FOLIATED CR MANIFOLDS

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Abstract

Let X be a (maximally complex) abstract CR manifold of dimension $4m + 1$ with nondegenerate Levi form. Suppose that X is foliated by compact complex manifolds of complex dimension $\geqslant m$. Then $m = 1$, the leaves are Riemann spheres, and X arises from a twistor construction.

Introduction

Consider for a moment the Hopf map

$$\pi: \mathbf{CP}_3 \to \mathbf{HP}_1 \approx S^4$$

obtained by remembering that a pair of quaternions is also a quadruple of complex numbers:

$$\pi\big([z_0, z_1, z_2, z_3]\big) = [z_0 + z_1 j, z_2 + z_3 j].$$

The inverse image of any point $x \in S^4$ is then a complex projective line $\mathbf{CP}_1 \subset \mathbf{CP}_3$, and if we take the inverse image $X = \pi^{-1}(M)$ of a hypersurface $M^3 \subset S^4$, we obtain a real hypersurface in $\mathbf{CP}_3$ foliated by compact complex curves. For example, if M is the equator $S^3 \subset S^4$ given by $\{[q_0, q_1] \in \mathbf{HP}_1 \mid \|q_0\| = \|q_1\|\}$, the $\mathbf{CP}_1$-foliated real hypersurface X is the real hyperquadric $\{[z_0, z_1, z_2, z_3] \in \mathbf{CP}_3 \mid |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2 = 0\}$. The Levi form (§0) of this hyperquadric is clearly nondegenerate, a fact which carries over for *any* choice of $M^3 \subset S^4$.

How general is the above family of examples? For instance, is it possible to find a real hypersurface in a (possibly noncompact) complex 3-manifold which has nondegenerate Levi form and is foliated by compact holomorphic curves of higher genus? The answer, as shall be shown herein, is *no*.

Another sort of generalization would be to consider hypersurfaces of higher dimension; for instance, the real hyperquadric

$$X^{13} = \left\{ [z_0, z_1, \cdots, z_7] \in \mathbf{CP}_1 \Big| \sum_{n=0}^{3} |z_n|^2 = \sum_{n=4}^{7} |z_n|^2 \right\}$$

contains many $\mathbf{CP}_3$'s; one might expect at first glance that, by replacing the quaternions in the above example with the octonians, X^{13} would be fibered over S^7 with fibers consisting of $\mathbf{CP}_3$'s. In fact, however, the nonassociative character of the Cayley numbers prevents this construction from going though. Indeed, we shall see (§1) that this failure is really dictated by topological considerations more primitive than those stemming from complex analysis. Indeed, the only circumstance under which a CR $(4m + 1)$-manifold (e.g. a real hypersurface in a complex $(2m + 1)$-manifold) with nondegenerate Levi form may be foliated by compact complex m-manifolds occurs when $m = 1$ and the leaves have genus 0. This is proved in §2. The remainder of this paper then shows how all such foliated CR manifolds arise from a twistor construction generalizing the above method of generating CR manifolds from hypersurfaces of S^4. In the sequel to this paper, coauthored with Simon Salamon, it will be shown that, excluding the exceptional case of the hyperquadric, any such foliation is necessarily unique—a result very much in the same spirit as a local result of Bryant [3] concerning families of curves in CR 5-manifolds.

0. Notation and conventions

Let X be a smooth $(2k + 1)$-manifold. A (maximally complex) CR structure on X is defined to be a complex vector bundle D of rank k which is a smooth subbundle of the complexified tangent bundle

$$D \subset \mathbf{C} \otimes_{\mathbf{R}} TX = \mathbf{C}TX$$

for which

(1) $D \cap \overline{D} = O_X$—i.e. D contains no nonzero real vectors; and

(2) $[C^\infty(D), C^\infty(D)] \subset C^\infty(D)$—i.e. the complex distribution D is involutive.

Example. Let X be a real hypersurface in a complex $(k + 1)$-manifold Z. Let $T^{0,1}Z \subset \mathbf{C}TZ$ be the $(-i)$-eigenspace of the complex structure, spanned in local holomorphic coordinates $z^0, \cdots, z^k$ by $\partial/\partial \bar{z}^0$, $\partial/\partial \bar{z}^1, \cdots, \partial/\partial \bar{z}^k$. Then $D = (T^{0,1}Z) \cap (\mathbf{C}TX)$ is a CR structure on X.

CR structures isomorphic to examples of the above kind are called *realizable* or *imbeddable*. Any real-analytic CR structure is realizable, but many smooth (C^∞) CR structures are not (Nirenberg [11], Jacobowitz and Treves [5], LeBrun [8]). For positive results concerning imbeddability, cf. Kuranishi [6].

Associated to any CR structure is a real-linear imbedding $TX \hookrightarrow E$ of the tangent bundle of X into a complex vector bundle $E \to X$ of rank $k + 1$; namely, set $E = CTX/D$. (For instance, if $X \subset Z$ is an imbedded CR manifold, $E = TZ|_X$.) In particular, there is a smooth distribution $H \subset TX$ of real $2k$-planes which are the maximal complex subspaces of $TX \subset E$. Equivalently, $H = (D + \overline{D}) \cap TX$. Thus, one invariant of the CR manifold X is the skew form

$$A: H \times H \to TX/H$$
$$(v, w) \mapsto [v, w] + H,$$

which measures the degree to which the distribution H fails to be integrable.

