# GAUSS PARAMETRIZATIONS
# AND RIGIDITY ASPECTS OF SUBMANIFOLDS

## MARCOS DAJCZER & DETLEF GROMOLL

The normal spherical image, i.e., the Gauss map, plays a crucial role in the geometry of a euclidean hypersurface. In general, the Gauss map is not invertible. Our starting point here is the observation that whenever the rank (or the relative nullity) is constant, then one has a representation by the inverse of the Gauss map on the normal bundle of its image, which we call *Gauss parametrization*. This has many interesting applications. In particular, it is useful in the study of rigidity problems.

Recall the classical theorem of Beez-Killing: A hypersurface is locally rigid in $\mathbf{R}^{n+1}$ if the rank of the Gauss map is at least 3. A main result in this paper is that any complete minimal hypersurface $M^n$ in $\mathbf{R}^{n+1}$ is rigid as a minimal submanifold of $\mathbf{R}^{n+p}$ for any $p \geq 1$, provided $n \geq 4$, and $M$ is irreducible; cf. Theorem 2.1. This result is global in nature and fails to be true locally. In fact, any simply connected minimal hypersurface of rank 2 has precisely a one-parameter associated family of minimal deformations, already in codimension 1. Some other rigidity results will be discussed in the last section. Our main theorem there implies that (locally irreducible) hypersurfaces with nonzero constant mean curvature are locally rigid; cf. Theorem 3.3. In [4], we will deal with further applications of the Gauss parametrization, for example, a classification of real Kaehler hypersurfaces.

It seems that the idea of Gauss parametrizations has been used systematically only in a special case by Sbrana. In a beautiful paper [19], he studied deformable hypersurfaces about five years before E. Cartan considered the problem.

## 1. Gauss parametrizations

In this section, after reviewing some basic facts, we will discuss a local classification of hypersurfaces with constant relative nullity in spaces of constant curvature.

Let $Q = Q_c^{n+1}$ denote the standard simply connected model space of dimension $n + 1$ with constant curvature $c$. Let $f: M^n \to Q$ be an isometric immersion of a connected riemannian manifold $M^n$ as a hypersurface. For a (local) normal field $\varphi \neq 0$, we have the second fundamental form $A_\varphi$ along $f$, acting on the tangent bundle $TM$ pointwise by selfadjoint transformations. The relative nullity at a point $x \in M$ is the number $\nu(x) = \dim \ker A_{\varphi(x)}$. The minimal relative nullity of $f$ is defined by $0 \leqslant \nu_0 = \min_{x \in M} \nu(x) \leqslant n$. The set $M_0 = \{x \in M | \nu(x) = \nu_0\}$ is nonempty and open in $M$. The nullity distribution $\Delta = \ker A_\varphi$ is smooth and involutive on $M_0$, and each leaf $L$ is totally geodesic in $M$ and $Q$. It is a well-known result that all leaves are complete if $M$ is complete. We refer to [6] as a general reference.

Let us now assume that $f$ has constant nullity, i.e., $M = M_0$, and set $\nu_0 = n - k$. For a "saturated" open connected subset $U$ (meaning each leaf in $U$ is maximal in $M$), we consider the quotient space $V$ of leaves in $U$, and the projection $\pi: U \to V$. Suppose the space $V$ is a $k$-dimensional smooth manifold. (Since the leaves are totally geodesic in $Q$, it always is, but may fail to be Hausdorff.) This condition is satisfied in two important cases:

(1) Locally, of course, if we choose for $U$ the saturation of some cross section.

(2) If all leaves through points in $U$ are complete. Then $\pi: U \to V$ is an "affine" vector (sphere) bundle, which for $c \leqslant 0$ admits global cross sections making it into a "linear" vector bundle.

Taking $U$ as above to be orientable, we have the unit normal field $\varphi$ (unique up to sign). Note that $\varphi$ is parallel along each leaf in $U$.

We first look at the situation $Q = \mathbf{R}^{n+1}$. The Gauss map $\varphi: U \to S^n$ induces an immersion $\bar{\varphi}: V \to S^n \subset \mathbf{R}^{n+1}$ so that $\bar{\varphi} \circ \pi = \varphi$:

$$\begin{array}{ccc} U & \xrightarrow{\;\varphi\;} & S^n \\ {\scriptstyle\pi}\downarrow & \nearrow & \\ V & {\scriptstyle\bar{\varphi}} & \end{array}$$

