

## EXAMPLES OF COMPLETE MANIFOLDS WITH POSITIVE RICCI CURVATURE

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*Dedicated to Wilhelm Klingenberg on his sixtieth birthday*

A long standing question in riemannian geometry has been: Does a complete manifold  $M^n$  with positive Ricci curvature  $\text{Ric}$  also admit a complete metric with nonnegative sectional curvature  $K$ ? It is generally believed that this is not always true, but counterexamples were not known. The answer is actually affirmative for the dimension  $n = 3$  (cf. [6], [16]). Note that  $K > 0$  is sometimes known to be obstructed when a metric with  $\text{Ric} > 0$  exists. Simple examples are  $S^k \times \mathbf{R}^l$  in the noncompact case [5], and  $\mathbf{R}P^k \times \mathbf{R}P^l$  in the nonsimply connected compact case for  $k, l \geq 2$ , as a consequence of Synge's Lemma [4].

Examples of complete manifolds with  $K \geq 0$  remain fairly scarce. One way or another, they can all be obtained using classical spaces and quotients of isometric group actions (cf. [3] for a detailed list of references). There are several additional methods to produce complete metrics with  $\text{Ric} > 0$ . Certain fiber bundles were treated in [14] and [15], and a large class of Brieskorn varieties in [7]. Finally, by Yau's work, Kaehler metrics with  $\text{Ric} \geq 0$  exist on any compact Kaehler manifold with first Chern class  $c_1 \geq 0$  (cf. [17]). Interesting examples arise as complete intersections in  $CP^{n+r}$ , notably hypersurfaces. In particular, the  $K3$ -surface (quartic) in  $CP^3$  admits a Ricci flat metric, but this is a true border line case: Since the  $\hat{A}$ -genus does not vanish, we have  $\text{Ric} \equiv 0$  whenever  $\text{Ric} \geq 0$  (cf. [8]). It follows that  $K \geq 0$  would imply  $K \equiv 0$ , which is impossible. Therefore one can distinguish at least between the conditions  $\text{Ric} \geq 0$  and  $K \geq 0$ , in a weak sense.

In this paper we present new classes of complete manifolds with  $\text{Ric} > 0$ . First of all we construct noncompact examples many of which cannot carry metrics with  $K \geq 0$ . This settles the above question in the noncompact case.

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Although there are no compact counterexamples as yet, we obtain series of closed manifolds with  $\text{Ric} > 0$  and Euler number  $\chi < 0$ . They are either counterexamples, or contradict the global Hopf conjecture, i.e.  $\chi \geq 0$  for  $K \geq 0$ . Spaces of the last type also arise from complete intersections. The lowest dimensional is the cubic in  $CP^4$  with  $c_1 > 0$ ,  $\chi = -6$ . The only known invariant so far to distinguish between  $\text{Ric} > 0$  and  $K \geq 0$  is the homotopy type in the noncompact situation, due to the structure theory in [2].

Our starting point is somewhat reminiscent of the discussion for Brieskorn varieties given in [7]. We will consider the "stable" geometry and topology of certain real algebraic varieties with codimension 2 in euclidean space  $\mathbf{R}^{m+p+q}$ . Let  $f(z)$  be any multihomogeneous polynomial in  $\mathbf{R}^m$  for which the origin is an isolated critical point, and let  $F(z, x, y) = f(z) + |x|^2 - |y|^2$ ,  $x \in \mathbf{R}^p$ ,  $y \in \mathbf{R}^q$ . Intersecting the zero set  $F = 0$  of this polynomial with a suitable ellipsoid in  $\mathbf{R}^{m+p+q}$ , we obtain a compact manifold  $V_0$  with positive Ricci curvature (in the induced metric), as soon as  $p + q$  is large compared to  $|p - q|$ .  $V_0$  bounds the set  $F \leq 0$  in the ellipsoid whose interior is denoted by  $V_-$ . The metric of  $V_-$  can be warped near the boundary to yield a complete metric of positive Ricci curvature, provided  $p - q$  and  $p + q$  are sufficiently large, depending on  $f$ .

The above warping problem is delicate in general (cf. also [9]). In §1 we discuss it to the extent needed for our asymptotic estimates. §2 deals with the class of examples for which the boundary problem can be solved. The geometric estimates are obtained in §3, topological invariants in §4, and finally, in §5, we look at some special examples.

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### 1. The boundary problem

Let  $(V, \partial V)$  be a compact riemannian manifold with boundary  $\partial V$  and metric  $\langle \cdot, \cdot \rangle$ . The distance function from the boundary is denoted by  $t$ . Near the boundary we consider the unit vector field  $T = -\text{grad } t$ . On  $\text{Int}(V)$  we define a complete warped metric  $g_\epsilon$  by

$$(1.1) \quad g_\epsilon(X, Y) = \varphi^2(t) \langle T, X \rangle \langle T, Y \rangle + \langle X^\perp, Y^\perp \rangle,$$

where  $\varphi(t) = 1 + \exp(1/t + 1/(t - \epsilon))$  for  $0 < t < \epsilon$ ,  $\varphi(t) = 1$  for  $t \geq \epsilon$ , and  $X^\perp$  denotes the component of  $X$  orthogonal to  $T$ . This metric is defined as soon as  $\epsilon$  is smaller than the injectivity radius of the normal exponential map of  $\partial V$ . For our estimates,  $\varphi$  could be replaced by any other function satisfying

the following conditions:

$$\begin{aligned}
 (1.2) \quad & \varphi(t) \geq 1, \\
 & \varphi(t) = 1 \quad \text{for } t \geq \varepsilon, \\
 & \varphi'(t) = -T\varphi < 0 \quad \text{for } 0 < t < \varepsilon, \\
 & \int_0^\varepsilon \varphi(t) dt = \infty \quad (\text{for completeness of } g_\varepsilon), \\
 & T\varphi/\varphi^3 \text{ is bounded.}
 \end{aligned}$$

Assuming that the Ricci curvature of  $V$  with respect to  $\langle , \rangle$  is positive, it is an interesting problem to find conditions on  $(V, \partial V)$  and  $\varphi$  under which  $\text{Int}(V)$  has positive Ricci curvature with respect to  $g_\varepsilon$ . As we shall see,  $\partial V$  must have  $\text{Ric}_{\partial V} \geq 0$  with respect to the metric induced from  $\langle , \rangle$ , and the mean curvature of  $\partial V$  with respect to the outside normal must be nonnegative. In [9], Ingram gave certain sufficient conditions in the case when  $\text{Int}(V)$  is an open submanifold of a euclidean sphere. They are complicated and have not been verified as yet in any interesting example. We will use a different approach by studying the asymptotic geometry of a sequence of open submanifolds in suitable ellipsoids of increasing dimension.