Closely related to this "Frobenius obstruction" A is the *Levi form* $\mathfrak{L}$. The latter is the sesquilinear form

$$\mathfrak{L}: D \times D \to \mathbf{C}(TX/H)$$
$$(v, w) \mapsto i[v, \overline{w}] + \mathbf{C}H.$$

for v, w any sections of D. Note that A is exactly twice the imaginary part of $\mathfrak{L}$ under the canonical identification of D and H; thus, these two forms are degenerate or nondegenerate under the same circumstances. Note that if $X \subset Z$ is an imbedded CR manifold, defined locally by $f = 0$, where f is a real function with $df \neq 0$ at X, then the Levi form $\mathfrak{L}$ is represented locally by $i\partial\overline{\partial}f$.

Now suppose that $\Sigma \subset X$ is a submanifold of dimension $2m$. If $CT\Sigma \cap D$ is of complex dimension m at all points of X, we will say that Σ is a complex submanifold of X (of complex dimension m). Indeed, the involutive distribution $T^{0,1}\Sigma = CT\Sigma \cap D$ defines a complex structure on Σ by the Newlander-Nirenberg theorem (cf. Nirenberg [11]) and, in the case of an imbedded CR manifold $X \subset Z$, $\Sigma \subset X$ is a complex submanifold iff it is a complex submanifold of Z in the usual sense. The importance of the Levi form may be seen in the following observation: If $\Sigma \subset X$ is a complex submanifold, then $T^{0,1}\Sigma \subset D$ is an isotropic subspace; i.e. if $v, w \in T^{0,1}\Sigma$, then $\mathfrak{L}(v, w) = 0$. (This follows immediately from the fact that $T\Sigma$ is involutive and $T\Sigma \subset H$.) In particular, if $\mathfrak{L}$ is a definite form, in which case (X, D) is called pseudo-convex, X contains no complex submanifolds. More generally, if $\mathfrak{L}$ is nondegenerate, then X can contain no complex submanifold of dimension exceeding $k/2$.

1. Foliated contact manifolds

Let X be a smooth $(2k + 1)$-manifold. A *contact structure* on X is a distribution of $2k$-planes $H \subset TX$ which is "as far from being integrable as possible"—i.e. such that the skew form

$$A: H \times H \to TX/H$$
$$(v, w) \mapsto [v, w] + H$$

is nondegenerate. Equivalently, any nonzero 1-form φ orthogonal to H over some region $U \subset X$ satisfies

$$\varphi \wedge (d\varphi)^{\wedge k} \neq 0$$

If $\Sigma \subset X$ is a smooth k-manifold, we will say that Σ is a *Legendrian submanifold* (Arnold [1]) iff Σ is everywhere tangent to the contact structure: $T\Sigma \subset H$.

Proposition 1. *Suppose that a contact manifold X^{2k+1} is foliated by* **compact** *Legendrian submanifolds. Then a generic leaf of the foliation is either a sphere S^k or a real projective space $\mathbf{RP}^k$. If k is even,* **any** *leaf must be either a sphere or a projective space.*

Proof. Let $X_0 \subset X$ be the union of all the leaves of trivial holonomy, let M denote the space of such leaves, and let $\pi: X_0 \to M$ be the quotient map. M is then a smooth $(k + 1)$-manifold in a unique way making π a smooth submersion. X_0 is open and dense in X (Edward, Millett and Sullivan [4]).

Let $Y = \mathbf{RP}(T^*M)$ be the Brassmannian of k-plane elements on M. Then Y is a contact manifold in a canonical way—namely, the contact structure H_Y is given by

$$H_Y\|_V = (\pi_{Y*})^{-1}[V],$$

where $V \in Y$ is a k-dimensional vector subspace of some tangent space of M, and $\pi_Y: Y \to M$ is the canonical projection. In other words, H_Y is the orthogonal space of the canonical form $\sum_{j=1}^{k+1} p_j \, dx^j$ of the cotangent bundle of M.

There is now a tautological smooth map $\alpha: X_0 \to Y$ given by $\alpha(x) = \pi_* H_X$. (Since the fibers of π are k-dimensional and tangent to the contact structure H, $\pi_*[H_X]$ has dimension $2k - k = k$; so the map α is well defined.) Moreover, the pull-back of H_Y is, by construction, precisely H. In particular, the derivative of α is nonsingular, since a typical $(2k + 1)$-form on Y would be $\varphi \wedge (d\varphi)^{\wedge k}$, where φ is some "contact form" orthogonal to H_Y.

Now α takes fibers of π to fibers of π_Y; hence, for $q \in M$,

$$\alpha|_{\pi^{-1}(q)}: \pi^{-1}(q) \to (\pi_Y)^{-1}(q)$$

is a local diffeomorphism between a generic leaf and $\mathbf{RP}^k$. Since the leaf is, by hypothesis, compact and connected, $\alpha|_{\pi^{-1}(q)}$ is a covering map, and $\pi^{-1}(q)$ must be either S^k or $\mathbf{RP}^k$.