Let us consider the normal bundle $\Lambda$ along the immersion $\bar{\varphi}$ in $S^n$ whose fibers have dimension $n - k$. The pullback of $\Lambda$ under the Gauss map $\varphi$ is canonically the nullity distribution on $M$ (by parallel transport in $\mathbf{R}^{n+1}$). The fiber of $\Lambda$ at $\varphi(x)$ is $\Delta_x$. Observe that we are using here that $\varphi$ is parallel along the leaves. A point in $\Lambda$ is represented by $(\bar{x}, v)$, where $\bar{x} \in V$, $v \in \Delta_x$, $\pi(x) = \bar{x}$. Here $\Delta_x$ is considered to be a linear subspace of $\mathbf{R}^{n+1}$, by parallel translation as usual. Whenever no confusion is possible, we will identify $f(x) = x$ as well as $\varphi(x) = \bar{\varphi}(\bar{x}) = \bar{x}$.

Any cross section $\xi\colon V \to U$ of the submersion $\pi\colon U \to V$ allows us to extend the restriction $\varphi|\xi(v)$ canonically to a diffeomorphism $\phi_\xi$ from $U$ onto an open neighborhood of the zero section in $\Lambda$. Such sections exist in the important cases (1) and (2) above. Simply parallel translate the leaf through $x$ in $U$ into the fiber of $\Lambda$ at $\varphi(x)$ such that $\xi(\bar{x})$ is mapped into 0. Explicitly we have

$$(1.1) \qquad \phi_\xi(x) = (\bar{x}, x - \xi(\bar{x})),$$

where $\bar{x} = \pi(x)$. The inverse $\psi_\xi = \phi_\xi^{-1}$ is then given by

$$(1.2) \qquad \psi_\xi(\bar{x}, v) = \xi(\bar{x}) + v.$$

Note that $\psi_\xi$ is well defined and smooth on the whole normal bundle over $\varphi \circ \xi(V)$, but may be singular outside the image of $U$ under $\phi_\xi$.

Now fixing a point $x_0 \in U$, there is a natural local cross section $\eta$ through $x_0$, defined as follows: Let $\eta(\bar{x})$ be the unique point on the leaf $L = \pi^{-1}(\bar{x})$ closest to $x_0$. Explicitly we have

$$(1.3) \qquad \eta(\bar{x}) = x_0 + \gamma(\bar{x})\bar{x} + \nabla\gamma(\bar{x}).$$

Here $\gamma(\bar{x}) = \langle x - x_0, \bar{x}\rangle$ is the support function ($x$ is any point on $L$), which is clearly well defined, since the right-hand side is constant along leaves, and $\nabla\gamma$ is the gradient of $\gamma$ on $V$. We will always consider $V$ with the metric induced by $\bar{\varphi}$ from $S^n$. To verify (1.3), observe first that for $y = \eta(\bar{x})$, the relative position vector $y - x_0$ is perpendicular to $\Delta_y$. The normal component of $y - x_0$ is $\gamma(\bar{x})\bar{x}$. It remains to show that $\nabla\gamma(\bar{x})$ is the component of $y - x_0$ tangential to $V$. But for any $x \in U$ and any $u$ in the orthogonal complement $\Delta_x^\perp$ of $\Delta_x$ in the tangent space of $M$ at $x$, we have

$$\langle x - x_0, \varphi_* u\rangle = u\langle x - x_0, \varphi(x)\rangle = u(\gamma \circ \pi) = (\pi_* u)\gamma$$
$$= \langle \nabla\gamma, \pi_* u\rangle = \langle \bar{\varphi}_* \nabla\gamma, \bar{\varphi}_* \pi_* u\rangle = \langle \bar{\varphi}_* \nabla\gamma, \varphi_* u\rangle.$$

This completes the argument.

We should mention at this point that the natural sections $\eta$ just constructed are global as soon as all leaves are complete. Taking $\xi = \eta$, we call the inverse $\psi = \psi_\eta$ of the extended Gauss map $\phi = \phi_\eta$ the *Gauss parametrization* of $M$ about $x_0$, which by (1.2) and (1.3) has the representation

$$(1.4) \qquad \psi(\bar{x}, v) = x_0 + \gamma(\bar{x})\bar{x} + \nabla\gamma(\bar{x}) + v.$$

Usually we will normalize $f$, and assume, after parallel translation in $\mathbf{R}^{n+1}$, that $x_0 = 0$.

We are now in a position to describe a classification of constant rank hypersurfaces in $\mathbf{R}^{n+1}$, at least locally, essentially by inverting the above process.