We need formulas for the Ricci tensor with respect to  $g_\varepsilon$  in terms of the data from  $\langle , \rangle$ . The sectional curvature, Ricci curvature, etc. of  $V$  with respect to  $\langle , \rangle$  are denoted by  $K, \text{Ric}$ , etc., and the corresponding data with respect to  $g_\varepsilon$  by  $\bar{K}, \bar{\text{Ric}}$ , etc. The second fundamental tensor of the hypersurface  $V_t$  at distance  $t$  from  $V_0 = \partial V$  with respect to the normal  $T$  and the metric  $\langle , \rangle$  is denoted by  $S_t$ , i.e.  $S_t X = \nabla_X T$ . Then it is easy to verify the following formulas:

$$(1.3) \quad \bar{\text{Ric}}(T, T) = \text{Ric}(T, T) + (T\varphi/\varphi)\text{tr } S_t,$$

$$(1.4) \quad \bar{\text{Ric}}(T, X) = \text{Ric}(T, X) \quad \text{for } \langle X, T \rangle = 0,$$

$$\begin{aligned}
 (1.5) \quad \bar{\text{Ric}}(X, X) = & \text{Ric}(X, X) + (T\varphi/\varphi^3 + (1 - 1/\varphi^2)\text{tr } S_t)\langle S_t X, X \rangle \\
 & + (1/\varphi^2 - 1)(K(X, T) + \langle S_t^2 X, X \rangle) \quad \text{for } \langle X, T \rangle = 0.
 \end{aligned}$$

Writing an arbitrary vector  $Z$  as  $Z = \alpha T + \beta X$ , where  $\langle X, T \rangle = 0$ ,  $\bar{\text{Ric}}(Z, Z)$  becomes a quadratic form in  $\alpha$  and  $\beta$  which is positive definite if and only if

$$(1.6) \quad \bar{\text{Ric}}(T, T) > 0,$$

$$(1.7) \quad \bar{\text{Ric}}(X, X) > 0,$$

$$(1.8) \quad \bar{\text{Ric}}(T, T)\bar{\text{Ric}}(X, X) - \bar{\text{Ric}}(X, T)^2 > 0.$$

Since  $\lim_{t \rightarrow 0} T\varphi(t)/\varphi(t) = \infty$ , (1.3) and (1.6) imply  $\text{tr } S_t \geq 0$  and in particular  $\text{tr } S_0 \geq 0$ , so the mean curvature of  $\partial V$  must be nonnegative. (1.5) and (1.7) imply in the limit as  $t \rightarrow 0$ ,

$$0 \leq \text{Ric}(X, X) + \text{tr } S_0 \langle S_0 X, X \rangle - K(X, T) - \langle S_0^2 X, X \rangle = \text{Ric}_{\partial V}(X, X).$$

Therefore the induced Ricci curvature of  $\partial V$  must be nonnegative.

### 2. The class of examples

Let  $F$  be a multihomogeneous polynomial on  $\mathbf{R}^n$ , i.e.  $F$  is the direct sum of homogeneous polynomials  $F_i$  on  $\mathbf{R}^{n_i}$  of degree  $l_i$ ,

$$2 \leq l_1 \leq l_2 \leq \dots \leq l_k, \quad \sum_{i=1}^k n_i = n, \quad F(u) = \sum_{i=1}^k F_i(u_i).$$

Since  $\nabla F_i$  is homogeneous of degree  $l_i - 1$ , the only singularity of  $F$  is at the origin as soon as all singularities of  $F$  are isolated. In this case, the  $F_i$  are singular exactly at the origin. For a given  $F$  as above, with an isolated singularity at the origin, we consider for any  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_i > 0$  and  $r > 0$ , the quadratic form

$$G(u) = \sum_{i=1}^k \alpha_i \|u_i\|^2 - r^2.$$

The gradients  $\nabla F$  and  $\nabla G$  are linearly independent on  $F^{-1}(0) \setminus \{0\}$ . This can be seen as follows: If  $a\nabla F = b\nabla G$ , then  $a\nabla F_i = 2b\alpha_i u_i$ . Since  $F_i$  is homogeneous of degree  $l_i$ , one obtains  $\langle \nabla F_i, u_i \rangle = l_i F_i(u_i)$ , and therefore

$$a \sum_{i=1}^k F_i(u_i) = 2b \sum_{i=1}^k \frac{\alpha_i}{l_i} \|u_i\|^2.$$

On  $F^{-1}(0)$  we have  $\sum F_i(u_i) = F(u) = 0$ , and therefore  $b = 0$ . The equation  $a\nabla F(u) = 0$  then implies  $a = 0$ , since  $\nabla F(u) \neq 0$  for  $u \neq 0$ .

As a consequence,  $V_0 = F^{-1}(0) \cap G^{-1}(0)$  is a smooth hypersurface of the ellipsoid  $G^{-1}(0)$ , and

$$V_- = F^{-1}(-\infty, 0) \cap G^{-1}(0), \quad V_+ = F^{-1}(0, \infty) \cap G^{-1}(0)$$

are open subsets with common boundary  $V_0$  in the ellipsoid.

The diffeomorphism type of  $V_0, V_+, V_-$  is independent of  $\alpha$  and  $r$ , since for different  $\alpha$  and  $r$  the corresponding manifolds are isotopic in  $\mathbf{R}^n$ . To study the topology of these objects one can therefore choose  $\alpha_i = 1$  and  $r$  sufficiently small. The geometry of course depends on  $\alpha$  and  $r$ . For our geometric estimates, we will choose  $r = 1$  and  $\alpha_i = 2/l_i$ . By this choice of  $\alpha_i$ , we have

$\langle \nabla F, \nabla G \rangle = 4F$  which vanishes on  $V_0$ , so the gradients  $\nabla F, \nabla G$  are orthogonal along  $V_0$ . As a consequence, the Gauss equations for  $V_0$  simplify considerably. But this choice of  $\alpha_i$  is not only convenient for computations; in fact, some of the estimates definitely do not work in the sphere.

