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ON 4-DIMENSIONAL s-COBORDISMS

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0. Introduction

The idea that topological problems can be converted into questions of algebra and homotopy theory underlies much of modern higher-dimensional topology of manifolds. The s-cobordism theorem, also called the Barden-Mazur-Stallings theorem constitutes one of the basic building blocks of this approach. Let W be a compact $(n + 1)$-manifold with boundary the disjoint union of manifolds M_0 and M_1. Then this theorem asserts that for $n \geqslant 5$, W is diffeomorphic, piecewise linearly homeomorphic, or homeomorphic, depending on the category, to a product $M_0 \times [0, 1]$, if and only if W has the homotopy type of this product and a certain algebraic invariant $\tau(W, M_0) \in Wh(\pi_1 W)$, the Whitehead torsion, vanishes. (See [12], [9] and [11] for the topological case.) This result, whose simply-connected version $\pi_1 W = \{e\}$ is just Smale's h-cobordism theorem, at least provides a direction of attack in the attempt to decide when two homotopy equivalent manifolds M_0 and M_1 are diffeomorphic—try to construct W or to measure the obstruction to doing so. Moreover, Freedman demonstrated that this s-cobordism theorem is valid for W a topological five-dimensional s-cobordism with fundamental group not too large, e.g. finite or polycyclic; i.e., he showed that such a five-dimensional W is homeomorphic to $M_0 \times [0, 1]$ under the same hypothesis on the vanishing of the Whitehead torsion.

This paper provides a family of orientable counterexamples to the s-cobordism theorem in dimension four. (It seems to have been fairly widely understood (cf. [14]) that the realization of a certain non-orientable, non-smoothable normal invariant yields a nonorientable (and nonsmoothable) example.) Let Q_r be the quaternion group of order 2^{r+1};

$$Q_r = \{ y, x | y^2 = x^{2r}, yxy^{-1} = x^{-1} \}.$$

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A representation of Q_r in SO(4) is given by mapping y to the matrix

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and x to the matrix

$$\begin{pmatrix} R(1/2^{r+1}) & 0 \\ 0 & R(-1/2^{r+1}) \end{pmatrix}, \qquad R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Under this representation, Q_r acts freely on the unit sphere S^3; let $M_r = S^3/Q_r$ be the quotient manifold, i.e. a standard linear three-dimensional space-form with fundamental group Q_r. Then this paper will exhibit 4-manifolds W with boundary homeomorphic to $M_r \times \{0,1\}$, so that for $r \geqslant 2$

(i) W satisfies the hypotheses of the s-cobordism theorem; and

(ii) W is not homeomorphic to $M_r \times [0,1]$.

The first section of the paper will construct $W = W(k)$ and verify (i), where k is any invertible knot in S^3 whose 2^{r+1}-fold branched cyclic cover is a homology sphere. A first approximation $V = V(k)$ to W is constructed by removing from $M_r \times [0,1]$ a neighborhood of a Klein bottle and replacing it by a bundle over the circle S^1 with fiber the complement of k and with monodromy the restriction of an inversion (an orientation preserving diffeomorphism of S^3 that leaves k invariant and reverses its orientation.) This smooth construction produces an s-cobordism with respect to local coefficients in the integral group ring $\mathbf{Z}[Q_r]$; unfortunately (i) is not satisfied because the fundamental group will be not Q_r but an extension of it by a perfect group. This problem can be corrected (smoothly) by spherical modification ("surgery") along circles, at the cost of changing the simple homotopy type, and even the homology. However, topological decomposition results of Freedman permit the recovery of the original simple homotopy type over $\mathbf{Z}[Q_r]$, without altering the now correct fundamental group, by deleting extraneous copies of $S^2 \times S^2$; hence the topological manifold $W(k)$ obtained by this construction will satisfy (i).

The second section of the paper is devoted to proving the non-triviality of W for suitable k. Let $\Delta_k(t)$ denote the Alexander polynomial of the knot k. Let $\mathbf{Z}[x] \subset \mathbf{Z}[Q_r]$ be the subring generated by 1 and x, i.e. the integral group ring of the subgroup of order 2^{r+1} generated by x.

Theorem (*see Theorem* 2.1). *Suppose that $W(k)$ is homeomorphic to $M_r \times [0,1]$. Then there exist $i \in \mathbf{Z}$ and $u \in \mathbf{Z}[x]$ such that*

$$\Delta_k(x) = \pm x^i u^2.$$

A crucial fact used in the proof is due independently to H. Rubinstein [15] and to the authors [4]. It asserts that every homeomorphism of M_r that is homotopic to the identity is actually isotopic to it. The basic idea—one-sided Heegard splittings (see (1.1))—has been extended and applied in a number of contexts [15]–[17].

It is probably well-known that the 2^{r+1}-fold branched cyclic cover of k is a homology sphere if and only if $\Delta_k(x)$ is a unit in $\mathbf{Z}[x]$. This can be derived from the calculation of the homology in [7], which can be interpreted as taking a type of norm of $\Delta_k(x)$. A proof can also be extracted from the proof of (2.2). For example, let k be the $(2^r - 1, 2^r + 1)$ torus knot.Torus knots are invertible; view the (p, q)-torus knot as the intersection of the locus $\{(z_1, z_2) \mid z_1^p + z_2^q = 0\}$ with a small sphere and apply complex conjugation to obtain the inversion. It is easy to check (3.2) that $\Delta_k(x)$ is a unit and not of the form $\pm x^i u^2$, for $r \geqslant 2$ and k the indicated torus knot. Hence $W(k)$ is not a product $M_r \times [0,1]$, $r \geqslant 2$.

Using methods beyond the scope of this paper, the converse to the above theorem can also be proven; i.e. that $W(k) = M_r \times [0,1]$ if and only if $\Delta_k(x) = \pm x^i u^2$. More generally, $W(k) = W(k')$ if and only if $\Delta_k(x)\Delta_{k'}(x)$ has this form. From these results and the construction of §1, one then obtains that the set S_r of homeomorphism classes of s-cobordisms of M_r to itself is in 1-1 correspondence

$$S_r = \mathbf{Z}[x]^{\times}/\{\pm x^i, \text{squares}\}.$$

It is well known that the group on the right is a $\mathbf{Z}_2$—vector space of rank $2^r - r - 1$; an explicit basis is given in [2]. Thus S_0 and S_1 are trivial, $S_2 = \mathbf{Z}_2$, $S_3 = (\mathbf{Z}_2)^4$, etc. The induced group structure on S_r can be given geometrically by pasting along one end.

It remains unknown whether any of the nontrivial $W(k)$ admit smooth structures. All known smoothing obstructions applicable to this situation vanish.