If $k = 2m$ is even, we may use this to conclude that an arbitrary leaf is also either a sphere or a projective space, since the density of X_0 in X implies that any exceptional leaf is covered by a leaf of the above type. If the exceptional leaf were oriented, the covering leaf would necessarily be a sphere, with deck transformations acting in an orientation preserving fashion; but since $\chi(S^{2m})$ $\neq 0$, such a map has fixed points, and, being a deck transformation, is therefore the identity map. Thus, an orientable leaf is a sphere. Similarly, an unorientable leaf is a projective space, as is deduced from the above by passing to a double-covering.   q.e.d.

For k odd, the reader may find it amusing to construct Legendrian foliations with compact leaves such that some exceptional leaf is, for instance, a Lens space. This may be readily done by lifting the standard action of $\mathbf{Z}_p$ on $\mathbf{C}^N = \mathbf{R}^{2N}$ to the cosphere bundle of $\mathbf{R}^{2N}$.

2. Holomorphic considerations

Let X now denote a CR $(4m + 1)$-manifold, and let $D \subset CTX$ be the distribution of complex $2m$-planes constituting the anti-holomorphic tangent space of X. $H = (D + \overline{D}) \cap TX$ will denote the underlying distribution of real $4m$-planes. We will assume that the Levi form of D is nondegenerate—i.e. that H is a contact structure.

Now assume that X is foliated by compact complex submanifolds of complex dimension m. Thus we assume that, for each leaf Σ, $T^{0,1}\Sigma \subset D$. As a consequence, $T\Sigma \subset H$ and the foliation is Legendrian. As Σ is even dimensional and oriented, it is, by Proposition 1, diffeomorphic to S^{2m}. The holonomy of all the leaves is therefore trivial, and we can realize the foliation as a smooth fibering $\pi: X \to M$.

This allows us to construct a map $\beta: X \to G_m(CTM)$ by $\beta(x) = \pi_*[D_X]$, where $G_m(CTM)$ is the Grassman bundle of complex m-planes in CTX; since $D \cap CT\Sigma = T^{0,1}\Sigma$, $\pi_*[D_x]$ has complex dimension $2m - m = m$; thus, β is well defined. Now let $q \in M$, $\Sigma = \pi^{-1}(q)$, and identify $G_m(CT_qM)$ with $G_m(\mathbf{C}^{2m+1})$ by choosing a basis for T_qM; the resulting map $\beta|_{\pi^{-1}(q)}: \Sigma \to G_m(\mathbf{C}^{2m+1})$ will be called β_q for brevity.

Proposition 2.   *The map* $\beta_q: \Sigma \to G_m(\mathbf{C}^{2m+1})$ *is a holomorphic immersion.*

Proof.   The complex vector bundle $CTX/CT\Sigma = \pi^*CT_qM \to \Sigma$ has a canonical flat connection which may be expressed as

$$\nabla_v(w + CT\Sigma) = [v,w] + CT\Sigma,$$

where w is a section of CTX and the vector field v on Σ is extended in an arbitrary way so as to be tangent to the foliation. In particular, this gives $CTX/CT\Sigma$ a holomorphic structure by taking

$$\bar{\partial}_v(w + CT\Sigma) = [v, w] + CT\Sigma$$

when v is a section of D; the flatness of ∇ guarantees that $\bar{\partial}^2 = 0$. Since the integrability condition guarantees that $[v, w]$ is a section of D if v and w are, this makes $D/T^{0,1}\Sigma$ a holomorphic subbundle of $\pi^* CT_q M$. Thus β_q is tautologically holomorphic.

To conclude that β_q is also an immersion, we must use the fact that the Levi form is nondegenerate. Indeed, since the restriction of the Levi form to Σ vanishes, it follows that for every $v \in T^{1,0}\Sigma$ there is an element w of $D/T^{0,1}\Sigma$ with $[v, w] + H$ nonzero—i.e., $\nabla_v(w + CT\Sigma) \notin D/T^{0,1}\Sigma$. Thus, the derivative of β_q has maximal rank.

Corollary 3. *In the above situation, $m = 1$; i.e., X is a 5-manifold and the leaves are Riemann spheres.*

Proof. By Proposition 2, any leaf Σ may be holomorphically immersed in a Kähler manifold; pulling back the Kähler form, Σ is itself Kähler. In particular, $H^2(\Sigma, \mathbf{R}) \neq 0$—the Kähler form is not exact! Since we have already concluded that Σ is diffeomorphic to S^{2m}, $m = 1$.

Proposition 4. *The degree of β_q: $S^2 \to G_2 (\mathbf{C}^3)$ is 2, and β_q is an imbedding as a nondegenerate conic in $\mathbf{CP}_2$.*

Proof. The Levi form sets up a sesquilinear pairing of $T^{0,1}S^2$ and $D/T^{0,1}S^2$, so they are isomorphic as smooth complex line bundles. But the line-bundle $D/T^{0,1}S^2$ is, by construction, $\beta_q^*[\mathcal{O}(-1)]$. Hence

$$2 = \chi(S^2) = c_1(T^{1,0}S^2)[S^2] = -c_1(T^{0,1}S^2)[S^2]$$
$$= -c_1\big(\beta_q^*[\mathcal{O}(-1)]\big)[S^2] = q^*\big(c_1[\mathcal{O}(1)]\big)[S^2].$$

As the cohomology class $c_1[\mathcal{O}(1)]$ generates $H^2(\mathbf{CP}_1, \mathbf{Z})$, $\beta_q^*(c_1[\mathcal{O}(1)])[S^2]$ is precisely the degree of β_q. The degree is therefore 2. By the classical Plücker formulae, β_q can have no double point, and so is an imbedding as a nonsingular conic.

