$ is closed under the J-operator. Let N be the holomorphic normal bundle on $\tilde{Z}_x$, i.e., $N = T^{1,0}(X)|_{\tilde{Z}_x}/T^{1,0}(\tilde{Z}_x)$. As a differentiable $\mathbf{C}$-vector bundle, N is isomorphic to $N^{1,0} \cong V^{\perp}|_{\tilde{Z}_x}$. It is well known that any Hermitian holomorphic quotient bundle of a Hermitian holomorphic vector bundle of semipositive curvature remains semipositive, so that N, with the induced metric, is semipositive on $\tilde{Z}_x$. It follows that the first Chern class of N is represented by a semiposditive closed $(1,1)$ form. Now, $N_{\mathbf{C}} = N^{1,0} \oplus N^{0,1} \cong N \oplus \overline{N}$ as differentiable $\mathbf{C}$-vector bundles, where $\overline{N}$ is the antiholomorphic vector bundle obtained by taking conjugates of transition functions of N. By defining the length of $\bar{v}$ to be that of v for $v \in N_x$, $\bar{v} \in \overline{N_x}$, we see that $c_1(\overline{N}) = c_1(N)$. It follows that

$$c_1(N_{\mathbf{C}}) = c_1(N^{1,0}) + c_1(N^{0,1}) = c_1(N) + c_1(\overline{N}) = 2c_1(N)$$

is represented by a semipositive closed $(1,1)$ form. Hence, the triviality of $N_{\mathbf{C}}$ as a differentiable $\mathbf{C}$-vector bundle implies that $c_1(N) = 0$ and that the curvature form of N is identically zero on $\tilde{Z}_x$. We assert that the flatness of the Hermitian holomorphic vector bundle N implies the proposition, i.e., $R_{\xi\bar{\xi}\zeta\bar{\zeta}} = 0$ for $\xi \in V_x$, $\zeta \in V_x^{\perp}$ and $x \in U$. To see this we examine the curvature form of N more closely. Consider the exact sequence

$$0 \to N^* \to T^{1,0}(X)^*|_{\tilde{Z}_x} \to T^{1,0}(\tilde{Z}_x)^* \to 0.$$

The flatness of N implies that of the dual bundle N^*. By the curvature decreasing property of Hermitian holomorphic vector subbundles, we have, denoting by $\Theta' = \Theta_{N^*}$ and $\Theta = \Theta_{T^{1,0}(X)^*}$ the curvature forms of N^* and $T^{1,0}(X)^*$ with the induced metrics respectively,

$$\Theta'(\xi,\bar{\xi};\zeta^*,\bar{\zeta}^*) \leqslant \Theta(\xi,\bar{\xi};\zeta^*,\bar{\zeta}^*)$$

for $\xi \in T_x^{1,0}(\tilde{Z}_x)$ and $\zeta^* \in N^*$. Now let $\{e_1,\cdots,e_n\}$ be a basis of $T_x^{1,0}(X)$ and $\{e_1^*,\cdots,e_n^*\}$ be the dual basis. Then, for $\zeta^* = \sum a_i e_i^*$ belonging to N_x^*, we have

$$\begin{aligned}\Theta'(\xi,\bar{\xi},\zeta^*,\bar{\zeta}^*) &\leqslant \Theta(\xi,\bar{\xi},\zeta^*,\bar{\zeta}^*) = \sum a_i a_j \Theta\left(\xi,\bar{\xi},e_i^*,\overline{e_j^*}\right)\\ &= -\sum a_i a_j \Theta_{T^{1,0}(x)}\left(\xi,\bar{\xi};\, e_j,\overline{e_i}\right) = -\sum a_i \bar{a}_j R_{\xi\bar{\xi}i\bar{j}}.\end{aligned}$$

Here the next to last inequality follows from the standard relation between curvatures of dual bundles and the last equality comes from the definition of $\Theta_{T^{1,0}(X)}$, the curvature tensor of $T^{1,0}(X)$ with the induced Hermitian matrix. It follows now that $\Theta'(\xi, \bar{\xi}, \zeta^*, \bar{\zeta}^*) \leqslant 0$ and that equality hold for all $\zeta^* \in N_x^*$ only if $R_{\xi\bar{\xi}\zeta\bar{\zeta}} = 0$ for all $\zeta = \sum a_i e_i$ such that $\sum a_i e_i^* \in N_x^*$. But $\sum a_i e_i^* \in N_x^*$ if and only if $(\sum a_i e_i^*)(\xi) = 0$ for all $\xi \in T_x^{1,0}(\tilde{Z}_x)$, i.e., if and only if $\langle \sum a_i e_i, \xi \rangle = 0$, so that

$$R_{\xi\bar{\xi}\zeta\bar{\zeta}} = 0 \quad \text{for all } \zeta \in V_x^{\perp},$$

proving Proposition (7.3).