**Theorem 1.5.** *Let $g$: $V^k \to S^n$ be any isometric immersion and $\gamma$ any function on $V$. On the normal bundle along $g$, consider the map $\psi$: $\Lambda \to \mathbf{R}^{n+1}$,*

$$(1.6) \qquad\qquad \psi(y, w) = \gamma y + \nabla\gamma + w.$$

*Then, on the open subset of regular points, $\psi$ is an immersed hypersurface with constant nullity $n - k$. Conversely, any such hypersurface can be obtained this way, at least locally.*

**Remark 1.7.**   In Theorem 1.5, the extension of $g$, constant along the fibers in $\Lambda$, is the Gauss map of (1.6). In particular, we observe that the Gauss map can be prescribed arbitrarily, and all hypersurfaces with some Gauss map $g$ are parameterized by an arbitrary ("support") function $\gamma$. All this applies to the special classical case $k = n$, i.e., nullity zero, where $\psi$ is the inverse of the Gauss map.

The last theorem is an immediate consequence of the following proposition and (1.4).

**Proposition 1.8.**   *With the notations as in Theorem 1.5:*

(i) *$\psi$ has maximal rank $n$ at $(y, w)$ if and only if the selfadjoint operator*

$$(1.9) \qquad\qquad \gamma(y) \cdot I + H_{\gamma(y)} - A_w = P$$

*on the tangent space of $V$ at $y$ is nonsingular, where $H_\gamma$ is the hessian in $V$, $A_w$ the second fundamental form of $g$ at $y$ relative to $w$.*

(ii) *At such points, $g$ is a unit normal field of $\psi$, the second fundamental form $A = A_g$ in $\mathbf{R}^{n+1}$ has rank $k$, and*

$$(1.10) \qquad\qquad A = -P^{-1} \quad on \ \Delta^\perp,$$

*where $\Delta^\perp$ is the orthogonal complement of the relative nullity distribution $\Delta$.*

*Proof.*   We compute the Jacobian of $\psi$. Clearly $\psi_*$ is the identity on the vertical component of the tangent space of $\Lambda$ at $(y, w)$, which we identify with the fiber $\Lambda_y$. Any transversal tangent vector can be written as $\xi_* b$, where $b$ is a tangent vector of $V$ at $y$, and $\xi$ is a local section of $\Lambda$ through $(y, w)$. We identify $g_* b = b$. Then,

$$\psi_* \xi_* b = (\psi \circ \xi)_* b = \nabla_b \psi \circ \xi = (\gamma I + H_\gamma - A_\xi)_y b + \alpha(b, \nabla\gamma) + \nabla_b^\perp \xi.$$

Here $\nabla$ is the derivative in $\mathbf{R}^{n+1}$, $\nabla^\perp$ the normal connection in $S^n$ and $\alpha$ the normal valued second fundamental form of $V$ in $S^n$. Thus

$$(1.11) \qquad\qquad \psi_* \xi_* b = (P + Q)b,$$

where $Q$ is a linear transformation from the tangent space of $V$ into the normal space. We conclude for the images that $\operatorname{im}\psi_* = \operatorname{im} P \oplus \Lambda_y$, which proves (i).

Since $g$ is perpendicular to $\operatorname{im}\psi_*$ for all $w \in \Lambda_y$, the extension $G$ of $g$, constant along the fibers, is the Gauss map of $\psi$. In particular, $\Lambda_y \subset \ker A$, so rk $A \leqslant k$. For vectors of the form $u = \psi_* \xi_* b$, we have $Au = -G_* u = -G_* \psi_* \xi_* b = (G \circ \psi \circ \xi)_* b = -b$, since $G \circ \psi \circ \xi$ is the identity on $V$. But by (1.11), $u = (P + Q)b$. Therefore, $APb = Au = -b$, so $AP = -I$ on $\Delta^\perp$, which is parallel to the tangent space of $V$ at $y$.

We now turn to a discussion of hypersurfaces in euclidean spheres $Q = S_c^{n+1}$, say $c = 1$.

**Corollary 1.12.** *Let* $g: V^k \to S^{n+1}$ *be any isometric immersion. On the unit normal bundle along $g$, consider the map* $\psi: \Lambda^1 \to S^{n+1}$,

$$(1.13) \qquad \qquad \psi(y, w) = w.$$

*Then*:

(i) *On the open subset of regular points, $\psi$ is an immersed hypersurface with constant nullity $n - k$.*

(ii) *Conversely, any hypersurface of $S^{n+1}$ with constant relative nullity can be obtained this way, at least locally.*

(iii) *$\psi$ has maximal rank $n$ at $(y, w)$ if and only if the second fundamental form of $g$ in direction $w$ is nonsingular. At such a point, the second fundamental form $A = A_g$ of $\psi$ in $S^{n+1}$ has rank $k$, and*

$$(1.14) \qquad \qquad A = A_w^{-1} \quad on \ \Delta^\perp.$$

**Remark 1.15.** The extension of $g$, parallel along the fibers in $\Lambda^1$, is the spherical Gauss map. $\psi$ is also known as the "polar" map (cf. [14]), where parts (i) and (iii) of the corollary appear already. Note that the image $g(V)$ is just the focal set corresponding to zero principal curvatures.