3. The first fundamental form

Starting with an abstract CR $(4m + 1)$-manifold foliated by compact complex m-manifolds, we have concluded that, provided the Levi form is nondegenerate, $m = 1$ and the leaves are curves of genus 0. Moreover, we have found

a smooth inclusion

$$\beta\colon X \to \mathbf{P}(CTM)$$

which takes the leaves of X holomorphically to conic curves; here the smooth 3-manifold M is the space of leaves. We will not identify X with its image under β. If we now use $\pi\colon \mathbf{P}(CTM) \to M$ to denote the canonical projection, so that its restriction $\pi|_X\colon X \to M$ becomes the projection *formerly* denoted by π, we may also recall that, from our construction of β,

$$\pi_*[D_x] = x \quad \forall x \in X.$$

In particular, X does not meet the real directions $\mathbf{R}\mathbf{P}(TX) \subset \mathbf{P}(CTX)$, since D is a CR structure and hence contains no nonzero real vectors.

Since X intersects each fiber of π in a conic, there is, near every $q \in M$, a complex symmetric form g on CTM such that X is given by

$$(*) \qquad\qquad X = \big\{\, [v] \in \mathbf{P}(CTM)\,|\,g(v,v) = 0 \,\big\}.$$

One should think of g as a complex metric—in local coordinates

$$g = \sum_{j,k=1}^{3} g_{jk}\, dx^j \otimes dx^k,$$

where the complex coefficients g_{jk} satisfy $g_{jk} = g_{kj}$, $\det[g_{jk}] \neq 0$, and

$$\sum_{j,k=1}^{3} g_{jk} u^j u^k \neq 0$$

for u^1, u^2, u^3 *real* numbers not all zero. As $\{g_{jk}\}$ is only determined only up to an overall complex scale factor, one might expect some difficulty in choosing g globally. Fortunately, however, this problem does not actually occur.

Fact 5. *The tensor g may be chosen globally in such a way that $(*)$ is satisfied.*

Proof. Since M is odd dimensional, there is a global nonzero real vector field u on M. (For M compact, this amounts to the observation that $\chi(M) = 0$ by $\mathbf{Z}_2$ Poincaré duality.) Locally, there is exactly one candidate for g satisfying $g(u,u) = 1$. Since two such local candidates agree on their common territory, this condition determines g globally. q.e.d.

Of course, X does not actually determine the tensor g; the ambiguity of a nonzero complex scale factor remains. Thus, if two tensors g and $\hat{g}$ are related by $\hat{g} = f \cdot g$ for f a smooth function with values in the nonzero complex numbers $\mathbf{C}_*$, we will write $\hat{g} \sim g$, and denote by $[g]$ the equivalence class of g

with respect to this equivalence relation. The class $[g]$ will be called the *first fundamental form* of X; this terminology, intended to conjure up associations from the theory of Riemannian hypersurfaces, will be justified in the appendix.

4. A rival CR structure

Given a CR 5-manifold X foliated by compact complex curves, we saw in §2 that X could be imbedded canonically in $\mathbf{P}(CTM)$, where M is the 3-manifold of leaves. Moreover, the image of X is a conic subbundle avoiding the real directions $\mathbf{RP}(TM) \subset \mathbf{P}(CTM)$. In this section, we review a method (LeBrun [8]) for giving such a subbundle a CR structure. This "rival" CR structure will turn out to be closely related to, but by no means generally identical with, the given CR structure. Rather, the difference between the two CR structures will be measured by the "second fundamental form" of X, as detailed in §5.

Let $X \subset \mathbf{P}(CTM)$ be a conic subbundle avoiding the real directions. Thus,

$$X = \left\{ [v] \in \mathbf{P}(CTM) | g(v,v) = 0 \right\}$$

for some complex symmetric tensor $g \in C^{\infty}(\odot^2 CT^*M)$ with nonzero determinant and no real null vectors. Let

$$g^{-1} \in C^{\infty}(\odot^2 CTM)$$

be the inverse of g, considered as a morphism $CTM \to CT^*M$, and let $\tilde{X} \subset \mathbf{P}(CT^*M)$ be the bundle of dual conics

$$\tilde{X} = \left\{ [\varphi] \in \mathbf{P}(CT^*M) | g^{-1}(\varphi, \varphi) = 0 \right\}$$

so that

$$\flat \colon \mathbf{P}(CTX) \to \mathbf{P}(CT^*M)$$
$$[v] \mapsto [g(v, \cdot)]$$

provides a diffeomorphism from X to $\tilde{X}$. (Notice that this diffeomorphism is, fiber by fiber, a biholomorphism.) More geometrically, this map is the classical planar duality map assigning to each point of a conic the corresponding tangent line.