We now fix a multihomogeneous polynomial  $f = \sum_{i=1}^k f_i$  on  $\mathbf{R}^m$ ,  $\text{degree}(f_i) = l_i$ ,  $2 \leq l_1 \leq l_2 \leq \dots \leq l_k$ . For integers  $p \geq 1, q \geq 1, n = m + p + q$ , we consider the multihomogeneous polynomial

$$(2.1) \quad F(z, x, y) = f(z) + \|x\|^2 - \|y\|^2, \quad x \in \mathbf{R}^p, y \in \mathbf{R}^q,$$

and correspondingly,

$$(2.2) \quad G(z, x, y) = \sum_{i=1}^k \frac{2}{l_i} \|z_i\|^2 + \|x\|^2 + \|y\|^2 - 1.$$

The manifolds  $V_0, V_+, V_-$  now depend on  $f, p, q$ . To emphasize this dependence, we shall write  $V_0(f, p, q), V_+(f, p, q), V_-(f, p, q)$  for convenience of notation. Choosing  $\epsilon > 0$  sufficiently small, we consider the warped metrics  $g_\epsilon$  of (1.1) on  $V_-$ . Since  $V_-$  is open in  $G^{-1}(0)$ , the data  $\nabla, K, \text{Ric}$ , etc., are now data of this ellipsoid, with its standard metric.

### 3. Geometric estimates

Our aim is to adjust  $\epsilon, p, q$  so that the warped metric  $g_\epsilon$  on  $V_-(f, p, q)$  has positive Ricci curvature. According to §1 we have to establish (1.6)–(1.8). From (1.3)–(1.5) it is obvious that we need estimates for  $\|S_i\|, \text{tr } S_i$ , and  $T\varphi/\varphi^3$ . Since an upper bound for  $T\varphi/\varphi^3$  depends on  $\epsilon$ , it is necessary to find a lower bound for the injectivity radius of the normal exponential map of  $V_0(f, p, q)$ , independent of  $p$  and  $q$ .

Let us start with estimates on the ellipsoid  $G^{-1}(0)$  defined by the function  $G$  of (2.2). An elementary calculation gives the extremal values of the sectional curvature,

$$(3.1) \quad K_{\max} = \frac{l_k}{2}, \quad K_{\min} = \frac{4}{l_k l_{k-1}},$$

independent of  $p, q$ . The Ricci curvature therefore satisfies

$$(3.2) \quad (n - 2)K_{\min} \leq \text{Ric}(Z, Z) \leq (n - 2)K_{\max}$$

for any unit tangent vector  $Z$  of  $G^{-1}(0)$ . Since  $\text{Ric}$  is positive definite symmetric, the following estimate holds for any pair of orthonormal tangent vectors  $X, T$ :

$$\mu_{\min}^2 \leq \text{Ric}(X, X)\text{Ric}(T, T) - \text{Ric}(X, T)^2 \leq \mu_{\max}^2,$$

where  $\mu_{\min}$  and  $\mu_{\max}$  are the minimum and maximum eigenvalues of Ric. This inequality and (3.2) imply

$$(3.3) \quad (n-2)^2 K_{\min}^2 \leq \text{Ric}(X, X)\text{Ric}(T, T) - \text{Ric}(X, T)^2 \leq (n-2)^2 K_{\max}^2.$$

Next we obtain estimates for curvature quantities of  $V_0$  in the ellipsoid. The tangent space of  $V_0$  at  $p \in V_0$  is the orthogonal complement of the span of the gradients  $\nabla F, \nabla G$  at  $p$ . Since  $\langle \nabla F_p, \nabla G_p \rangle = 0$ , the second fundamental tensor  $S_0$  of  $V_0$  is given by

$$(3.4) \quad S_0 X = \|\nabla F\|^{-1} (\nabla_X \nabla F)^{\mathcal{F}} = \|\nabla F\|^{-1} (H_F X)^{\mathcal{F}},$$

where  $(\ )^{\mathcal{F}}$  denotes the projection to the tangent space of  $V_0$ , and  $\nabla$  is the derivative of  $\mathbf{R}^n$ . Note that  $\|\nabla F\|$  is bounded, and bounded away from zero on  $V_0(f, p, q)$ , independent of  $p, q$ . The eigenvalues of the hessian  $H_F$  of  $F$  are the eigenvalues of  $H_f$  and the values 2, -2. Therefore, there is a constant  $C_0$  such that

$$(3.5) \quad \|S_0\| < C_0, \quad \text{independent of } p \text{ and } q.$$

The mean curvature  $\text{tr } S_0$  of  $V_0$  is given by

$$(3.6) \quad \begin{aligned} \text{tr } S_0 &= \|\nabla F\|^{-1} \text{tr}(H_F)^{\mathcal{F}} \\ &= \|\nabla F\|^{-1} \left[ \text{tr } H_f + 2(p - q) - \frac{\langle H_F \nabla F, \nabla F \rangle}{\|\nabla F\|^2} - \frac{\langle H_F \nabla G, \nabla G \rangle}{\|\nabla G\|^2} \right], \end{aligned}$$

from which we get estimates

$$(3.7) \quad \begin{aligned} &\|\nabla F\|^{-1} (\text{tr } H_f + 2(p - q) - 2\|H_f\|) \\ &\leq \text{tr } S_0 \leq \|\nabla F\|^{-1} (\text{tr } H_f + 2(p - q) + 2\|H_f\|). \end{aligned}$$

As an immediate consequence we have:

**Lemma 1.** *There is an integer  $s$  such that for  $p, q$  which  $p - q \geq s$ , the mean curvature  $\text{tr } S_0$  of  $V_0(f, p, q)$  is positive. For a fixed  $s_0 \geq s$ ,  $\text{tr } S_0$  is bounded for all  $p, q$  in terms of  $p - q = s_0$ .*

This was observed already in [9].

Next we give an estimate for the Ricci curvature of  $V_0$ . By the Gauss equation for  $V_0$  in the ellipsoid  $G^{-1}(0)$ , the Ricci curvature of  $V_0$  in direction  $X$  is found to be

$$(3.8) \quad \begin{aligned} \text{Ric}_{V_0}(X, X) &= \text{Ric}(X, X) - K(X, \nabla F / \|\nabla F\|) \\ &\quad + \text{tr } S_0 \langle S_0 X, X \rangle - \|S_0 X\|^2. \end{aligned}$$

Using (3.2) and (3.6) we obtain

$$\text{Ric}_{V_0}(X, X) \geq (n - 3)K_{\min} - \|H_F\|/\|\nabla F\|^2(|\text{tr } H_F| + 2|p - q| + 3\|H_F\|).$$

Since  $n = m + p + q$ , this proves:

**Theorem 1.** *For any integer  $s$  there is an integer  $r$  such that for any  $p, q$  satisfying  $p - q = s, p + q \geq r$ , the Ricci curvature of  $V_0(f, p, q)$  is strictly positive.*

This result is also contained in [9].