Added in proof. Some remarkable recent work of Donaldson shows that, in one higher dimension than considered here, there are smoothly non-product s-cobordisms. As these are simply-connected, the results of Freedman on 5-dimensional h-cobordisms, quoted above, imply that they are topological products.

1. Construction of s-cobordisms

Let k be an invertible knot in S^3. Let $X = X(k)$ be the complement of the interior of a tubular neighborhood of k. Then $\partial X = S^1 \times S^1$, where $S^1 \times \{x\}$ is a meridian and $\{x\} \times S^1$ a longitude with linking number zero with k. The invertibility of k is equivalent to the existence of a diffeomorphism $\psi\colon X \to X$ with $\psi(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$ for $z_i \in S^1 = \{z | z \text{ a complex number of modulus } 1\}$. Let $S^1 \times_\psi X$ denote the bundle over S^1 with fibre X and monodromy ψ; i.e., the mapping torus.

Now consider the 1-sided Heegard decomposition

$$M_r = N(K) \cup H. \tag{1.1}$$

Here K is the Klein bottle, H is a solid torus, $N(K)$ is the interval bundle associated to the orientable double cover of K and $\partial H = \partial N(K)$. To see this decomposition, let $S^3 = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 = 1\}$. Then the torus $\{(z_1, z_2) | |z_1| = |z_2| = 1/2\}$ is invariant under the representation of Q_r defined above, K is its quotient, and H is the quotient of the complement a neighborhood of this torus.

The embedding $K = K \times \{1/2\} \subset M_r \times [0,1]$ has the neighborhood $N(K) \times [1/4, 3/4]$. As K is itself a bundle over S^1 with fiber S^1 and monodromy $z \to \bar{z}$, it is not difficult to see that this neighborhood can be described as $S^1 \times_{\psi_0} X_0$, where $X_0 = S^1 \times D^2$ is the complement of the trivial knot and $\psi_0(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$ for $z_1 \in S^1 = \{z | |z| = 1\}$ and $z_2 \in D^2 = \{z | |z| \leqslant 1\}$. Let

$$Y = M_r \times [0,1] - \operatorname{Int}\left(S^1 \times_{\psi_0} X_0\right).$$

Then ∂Y consists of $M_r \times \{0,1\}$ and the component

$$\partial_0 Y = \partial\left(S^1 \times_{\psi_0} X_0\right) = S^1 \times_{(\psi_0 | \partial X_0)} (\partial X_0) = S^1 \times_{(\psi | \partial X)} \partial X = \partial\left(S^1 \times_\psi X\right).$$

Hence we may define the union along the indicated boundary component,

$$V = V(k) = Y \cup_{\partial_0 Y} S^1 \times_\psi X. \tag{1.2}$$

We wish to prove that V is an s-cobordism with local coefficients in $\mathbf{Z}[Q_r]$. Even to make sense of this statement, a homomorphism $\pi_1 V \to Q_r$ is needed. To construct one, first recall that as X is a knot complement, it has a (smooth) map $f\colon X \to X_0$ with $f | \partial X = \text{identity}$. The existence of f is a standard fact (using $S^1 = K(\mathbf{Z}, 1)$ and Alexander duality). Equally standard arguments prove the fact that $\psi_0 f$ and $f\psi$ are homotopic relative the boundary. The homotopy gives a fiber map

$$T(f)\colon S^1 \times_\psi X \to S^1 \times_{\psi_0} X_0,$$
$$T(f) | \partial\left(S^1 \times_\psi X\right) = \text{identity}.$$

Hence a well-defined continuous map

$$g\colon V \to M_r \times [0,1]$$

is obtained by setting $g\,|\,Y = \mathrm{id}_Y$ and $g\,|\,S^1 \times_\psi X = T(f)$. The induced map

$$g_*\colon \pi_1 V \to \pi_1(M_r \times [0,1]) = Q_r$$

is the desired homomorphism.

(1.3) Proposition. *Suppose the 2^{r+1}-st branched cycle cover of k is a homology sphere. Then g is a simple homology equivalence over* $\mathbf{Z}[Q_r]$.

We recall what the conclusion means [3]: The map g is covered by $\tilde{g}$: $\tilde{V} \to S^3 \times [0,1]$ where $\tilde{V}$ is the covering space corresponding to the kernel of the map g_* on fundamental groups. The chain complex $C_*(\tilde{g})$ can be defined as the cellular chains of the mapping cylinder of $\tilde{g}$, relative $\tilde{V}$; it will be a based chain complex over $\mathbf{Z}[Q_r]$. Here Q_r acts by covering translations and a basis is given by choosing lifts of the relative cells of the mapping cylinder of g, relative the subspace V. The statement that g is a homology equivalence over $\mathbf{Z}[Q_r]$ is equivalent to the statement that $C_*(\tilde{g})$ is acyclic. In this case, the Whitehead torsion

$$\tau(g) = \tau(C_*(\tilde{g})) \in Wh(Q_r)$$

is defined, and the proposition asserts its vanishing.

Since $\partial V = M_r \times \{0,1\}$ and $g \circ h = \mathrm{id}_{M_r}$ for h the inclusion of either component, it follows from (1.3) that h is also a $\mathbf{Z}[Q_r]$ homology equivalence. Further (compare [12]),

$$0 = \tau(g \circ h) = \tau(g) + \tau(h).$$

Thus $\tau(h) = 0$, $\tau(h)$ defined with respect to the homomorphism $g_*\colon \pi_1 V \to Q_r$; i.e.,

(1.4) Corollary. *If the 2^{r+1}-st branched cyclic cover of k is a homology sphere, then $(V, M_r \times 0, M_r \times 1)$ is an s-cobordism over* $\mathbf{Z}[Q_r]$.

Proof of 1.3. Since $g\,|\,Y$ is the identity, by excision (or the Meyer-Vietoris sequence) for homology with local coefficients and additivity for Whitehead torsion [12, §3] it suffices to prove that

$$T(f)\colon S^1 \times_\psi X \to S' \times_{\psi_0} X_0$$

is a simple homology equivalence over $\mathbf{Z}[Q_r]$ with respect to the map $\pi_1(S^1 \times_{\psi_0} X_0) \to \pi_1(M_r \times [0,1]) = Q_r$ induced by inclusion. But

$$\pi_1\left(S^1 \times_{\psi_0} X_0\right) = \pi_1(K) = \{\, t, u \,|\, tut^{-1} = u^{-1} \},$$

and the map induced by inclusion sends t to y and u to x. Let $H = \{\, s, v \,|\, svs^{-1} = v^{-1}, v^{2^{r+1}} = 1 \}$. Then there is a factorization

$$\begin{tikzcd} \pi_1\left(S \times_{\psi_0} X_0\right) \arrow[r] \arrow[dr, "\mu"'] & Q_r \\ & H \arrow[u, "\lambda"'] \end{tikzcd}$$

where $\lambda(s) = y$, $\lambda(v) = x$, $\mu(t) = s$, $\mu(u) = v$. Hence it suffices to prove the stronger result: $T(f)$ is a simple homology equivalence over $\mathbf{Z}[H]$.