Let L be the tautological complex line bundle on $\mathbf{P}(CT^*M)$ with fiber at $[\varphi]$ consisting of all complex multiples of φ; thus, the restriction of L to a typical fiber of $\hat{\pi} \colon \mathbf{P}(CT^*M) \to M$ is the usual Hopf bundle over $\mathbf{P}_2$ with Chern class -1. Let θ be the canonical L^*-value 1-form on $\mathbf{P}(CT^*M)$ defined by

$$\theta_{[\varphi]} = \hat{\pi}^*\varphi,$$

and notice that $\theta \wedge d\theta$ is a well-defined 3-form with values in $L^{*\otimes 2}$. Then, letting $j\colon \tilde{X} \to \mathbf{P}(CT^*M)$ denote the inclusion map, we may let $\tilde{Д} \subset CTX$ denote the set of vectors $u \in CTX$ satisfying

$$u \lrcorner j^*(\theta \wedge d\theta) = 0.$$

Then $\tilde{Д}$ is a CR structure on $\tilde{X}$ (LeBrun [8]) and thus determines a CR structure $Д$ on X via the diffeomorphism $\flat$; this is the promised "rival" CR structure. Moreover, this CR structure satisfies the following two conditions:

(1) The fibers of the projection $\pi\colon X \to M$ are holomorphic curves with respect to $Д$; and

(2) $\pi_*(Д_x) = x$ $\forall x \in X \subset \mathbf{P}(CTM)$.

Note that the second condition states that, should we choose to construct $\beta\colon X \to \mathbf{P}(CTM)$ for the rival CR structure $Д$, β simply becomes the identity map $X \to X$. This fact is critical to the classification theorem of the next section.

One should ask what goes wrong with the construction of $Д$ when X is *not* conic. Indeed, all that is important for the construction is that (1) S avoids $\mathbf{RP}(TM)$, so that $Д$ contains no real nonzero vectors; (2) X intersects each $\mathbf{CP}_2$ fiber of $\mathbf{P}(CTM)$ in a holomorphic curve; and (3) the tangent-line map $\flat\colon X \to \tilde{X}$ is a diffeomorphism. If one does not stipulate (as we have done) that the fiber curves are *compact*, one may create many other CR manifolds in this way. If the curves *are* compact, however, they will necessarily have inflection points (by the classical Plücker formulae) unless the degree is 2; this gives rise to cusps in the dual locus $\tilde{X}$, and the CR structure $Д$ fails to be defined at the singular points. In this way, however, one may construct many interesting examples of foliated CR manifolds with leaves that are *punctured* Riemann surfaces of positive genus.

5. The second fundamental form

Given an arbitrary CR 5-manfiold (X, D) with nonnegative Levi-form foliated by compact holomorphic curves, we have shown that the first fundamental form of X determines a "rival CR structure" $Д$ on X such that

(1) The leaves are also holomorphic with respect to $Д$:

$$T^{0,1}\Sigma = CT\Sigma \cap D = CT\Sigma \cap Д$$

for each leaf Σ; and

(2) $\pi_*Д_x = \pi_*D_x$ for all $x \in X$, where $\pi\colon X \to M$ is again the projection to the leaf manifold M.

For simplicity, let $V^{0,1}$ and $V^{0,1} \subset CTX$ denote, respectively, the antiholomorphic and holomorphic tangent spaces of the leaves; thus $V^{1,0} + V^{0,1} = \mathrm{Ker}[\pi_*: CTX \to CTM]$. Statements (1) and (2) are then equivalent to the observation that

$$D + V^{1,0} = \mathrm{\Pi} + V^{1,0}.$$

The inclusion $D \hookrightarrow \mathrm{\Pi} + V^{1,0}$ therefore gives rise to a morphism

$$\gamma: D/V^{0,1} \to V^{1,0}$$

which vanishes identically precisely if $D \equiv \mathrm{\Pi}$. Now we have already noticed (§2) that $D/V^{0,1}$ is leaf-wise a holomorphic line-bundle of Chern class -2; indeed, $V/V^{0,1} = \beta^*L$.

Proposition 6. *The bundle morphism γ is leaf-wise holomorphic. Conversely, every leaf-wise holomorphic morphism $\beta^*L \to V^{1,0}$ arises for some D.*

Proof. Let $u \in C^\infty(V^{1,0})$ be a nonzero leaf-wise holomorphic vector field over some open set of X; let $w \in C^\infty(D)$ and $\tilde{w} \in C^\infty(\mathrm{\Pi})$ represent the same leaf-wise holomorphic section of $D/V^{0,1} = \mathrm{\Pi}/^{0,1} = \beta^*L$. Thus, for appropriate complex-valued functions f_1, f_2, f_3, f_4 we have

$$[\bar{u}, \tilde{w}] = f_1 \bar{u}, \quad [\bar{u}, w] = f_2 \bar{u}, \qquad w = \tilde{w} + f_3 u + f_4 \bar{u}.$$

Note that $\gamma([w]) = f_3 u$, so the claim is that $\bar{u} f_3 = 0$. But

$$[\bar{u}, f_3 u + f_4 \bar{u}] = [\bar{u}, w] - [\bar{u}, w] = f_1 \bar{u} + f_2 \bar{u},$$

and so

$$(\bar{u} f_3) u = (f_1 - f_2 - \bar{u} f_4) \bar{u}.$$

Since u and $\bar{u}$ are linearly independent, it follows that γ is leaf-wise holomorphic.