*Proof.* Extend $\psi$ to all of $\Lambda$ by (1.13), thus parameterizing the cone of $\psi$ in $\mathbf{R}^{n+2}$ through 0, which has constant nullity $n - k + 1$, by Theorem 1.5. This immediately implies (i). To verify (iii), apply Proposition 1.8, where $\gamma = 0$. It remains to prove (ii). Take a local cross section $\xi$ and the corresponding parametrization $\psi_\xi$ as in (1.2) of the cone over the hypersurface. Since all leaves contain the origin, $\xi$ is a section in the normal bundle of the Gauss image. Now we have a new parametrization $\psi$ of the cone with $\psi(y, w) = \psi_\xi(y, w - \xi(x)) = w$, which we restrict to $\Lambda^1$.

The case where $Q_c$ is the hyperbolic space $H^{n+1}$ can be dealt with similarly. Since we will only be concerned with the case $c \geqslant 0$ in what follows, we just outline the local classification of hypersurfaces with constant nullity. Let $g: V^k \to L_1^{n+1}$ be an isometric immersion of any riemannian manifold into the Lorentzian unit sphere in flat Lorentzian space $L_0^{n+2}$. Consider the unit normal

bundle $\Lambda_1$ along $g$, i.e., the set of pairs $(y, w)$ where $y \in V$ and $w$ is perpendicular to $V$ at $y$ in $L_0^{n+2}$, of length $-1$. Then $\psi: \Lambda_1 \to H^{n+1} \subset L_0^{n+2}$ with $\psi(y, w) = w$ is the "polar" map, and everything works as in Corollary 1.12.

We conclude this section with a discussion of some examples.

(a) Theorem 1.5 gives a nice description of flat hypersurfaces in $\mathbf{R}^{n+1}$ without totally geodesic points. They have constant relative nullity $n - 1$, and thus can be locally parameterized by a regular curve $c$ in $S^n$ and a function $\gamma$,

$$\psi(c, w) = \gamma c + \gamma' \frac{c'}{\|c'\|} + w.$$

(b) There are many nontrivial examples of complete hypersurfaces in $\mathbf{R}^{n+1}$ with relative nullity $n - 2$, which are ruled by euclidean spaces of dimension $n - 1$. The Gauss image $V^2$ in $S^n$ is a ruled surface with constant nullity 1 that can never be complete. The example in [18, p. 623] is of this type.

(c) Examples of compact hypersurfaces with constant positive relative nullity in euclidean spheres are Cartan's minimal isoparametric hypersurfaces with three distinct principal curvatures, see [15]. They are precisely the polar images of the standard imbeddings of the projective planes, as considered in [7] and [22].

## 2.  Rigidity of minimal submanifolds

The main purpose of this section is to prove

**Theorem 2.1.**   *Let $M^n$ be a complete riemannian manifold, $n \geqslant 4$, which does not have euclidean space $\mathbf{R}^{n-3}$ as a factor. Then, any minimal immersion $f$: $M^n \to \mathbf{R}^{n+1}$ ($S^{n+1}$) is rigid in the following strong sense: Any other minimal immersion $g$: $M^n \to \mathbf{R}^{n+p}$ ($S^{n+p}$) is congruent to $f$ in $\mathbf{R}^{n+p}$ ($S^{n+p}$) through a rigid motion, for any $p \geqslant 1$. (In the spherical case, the assumption on euclidean factors is not needed.)*

Important ingredients in the proof are the following results.

**Lemma 2.2.**   *Let $f$: $M^n \to \mathbf{R}^{n+1}$ ($S^{n+1}$) be an immersion with constant relative nullity $0 \leqslant \nu_0 \leqslant n$. Then $f$ is minimal iff the Gauss parametrization satisfies*

(i) *In* $\mathbf{R}^{n+1}$:        $\operatorname{tr}(\gamma I + H_\gamma - A_w)^{-1} = 0$

(ii) *In* $S^{n+1}$:        $\operatorname{tr} A_w^{-1} = 0$

*for all $(y, w)$. In particular, for $\nu_0 = n - 2$, $f$ is minimal if and only if the Gauss image $V^2$ is a minimal surface in the sphere and*

(2.3)                    $\Delta\gamma + 2\gamma = 0$   *on* $V^2$.