We finally turn to the estimates for  $\|S_t\|, \text{tr } S_t$ , and the injectivity radius of the normal exponential map of  $V_0$ . For these estimates, a basic differential equation for the second fundamental tensor  $S_t$  is useful. Consider for any  $p \in V_0$  the geodesic  $\gamma(\tau) = \exp(-\tau T_p)$ , where  $T = -\text{grad } t$  as before, and the tensor field  $\tau \rightarrow S_\tau \circ \gamma(\tau)$  with covariant derivative  $S'_\tau$  on the normal bundle along  $\gamma$ . One has

$$(3.9) \quad S'_t = R_T + S_t^2,$$

where  $R_T X = R(X, T)T$ . This can be checked by taking the second derivative of variational Jacobi fields  $X$  along  $\gamma$ , satisfying

$$(3.10) \quad \langle X, \dot{\gamma} \rangle = 0 \quad \text{and} \quad X'_t = -S_t X_t.$$

The following proposition contains the estimates needed.

**Proposition 1.** *Let  $s$  be as in Lemma 1 and  $p - q \geq s, C_0$  as in (3.5). Then, independent of  $p$  and  $q$ ,*

(a) *A lower bound for the injectivity radius of the normal exponential map of  $V_0(f, p, q)$  is given by*

$$\rho = K_{\max}^{-1/2} \tan^{-1}(K_{\max}^{1/2}/C_0),$$

(b)  $\|S_t\|$  *is bounded for*  $0 \leq t \leq \rho/2$ ,

(c)  $\text{tr } S_t > 0$  *for*  $0 \leq t \leq \rho$ , *and there is a constant*  $C_1$  *such that*  $\text{tr } S_t \leq \text{tr } S_0 + (n - 2)C_1 t$  *for*  $0 \leq t \leq \rho/2$ .

*Proof.* (a) the injectivity radius  $\rho_0$  and the focal radius of the normal exponential map coincide. Otherwise, the boundary of the tubular neighborhood of radius  $\rho_0$  about  $V_0$  will intersect itself tangentially somewhere in  $V_-$ , thus giving rise to a geodesic of length  $2\rho_0$ , locally minimizing the distance between points on  $V_0$ . But since the Ricci curvature of the ellipsoid is positive and  $\text{tr } S_0 > 0$ , standard variational techniques [12] lead to a contradiction. This is completely analogous to Klingenberg's basic argument to estimate the injectivity radius from a point (cf. [11]).

A lower bound for the focal radius is now easily obtained by basic comparison: Since  $K \leq K_{\max}$  and  $S_0 \leq C_0$  (meaning  $C_0 \cdot I - S_0$  is a nonnegative operator), the first focal point of  $V_0$  along any normal unit speed geodesic in  $V_-$  cannot come before  $\rho$  (cf. [18]). Part (b) contains another explicit argument.

(b) Consider the solution of the differential equation

$$(3.11) \quad h' = K_{\max} + h^2, \quad h(0) = h_0 = C_0,$$

given by  $h_t = K_{\max}^{1/2}(C_0 + K_{\max}^{1/2} \tan tK_{\max}^{1/2})(K_{\max}^{1/2} - C_0 \tan tK_{\max}^{1/2})^{-1}$ . Note that

$$(3.12) \quad h_0 \pm S_0 > 0.$$

By (3.9) and (3.11),

$$(3.13) \quad (h + S)'_t > 0$$

whenever  $S_t$  is defined. Similarly,

$$(3.14) \quad (h - S)' = (h + S)(h - S) + (K_{\max} - R_T) > 0,$$

certainly if  $h \pm S > 0$ . Now let  $[0, t_0)$  be the largest interval on which  $h, S$  are defined and  $h - S$  is positive. Then by (3.12)–(3.14),

$$(3.15) \quad (h \pm S)_t \geq (h \pm S)_0 > 0$$

and

$$(3.16) \quad \|S_t\| \leq h_t,$$

$0 \leq t < t_0$ . Then  $t_0 = \rho$ , the first singular point of  $h$ : If  $t_0 < \rho$ , we conclude from (3.16) and (3.10) that  $t_0$  is smaller than the focal radius of  $V_0$ . Therefore,  $S_{t_0}$  is defined, and by (3.15),  $(h - S)_{t_0}$  is positive, contradicting the choice of  $t_0$ . Now (b) is an immediate consequence of (3.16). The last argument simplifies only slightly if we assume the estimate in (a).

(c) The equation (3.9) implies  $0 \leq (\text{tr } S_t)' \leq (n - 2)(K_{\max} + \|S_t\|^2)$ . Since  $\text{tr } S_0 > 0$  and  $\|S_t\|$  is bounded by (b), one can choose  $C_1$  as an upper bound for  $K_{\max} + \|S_t\|^2$ .

We are now in a position to prove our main result concerning the estimates for Ricci curvature.

**Theorem 2.** *Let  $f$  be a multihomogeneous polynomial. Fix  $s$  so that the mean curvature  $\text{tr } S_0$  of  $V_0(f, p, q)$  is positive for  $p - q \geq s$ . Then there is a number  $\epsilon > 0$  and an integer  $r$ , such that  $V_-(f, p, q)$  has positive Ricci curvature with respect to the warped metric  $g_\epsilon$  for any  $p, q$  satisfying  $p - q = s$  and  $p + q \geq r$ .*

*Proof.* Let  $\rho$  and  $C_1$  be as in Proposition 1, and  $\delta = \frac{1}{2}K_{\min}^2 \leq \frac{1}{2}K_{\min}$ , compare (3.1). By (b) of Proposition 1 we can choose  $0 < \epsilon < \rho/2$ , so that  $\delta \leq K_{\min}^2 - \epsilon C_1 \|S_t\| K_{\max}$  and  $\delta \leq K_{\min} - \epsilon C_1 \|S_t\|$  for  $0 \leq t \leq \epsilon$ .



Using (c) we obtain for any  $X$  with  $\|X\| = 1, \langle X, T \rangle = 0$ :

$$\left| \left( \frac{T\varphi}{\varphi^3} + \left( 1 - \frac{1}{\varphi^2} \right) \text{tr } S_t \right) \langle S_t X, X \rangle + \left( \frac{1}{\varphi^2} - 1 \right) (K(X, T) + \langle S_t^2 X, X \rangle) \right| \leq (n - 2) \varepsilon C_1 \|S_t\| + A,$$

where  $A = T\varphi/\varphi^3\|S_t\| + \text{tr } S_0\|S_t\| + K_{\max} + \|S_t\|^2$  is uniformly bounded, independent of  $p + q, 0 \leq t \leq \varepsilon$ .