View $X_0 \subset S^1 \times_{\psi_0} X_0$ as the fiber over the basepoint, and similarly for X. Then there is an obvious commutative diagram

$$\begin{array}{ccc} \pi_1 X_0 & \longrightarrow & C \\ \downarrow & & \cap \\ \pi_1\left(S^1 \times_{\psi_0} X_0\right) & \longrightarrow & H, \end{array}$$

where C is the subgroup generated by v and the top map sends the generator u of $\pi_1 X_0$ to v. We first assert f: $X \to X_0$ is a homology equivalence over $\mathbf{Z}[C]$. It is easy to see that this is equivalent to the assertion that the (unbranched) 2^{r+1}-fold cyclic cover $\tilde{X}$ *of* X has the (integral) homology of a circle. This assertion follows by Alexander duality from the hypothesis that the 2^{r+1}-st branched cyclic cover of k is a homology sphere.

Let $\tilde{f}$ be a map on 2^{r+1}-fold cyclic covers lying over f. Then $\tilde{f} = \tilde{T}(f) \mid X$, where $\tilde{T}(f)$ is a covering map of $T(f)$ on covering spaces corresponding to $\ker\mu$ and $\ker(T(f)_*\mu)$. Let (C_*, ∂) be the chain complex of $\tilde{f}$ (see the paragraph following 1.3); we assume f, ψ, ψ_0, and the homotopy used to construct $T(f)$ all cellular. Then as f is a homology equivalence over $\mathbf{Z}[C]$, (C_*, ∂) is an acyclic chain complex. From an explicit cell decomposition it is not hard to see that $C_*(\tilde{T}(f), \Delta)$ can be constructed as follows:

$$C_i(\tilde{T}(f)) = (C_i \oplus C_{i-1}) \otimes_{\mathbf{Z}[C]} \mathbf{Z}[H]; \qquad \Delta | C_i = \partial_i,$$

and for $X \in C_{i-1}$,

$$\Delta_i(x) = \pm(x - \psi_* x) \oplus \partial_{i-1}(x) \in (C_{i-1} \oplus C_{i-2}) \otimes_{\mathbf{Z}[C]} \mathbf{Z}[H],$$

ψ_* induced on the relative chains by (ψ, ψ_0). Thus Δ_i looks like

$$\begin{pmatrix} \partial_i & 0 \\ * & \partial_{i-1} \end{pmatrix}$$

with respect to the sum decomposition of $C_i(\tilde{T}(f))$.

It is an easy exercise from the definition of torsion to conclude that $C_*(\tilde{T}(f))$ is acyclic and

$$\tau(T(f)) = \tau(C_*(\tilde{T}(f))) = 0.$$

This completes the proof of (1.3).

Let K be the kernel of

$$g_*: \pi_1 V \to \pi_1(M_r \times [0,1]) = Q_r;$$

then $K = \pi_1 \tilde{V}$. By (1.3), $\tilde{g}$ induces isomorphism of homology groups. Hence $K/[K, K] = H_1(V) = 0$. Let $x_1, \cdots, x_m$ be normal generators of K, and represent them by disjoint smooth framed embeddings

$$\phi_i: S^1 \times D^3 \to \operatorname{Int}(V), \qquad 1 \leqslant i \leqslant m.$$

Let V be obtained from $V \times [0,1]$ by attaching handles along the images of the ϕ_i; i.e.,

$$U = V \times [0,1] \cup_{(\phi_1 \cup \cdots \cup \phi_m) \times \{1\}} \left(\bigcup_1^m h_i \right),$$

where h_i is a copy of $D^2 \times D^3$. The manifold V is parallelizable (because M_r is, $H_*(V, M_r \times \{0\}) = 0$, and BSO is a simple space). By well-known arguments [10], for a suitable choice of the framings (i.e. the extension of $\phi_i \mid S^1 \times 0$ to $S^1 \times D^3$) it may be arranged that U is also parallelizable. By Van-Kampen's theorem,

$$\pi_1 U = (\pi_1 V)/K = Q_r,$$

with $M_r \times i \subset V \subset U$, $i = 0, 1$, inducing an isomorphism on π_1.

Clearly

$$\partial U = (V \times 0) \cup (\partial V \times [0,1]) \cup W',$$

where

$$V' = \left(V - \bigcup_1^m \phi_i (S^1 \times \mathring{D}^3) \right) \cup_\partial \left(\bigcup_1^m \bar{h}_i \right),$$

$\bar{h}_i$ the copy of $D^2 \times S^2$ in ∂h_i. Since U is obtained from V' by attaching 3-handles, $\pi_1 V' = \pi_1 U = Q_r$.

Now consider

$$L = \pi_2(V') = H_2(\tilde{V}') = H_2(V'; \mathbf{Z}[Q_r]),$$

$\tilde{V}$ the universal covering space. Intersection numbers yields a Hermitian pairing over $\mathbf{Z}[Q_r]$

$$\langle \, , \rangle : L \times L \to \mathbf{Z}[Q_r].$$

Elements $e_1, \cdots, e_m$, $f_1, \cdots, f_m$ in L can be defined as follows: f_j is represented by the copy of $0 \times S^2$ contained in $\bar{h}_j$. To define e_j, let

$$V_0 = V - \bigcup_1^m \phi_i (S^1 \times \mathring{D}^3),$$

and let $\tilde{V}_0 \subset \tilde{V}$ be the induced covering. By general position, $\pi_1 V_0 \to \pi_1 V$ induced by inclusion is an isomorphism. It follows that the circle $\phi_j(S^1 \times z)$, $z \in \partial D^3$, lifts to $\tilde{V}_0$. Since

$$H_1(\tilde{V}_0) = (\pi_1 \tilde{V}_0)_{ab} = K/[K, K] = 0,$$

this lift will bound an orientable surface S_j' *in* $\tilde{V}_0$. It also bounds a disk d_j *in* $\tilde{V}'$ lying over the copy of $D^2 \times z$ *in* $\bar{h}_j$. The union $S_j' \cup d_j = S_j$ is a closed surface and represents an element

$$e_j' \in H_2(\tilde{V}') = L.$$

Let

$$e_j = e_j' - \langle e_j', e_j' \rangle f_j.$$

(1.5) Proposition. $H_2(V'; \mathbf{Z}[Q_r]) = \pi_2(V')$ *is a free* $\mathbf{Z}[Q_r]$*-module with basis* $\{e_1, \cdots, e_m, f_1, \cdots, f_m\}$,

$$\langle e_j, e_i \rangle = \langle f_j, f_i \rangle = 0 \quad \text{for } 1 \leqslant i, j \leqslant m,$$
$$\langle e_i, f_j \rangle = \delta_{ij}.$$

That the intersection numbers are as indicated is apparent from the definition of the e_i and f_i. That they form a basis is a standard handlebody-theoretic argument involving, in this case, the exact homology sequences of the pairs $(U, V \times 0)$ and (U, V') over $\mathbf{Z}[Q_r]$. We leave the details to the reader.