*The latter condition is redundant in $S^{n+1}$.*

*Proof.* This follows from Proposition 1.8 and Corollary 1.12. Observe also that for a selfadjoint linear transformation $A \neq 0$ on $\mathbf{R}^2$, $\text{tr}\, A = 0$ iff $\text{tr}\, A^{-1} = 0$.

For example, the restriction of any linear function on $\mathbf{R}^{n+1}$ to $V^2$ satisfies (2.3). The second part of Lemma 2.2 gives a local classification of minimal hypersurfaces with constant relative nullity $n - 2$.

**Lemma 2.4.** *Let $f\colon M^n \to \mathbf{R}^{n+1}$ be an immersion with constant relative nullity $\nu_0 = n - 2$. If the leaves are complete and the mean curvature $H$ does not change sign (along leaves), then the Gauss image $V^2$ is minimal.*

*Proof.* Using the global Gauss parametrization, let $P_w = \gamma I + H_\gamma - A_w$. Now $H = -\text{tr}\, P_w^{-1} = -\text{tr}\, P_w \cdot \det P_w^{-1}$. Since $\det P_w \neq 0$, and $\text{tr}\, P_w$ is linear in $w$, $H$ must change sign along a leaf unless $\text{tr}\, A_w = 0$.

**Theorem 2.5.** *Let $f\colon M^n \to \mathbf{R}^{n+1}$ $(S^{n+1})$ be an isometric immersion of a complete manifold with constant relative nullity $\nu_0 = n - 2$ everywhere. Suppose the mean curvature $H$ does not change sign (locally). Then $f(M)$ splits as a euclidean product $L^3 \times \mathbf{R}^{n-3}$, where $L^3 \subset \mathbf{R}^4$ and $\nu_0 = 1$. In the spherical case, $f(M)$ is totally geodesic, provided $n \geqslant 4$.*

*Proof.* We use the global Gauss parametrization on the (unit) normal bundle of the leaf space $V^2$, which is regular at all points $(y, w)$. In the spherical case, $A_w$ is invertible and $H(w) = \text{tr}\, A_w^{-1} = -\text{tr}\, A_{-w}^{-1} = -H(-w)$, thus, by assumption, $\text{tr}\, A_w = 0$. But the space of symmetric $2 \times 2$-matrices with trace 0 has dimension 2, and the codimension of the Gauss image is at least 3, so for dimension reasons, $A_w$ must be singular for some unit normal vector $w$ at any point $y \in V$.

In the euclidean case, $V^2$ is minimal by Lemma 2.4. Suppose that for some point $y \in V$ the normal space of $V$ at $y$ contains a 2-dimensional linear subspace $E$ such that $A_w \neq 0$ for $0 \neq w \in E$, i.e. $w \to A_w$ is a linear isomorphism $E \to F$, where $F$ is the space of selfadjoint endomorphisms of the tangent space of $V$ at $y$ with trace 0. The image of $w \to \gamma(y)I + H_{\gamma(y)} - A_w = B - A_w$ is the affine plane of all symmetric matrices with same trace, $\text{tr}\, B$, which always contains singular elements. Therefore we have shown that for all $y \in V$, the kernel of the transformation $w \to A_w$ has dimension $\geqslant n - 1$, i.e. the first normal space has dimension $\leqslant 1$. Then by standard results on submanifolds and using the analyticity of $V$, we conclude that the minimal surface $V$ is either totally geodesic, or is contained in a totally geodesic $S^3 \subset S^n$; cf. [3]. In the first case, the normal bundle $\Lambda$ of $V$ is parallel in $\mathbf{R}^{n+1}$, and in the second case, $\Lambda = \Lambda_1 \oplus \Lambda_{n-3}$ splits, where $\Lambda_1$ is the normal bundle of $V$ in $S^3$, and the orthogonal complement $\Lambda_{n-3}$ is parallel in $\mathbf{R}^{n+1}$. Now the claim follows easily from (1.4).

We should mention that for hypersurfaces in $S^{n+1}$ the statement of the last theorem is trivial for $n \geqslant 5$, since any two leaves would necessarily intersect. The conclusion is in general false for $n = 3$, and it is sharp in the euclidean case, at least if we only assume that the leaves are complete, which is sufficient to prove the theorem. Examples will be discussed after the proof of Theorem 2.1. Note that in view of [9], the euclidean case is only interesting for scalar curvature $s < 0$. It is also clear from the proof of Lemma 2.4 that the assumption on the mean curvature can be weakened to the condition that $H/s$ is bounded from above (below).