The last estimate, (1.5), and (3.2) yield

$$\overline{\text{Ric}}(X, X) \geq (n - 2)(K_{\min} - \varepsilon C_1 \|S_t\|) - A \geq (n - 2)\delta - A.$$

Similarly, (1.3)–(1.5), (3.2), and (3.3) give us

$$\begin{aligned} \overline{\text{Ric}}(X, X) \overline{\text{Ric}}(T, T) - \overline{\text{Ric}}(X, T)^2 &\geq \overline{\text{Ric}}(X, X) \frac{T\varphi}{\varphi} \text{tr } S_t + (n - 2)^2 (K_{\min}^2 - \varepsilon C_1 \|S_t\| K_{\max}) - (n - 2) K_{\max} A \\ &\geq \overline{\text{Ric}}(X, X) \frac{T\varphi}{\varphi} \text{tr } S_t + (n - 2)^2 \delta - (n - 2) K_{\max} A. \end{aligned}$$

By choosing  $p + q$  and hence  $n = m + p + q$  large enough we see first that  $\overline{\text{Ric}}(X, X)$  and in turn  $\overline{\text{Ric}}(X, X)\overline{\text{Ric}}(T, T) - \overline{\text{Ric}}(X, T)^2$  become positive. For this one should note  $(T\varphi/\varphi)\text{tr } S_t > 0$  by (1.2) and (c) of the proposition.

#### 4. The topology of the examples

In this section we study the topology of the manifolds  $V_0(f, p, q), V_-(f, p, q), V_+(f, p, q)$  introduced in §2. In the special case when  $p = 1$  or  $q = 1$ , some of our results follow from Proposition 4 in [10]. For topological conclusions it is not essential that  $f$  is a multihomogeneous polynomial on  $\mathbf{R}^m, f$  may be any real analytic function with an isolated singularity at the origin,  $f(0) = 0$ . However, the function  $G$  of (2.2) will be replaced by

$$(4.1) \quad G(z, x, y) = \|z\|^2 + \|x\|^2 + \|y\|^2 - \varepsilon^2,$$

with  $\varepsilon$  sufficiently small. For a multihomogeneous  $f$ , the topology of  $V_0, V_-, V_+$  is not changed, as pointed out in §2. For an analytic  $f$ , the topology of  $V_0, V_-, V_+$  is independent of  $\varepsilon$ , as soon as  $\varepsilon$  is small enough.

It will be more convenient here to work with the closures  $\overline{V}_{\pm}$  of  $V_{\pm}$ . The sets  $V_0, \overline{V}_-, \overline{V}_+$  consist of all  $(z, x, y) \in \mathbf{R}^{m+p+q}$  satisfying  $f(z) + \|x\|^2 - \|y\|^2 = 0, \leq 0, \geq 0$  respectively, and  $\|z\|^2 + \|x\|^2 + \|y\|^2 = \varepsilon^2$ . Furthermore, we consider  $W_0, W_-, W_+$  given by all  $z \in \mathbf{R}^m$  with  $f(z) = 0, \leq 0, \geq 0$  respectively, and  $\|z\|^2 = \varepsilon^2$ . We also need  $U_0, U_-, U_+$  given by all  $(z, x) \in \mathbf{R}^{m+p}$  satisfying  $f(z) + \|x\|^2 = 0, \leq 0, \geq 0$  respectively, and  $\|z\|^2 + \|x\|^2 = \varepsilon^2$ .

Clearly, the topology of  $V_0, V_-, V_+$  only depends on  $f, p, q$ . However, it is difficult in general to determine the invariants of an arbitrary  $f$  which will enter the computations. They are reflected in the topology of  $W_0, W_-, W_+$ . We will show that the topology of  $V_0, V_-, V_+$  is determined by the topology of  $W_0, W_-, W_+$ . For some functions  $f$ , the topology of  $W_0, W_-, W_+$  and hence of  $V_0, V_-, V_+$  can be computed.

For technical reasons we introduce the set  $\tilde{W}_-$  given by all  $z \in \mathbf{R}^m$  satisfying  $f(z) + \varepsilon^2 - \|z\|^2 = 0$  and  $\|z\|^2 \leq \varepsilon^2$ , as well as the sets  $C_-, C_+$  consisting of all  $z \in \mathbf{R}^m$  with  $f(z) + \varepsilon^2 - \|z\|^2 \leq 0, \geq 0$  respectively, and  $\|z\|^2 \leq \varepsilon^2$ . They will be needed in the following proposition.

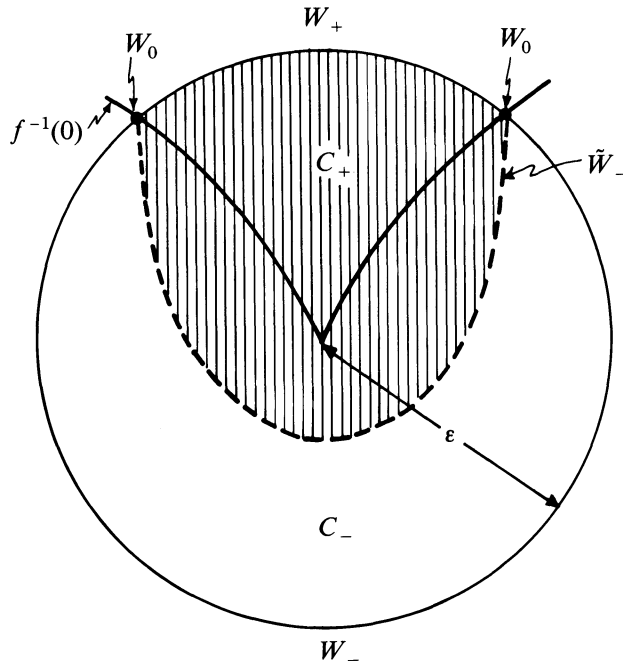


Figure 1 indicates the location of the last sets in  $\mathbf{R}^m$ , when  $m = 2$ .  $f^{-1}(0)$  is the curve having a singularity at the origin.  $W_0$  consists of the two points where  $f^{-1}(0)$  intersects the circle of radius  $\varepsilon$ .  $W_+, W_-$  are arcs of this circle.  $\tilde{W}_-$  is the dashed curve dividing the disc of radius  $\varepsilon$  into the two regions  $C_-, C_+$ .

**Proposition 2.** *For sufficiently small  $\varepsilon > 0$ , there is a continuous function  $\tau: W_- \rightarrow (0, 1]$  so that*

- (a)  $\tau_z \cdot z \in \tilde{W}_-$ ,
- (b)  $\tau_z = 1$  if and only if  $z \in W_0$ ,

(c)  $C_- = \bigcup_{z \in W_-} \{tz \mid \tau_z \leq t \leq 1\}$ ,

(d)  $C_+$  is homeomorphic to the  $m$ -cell  $D_\epsilon^m$  of radius  $\epsilon$ .