Now the results of Freedman apply to yield a topological connected sum decomposition

$$V' = W(k) \# m(S^2 \times S^2)$$

where the jth copy of $S^2 \times z$ represents e_j, and the jth copy of $z \times S^2$ represents f_j.

(Strictly speaking, one should consider a self intersection form μ: $L \to \mathbf{Z}[Q_r]/\{\xi - \bar{\xi}\}$, defined using a framing of V', and with $\langle\ ,\ \rangle$ as associated bilinear pairing. But

$$\mu(x) + \overline{\mu(x)} = \langle x, x \rangle;$$

it follows in this case that $\mu(x) \equiv 0$ if and only if $\langle x, x \rangle = 0$. Hence $\mu(e_i) = \mu(f_i) = 0$ *automatically*.)

(1.6) Proposition. $(W(k), M_r \times \{0\}, M_r \times \{1\})$ *is an s-cobordism, and there exists a (topological) s-cobordism Z relative the boundary from V to* $W(k)$*, over* $\mathbf{Z}[Q_r]$.

Proof. Let P be obtained from $V' \times [0,1]$ by attaching 3-handles along the spheres $(S^2 \times z) \times \{1\}$ in the above decomposition of V'; thus P is homeomorphic to the boundary connected sum of $W(k) \times [0,1]$ with m copies of $D^3 \times S^2$. Let $(V' = V' \times 0)$

$$Z = U \cup_{V'} P.$$

Clearly Z is a relative boundary cobordism from V to $W = W(k)$. We claim that Z is an s-cobordism over $\mathbf{Z}[Q_r]$.

In fact, consider the inclusion $W \subset Z$. Then Z is obtained from W by first attaching 2-handles, and then attaching 3-handles along 2-spheres in V' representing the classes f_i above. Since $\langle e_i, f_j \rangle = \delta_{ij}$, the attaching sphere of the jth 3-handle and the dual sphere (the "left-hand sphere") of the jth 2-handle intersect algebraically (over $\mathbf{Z}[Q_r]$) in δ_{ij}. It is a standard argument in this case ([12], [9]) that $W \subset Z$ is a homotopy equivalence with vanishing torsion. Indeed, $\pi_1 W = \pi_1 Z = Q_r$, the chain complex $C_*(Z, W, \mathbf{Z}[Q_r])$ is nonzero except in dimensions 2 or 3, and, with respect to a basis of 2-cells and 3-cells given by the handles, the boundary map of this chain complex has the identity matrix. Hence Z is an s-cobordism over $\mathbf{Z}[Q_r]$. In particular by duality, $V \subset Z$ will be a simple homology equivalence over $\mathbf{Z}[Q_r]$. Since V itself is an s-cobordism over $\mathbf{Z}[Q_r]$, it follows easily that W is also. Since $\pi_1 W = Q_r$, W is therefore an actual s-cobordism of $M_r \times \{0\}$ and $M \times \{1\}$.

2. Nontriviality of $W(k)$

Let Δ_k denote the Alexander polynomial of the knot k; then $\Delta_k(x) \in \mathbf{Z}[x \mid x^{2^{r+1}} = 1]$.

(2.1) Theorem. *Suppose that $W(k)$ is homeomorphic to $M_r \times [0,1]$. Then there exist i and $u \in \mathbf{Z}[x \mid x^{2^{r+1}} = 1]$ such that*

$$\Delta_k(x) = \pm x^i u^2.$$

The first step in proving (2.1) is to identify $\Delta_k(x)$ as a suitable type of Reidemeister-Whitehead torsion. To do this, we return to the cobordism $V = V(k)$ and $g\colon V \to M_r \times [0,1]$ defined just prior to (1.3). Let $R = \mathbf{Z}[x \mid x^{2^{r+1}} = 1]$, and let $\langle x \rangle$ be the cyclic group generated by x.

(2.2) Proposition. *V contains a proper, smooth codimension submanifold L, with (interval bundle) neighborhood $N(L)$, meeting the boundary transversely, so that*

(i) $(N(L), L) \cap M \times \{i\} = (N(K), K) \times \{i\}$, $i = 0, 1$;

(ii) $(g_*)\pi_1(V - \operatorname{Int} N(L)) \subset \langle x \rangle$, *the subgroup of Q_r generated by x,*

(iii) $H_*(V - \operatorname{Int}(N(L), H \times \{i\}; R) = 0$, $i = 0, 1$; *and*

(iv) $\tau(V - \operatorname{Int} N(L), H \times \{i\}) \equiv \Delta_k(x) \operatorname{mod}\{\pm x^i\}$.

Here the homology groups in (iii) are defined using the local coefficients in $R = \mathbf{Z}\langle x \rangle$ provided by (ii). In view of (iii), the torsion in (iv) is defined; recall that the determinant map proves an identification (see [1], [12]),

$$Wh(\langle x \rangle) = R^{\times}/\{\pm x^i\}, \qquad R^{\times} = \textit{units of } R.$$

Proof of Proposition 2.2. K can be moved by an ambient isotopy of M_r to a Klein bottle $K' \subset M_r$ that meets K transversely in a circle C which, suitably oriented, represents $x \in Q_r = \pi_1 M_r$. If K is viewed in the usual way as a circle bundle over S^1, then $K \cap K' = C$ may be taken to be a fiber, and $K' \cap N(K)$ an annulus $C \times [-1,1]$ with $C = C \times \{0\}$. This can be seen by considering the union of $N(K)$ with a boundary collar of H as the total space a circle bundle ξ over the Mobius band (pictured in Fig. 1). Then K and K' will be the total spaces of the restrictions of this bundle to the curves $\lambda = S^1$ and λ' pictured in Fig. 1; $N(K)$ will be the total space of the restriction to the indicated sub-Mobius band. Hence we obtain a new decomposition

$$M_r = N(K') \cup H',$$

isotopic to the original decomposition of (1.1)