(7.4) Invariance of V under parallel transport. Recall that $\mathscr{M}|_U$ is invariant under parallel transport along curves on $\tilde{Z}_x$, which implies in particular that $V|_U$ is invariant under parallel transport along any curves (not necessarily geodesics) $\gamma(t)$ such that $\dot{\gamma}(t) \in V_{\gamma(t)}$. In this subsection we assert the stronger statement:

Proposition (7.4). *The bundle V of maximal subspaces, defined by $V = \bigcup_{x \in X} V_x$ and $V_x =$ **C**-linear span of $\mathscr{M}_x$, is invariant under parallel transport along any smooth curve l on X. In particular, V is a bona fide differentiable vector bundle on X.*

Proof. (I) First we assert that it suffices to prove that $V|_U$ is invariant under parallel transport along geodesics. First of all, we contend that the latter statement would imply that maximal directions remain maximal directions when translated by parallel transport along any geodesic passing through U. At each $x' \in X$ there exists a geodesic γ joining x' to some point $x \in U$. Adjacent geodesics emanating from x will also intersect U so that maximal directions at x' remain maximal when translated by parallel transport along some open cone of geodesics emanating from x' and hence along all geodesics emanating from x', by the identity theorem for real-analytic functions. It follows that the bundle V of maximal subspaces is invariant under translation by parallel transport along any geodesic. In particular, V is a differentiable vector subbundle of $T^{1,0}(X)$. Let $l(t)$, $-\delta < t < \delta$, be any smooth curve. Suppose $\chi \in V_{l(0)}$ and

$$\chi(t) = \xi(t) + \zeta(t), \qquad \xi(t) \in V_{l(t)}, \zeta(t) \in V_{l(t)}^{\perp},$$

is the decomposition of the parallel transport $\chi(t) \in T_{l(t)}^{1,0}$ according to the orthogonal decomposition $T^{1,0}(X) = V \oplus V^{\perp}$. From the invariance of V under parallel transport along geodesics it follows readily that $\|\zeta(t)\| = O(t^2)$, $\|\cdot\|$ denoting the length. To show that $\chi(t) \in V_{l(t)}$ it suffices to show that $d\|\zeta(t)\|^2/dt \equiv 0$. Let $\xi_t(s)$ and $\zeta_t(s)$ denote the translation of $\xi(t)$ and $\zeta(t)$ to $l(s)$, for s sufficiently close to t, by parallel transport. Obviously, $\xi_t(t) = \xi(t)$,

$\zeta_t(t) = \zeta(t)$ and $\xi_t(s) + \zeta_t(s) = \chi(s)$. Let $\xi_t(s) = \xi_t'(s) + \zeta_t'(s)$ and $\zeta_t(s) = \xi_t''(s) + \zeta_t''(s)$ denote the decompositions of ξ_t and ζ_t according to the decomposition $T^{1,0}(X) = V \oplus V^{\perp}$. Then

$$\zeta(s) = \zeta_{t_0}'(s) + \zeta_t''(s).$$

We have

$$\frac{d}{dt}\|\zeta\|^2(t_0) = 2\operatorname{Re}\left\langle \frac{\nabla\zeta}{ds}, \zeta\right\rangle(t_0)$$
$$= 2\operatorname{Re}\left\langle \frac{\nabla}{ds}\zeta_{t_0}', \zeta\right\rangle(t_0) + 2\operatorname{Re}\left\langle \frac{\nabla}{ds}\zeta_{t_0}'', \zeta\right\rangle(t_0).$$

Just as $\|\zeta(t)\| = O(t^2)$ we also have $\|\zeta_t'(s)\| = O((t-s)^2)$, so that $(\nabla/ds)\zeta_{t_0}'(t_0) = 0$. To estimate $(\nabla/ds)\zeta_{t_0}''(t_0)$ we observe first of all

Lemma. *For any t such that $-\delta < t_0 < \delta$ and any $\mu_t \in T_{l(t)}^{1,0}(X)$, let*

$$\mu_t(s) = \tilde{\xi}(s) + \tilde{\zeta}(s)$$

be the decomposition of the translation $\mu(s)$ of $\mu(t) = \mu$ by parallel transport along l. Then

$$\left\|\frac{\nabla}{ds}\tilde{\zeta}(t)\right\| \leqslant K\|\mu_t\|$$

with a positive constant K independent of t_0.