In particular,  $W_-$  and  $\tilde{W}_-$  are homeomorphic and  $(W_-, W_0)$ ,  $(\tilde{W}_-, W_0)$  are strong deformation retracts of  $(C_-, W_0)$ .

**Remark.** In fact,  $\tau$  is differentiable and  $\tilde{W}_-$  is a smooth hypersurface with boundary  $W_0$ , diffeomorphic to  $W_-$ .  $C_-$  is a topological manifold, whose boundary  $W_- \cup \tilde{W}_-$  is the double of  $W_-$ .

*Proof.*  $\epsilon$  will be determined so that

$$(4.2) \quad \langle \nabla F, I \rangle < 2\|I\|^2 \text{ on } C_-, \text{ and } \nabla f, I \text{ independent on } W_0,$$

where  $I$  denotes the position vector field. Assuming this estimate we proceed as follows: As a consequence of (4.2), we conclude that  $\tilde{W}_-$  is a differentiable manifold with boundary  $W_0$ . Furthermore, the position field  $I$  is transversal to  $\tilde{W}_-$ , and the function  $g(z) = f(z) + \epsilon^2 - \|z\|^2$  decreases radially in  $C_-$ . For  $z \in W_-$  we define  $\tau(z) = \max\{t \mid 0 < t \leq 1, tz \in \tilde{W}_-\}$ . Note  $g(z) \leq 0$  and  $g(0) = \epsilon^2 > 0$ . By transversality,  $\tau$  is continuous, also  $\{tz \mid \tau_z \leq t \leq 1\} \subset C_-$ ,  $\tau_z z \in \tilde{W}_-$ , and (b) holds. To check (c), take any  $z \in C_-$ , so  $g(z) \leq 0$ . Since  $g$  decreases along the radial ray through  $z$ , the ray stays in  $C_-$  until it meets  $W_-$  at the point  $\epsilon z/\|z\|$ . To prove (d), we give a homeomorphism  $h$  of the  $\epsilon$ -ball  $D_\epsilon^m$  with  $C_+$ . Let  $h(0) = 0$ . For  $z \neq 0$ , let  $h(z) = z$  if  $\epsilon z/\|z\| \in W_+$  and  $h(z) = \tau(z/\|z\|)z$  if  $\epsilon z/\|z\| \in W_-$ .  $h$  is a homeomorphism which carries  $W_-$  to  $\tilde{W}_-$ .

To establish the first statement of (4.2), it suffices to find  $\epsilon$  so that

$$(4.3) \quad \langle \nabla f, I \rangle < 2f(z) + 2\epsilon^2 \quad \text{for } z \in C_-,$$

since  $f(z) + \epsilon^2 \leq \|z\|^2$  on  $C_-$ . Write  $f = f_d + (f - f_d)$ , where  $f_d$  is the lowest order term of the Taylor expansion about the origin, of degree  $d$ . Note that  $d \geq 2$ , since the origin is a singular point of  $f$ . Then (4.3) becomes

$$(4.4) \quad (d - 2)f_d - 2(f - f_d) + \langle \nabla(f - f_d), I \rangle < 2\epsilon^2.$$

The left-hand side of (4.4) is of order  $\geq 3$ , therefore if  $\epsilon > 0$  is sufficiently small, the inequality holds for all  $z \in C_-$ .

The second statement of (4.2) follows immediately from §2, when  $f$  is multihomogeneous. The general algebraic case is contained in Corollary 2.9 of [13]. For analytic functions, one can use the ‘‘Curve Selection Lemma’’ (cf. Theorem 1 of [1], for example).

The following lemma contains all the topological information we need.

- Lemma 2.** (a)  $U_-$  is homotopy equivalent to  $W_-$ ,  
 (b)  $U_0$  is homeomorphic to

$$(W_- \times S^{p-1}) \cup (W_0 \times D^p) = \partial(W_- \times D^p),$$

(c)  $U_+$  is homeomorphic to  $(D_\epsilon^m \times S^{p-1}) \cup (W_+ \times D^p)$ ,  
 where  $D_\epsilon^m$  is the closed  $m$ -cell of radius  $\epsilon$  and  $W_+ \subset \partial D_\epsilon^m = S_\epsilon^{m-1}$ .

- (a')  $\bar{V}_+$  is homotopy equivalent to  $U_+$ ,  
 (b')  $V_0$  is homeomorphic to  $\partial(U_+ \times D^q)$ ,  
 (c')  $\bar{V}_-$  is homeomorphic to  $(D_\epsilon^{m+p} \times S^{q-1}) \cup (U_- \times D^q)$ .

*Proof.* We will prove (a), (b) and (c). The corresponding statements (a'), (b'), (c') can be obtained similarly by observing that the roles of  $U_-$  and  $U_+$  in the proof of Proposition 2 are interchanged if  $f(z) + \|x\|^2$  is replaced by  $f(z) - \|x\|^2$ .

(a)  $U_- = \{(x, z) \mid z \in C_-, \|x\|^2 = \epsilon^2 - \|z\|^2\} = \cup\{z\} \times S_z^{p-1}, z \in C_-$ , where  $S_z^{p-1}$  denotes the sphere of radius  $(\epsilon^2 - \|z\|^2)^{1/2}$  in  $\mathbf{R}^p$ , i.e.  $U_-$  is a singular sphere bundle over  $C_-$ . The spheres  $S_z^{p-1}$  degenerate to points on  $W_-$ . For  $z \in W_-$ , the union of the sets  $\{tz\} \times S_{tz}^{p-1}, \tau(z) \leq t \leq 1$ , is homeomorphic to  $\{z\} \times D_z^p$ , where  $D_z^p$  denotes the  $p$ -cell of radius  $\epsilon(1 - \tau_z)$ . Using (c) of Proposition 2, one can see that  $U_-$  is homeomorphic to the singular disc bundle  $\cup\{z\} \times D_z^p, z \in W_-$ , over  $W_-$ . The cells degenerate to points on  $W_0 = \partial W_-$ . Clearly this bundle is homotopy equivalent to  $W_-$ .

For the proof of (b), note that  $U_0 = \{(z, x) \mid z \in \tilde{W}_-, \|x\|^2 = \epsilon^2 - \|z\|^2\} = \cup\{z\} \times S_z^{p-1}, z \in \tilde{W}_-$ , which is homeomorphic to  $(W_- \times S^{p-1}) \cup (W_0 \times D^p)$ . This can be seen by using a collar neighborhood of  $W_0$  in  $W_-$ . Observe again that the spheres  $S_z^{p-1}$  degenerate to points on  $W_0 = \partial W_-$ .  $D^p$  is the union of the spheres along a normal geodesic in such a collar, starting orthogonal from  $W_0$ .