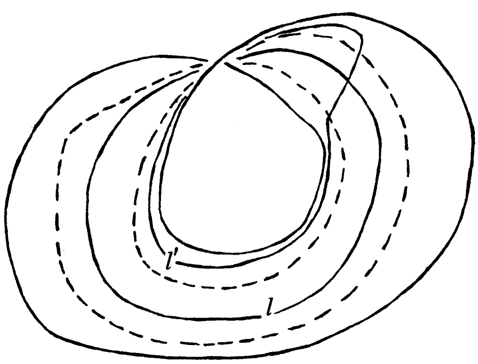

FIG. 1

Now, it is clear that, with the canonical identification $N(K) \times [1/4, 3/4] = S^1 \times_{\psi_0} X_0$, we have that the intersection

$$\left(K' \times [0,1]\right) \cap \left(S^1 \times_{\psi_0} X_0\right)$$

is just the fiber X_0 lying over the point $\lambda \cap \lambda'$ of S^1. Hence, viewing X as the fiber of $S^1 \times_{\psi_0} X$ over $\lambda \cap \lambda'$ we may form the union along $\partial X = \partial X_0$

$$L' = \left(\left(K' \times [0,1]\right) \cap Y\right) \cup_{\partial X} X.$$

(*Recall* $Y = M_r \times [0,1] - \operatorname{Int}(S^1 \times_{\psi_0} X_0)$.) It is clear that

$$L' = g^{-1}\left(K' \times [0,1]\right);$$

hence for $N(L')$ any regular neighborhood of L', $g_*(\pi_1(V - \operatorname{Int} N(L'))$ is contained in the image of $\pi_1(H') = \pi_1(M - K' \times [0,1])$. It is also clear that this image is precisely $\langle x \rangle$, since H' and H are isotopic and the core of H represents x. Hence L' satisfies (ii).

An explicit construction of $N(L')$ goes as follows:

First, let $N(K')$ be the total space of the restriction of ξ to a small Mobius band with λ' as core and meeting $\lambda = S^1$ in an interval I' centered at $\lambda \cap \lambda'$. Then

$$\left(N(K') \times [0,1]\right) \cap \left(S^1 \times_{\psi_0} X_0\right)$$

is just the restriction of the bundle $S^1 \times_{\psi_0} X_0$ to the interval I'. Hence we may define

$$N(L') = \left(\left(N(K') \times [0,1]\right) \cap Y\right) \cup E,$$

where E is the part of $S^1 \times_\psi X$ lying over I, and the union is taken along the common part of the boundary, namely the part of $S^1 \times_\psi (\partial X) = S^1 \times_{\psi_0}(\partial X_0)$ lying over I'. It is obvious that

$$\left(N(L'), L'\right) \cap M_r \times \{i\} = \left(N(K'), K'\right) \times \{i\}.$$

It also follows that $V - \operatorname{Int}(N(L'))$ is obtained as the union

$$\left(\left(H' \times [0,1]\right) \cap Y\right) \cup F,$$

where $H' = M_r - \operatorname{Int} N(K')$, F is the part of $S^1 \times_\psi X$ lying over $S^1 - \operatorname{Int} I'$, and the union is along the part of $S^1 \times_\psi(\partial X_0)$ lying over $S^1 - \operatorname{Int} I'$. By comparison, the product $M \times [0,1] - \operatorname{Int}(N(K') \times [0,1]) = H' \times [0,1]$ has the decomposition

$$\left(\left(H' \times [0,1]\right) \cap Y\right) \cap F_0,$$

where F_0 is the part of $S^1 \times_{\psi_0}(\partial X)$ lying over $S^1 - \operatorname{Int} I'$, and the union is along the part of $S^1 \times_{\psi_0}(\partial X_0)$ lying over this interval. Further $g|(H' \times [0,1]) \cap Y$ is the identity, whereas $g|F\colon F \to F_0$ is a bundle map.

In fact, $F = X \times [0,1]$ and $F_0 = X_0 \times [0,1]$, as these are bundles over intervals. As noted in the proof of 1.3, the canonical map $f\colon X \to X_0$ is a homology equivalence over $R = \mathbf{Z}\langle x\rangle$, as consequence of the hypothesis on the 2^{r+1}-fold branched cyclic cover of k. It follows that $g|F$, which restricts on each fiber to a map homotopic to f, has the same property. Hence by excision (or the Mayer-Vietoris sequence for homology with local coefficients) and the exact sequence of a pair, g restricts to a homology equivalence over R of

$$\left(V - \operatorname{Int} N(L'), H' \times \{i\}\right) \quad \text{with} \left(H' \times [0,1], H' \times \{i\}\right), \; i = 0, 1.$$

Hence

$$H_j\left(V - \operatorname{Int}(L'), H' \times \{i\}; R\right) = 0 \quad \text{for all } j.$$

Since $H' \times \{i\} \subset H' \times [0,1]$ is a simple homology equivalence over R, the additivity of torsions over unions and compositions are easily applied to deduce that in $R^{\times}/\{\pm x^i\}$

$$1 \equiv \tau\big(V - \operatorname{Int} N(L'), H \times \{i\}\big)\tau(f).$$

Let $m \subset \partial X_0$ be a meridian of the knot k. Then $\tau(X, m)$ is defined and the additivity with respect to compositions gives

$$1 \equiv \tau(X, m)\tau(f).$$

Hence

$$\tau\big(V - \operatorname{Int} N(L'), H \times \{i\}\big) = \tau(X, m) \in R^{\times}/\{\pm x^i\}.$$

An argument similar to [13, §5] shows that

$$\tau(X, m) \equiv \Delta_k(x).$$

This completes the proof of (2.2), but with L', K', H', etc. in place of L, K, H,. To obtain L with the desired properties, just apply $\lambda \times \mathrm{id}_{[0,1]}$ to L', where λ is the final stage of an ambient isotopy throwing K' onto K and H' onto H.

From now on we assume the hypothesis of (2.1), viz. that $W = W(k)$ is homeomorphic to $M \times [0,1]$. Let

$$h\colon \big(W, M \times \{0\}, M \times \{1\}\big) \to \big(M \times [0,1], M \times \{0\}, M \times \{1\}\big)$$

be a homeomorphism. After composition with $(h \mid M \times \{0\}^{-1} \times \mathrm{id}_{[0,1]}$, it may be assumed that $h \mid M \times \{0\}$ is the identity. It follows (since $M \times \{i\} \subset W$ is a homotopy equivalence) that $h \mid M \times \{1\}$ is homotopic to the identity. Hence, by [15], [4], $h \mid M \times \{1\}$ is isotopic to the identity. By using this isotopy in a collar neighborhood, we may arrange for the following crucial extra condition on h: $h \mid M \times \{1\}$ is the identity.