Proof of Lemma. Suppose $\mu' = c\mu$, and $\mu_t'(s) = \tilde{\xi}'(s) + \tilde{\zeta}'(s)$ is the corresponding decomposition of μ'. Then, obviously $\tilde{\zeta}'(s) = c\tilde{\zeta}(s)$ so that

$$\frac{\nabla}{ds}\tilde{\zeta}'(t) = c\frac{\nabla}{ds}\tilde{\zeta}(t).$$

Now let K be the supremum of all $\|(\nabla/ds)\tilde{\zeta}(t)\|$ obtained from all possible t with $-\delta < t < \delta$ and from all possible $\mu_t \in T_{l(t)}^{1,0}(X)$ of unit length. K is clearly finite by the real-analyticity of $\mu_t(s)$ jointly in μ_t and s, when $\mu_t(s)$ is defined on $T^{1,0}(X)|_{\hat{l}} \times (-2\delta, 2\delta)$, where $\hat{l}$ is an extension of l to $(-2\delta, 2\delta)$, assumed to lie within the cut-locus of $x \in X$. The Lemma is obviously valid with this constant K. Given the Lemma, we can now estimate

$$\left|\frac{d}{dt}\|\zeta\|^2(t_0)\right| = 2\left|\operatorname{Re}\left\langle \frac{\nabla}{ds}\zeta_{t_0}'', \zeta\right\rangle(t_0)\right|$$
$$\leqslant K\left\|\zeta_{t_0}(t_0)\right\|\,\|\zeta(t_0)\| = K\|\zeta(t_0)\|^2.$$

To show that $d\|\zeta(t)\|^2/dt \equiv 0$ and hence that V is invariant under parallel transport along any curve it suffices therefore to show that any real-analytic function $f(t)$ defined on $(-\delta, \delta)$ satisfying $|df/dt| \leqslant K|f|$, $f(0) = 0$, must necessarily be identically zero. (Observe that $d\|\zeta\|^2(0)/dt = \langle\zeta(0), (\nabla/dt)\zeta(0)\rangle = 0$.) In fact, if $f = c_m t^m + O(t^{m+1})$, $c_m \neq 0$,

$$|df/dt| = \left|mc_m t^{m-1} + O(t^m)\right| = m|c_m|t^{m-1},$$

which clearly dominates $K|f|$ in a neighborhood of 0 for any constant K, proving the assertion that Proposition (7.4) can be reduced to the corresponding statement on the open set U for geodesics γ.

(II) The proof of the reduction on (I) implies that to prove Proposition (7.4) it suffices to show that if $\alpha \in \mathscr{M}_x$, $x \in U$, and $\gamma(t)$, $-\delta < t < \delta$, is any geodesic on U with $\gamma(0) = x$, then, for the decomposition $\alpha(t) = \xi(t) + \zeta(t)$ of the translation $\alpha(t)$ of $\alpha = \alpha(0)$ by parallel transport along γ according to the decomposition $T^{1,0}(U) = V|_U \oplus V^\perp|_U$, we have

$$\|\zeta(t)\| = O(t^2).$$

In fact, this would imply that $\alpha(t) \in V_{\gamma(t)}$, so that $V_{\gamma(0)}$ is translated to $V_{\gamma(t)}$ since $V_{\gamma(0)}$ is generated as a **C**-linear space by the space of maximal directions $\mathscr{M}_{\gamma(0)} = \mathscr{M}_x$.