For (c), we first describe  $U_+$  as a singular sphere bundle over the cell  $C_+$ :  $U_+ = \{(z, x) \mid f(z) + \epsilon^2 - \|z\|^2 \geq 0, \|x\|^2 = \epsilon^2 - \|z\|^2\} = \cup\{z\} \times S_z^{p-1}, z \in C_+$ . The spheres over points of  $W_+ \subset C_+$  degenerate to points. Since  $\tilde{W}_-$  intersects the  $\epsilon$ -sphere transversally in  $W_0$ , there is a nonvanishing vector field in a neighborhood of  $W_+$  in  $C_+$ , which is transversal to  $W_+$  and tangent along  $\tilde{W}_-$ . Using the flow of this vector field, a neighborhood  $N_\delta$  of  $W_+$  in  $C_+$  is seen to be homeomorphic to  $W_+ \times [0, \delta]$ , such that  $N_\delta \cap \tilde{W}_- \approx W_0 \times [0, \delta]$ . The part of the degenerate sphere bundle over  $N_\delta$  is homeomorphic to  $\cup(w, t) \times S_t^{p-1}, (w, t) \in W_+ \times [0, \delta]$ .  $S_t^{p-1}$  is the sphere of radius  $t$ . The homeomorphism carries  $\{z\} \times S_z^{p-1}$  to  $(w, t) \times S_t^{p-1}$ . The closure of the complement of  $N_\delta$  in  $C_+$  is still homeomorphic to  $C_+$ , and  $\cup\{t\} \times S_t^{p-1}, t \in [0, \delta]$ , is homeomor-

phic to  $D^p$ . From this we obtain that  $U_+$  is homeomorphic to  $(C_+ \times S^{p-1}) \cup (W_+ \times D^p)$ . By (d) of Proposition 2,  $C_+$  is homeomorphic to  $D_\epsilon^m$ . The homeomorphism  $D_\epsilon^m \rightarrow C_+$  given there leaves  $W_+$  fixed. This completes the proof of (c).

As an immediate consequence we have

**Corollary 1.** *Let  $k_0 = \min\{p - 1, q - 1\}$ .*

(a)  $U_+$  and  $V_+$  are  $(p - 1)$ -connected,

(b)  $V_-$  is  $(q - 1)$ -connected,

(c)  $V_0$  is  $k_0$ -connected.

*Proof.* For  $0 \leq k < p$ , a map  $S^k \rightarrow U^+ \approx (D_\epsilon^m \times S^{p-1}) \cup (W_+ \times D^p)$  is homotopic to a map  $S^k \rightarrow D_\epsilon^m \times S^{p-1}$ , since it can be approximated by a map with an image not intersecting  $W_+ \times \{0\}$ , for dimension reasons. But then it is homotopic to a constant, since it can be first deformed into  $\{w\} \times S^{p-1}$ , where  $w \in W_+$ , and then to a point in  $\{w\} \times D^p$ . The argument for  $V_-$  is analogous.

For  $0 \leq k < k_0$ , a map  $S^k \rightarrow V_0 \approx (U_+ \times S^{q-1}) \cup (U_0 \times D^q)$  is homotopic to a map  $\psi: S^k \rightarrow U_+ \times S^{q-1}$ , as above. The map  $\pi \circ \psi$ , where  $\pi: U_+ \times S^{q-1} \rightarrow U_+$  denotes the projection, is homotopic to a constant mapping of  $S^k$  to some point  $u_0 \in U_0$ , since  $U_+$  is  $(p - 1)$ -connected. Hence  $\psi$  is homotopic to a map  $S^k \rightarrow \{u_0\} \times S^{q-1}$ , which in turn is homotopic to a constant in  $\{u_0\} \times D^q$ .

Let  $A$  denote any of the three  $W, U, \bar{V}$ .  $A_0, A_+, A_-$  are submanifolds of a sphere  $S^\nu$ ,  $A_+ \cup A_- = S^\nu$ ,  $A_+ \cap A_- = A_0$ , where  $\nu = m - 1$  for  $A = W$ ,  $\nu = m + p - 1$  for  $A = U$  and  $\nu = m + p + q - 1$  for  $A = \bar{V}$ . In the following  $H$  means homology,  $\tilde{H}$  reduced homology, with coefficients in a field of characteristic zero. Now we have

**Lemma 3.**  $\tilde{H}_k(A_+) \simeq \tilde{H}_{\nu-k-1}(A_-), H_k(A_0) \simeq H_k(A_-) \oplus H_{\nu-k-1}(A_-)$ .

*Proof.* By duality, for any closed  $A \subset S^\nu$ ,  $\tilde{H}_k(A) \simeq \tilde{H}_{\nu-k-1}(S^\nu - A)$ . Hence

$$\tilde{H}_k(A_+) \simeq \tilde{H}_{\nu-k-1}(S^\nu - A_+) \simeq \tilde{H}_{\nu-k-1}(A_-).$$

The last isomorphism holds, since  $A_0 = \partial A_+ = \partial A_-$  has a collar neighborhood in  $S^\nu$ , and  $S^\nu = A_+ \cup A_-$ . Similarly

$$\begin{aligned} \tilde{H}_k(A_0) &\simeq \tilde{H}_{\nu-k-1}(S^\nu - A_0) \simeq \tilde{H}_{\nu-k-1}(A_+) \oplus H_{\nu-k-1}(A_-) \\ &\simeq \tilde{H}_k(A_-) \oplus H_{\nu-k-1}(A_-), \end{aligned}$$

and thus  $H_k(A_0) \simeq H_k(A_-) \oplus H_{\nu-k-1}(A_-)$ .

**Corollary 2.** *The Euler characteristics satisfy*

$$\begin{aligned}\chi(A_+) - 1 &= (-1)^{\nu-1}(\chi(A_-) - 1), \\ \chi(A_0) &= (1 + (-1)^{\nu-1})\chi(A_-), \\ \chi(A_+) &= \chi(A_-), \quad \chi(A_0) = 2\chi(A_-) \quad \text{for } \nu \text{ odd.}\end{aligned}$$

Lemma 3 and (a), (a') of Lemma 2 show that the homology of all the nine spaces  $W_0, W_+, W_-, U_0, U_+, U_-, V_0, V_+, V_-$  is determined by the homology of any one of these spaces. Since we are mainly interested in  $V_0$  and  $V_-$ , we only note:

**Theorem 3.**

$$\begin{aligned}\tilde{H}_k(V_-) &\simeq \tilde{H}_{k-q}(W_-), \\ H_k(V_0) &\simeq H_k(V_-) \oplus H_{m+p+q-k-2}(V_-), \\ \chi(V_-) &= (-1)^q(\chi(W_-) - 1) + 1, \\ \chi(V_0) &= (1 + (-1)^{m+p+q})\chi(V_-).\end{aligned}$$

*If  $m$  is even and  $p, q$  are odd, then*

$$\chi(V_0) = 4 - 2\chi(W_-) = 4 - \chi(W_0).$$

*Proof.* The proof is obvious from (a), (a') in Lemma 2, Lemma 3, and its Corollary 2.