Let Z be the relative the boundary s-cobordism of V and W, as in (1.6); thus

$$\partial Z = V \cup_{M, \times \{0,1\}} W.$$

Hence ∂Z has the codimension one submanifold

$$\bar{L} = L \cup_{K \times \{0,1\}} h^{-1}(K \times [0,1]),$$

with the interval bundle neighborhood

$$\overline{N(L)} = N(L) \cup_{N(K)\times\{0,1\}} h^{-1}(N(K)\times[0,1])$$

(2.3) Proposition. *There exists a proper codimension one submanifold J of Z, meeting ∂Z transversely, with interval bundle neighborhood $N(J)$, such that*

(i) $\partial J = J \cap \partial Z = \bar{L}$,

(ii) $N(J) \cap \partial Z = \overline{N(L)}$, *and*

(iii) $(Z - \operatorname{Int} N(J),\ S(J))$ *is 2-connected.*

(*In* (iii), $S(J)$ *denotes the associated S^0-bundle of $N(J)$, easily seen in this case to be just the orientable double cover of J.*)

The detailed proof of (2.3) will be omitted. Using the homeomorphism h, the smooth structure on V extends to one on ∂Z with V as a codimension zero submanifold and $\bar{L}$ as a smooth submanifold. Under the assumption that this smooth structure extends to a smooth structure on Z, the proof of (2.3) is entirely standard in nature. One first recalls that, by a version of the Thom-Pontrjagin construction, $\bar{L} = f^{-1}(RP^{N-1})$, for $f\colon \partial Z \to RP^N$ a smooth map transverse regular to RP^{N-1}, N large. Using the fact that homotopy classes of maps into real projective space RP^N are in 1-1-correspondence with $H^1(-; Z_2)$, one verifies that f extends to $F\colon Z \to RP^N$; make F transverse to RP^{N-1} without changing $F\,|\,\partial Z$, and let $J' = F^{-1}(RP^{N-1})$. Then $J' \subset Z$ will satisfy (i) and (ii), and J' can then be altered to J satisfying (iii) as well by the well-known codimension-one surgery technique of handle exchanges of one-and two-handles, which is perfectly valid in a 5-manifold.

Actually, a similar argument holds as well in the topological category. However, this requires topological transversality and, in particular, since $\dim J = 4$, the work of Freedman again. Hence we indicate briefly a method that avoids 4-dimensional topological transversality. The Kirby-Siebenmann obstruction $\theta(Z,\partial Z) \in H^4(Z,\partial Z;\mathbf{Z}_2)$ measures precisely the obstruction to extending the smooth structure on ∂Z. It can be shown that there is a topological s-cobordism of $M_r \times [0,1]$ to itself, relative the boundary, realizing a given $K - S$ obstruction in $H^4(M\times[0,1]^2, \partial(M\times[0,1]^2), \mathbf{Z}_2)$. The construction involves the topological theory of surgery, Kirby-Siebenmann (Lashof-Rothenberg) theory in higher dimensions (i.e., at least 5) and some calculations of L-groups and normal invariants. This s-cobordism can be glued to Z along $W = M_r \times [0,1]$ to kill $\theta(Z,\partial Z)$; in other words Z may be modified to a smooth s-cobordism, rel ∂, from V to W, and the smooth argument outlined above then applies. [Thus, for the rest of §2, we could work entirely in the smooth category. A completely different approach, modelled on [5], would involve taking products with CP^2 and applying high dimensional topological methods.]

Let J be as in (2.3), and let $Z_0 = Z - \operatorname{Int} N(J)$. Let h be surpressed from the notation from now on; then

$$\partial Z_0 = (V - \operatorname{Int} N(L)) \subset S(J) \subset H \times [0,1].$$

Further, by (2.3)(iii) and Van-Kampen's theorem, $\pi_1 J = Q_r$, $\pi_1 Z_0$ is cyclic of order 2^{r+1} and $\pi_1(H \times \{i\}) \to \pi_1 Z_0$ is surjective. Hence we may view $\pi_1 Z_0$ as the subgroup $\langle x \rangle$ of $\pi_1 Z = \pi_1 W = Q_r$ generated by x.

For the next part of the argument we will need to consider the following exact sequences of chain complexes over $R = \mathbf{Z}[x \mid x^{2^{r+1}} = 1]$; the coefficients R will be surpressed from the notation for the rest of this section

$$\begin{aligned} &0 \to C_*(V - \operatorname{Int} N(L), H \times \{0\}) \to C_*(Z_0, H \times \{0\}) \\ &\quad \to C_*(Z_0, V - \operatorname{Int} N(L)) \to 0; \end{aligned} \tag{2.4.1}$$

$$\begin{aligned} &0 \to C_*(S(J) \cup H \times [0,1], H \times \{0\}) \to C_*(Z_0, H \times \{0\}) \\ &\quad \to C_*(Z_0, S(J) \cup H \times [0,1]) \to 0. \end{aligned} \tag{2.4.2}$$

[As noted above, we can be working in the smooth category, in which case these chain complexes are defined from handle decompositions in the standard way. In the topological category, one may apply [11] to obtain handle decomposition of products of all the indicated pairs with a (given) disk, and use the corresponding chain complexes for the exact sequences (2.4.1) and (2.4.2).]

(2.5) Proposition. *$H_i(Z_0, H \times \{0\}) = H_i(Z_0, V - \operatorname{Int} N(L))$ is zero for $i \neq 2$ and is a stably free R-module for $i = 2$. $H_i(Z_0, S(J) \cup H \times [0,1]) = 0$ for $i \neq 3$ and is a stably free R-module for $i = 3$.*

Proof. By (2.2)(iii), $H_*(V - \operatorname{Int} N(L), H \times \{i\}) = 0$. Hence, by the homology sequence of (2.4.1), $H_i(Z_0, H \times \{0\})$ and $H_i(Z_0, V - \operatorname{Int} N(L))$ are isomorphic. Since everything is connected and the composite

$$\pi_1(H \times \{i\}) \to \pi_1(V - \operatorname{Int} N(L)) \to \pi_1 Z_0 = \langle x \rangle$$

is surjective, the groups $H_i(Z_0, V - \operatorname{Int}(L))$ vanish for $i = 0, 1$.

Consider the triple

$$S(J) \subset S(J) \cup H \times [0,1] \subset Z_0.$$

By excision,

$$H_i(S(J) \cup H \times [0,1], S(J)) = H_i(H, \partial H),$$

which is clearly trivial for $i \leqslant 1$. By (2.3)(iii) (and the Hurewicz theorem) $H_i(Z_0, S(J)) = 0$ for $i \leqslant 2$. Hence, by the long exact homology sequence of the triple, $H_i(Z_0, S(J) \cup H \times [0,1]) = 0$ for $i \leqslant 2$. By Poincare duality,

$$H^i(Z_0, S(J) \cup H \times [0,1]) = H_{5-i}(Z_0, V - \operatorname{Int} N(L)).$$

Hence these cohomology groups vanish for $i > 3$, and similarly for the homology groups. Hence, by the argument of [18, 2.3], with minor modifications, it follows that $H_3(Z_0, S(J) \cup H \times [0,1])$ is stably free. By Poincare duality, the assertions about $H_i(Z_0, V - \mathrm{Int}(L))$ also follow.