*Proof of Theorem* 2.1. If there exists a point in $M$ with relative nullity $\nu > n - 2$, the result was proved in [1], and holds in fact already locally. If $\nu = n$ everywhere, the conclusion is trivial. Thus it remains to consider the case when $\nu = n - 2$ on a (necessarily) open and dense subset $U$, which automatically contains a complete totally geodesic leaf through each of its points. By Theorem 2.5, each connected component of $U$ splits, and therefore, $f$ splits globally, by analyticity, which we had excluded.

We make a few remarks in connection with Theorem 2.1. The rigidity as stated is known to be false for minimal surfaces. But that case can be completely analyzed (cf. [13]), which stimulated some of this work. If $M^n = L^2 \times \mathbf{R}^{n-2}$ splits isometrically, we conclude from the proof of Theorem 2.5 that the immersion $f = f_2 \times I_{n-2}$ splits, where $I_{n-2}$ is the identity, and the classification is reduced to [13]. So in particular, if $f$ is substantial, the codimension $p$ can only be 1 or 4. If $M^n = L^3 \times \mathbf{R}^{n-3}$, it follows again that $f = f_3 \times I_{n-3}$ splits, and the classification is reduced to understanding complete minimal hypersurfaces $M^3 \subset \mathbf{R}^4$, with nullity 1. Their Gauss parametrization has singularities precisely at points where the hessian of $\gamma$ and the second fundamental tensor of the Gauss image commute. We do not know whether or not such irreducible examples exist. It follows, for example, that their Gauss image cannot be any compact immersed minimal surface of genus $\neq 1$, or the Clifford torus. We now describe examples of hypersurfaces $M^3 \subset \mathbf{R}^4$ with complete leaves, which do not split off minimal factors $L^2$. Consider the Clifford torus in $S^3$, parameterized on $V = \mathbf{R}^2$ by

$$g(t, s) = \frac{1}{\sqrt{2}} \left( e^{i\sqrt{2} t}, e^{i\sqrt{2} s} \right).$$

For any function $\gamma$ on $W \subset \mathbf{R}^2$, the Gauss parametrization $\psi$ is nonsingular on the whole fibers of the normal bundle of $g(W)$ iff $(\Delta\gamma + 2\gamma)^2 < 4\gamma_{ts}^2$, which follows easily from (1.9). Taking a $\gamma$ with $\Delta\gamma + 2\gamma = 0$, $\psi$ will be minimal by (2.3), and nonsingular wherever $\gamma_{ts} \neq 0$.

In the spherical case, Theorem 2.1 is false for $n = 3$. Take any minimal surface $V^2 \subset S^4$ with everywhere nonzero normal curvature. This is equivalent to $\det A_w \neq 0$ for all $w$. Then the polar map (1.13) is nonsingular everywhere and provides examples, which are even complete if $V^2$ is complete.

At this point we wish to remark that Lemma 2.2 can be viewed as a generalization of a classical result for minimal surfaces in $\mathbf{R}^3$. The local Gauss parametrization of any hypersurface $M^n \subset \mathbf{R}^{n+1}$ with constant nullity $\nu = 0$, i.e. nonvanishing Gauss-Kronecker curvature, is $\psi(y) = \gamma y + \nabla \gamma$, on the Gauss image $V$, which is an open subset of $S^n$. So $M$ is parameterized by precisely one function $\gamma$, and $M$ is minimal iff $\operatorname{tr}(\gamma I + H_\gamma)^{-1} = 0$. For $n = 2$, this means $\Delta\gamma + 2\gamma = 0$ on $V$, which is equivalent to the result in [2, p. 57], as pointed out to us by R. Schoen. Incidentally, for $n = 3$, $M^3$ has constant scalar curvature iff $\Delta\gamma + 3\gamma = 0$.

We conclude this section with a discussion of the local rigidity problem for a minimal hypersurface $f: M^n \to Q_c^{n+1}$. It suffices to analyze the situation when $M^n$ has constant nullity $n - 2$. Although Lemma 2.2 gives an explicit description in terms of minimal surfaces, it is not clear whether or not they can be deformed (as minimal hypersurfaces). We will show that (as in $Q_c^3$) there is a one-parameter *associated family* $f_\theta: M^n \to Q_c^{n+1}$ of minimal immersions, $\theta \in S^1$, which describes all possible deformations, if $M$ is simply connected.

Choose a global unit normal $N$, and let $A$ be the second fundamental form in direction $N$. In the orthogonal complement $\Delta^\perp$ of the nullity distribution $\Delta$, fix an orientation. Take any function $\theta: M \to S^1$ and consider the tensor field $R_\theta$, which is the identity on $\Delta$ and the rotation through $\theta$ in $\Delta^\perp$. The tensor field $A_\theta = R_\theta A = R_{\theta/2} A R_{-\theta/2}$ is selfadjoint and $\operatorname{tr} A_\theta = 0$. Clearly, $A_\theta$ satisfies the Gauss equation.