## 5. The special examples

From the geometric point of view, interesting examples arise when  $V_0$  has negative Euler characteristic, and  $V_-$  is not a vector bundle over a closed manifold. According to Theorem 3,  $\chi(V_0)$  is negative as soon as  $m$  is even,  $p, q$  are odd, and  $\chi(W_0) > 4$ . The simplest examples of this type occur when  $f$  is a function on  $\mathbf{C} \simeq \mathbf{R}^2$  so that  $W_0$  consists of more than 4 points on the unit circle.

For an integer  $l \geq 2$ , we consider the function  $f: \mathbf{C} \rightarrow \mathbf{R}$ ,  $f(z) = \operatorname{Re}(z^l)$ . Then  $W_0$  consists of  $2l$  points on the circle. Both  $W_-$  and  $W_+$  are unions of  $l$  disjoint arcs on the circle. In this context we also write  $V(l, p, q)$  instead of  $V(f, p, q)$ .

**Theorem 4.** (a)  $\chi(V_0) = 4 - 2l$  for  $p, q$  odd.

(b) For  $q \geq 2$ ,  $V_-(l, p, q)$  is a simply connected manifold, which is not of the homotopy type of any closed manifold, if  $l \geq 3$ . In particular,  $V_-$  does not admit any complete metric of nonnegative sectional curvature.

(c) For any integer  $l$ , there is an integer  $s_0$  such that for any fixed  $s \geq s_0$ , there exist  $\epsilon > 0$  and an integer  $r$  so that whenever  $p - q = s$  and  $p + q \geq r$ , the following holds:

- (i)  $V_0(l, p, q)$  has positive Ricci curvature with respect to the natural metric of §2;
- (ii)  $V_-(l, p, q)$  has positive Ricci curvature with respect to the warped metric  $g_\epsilon$  introduced in §1.

*Proof.* (a) is obvious from Theorem 3.

(b)  $V_-$  is simply connected for  $q \geq 2$  by Corollary 1. The Betti numbers of  $V_-$  can be computed as follows:  $b_0(W_-) = l$ ,  $b_k(W_-) = 0$  for  $k \geq 1$ , since  $W_-$  consists of  $l$  arcs on the circle. From Theorem 3 we obtain  $b_0(V_-) = 1$ ,  $b_q(V_-) = l - 1$ , and  $b_k(V_-) = 0$  for  $k \neq 0, q$ . Since  $V_-$  is simply connected, any closed manifold of the same homotopy type must be orientable and hence satisfy Poincaré duality. This is excluded by the Betti numbers, as soon as  $l \geq 3$ .

(c) is the statement of Theorem 2 in the case of the special examples.

**Remarks.** (i) It is a curious fact that some  $V_0(l, p, q)$  contain an exotic Brieskorn sphere of codimension  $p - q + 1$  with positive Ricci curvature, given by the equations

$$z'_0 + z_1^2 + \dots + z_q^2 = 0, \quad \frac{2}{l}|z_0|^2 + |z_1|^2 + \dots + |z_q|^2 = 1,$$

where  $z_0, \dots, z_q$  are complex variables (cf. [7]). In this context  $z_0$  corresponds to the variable  $z$  of  $f(z)$ , and  $z_k = x_k + iy_k$ , where  $x_k, y_k$  are the variables in our equations,  $1 \leq k \leq q$ .

(ii) Lemma 2 can be used to show that  $V_0$  is homeomorphic to a manifold obtained from  $S^{p+1} \times S^{q-1}$  by surgery. Take  $l$  disjoint  $(p + 1)$ -cells  $D^{p+1}$  in  $S^{p+1}$ , remove  $\text{int}(D^{p+1} \times S^{q-1})$  from  $S^{p+1} \times S^{q-1}$ , and attach  $l$  disjoint copies of  $S^p \times D^q$  along the common boundary:

$$V_0 \approx \{S^{p+1} \times S^{q-1} - l(D^{p+1} \times S^{q-1})\} \cup l(S^p \times D^q).$$

This can be seen as follows. According to Lemma 2,  $V_0 = \partial\{[(D^m \times S^{p-1}) \cup (W_+ \times D^p)] \times D^q\}$ . Since  $W_+ = S^{m-1} \cdot \text{int}(W_-)$ , we have

$$\begin{aligned} V_0 &= \partial\{[D^m \times S^{p-1} \cup S^{m-1} \times D^q - \text{int}(W_- \times D^p)] \times D^q\} \\ &= \{D^m \times S^{p-1} \cup S^{m-1} \times D^p - \text{int}(W_- \times D^p)\} \\ &\quad \times S^{q-1} \cup (\partial(W_- \times D^p)) \times D^q. \end{aligned}$$

Now  $D^m \times S^{p-1} \cup S^{m-1} \times D^p = \partial(D^m \times D^p) \approx S^{m+p+1}$ .

In our examples,  $m = 2$  and  $W_-$  consists of  $l$  disjoint 1-cells, so  $W_- \times D^p$  is a union of  $l$  disjoint  $(p + 1)$ -cells.

(iii) The number of diffeomorphism types of compact manifolds  $V_0(f, p, q)$  with positive Ricci curvature is increasing rapidly with the dimension  $n$ , although still finite for any fixed  $n$ . The possible homotopy types seem quite general, in a stable sense (cf. §3.) Similar considerations apply to the complete examples  $V_-(f, p, q)$ . They are of finite type by construction. We finally point out that the  $V_0$  arise metrically as submanifolds of euclidean spaces, with the smallest interesting codimension 2. A positively Ricci curved hypersurface in  $\mathbf{R}^n$  has necessarily positive sectional curvature, and is therefore the boundary of a strictly convex body. Furthermore, it is easy to see that  $V_-$  with the metric  $g_\varepsilon$  arises isometrically with optimal codimension 2 in  $\mathbf{R}^{n+1}$  as the graph of a function on  $V_-$  in the ellipsoid.

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