In view of (2.5), let b be a stable[1] basis for $H_2(Z_0, H \times \{0\})$, and let b also denote the image of this basis under the isomorphism of (2.5), i.e., the isomorphism to which the homology exact sequence of (2.4.1) reduces. With respect to these bases, Whitehead torsions τ_b are defined in $Wh(\langle x \rangle) = R^{\times}/\{\pm x^i\}$, and by [18, 3.2],

(2.6.1) $\tau_b(Z_0, H \times \{i\}) = \tau(V - \mathrm{Int}\, N(L), H \times \{0\})\tau_b(Z_0, V - \mathrm{Int}\, L).$

On the other hand, the homology exact sequence of (2.4.2) reduces to

$$\begin{aligned} &0 \to H_3(Z_0, S(J) \cup H \times [0,1]) \xrightarrow{\partial} H_2(S(J) \cup H \times [0,1], H \times \{0\}) \\ &\xrightarrow{j^*} H_2(Z_0, H \times \{0\}) \to 0. \end{aligned} \tag{2.6.2}$$

It follows that the middle group is stably free and is the only non-vanishing group of $(S(J) \cup H \times [0,1], H \times \{0\})$. By duality

$$\begin{aligned} H_3(Z_0, S(J) \cup H \times [0,1]) &= H^2(Z_0, V - \mathrm{Int}\, N(L)) \\ &= \mathrm{Hom}_R(H_2(Z_0, V - \mathrm{Int}\, N(L)), R) \end{aligned}$$

has the dual basis b^*. Hence the middle group has the basis $c = b^*b$ (see [12] for the notation), and from (2.4.2) and [12, 3.2] we obtain

(2.6.3) $\tau_b(Z_0, H \times \{0\}) = \tau_c(S(J) \cup H \times \{0\})\tau_{b^*}(Z_0, S(J) \cup H \times [0,1]).$

By duality for Whitehead torsion

$$\tau_{b^*}(Z_0, S(J) \cup H \times [0,1]) = \tau_b(Z_0, V - \mathrm{Int}\, N(L))^{-},$$

where R has the conjugation

$$\left(\sum a_i x^i\right) = \sum a_i x^{-i}.$$

This conjugation induces the identity on $R^{\times}/\{\pm x^i\}$; see [1]. Hence from (2.6.1) and (2.6.3) we obtain

(2.7) $\tau(V - \mathrm{Int}\, N(L), H \times \{0\}) = \tau_c(S(J) \cup H \times [0,1], H \times \{0\}).$

[1]Actual basis can be obtained by stabilizing using handle exchanges of trivial 2-handles to replace J by its connected sum with copies of $S^2 \times S^2$.

But the inclusion

$$(S(J), S(K \times [0,1])) \subset (S(J) \cup H \times [0,1], H \times \{0\})$$

induces an "excision isomorphism"; in fact, the relative chain complexes are identical. Hence

$$\tau(V - \operatorname{Int} N(L), H \times \{0\}) = \tau_c(S(J), S(K \times [0,1])). \tag{2.8}$$

Now $t \in Q_r$ operates on $S(J)$ by the covering translation of a double cover. In fact it is obvious from the definitions that $C_*(S(J), S(K \times [0,1]))$ can be obtained from $C_*(J, K \times [0,1]; \mathbf{Z}[Q_r])$ by restricting the module structure to the subring R. In particular,

$$H_i(S(J), S(K \times [0,1])) = H_i(J, K \times [0,1]; \mathbf{Z}[Q_r])$$

as R-modules, and similarly for cohomology. It follows from [18, §2] again that the only non-vanishing group $H_2(J, K \times [0,1]; \mathbf{Z}[Q_r])$ is stably free over $\mathbf{Z}[Q_r]$. Moreover, if $d = \{d_1, \cdots, d_q\}$ is a stable basis, then $\{d, td\} = \{d_1, \cdots, d_q, td_1, \cdots, td_q\}$ will be a stable basis over R of $H_2(S(J), S(K \times [0,1]))$. Again, $\tau_d(J, K \times [0,1])$ is defined. Recall the transfer map

$$tr\colon Wh(Q_r) \to Wh(\langle x \rangle).$$

induced by restriction of module structures from $\mathbf{Z}[Q_r]$ to R.

(2.9) Lemma. *$tr(\tau_d(J, K \times [0,1])) = \tau_{\{d,td\}}(S(J), S(K \times [0,1]))$.*

(2.10) Lemma. *Let $\gamma\colon Wh(\langle x \rangle) \to Wh(Q_r)$ be induced by inclusion. Then $tr(\gamma(\xi)) \equiv \xi^2$.*

The first lemma is an exercise in the definitions. For (2.10), recall [1] that every element ξ of $Wh(\langle x \rangle)$ is represented by a symmetric unit $u \in R^\times$; i.e., $u = \bar{u}$. So $tr(\gamma(\xi))$ is represented by the automorphism of $\mathbf{Z}[Q_r]$ as a right R-module given by left multiplication by u. With respect to the basis $\{1, t\}$ of $\mathbf{Z}[Q_r]$ over R, this has the matrix

$$\begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}$$

as $ut = t\bar{u}$. The result follows.

Now consider the short exact (Meyer-Vietoris) sequence of chain complex over $\mathbf{Z}[Q_r] = \Lambda$,

$$\begin{aligned} &0 \to C_*(S(J), S(K \times [0,1]), \Lambda) \\ &\to C_*(Z_0, H \times \{0\}; \Lambda) \oplus C_*(N(J), N(K) \times [0,1]; \Lambda) \\ &\to C_*(Z, M_r \times [0,1]; \Lambda) \to 0. \end{aligned} \tag{2.11}$$

The long exact sequence of this short exact sequence reduces to an isomorphism ($\otimes = \otimes_R$).

$$(2.12)\qquad \begin{aligned}&(j_* \otimes 1, \delta)\colon H_2(S(J), S(K \times [0,1])) \otimes \Lambda \\ &\qquad \to (H_2(Z_0, H \times \{0\}) \otimes \Lambda) \oplus H_2(J, K \times [0,1]; \Lambda).\end{aligned}$$

Here j_* is as in the homology exact sequence (2.6.2) and δ is the extension to Λ-modules of the identification of R-modules

$$H_2(S(J), S(K \times [0,1])) = H_2(J, K \times [0,1]; \Lambda);$$

note that

$$(J, K \times [0,1]) \to (N(J), N(K) \times [0,1])$$

is a simple homotopy equivalence.