Conversely, every leaf-wise holomorphic morphism $\gamma: \beta^*L \to V^{1,0}$ arises from a unique CR struction D. Indeed, if we set

$$D = \big\{ \tilde{w} + \gamma([\tilde{w}]) \,|\, \tilde{w} \in \mathrm{\Pi} \big\},$$

D is automatically involutive, as may be seen by reversing the steps of the above argument.   q.e.d.

Up to this point, no strong use has been made of the compactness of the leaves of the given foliation; γ is in fact defined for an *arbitrary* foliation away from the flex points of β (where $\mathrm{\Pi}$ ceases to be defined). However, the compactness *does* have an important consequence—it allows us to explicitly parametrize all morphisms γ, as will be explained below.

Let X have leaf manifold M and first fundamental form $[g]$. Then a pair (g, e), where $g \in [g]$ and $e \in C^\infty(\Lambda^3 CT^*M)$ is a complex 3-form, is called an *oriented first fundamental form* if for every triple v_1, v_2, $v_3 \in CTM$ satisfying

$g(v_j, v_k) = \delta_{jk}$ we have $e(v_1, v_2, v_3) = \pm 1$. M admits an oriented first fundamental form iff $w_1 \cup w_1 = 0$, for the latter is the condition for the existence of a nonzero complex 3-form $\hat{e}$, and if $\hat{g}$ is an element of $[g]$ we may then take

$$g = (\det \hat{g})^{-1}\hat{g} \quad \text{and} \quad e = (\det \hat{g})^{-1}\hat{e},$$

where $\det \hat{g}$ is computed relative to the 3-form $\hat{e}$. In particular, there are oriented first fundamental forms if M is either orientable (which we may arrange by passing to a double cover) or spin (in which case the Wu relations give $w_1 \cup w_1 = w_2 = 0$).

Given an oriented first fundamental form, the bundle morphism γ can be described explicitly by a trace-free quadratic form II, called the *second fundamental form* of (X, D). Indeed, if we are given a rank 2 tensor II, we may obtain a morphism γ by

$$\gamma([v]) = v \times \text{II}(v, \cdot)$$

for $[v] \in X \subset \mathbf{P}(CTM)$; more precisely, in any local frame

$$(\gamma([v]))^a = e^a_{bc}\text{II}^c_d v^b v^d \quad \forall v \neq 0 \text{ s.t. } g(v, v) = 0,$$

where indices are raised with g^{ab} and the summation convention applies. This expression is to be read as a vertical projective vector field, and one should remark immediately that it is tangent to X because it is orthogonal to v.

Notice that the above expression is independent of the trace II^a_a of II (which we shall therefore take to vanish) because e is skew. Further, because the above is to be interpreted projectively, the expression on the right is to be read *modulo* v; the skew part of II therefore also makes no contribution, since $v \times \text{II}(v, \cdot)$ is orthogonal to v and $\text{II}(v, \cdot)$, and hence would be a multiple of v for II skew and v null. On the other hand, once we impose the conditions $\text{II}_{ab} = \text{II}_{ba}$ and $\text{II}^a_a = 0$, we get an injection from tensors II to vector fields γ on X which are vertical, fibre-wise holomorphic, and of homogeneity 1 in v.

It follows by counting dimensions that *every* γ arises from some II. Indeed, the space of trace-free quadratic forms II at some $q \in M$ has dimension $\binom{3+1}{2} - 1 = 5$, while the dimension of

$$H^0(\pi^{-1}(q), \mathfrak{O}(V^{1,0}) \otimes \beta^*L^*) \cong H^0(\mathbf{P}_1, \mathfrak{O}(4))$$

is also 5. Thus, for every (g, e, II) on an arbitrary 3-manifold M we get a CR 5-manifold with nondegenerate Levi form foliated by $\mathbf{P}_1$'s, and every such CR manifold either arises from some (M, g, e, II) or is double-covered by a CR manifold which does. Note that (g, e, II) and $(\Omega^2 g, \Omega^3 e, \Omega \text{II})$ produce the *same* CR manifold for any nonzero complex-valued function Ω on M; the orbit $[g, e, \text{II}]$ of $g, e, \text{II})$ under this action will be called the *first and second fundamental forms* of the corresponding X. In the case where M is not

orientable,, we may still specify $(g, e \otimes \mathrm{II})$ globally, and thereby avoid the need to double-cover X; in this case, we will refer to the orbit of $(g, e \otimes \mathrm{II})$ under the action $(g, e \otimes \mathrm{II}) \to (\Omega g, \Omega^2 e \otimes \mathrm{II})$ as the first and second fundamental forms.

We will refer to the above construction of $\mathbf{P}_1$-foliated CR manifolds from first and second fundamental forms on a 3-manifold as the *twistor construction*; the reason for this terminology will become clear in the appendix. Collecting our results, we have proved the following:

Theorem 7 (*Classification Theorem*). *Let X be a CR $(4m + 1)$-manifold foliated by compact complex m-manifolds. If the Levi form of X is nondegenerate, $m = 1$, the leaves are Riemann spheres, and X arises via the twistor construction from a 3-manifold M equipped with first and second fundamental forms. The latter are uniquely determined by X, and any pair of quadratic forms conversely determines a unique CR manifold.*

Appendix: The geometry of fundamental forms

In this appendix, we will relate the fundamental forms of a foliated CR manifold (as defined in this paper) to the familiar first and second fundamental forms of a hypersurface in a smooth 4-manifold.