Suppose now $\zeta(t) = ct\tilde{\zeta}(t) + O(t^2)$ with $\|\tilde{\zeta}(t)\| = 1$, where $O(t^2)$ stands for a vector-valued function of length of order $O(t^2)$. Then,

$$\begin{aligned}
&R\big(\alpha(t), \overline{\alpha(t)}, \alpha(t), \overline{\alpha(t)}\big) \\
&\quad = R\big(\xi(t) + \zeta(t), \overline{\xi(t) + \zeta(t)}, \xi(t) + \zeta(t), \xi(t) + \overline{\zeta(t)}\big) \\
&\quad = R\big(\xi(t), \overline{\xi(t)}, \xi(t), \overline{\xi(t)}\big) + 4\operatorname{Re} R\big(\xi(t), \overline{\xi(t)}, \xi(t), \overline{\zeta(t)}\big) \\
&\qquad + 4R\big(\xi(t), \overline{\xi(t)}, \zeta(t), \overline{\zeta(t)}\big) + 2\operatorname{Re} R\big(\xi(t), \overline{\zeta(t)}, \xi(t), \overline{\zeta(t)}\big) \\
&\qquad + 4\operatorname{Re}\big(\xi(t), \overline{\zeta(t)}, \zeta(t), \overline{\zeta(t)}\big) + O(t^4).
\end{aligned}$$

By Proposition (7.3) $R(\xi(t), \overline{\xi(t)}, \zeta(t), \overline{\zeta(t)}) = 0$. We claim that actually for any $x \in U$, any $\xi \in V_x$ and any $\zeta \in V_x^\perp$ we have

$$(*) \qquad R_{\xi\bar{\xi}\chi\bar{\chi}} = 0 \quad \text{for all } \chi \in T_x^{1,0}(X).$$

To see this, suppose $\xi = \alpha \in \mathscr{M}_x$. From Proposition (7.3) we have $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ for all $\zeta \in V_x^\perp$, so that $\zeta \in \mathscr{N}_\alpha$. However, by Proposition (2.2.1) now we have

$$R_{\alpha\bar{\zeta}\chi\bar{\chi}} = 0 \quad \text{for all } \chi \in T_x^{1,0}(X),$$

which yields $(*)$ since V_x is the linear span of $\mathscr{M}_x$. It follows now from $(*)$ that

$$R\big(\alpha(t), \overline{\alpha(t)}, \alpha(t), \overline{\alpha(t)}\big) = R\big(\xi(t), \overline{\xi(t)}, \xi(t), \overline{\xi(t)}\big) + O(t^4).$$

On the other hand, if $\zeta(t) = ct\tilde{\zeta}(t) + O(t^2)$, $\|\tilde{\zeta}(t)\| = 1$, we have

$$\|\xi(t)\|^2 = 1 - c^2t^2 + O(t^4),$$

so that

$$R\big(\alpha(t),\,\overline{\alpha(t)},\,\alpha(t),\,\overline{\alpha(t)}\big) \leqslant \big(1 - c^2t^2 + O(t^4)\big)^2 R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(0) + O(t^4)$$
$$= (1 - 2c^2t^2)R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(0) + O(t^4).$$

If $c \neq 0$ we would have $R_{\alpha\bar{\alpha}\alpha\bar{\alpha},\eta}(0) = 0$ and $R_{\alpha\bar{\alpha}\alpha\bar{\alpha},\eta\eta}(0) \leqslant -4c^2R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(0)$ for $\eta = \gamma(0)$. But we have simply, from Berger's formula, $R_{\alpha\bar{\alpha}\alpha\bar{\alpha},\eta\eta}(0) = 0$. This proves $c = 0$, so that in the decomposition $\alpha(t) = \xi(t) + \zeta(t)$ we have

$$\|\zeta(t)\| = O(t^2)$$

implying by the reduction method of (I) that V is a differentiable **C**-vector bundle invariant under parallel transport along any smooth curve on X, proving Proposition (7.4).

(7.5) Integral submanifolds of $\operatorname{Re} V^{\perp}$. Recall that V is a distribution of tangent vectors of type $(1,0)$ invariant under parallel transport and for each $x \in X$ there is a compact totally geodesic complex submanifold $\tilde{Z}_x$ which is locally symmetric. (Such a Z_x exists now for each $x \in X$ because one can take the open set U to be X, since we know now that $V = \bigcup_{x \in X} V_x$ is a differentiable **C**-vector bundle.) The $\tilde{Z}_x$ are integral submanifolds of $\operatorname{Re} V$. According to the orthogonal decomposition $T^{1,0}(X) = V \oplus V^{\perp}$ we can divide vectors of $T^{1,0}(X)$ into types $\xi \in V$ and $\zeta \in V^{\perp}$. In (7.2) we deduced that X is locally symmetric if $\nabla_{\zeta} R_{\zeta\bar{\zeta}\zeta\bar{\zeta}} = 0$ for all $\zeta \in V^{\perp}$.