Now, with respect to basis $(b_* b) \otimes 1$ and $\{b \otimes 1, d\}$, the isomorphism $(j_* \otimes 1, \delta)$ has matrix of the form

$$\begin{bmatrix} 0 & A_1 \\ I & A_2 \end{bmatrix}$$

as j_* vanishes on each element of ∂b^*; see (2.6.2). Hence A_1 is invertible; i.e. the image ∂b^* *of* b^* is a basis of $H_2(J, K \times [0,1]; \Lambda)$, over Λ. Thus $\{\partial b^*, t(\partial b^*)\}$ is a basis of $H_2(S(J), S(K \times [0,1]))$ over R.

With respect to the basis $b^*b \otimes 1$ and $(b \otimes 1, \partial b^*)$, the isomorphism of (2.12) obviously has zero torsion; hence [12, 3.2] applied to (2.11) yields

$$\gamma\tau_{b^*b}(S(J), S(K \times [0,1] = \gamma\tau_b(Z_0, H \times \{0\})\tau_{\partial b^*}(J, K),$$

since $\tau(Z, M_r \times [0,1]) = 1$ as Z is an s-cobordism. Hence $\tau_{\partial b^*}(J, K)$ is in the image of γ. Hence

$$\tau_{\{\partial b^*, t\partial b^*\}}(S(J), S(K \times [0,1]) = tr\tau_{\partial b^*}(J, K \times [0,1])$$

is a square, by (2.10), in $R^\times/\{\pm x^i\}$.

Applying the quotient notation of [12] for the element in a Whitehead group represented by a change of basis matrix, we easily obtain from the definitions

$$\tau_{b^*b}(S(J), S(K \times [0,1])) = \tau_{\{\partial b^*, t\partial b^*\}} S(K \times [0,1))[b^*b/\{\partial b^*, t\partial b^*\}].$$

Hence, by (2.8) ($c = b^*b$) and (2.2), the proof of (2.1) is reduced to showing that

$$[b^*b/\{\partial b^*, t(\partial b^*)\}] = [b/j_* t(\partial b^*)]$$

is a square in $R^\times/\{\pm x^i\}$ ($= Wh(\langle x \rangle)$.).

To prove this, consider the bilinear pairing ϕ on $H_3(Z_0, S(J) \cup H \times [0,1])$ given by

$$\phi(\alpha, \beta) = \langle Dj^*(t\partial\alpha), \beta \rangle,$$

where D is the appropriate Poincare duality isomorphism, and $\langle -,- \rangle$ is the evaluation of cohomology on homology. Then the change of basis matrix B expressing $j_*(t(\partial b^*))$ in terms of b is clearly the matrix for ϕ with respect to the basis b^*. On the other hand, it is a well-known argument to show that $\phi(\alpha,\beta) \in R$ is the intersection number of the homology classes $t(\partial,\alpha)$ and $\partial\beta$. Hence

$$\begin{aligned}\phi(\beta,\alpha) &= t(\partial\beta)\cdot(\partial\alpha) = -\left(t^2(\partial\beta)\cdot t(\partial\alpha)\right)^- \\ &= -x^{2^{r+1}}(t(\partial\alpha)\cdot(\partial\beta)) = -x^{2^{r+1}}\phi(\alpha,\beta),\end{aligned}$$

as on $S(J)$ the operation given by t reverses orientation and carries $x \in \pi_1 S(J)$ to x^{-1}. Hence

$$B = -x^{2^r}B^t, \tag{2.13}$$

B^t = *transposed of* B. In [6], it is proven that for an invertible matrix over R satisfying (2.13), $\det B = \pm u^2$, some $u \in R^\times$. Thus $[b/j_*t(\partial b^*)] = \det B \in (\langle x \rangle) = R^\times/\{\pm x^i\}$ is a square.

3. Further results

With only minor modifications, the results of §2 can be improved to the following:

(3.1) Theorem. *Let k and k' be invertible knots with $\Delta_k(x)$ and $\Delta_k(x)$ units in $R = \mathbf{Z}[x \mid x^{2^{r+1}} = 1]$. Then the s-cobordisms $W(k)$ and $W(k')$ from M_r to itself are homeomorphic only if, for some i and $u \in R^\times$,*

$$\Delta_k(x)\Delta_k(x) = \pm x^i u^2.$$

Using more sophisticated, surgery-theoretical methods, beyond the scope of this paper, the converse to this theorem can also be proven: If $\Delta_k(x)\Delta_{k'}(x) = \pm x^i u^2$, then $W(k)$ and $W(k')$ are homeomorphic.

One particularly nice example is given by letting k be the $(p,q) = (2^r - 1, 2^r + 1)$ torus knot. It is well-known that k is invertible. For example, view k as the intersection of a small sphere with the complex algebraic variety in C^2 given by $z_1^p + z_2^q = 0$; then complex conjugation in each co-ordinate provides the inversion. Further

$$\Delta_k(T) = (T^{pq} - 1)(T - 1)/(T^p - 1)(T^q - 1)$$

(3.2) Proposition. $\Delta_k(x)$ *is a unit, but* $\Delta_k(x) \neq \pm x^i u^2$.

We leave the proof mostly as an exercise. One method is to use the fibered square:

$$\begin{array}{ccc} R & \longrightarrow & {}^{\varepsilon}R/(1 + x + \cdots + x^{2^{r+1}}) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}/2^{r+1}\mathbf{Z} \end{array}$$

Then $\varepsilon(\Delta_k(x))$ is the product of cyclotomic units, and $a(\delta_k(x)) = \Delta_k(1) = 1$. From this it follows that $\Delta_k(x)$ is a unit; i.e. the 2^{r+1}-fold branched cyclic cover of k is a homology sphere. To see it does not have the form $\pm x^i u^2$, it suffices to pass to the ring $R \otimes \mathbf{Z}_2$. In this ring, $x^i u^2$ always has the form $x^\varepsilon \Sigma a_i x^{2i})$, $\varepsilon = 0$ *or* 1. It is easy to see that $\Delta_k(x)$ does not have this form. [For example, for $r = 2$, the (3, 5)-torus knot yields

$$\Delta_k(x) = -x^7 + x^5 + x^4 + x^3 - x + 2.]$$

Every element of $R^\times/\{\pm x^i\}$ can be represented by a self-conjugate polynomial p with $p(1) = 1$, and every such polynomial is the Alexander polynomial of a knot. In fact, one can obtain every element of $R^\times$ from invertible knots. Hence there actually are at least $2^{2^r - r - 1}$ distinct s-cobordisms of M_r with itself. For example, for $r = 2$, we have M_2 and $W(K)$, k the (3, 5)-torus knot, whereas for $r = 1$, granting the converse of (3.1), every s-cobordism of M_r to itself is homeomorphic to $M_1 \times [0, 1]$.

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