**Lemma 2.6.** *$A_\theta$ satisfies the Codazzi equation if and only if $\theta$ is constant.*

*Proof.* This is a straightforward computation.

Now $A_\theta$ determines the minimal immersion $f_\theta$, where $f_0 = f$, and the $f_\theta$ are mutually not congruent (as maps). We have the following extension of Schwarz's classical result on minimal surfaces.

**Theorem 2.7.** *Let $f$ be a minimal isometric immersion of a simply connected riemannian manifold $M^n$ into $Q_c^{n+1}$, with constant relative nullity $n - 2$. Then any other minimal isometric immersion $\bar{f}: M^n \to Q_c^{n+1}$ is congruent to some $f_\theta$ in the associated family of $f$.*

*Proof.* By [1], $\bar{f}$ must have constant nullity $n - 2$ and the same nullity distribution $\Delta$. Let $\bar{A}$ be the second fundamental tensor of $\bar{f}$. Since by the Gauss equation, $\det \bar{A}|\Delta^\perp = \det A|\Delta^\perp$ and $\operatorname{tr} \bar{A} = \operatorname{tr} A = 0$, it follows that $\bar{A} = R_\theta A$ for some function $\theta: M \to S^1$. By Lemma 2.6, $\theta$ must be constant.

Perhaps the simplest nontrivial example of an associated family of minimal hypersurfaces in $\mathbf{R}^4$ are the cones over the associated family of a (simply connected) minimal surface in $S^3$.

The associated family $f_\theta$ of the minimal immersion $f\colon M^n \to \mathbf{R}^{n+1}$ can be extended to a 2-parameter family of minimal immersions $f_{\theta,\varphi}\colon M^n \to \mathbf{R}^{2n+2}$, with the same constant relative nullity $n-2$, by

$$f_{\theta,\varphi} = \cos\varphi f_\theta \oplus \sin\varphi f_{\theta+\pi/2};$$

cf. [13] in the case $n = 2$. Finally we mention that the construction of associated families can be carried out the same way for minimal immersions $M^n \to \mathbf{R}^{n+p}$ with constant relative nullity $n-2$, keeping the normal bundle with its connection fixed.

### 3.  Some other results

We will discuss some further rigidity aspects of hypersurfaces in euclidean space.

**Theorem 3.1.** *Let $f\colon M^n \to \mathbf{R}^{n+1}$ be an isometric immersion, $M$ complete, without flat points. If the mean curvature $H$ does not change sign (locally), then $f$ is rigid on an open subset, or $f(M)$ splits isometrically as $L^3 \times \mathbf{R}^{n-3}$.*

*Proof.*   This follows immediately from Theorem 2.5.

**Theorem 3.2.** *Let $M^n$ be complete with scalar curvature $s$ bounded away from zero, and $f\colon M^n \to \mathbf{R}^{n+1}$ an isometric immersion such that the mean curvature $H$ is bounded from above (below), or just bounded if $M$ is nonorientable. Then $f$ is rigid on an open subset.*

*Proof.*   It suffices to consider the case when the relative nullity $\nu_0 = n - 2$ is constant everywhere. We first show that the Gauss image $V^2$ is totally geodesic, i.e. the nullity distribution is parallel, and $f(M)$ splits as $L^2 \times \mathbf{R}^{n-2}$. Take the Gauss parametrization on $V^2$, and suppose for some $(y, w)$, $\|w\| = 1$, we have $A_w \neq 0$. Let $P_t = \gamma I + H_\gamma - tA_w$. Using an orthonormal basis of principal directions, let $\lambda_1 \neq 0$, $\lambda_2$ be the principal curvatures and $h_{ij}$ the components of the hessian $H_\gamma$. Now

$$s^{-1} = \det P_t = (\gamma + h_{11} - t\lambda_1)(\gamma + h_{22} - t\lambda_2) - h_{12}^2$$

is bounded in $t$. Thus $\lambda_2 = 0$ and $\gamma + h_{22} = 0$. But

$$H = -\operatorname{tr} P_t^{-1} = -\operatorname{tr} P_t \cdot \det P_t^{-1} = (\Delta\gamma + 2\gamma - t\lambda_1) \cdot h_{12}^{-2},$$

which is not bounded. Since $L^2 \subset \mathbf{R}^3$ is complete, with Gauss curvature $K$ bounded away from zero, by Efimov's Theorem [11], $K$ must be positive, and $L^2$ is compact and rigid in $\mathbf{R}^3$ by Minkowski's Theorem [20]. It follows that $f$ is rigid.