Since V is invariant under parallel transport, the same applies to $V^{\perp}$. The arguments in Proposition (7.1) for V now apply to $V^{\perp}$ to show that $\operatorname{Re} V^{\perp}$ is an integrable distribution of tangent vectors. For each $x \in X$, let $Z_x^{\perp}$ be the leaf passing through x of the foliation defined by the distribution $\operatorname{Re} V^{\perp}$. The arguments of Proposition (7.1) imply that $Z_x^{\perp}$ is totally geodesic and complex analytic. From Proposition (7.3) we know that $R_{\xi\bar{\xi}\zeta\bar{\zeta}} = 0$ whenever $\xi \in V_x$ and $\zeta \in V_x^{\perp}$. It follows by the hypothesis of the Main Theorem that $Z_x^{\perp}$ can be regarded as a complete Kähler-Einstein manifold of positive Ricci curvature. By the theorem of Bonnet-Meyers each $Z_x^{\perp}$ is compact. Now it is obvious how one can prove the Main Theorem by induction on the complex dimension of X. In fact, $Z_x^{\perp}$ satisfies the hypothesis on X in the Main Theorem. We can therefore assume as an induction hypothesis that $Z_x^{\perp}$ is a Hermitian symmetric manifold unless $V = T^{1,0}(X)$, in which case there is nothing to prove. Hence, $\nabla_{\zeta} R_{\zeta\bar{\zeta}\zeta\bar{\zeta}} = 0$ whenever $\zeta \in V^{\perp}$, proving $\nabla R \equiv 0$ on X for the curvature tensor R, completing the proof of the Main Theorem.

Remarks. (i) By a result of Kobayashi [11], all locally symmetric compact complex manifolds of positive Ricci curvature must be simply connected. Hence, the manifold X in the Main Theorem is globally symmetric.

(ii) It is clear that the proof of the Main Theorem implies immediately the more general case when the Ricci tensor of X is only assumed to be parallel and positive. Positivity of the Ricci tensor is only usd in proving compactness of certain integral submanifolds, using the theorem of Bonnet-Meyers.

8. A Generalization of the Main Theorem

(8.1) Recall Corollaries 1 and 2 of §0 which asserts that the Main Theorem can be generalized to the case when the Ricci tensor is parallel and the Kähler manifold X is complete, possibly noncompact. By results of Bishop & Goldberg [4]–[6], for a compact Kähler manifold X of nonnegative holomorphic bisectional curvature, the Ricci tensor of X is parallel if and only if X has constant scalar curvature. To complete the present article it suffices to prove Corollary 2, where X is only assumed to be complete and the Ricci tensor of X is assumed parallel, in place of being Kähler-Einstein of positive Ricci curvature.

The only places where the positivity of the Ricci curvature is used are (7.2) and (7.5), where we applied the theorem of Bonnet-Meyers. The Kähler-Einstein condition was only used in obtaining formulas for computing $\nabla R_{i\bar{j}k\bar{l}}$, etc. But it is clear that these formulas (obtained by commutation) would still hold if $\nabla(\mathrm{Ric}) \equiv 0$, i.e. the Ricci tensor is parallel.