Let N be an oriented smooth 4-manifold, and let h be a Riemannian metric on N with Weyl curvature W satisfying $W = *W$, where $*$ is the Hodge star operator and where W is treated as a bundle valued 2-form. Let $\Lambda^2_\pm$ denote the bundle of (anti-) self-dual 2-forms on N, and let $Z \subset \Lambda^2_-$ be the sphere bundle of Λ^2_-; thus,

$$Z := \left\{ \omega \in \Lambda^2 T^*N \,\middle|\, \omega = -*\omega, (\omega, \omega) = 1 \right\},$$

where $(\,,\,)$ is the inner product induced by h. Then Z carries a canonical complex structure [Atiyah et al. [2]]. This complex 3-manifold is called the *twistor space* of N.

Before we describe this complex structure, let us first provide an alternative picture of Z. Recall that an *almost complex structure* on a manifold N is a section J of $\mathrm{End}\, TN$ satisfying $J^2 = -1$. Every such structure induces a canonical orientation on N, namely the one determined by $e^1 \wedge J^*e^1 \wedge \cdots \wedge e^m \wedge J^*e^m$, where $\{e^1, \cdots, e^m\}$ is a generic set of covectors and $\dim N = 2m$. If we are given a Riemannian metric h, the principal $\mathrm{GL}(2m)$ bundle $\mathrm{End}\, TN$ has a corresponding subbundle $O(TN)$ with fiber $O(2m)$ consisting of endomorphisms which are orthogonal with respect to h—a condition which is invariant under conformal rescaling of h. An almost

complex structure is then said to be *orthogonal* if it is a section of $O(TN)$. Finally, we will say that an almost complex structure J is positive (respectively, negative) if the orientation it determines is (not) the given one. Then the bundle Z is precisely the bundle whose sections are the negative orthogonal complex structures on the Riemannian 4-manifold N:

$$Z = \left\{ J \in O(TN) | J^2 = -1, J < 0 \right\}.$$

Indeed, there is a one-to-one correspondence between unit anti-self-dual 2-forms ω and negative orthogonal almost complex structures J given by requiring that

$$h(Ju, v) = \sqrt{2}\, \omega(u, v)$$

for all tangent fields u and v; similarly, positive orthogonal almost complex structures correspond to unit self-dual 2-forms. This is most easily seen by choosing a proper orthonormal frame of the form $e_1, e_2 = J(e_1)$, e_3, $e_4 = \pm J(e_3)$, noticing that the corresponding unit (anti-) self-dual form is then $\omega = \frac{1}{2}(e^1 \wedge e^2 \pm e^3 \wedge e^4)$, and that the forms so constructed sweep out, as e_2 varies, an S^2, which thus must be the entire unit sphere in $\Lambda^2_\pm$.

To construct the complex structure of Z, let p: $Z \to N$ be the canonical projection with fiber S^2, let $V \subset TZ$ denote the vertical space $\ker[p_*$: $TZ \to TN]$, and let $E \subset TZ$ be the horizontal space of the Levi-Civita connection; thus $TZ = E \oplus V$. We will construct endomorphisms $J_1 \in \operatorname{End} E$ and $J_2 \in \operatorname{End} V$ with $J_1^2 = -1$, $J_2^2 = -1$, and let the almost complex structure of Z be $J_1 \oplus J_2$. The integrability condition of this almost complex structure will then be precisely $W = *W$.

To define J_1, notice that $E = p^*TN$, and hence $\operatorname{End} E = p^*\operatorname{End} TN$. But the points of Z *are* endomorphisms of TN; hence there is a canonical section $p^*\operatorname{End} TN$, namely $J_1|_J = J$. By construction, $J_1^2 = -1$.

To define J_2, notice that each fiber $p^{-1}(q)$ is a metric 2-sphere, and in particular has a complex structure once an orientation is specified; this complex structure on the fibers will be J_2. The vector bundle Λ^2_- has a natural orientation because the identification of 2-forms with endomorphisms of TN gives each fiber a Lie algebra structure isomorphic to $su(2)$, so that $e_1 \wedge e_2 \wedge [e_1, e_2]$, $e_1, e_2 \in \Lambda^2_-$, defines an orientation class; hence the fibers of $Z \subset \Lambda^2_-$ may be given a standard orientation by contracting this orientation with the *inward* pointing normal vector field. This defines J_2.

As stated before, $J_1 \oplus J_2$ is integrable if and only if $W = *W$. (By contrast, $J_1 \oplus (-J_2)$ is *never* integrable.) Moreover, the resulting complex structure depends only on the conformal class of h. Note that, by construction, each fiber $p^{-1}(q)$, $q \in N$, is a holomorphic curve $\mathbf{CP}_1 \subset Z$.

The basic example of this construction is obtained by taking (N, h) to be S^4 with its usual metric. In this case, Z is biholomorphic to $\mathbf{CP}_3$, and $p\colon Z \to N$ is the Hopf map $\mathbf{CP}_3 \to \mathbf{HP}_1$.