**Theorem 3.3.** *Any isometric immersion* $f\colon M^n \to \mathbf{R}^{n+1}$ *with constant mean curvature* $H \neq 0$ *is rigid* ($M$ *connected*), *unless* $f(M) \subset L^2 \times \mathbf{R}^{n-2}$ *or* $f(M) \subset S^1 \times \mathbf{R}^{n-1}$ *splits, where* $L^2 \subset \mathbf{R}^3$ *has constant mean curvature* $H$.

In the latter case there exists in general a one-parameter family of deformations, with the same constant mean curvature; cf. [12]. The above result relates to [5]; it also answers a question raised in [10].

*Proof.* As before, assume $v_0 = n - 2$, and the Gauss image is not totally geodesic. Since $H = -\operatorname{tr} P_t^{-1} = -\operatorname{tr} P_t \cdot \det P_t^{-1}$ is constant in $t$, it follows that

(i) $\det A_w = 0$,

(ii) $H[(\gamma + h_{22})\lambda_1 + (\gamma + h_{11})\lambda_2] = -\operatorname{tr} A_w$,

(iii) $\operatorname{tr} P_0 = -H \cdot \det P_0$.

We conclude $h_{12}^2 = -H^{-2}$, which is a contradiction. The case $v_0 = n - 1$ is easier and similar. To complete the argument, it is enough to use that $f$ is analytic.

We finally present a result that improves a well-known rigidity theorem; cf. [8].

**Theorem 3.4.** *Let* $M^n$ *have constant scalar curvature* $s \neq 0$, $M$ *connected*, $f\colon M^n \to \mathbf{R}^{n+1}$ *an isometric immersion with relative nullity* $n - 2$. *Then* $f(M) \subset L^2 \times \mathbf{R}^{n-2}$ *splits. Furthermore,* $f(M) = S^2 \times \mathbf{R}^{n-2}$ *if* $M$ *is complete.*

*Proof.* Suppose the Gauss image $V^2$ is not totally geodesic. Now $\det P_t = s^{-1}$ is constant, so

(i) $\det A_w = 0$,

(ii) $(\gamma + h_{22})\lambda_1 + (\gamma + h_{11})\lambda_2 = 0$,

(iii) $\det P_0 = s^{-1}$.

It is an easy consequence of (i) that $V^2$ has constant relative nullity 1 in $S^n$, in a neighborhood of $y$. Therefore locally, $V^2$ is a ruled surface in $S^n$, with constant curvature 1. Let $X$, $Y$ be orthonormal fields on $V^2$, where $Y$ is tangent to the ruling. We have $\nabla_Y Y = \nabla_Y X = 0$, and $[X, Y] = \nabla_X Y$ is collinear with $X$, in the connection of $V^2$. We conclude from (ii) that $\gamma + YY\gamma = 0$, so

$$-X\gamma = X\langle \nabla_Y \nabla\gamma, Y\rangle = \langle \nabla_X \nabla_Y \nabla\gamma, Y\rangle + \langle \nabla_Y \nabla\gamma, \nabla_X Y\rangle$$

$$= \langle R(X, Y)\nabla\gamma, Y\rangle + \langle \nabla_Y \nabla_X \nabla\gamma, Y\rangle + 2\langle \nabla_{[X,Y]}\nabla\gamma, Y\rangle$$

$$= -X\gamma + 2\langle \nabla_{[X,Y]}\nabla\gamma, Y\rangle,$$

since $\langle \nabla_X \nabla\gamma, Y\rangle = h_{12}$ is constant by (iii). Therefore, $\langle \nabla_{[X,Y]}\nabla\gamma, Y\rangle = 0$, and it follows that $[X, Y] = 0$. Observe, $h_{12}^2 = -s^{-1} \neq 0$. Since $V^2$ is not flat, this is a contradiction. If $M$ is complete, $f(M) = L^2 \times \mathbf{R}^{n-2}$ splits globally, and the last claim is obvious.

An important application of the last theorem is the result of Takahashi that any homogeneous hypersurface in $\mathbf{R}^{n+1}$ is isometric to $S^m \times \mathbf{R}^{n-m}$, $0 \leqslant m \leqslant n$; see [16], [21]. An elementary argument for the case of relative nullity $\nu_0 \neq n - 2$ was given in [17].

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I.M.P.A., RIO DE JANEIRO, BRAZIL
STATE UNIVERSITY OF NEW YORK, STONY BROOK