Proof of Corollary 2 (*and hence Corollary* 1). The point of the proof is simply to split off the directions where the Ricci tensor vanishes. Define $W_x \subset T_x^{1,0}(X)$ to be the subspace of all $\chi \in T_x^{1,0}(X)$ such that $\mathrm{Ric}(\chi, \chi) = 0$. Clearly W_x is a **C**-vector subspace of $T_x^{1,0}(X)$ since the Ricci form is a Hermitian symmetric bilinear form on X. Since the Ricci tensor $W = \bigcup_{x \in X} W_x$ is a differentiable vector bundle on X invariant under parallel transport, by the arguments of Proposition (7.2) $\mathrm{Re}\, W$ as an integral distribution of real tangent vectors. The leaves L_x of the foliation defined by $\mathrm{Re}\, W$ are flat since they are totally geodesic, the Ricci tensor on L is everywhere zero and holomorphic bisectional curvatures of L_x are nonnegative. Let $W_x^{\perp}$ be the orthogonal complement of W_x in $T_x^{1,0}(X)$. Then $W = \bigcup_{x \in X} W_x^{\perp}$ is invariant under parallel transport. Denote by Z_x a leaf of the foliation defined by $\mathrm{Re}\, W^{\perp}$. Then Z_x carries positive Ricci curvature by the definition of $W_x^{\perp}$. Moreover, the Ricci tensor of Z_x is parallel since $R_{\xi\bar{\xi}\chi\bar{\chi}} = 0$ for all $\xi \in W_x^{\perp}$ and $\chi \in W_x$. By remark (ii) of (7.5) we conclude that each Z_x is a (global) Hermitian symmetric space, so that $\nabla_{\xi} R_{\xi\bar{\xi}\xi\bar{\xi}} = 0$ for all $\xi \in W^{\perp}$. Since W is invariant under parallel transport and $R_{\chi\bar{\chi}\chi\bar{\chi}} = 0$ for all $\chi \in W$, it follows that $\nabla_{\xi} R_{\chi\bar{\chi}\chi\bar{\chi}} = \nabla_{\chi} R_{\chi\bar{\chi}\chi\bar{\chi}} = 0$ for $\xi \in W_x^{\perp}$ and $\chi \in W_x$. The only other terms of

∇R, up to conjugation and permutation of indices, are of the types $\nabla_\mu R_{\xi\bar{\chi}\mu\bar{\mu}}$, where $\mu \in T_x^{1,0}(X)$ is arbitrary. It suffices therefore to show that

$$(*) \qquad R_{\xi\bar{\chi}\mu\bar{\mu}} = 0 \quad \text{for all } \mu \in T_x^{1,0}(X).$$

To prove $(*)$ it is equivalent to prove $R_{\chi\bar{\mu}\mu\bar{\mu}} = 0$ for all $\mu \in T_x^{1,0}(X)$ since $R_{\chi\bar{\chi}\mu\bar{\mu}} = 0$ for all $\chi \in W_x$ and $\mu \in T_x^{1,0}(X)$. Let $\mu \in T_x^{1,0}(X)$ and consider the function

$$F(\varepsilon) = R\big(\chi + \varepsilon\mu, \overline{\chi + \varepsilon\mu}, \chi + \varepsilon\mu, \overline{\chi + \varepsilon\mu}\big)$$

defined for ε real. Then, from $F(\varepsilon) \geqslant 0$ we obtain by variation formulas that $R_{\chi\bar{\chi}\chi\bar{\mu}} = 0$ and $4R_{\chi\bar{\chi}\mu\bar{\mu}} + 2\operatorname{Re} R_{\chi\bar{\mu}\chi\bar{\mu}} \geqslant 0$. But since $R_{\chi\bar{\chi}\mu\bar{\mu}} = 0$ and we can always assume $\operatorname{Re} R_{\chi\bar{\mu}\chi\bar{\mu}} \leqslant 0$ if μ is replaced by $e^{i\theta}\mu$ for an appropriate real angle θ, it follows that $R_{\chi\bar{\mu}\chi\bar{\mu}} = 0$. Computing now the third variation of F against ε at 0, we obtain $R_{\chi\bar{\mu}\mu\bar{\mu}} = 0$, proving $(*)$, thus showing that $\nabla R \equiv 0$ on X and proving Corollary 2 (and hence Corollary 1).

Concluding remarks. (i) By Koszul [13] and Lichnerowicz [16] every compact homogeneous Kähler manifold X carries a Kähler metric with parallel and semipositive Ricci tensor. Analogous to the situation of Gray [8] our theorem shows that such a Kähler metric on X cannot have nonnegative holomorphic bisectional curvature everywhere unless X is Hermitian symmetric (cf. Lichnerowicz [14], [15]).

(ii) As was indicated in Gray [8], every compact homogeneous space X of the form G/T, where G is a compact Lie group and T is a maximal torus of G, admits an Einstein and bi-invariant metric of nonnegative sectional (and hence nonnegative bisectional) curvature. This metric is in general not Kählerian.

(iii) See Ausländer [1] for an example of compact flat Kähler manifolds which are not homogeneous. Hence, in the formulation of Corollaries 1 and 2, we can only conclude that X is locally symmetric.

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