Now, in the above situation, we obtain a $\mathbf{CP}_1$-foliated CR manifold by considering $X = p^{-1}(M) \subset Z$ for $M \subset N$ any smooth hypersurface. Let g denote the metric on M induced by h (i.e. the classical first fundamental form of $M \subset N$) and let B denote the classical second fundamental form of M; let II denote the trace-free part of B.

Theorem. *Using the above definitions, the first and second fundamental forms of the foliated CR manifold $X = p^{-1}(M)$ are $[g, \mathrm{II}]$—i.e. the conformal parts of the first and second fundamental forms of the imbedding $M \hookrightarrow N$.*

Proof. Choose a local normal vector field ξ along M, and give M the corresponding orientation. There is an isometric identification ψ of $\Lambda^2_-|_M$ with TM given by $\psi(\omega) = \sqrt{2}(\xi \lrcorner \omega)^{\#}$, where $\#$ is the identification of T^*M and TM induced by the metric. The usual connection on Λ^2_- thereby induces a (torsion) metric connection $\hat{\nabla}$ on M related to the usual connection ∇ by $\hat{\nabla}_u v = \nabla_u v - v \times B(u, \cdot)$, where B is the second fundamental form of $M \subset N$ and there $\times$ denotes the 3-space cross-product; indeed, if $\tilde{\nabla}$ denotes the Levi-Civita connection of N, t orthogonal projection to TM, and if vectors and 1-forms are identified via h, we have

$$\hat{\nabla}_u v := \sqrt{2}\,\xi \diagdown \tilde{\nabla}_u \frac{1}{\sqrt{2}}(\xi \wedge v - *\xi \wedge v)$$

$$= \xi \diagdown [\xi \wedge \tilde{\nabla}_u v + (\tilde{\nabla}_u \xi) \wedge v - *(\xi \wedge \tilde{\nabla}_u v) + *(v \wedge \tilde{\nabla}_u \xi)]$$

$$= (\tilde{\nabla}_u v)_t - \xi \diagdown *(v \wedge B(u, \cdot)) = \nabla_u v - v \times B(u, \cdot).$$

Now since $\Lambda^2_-|_M = TM$, we may identify $X = p^{-1}(M)$ with the sphere bundle of M. Let $\hat{E} \subset TX$ be the horizontal space of $\hat{\nabla}$, so that $\hat{E} = E \cap TX$, and let $\eta \in C^\infty(E)$ be the canonical horizontal field satisfying

$$p_* \eta|_v = v.$$

Then $\hat{E} \cap J_1 \hat{E}$ is the perpendicular space of η, and J_1 acts on this space by $u \to u \times \eta$, where the cross-product has been pulled back to $\hat{E} = p^*TM$; this is an immediate consequence of the definitions of ψ and J_1. In particular,

$$\hat{u} = p_*(u - iJ_1 u)$$

satisfies $g(\hat{u}, \hat{u}) = 0$ for all such u; the image of X under β is thus precisely the null locus of g, and the first fundamental form of X is $[g]$.

Let us now identify CT^*M with CTM via #, let $j\colon X \hookrightarrow CTM$ denote the inclusion of

$$\hat{X} := \left\{ u \in CTM \mid g(u, u) = 0 \right\}$$

into the complex tangent bundle, and, for elements of $CT(CTM)$, let the operations $\perp$, $\|$ denote projection to the horizontal space *of the Levi-Civita connection* ∇ and to the (1.0) part of the vertical space, respectively. Then the canonical form satisfies

$$\theta(v)\big|_u = g(u, v^\perp) \quad \text{and} \quad d\theta(v, w) = g(v^\|, w^\perp) - g(v^\perp, w^\|).$$

Hence if ζ is the canonical complex vector field satisfying $\zeta = \zeta^\perp$ and $\zeta^\perp\big|_u = u$, we have

$$\zeta \diagdown j^*(\theta \wedge d\theta) \equiv 0 \quad \text{on } \hat{X}.$$

As a consequence, the projective image of ζ (in the tangent bundle of $\beta(X)$) is an element of Д. On the other hand, if $\hat{\zeta}$ is horizontal with respect to $\hat{\nabla}$ and satisfies $(\hat{\zeta} - \zeta)^\perp \equiv 0$, we have

$$(\hat{\zeta} - \zeta)^\| = \hat{\zeta}^\| = \hat{\zeta}^\perp \times B(\hat{\zeta}^\perp, \cdot),$$

since $\hat{\nabla}_u u = 0 \Leftrightarrow \nabla_u u = u \times B(u, \cdot)$. However, the trace part of B has no effect on the right-hand side of the above equation. But when projected to X, $\hat{\zeta}$ is a element of $C \otimes \hat{E}$ whose projection to CTM gives the map β; as a consequence, $\hat{\zeta}$ has an element of D as its projective image. Since $\zeta^\| = \zeta^\perp \times \mathrm{II}(\hat{\zeta}^\perp, \cdot)$, it follows that II represents the second fundamental form of X. q.e.d.

This theorem should provide sufficient motivation for the terminology "first and second fundamental form" used in this paper. Indeed, techniques similar to those in LeBrun [7] may be used to show that any given foliated CR manifold arises via the construction used in this appendix provided that the fundamental forms are real and analytic. Note, however, that the forms that arise in this construction *are* always real, although generalizations, in which complex forms arise, are made possible by the consideration of totally real 3-manifolds in the complex half-flat 4-manifolds of Penrose [9].

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