J. DIFFERENTIAL GEOMETRY
21 (1985) 63–72

ASYMPTOTIC BEHAVIOR OF CONVEX SETS IN THE HYPERBOLIC PLANE

E. GALLEGO & A. REVENTÓS

1. Introduction

In this note we study the following problem posed by L. A. Santaló and I. Yañez in [5]:

"*Let $C(t)$ be a family of bounded closed convex sets in the hyperbolic plane, depending on the parameter t $(0 \leqslant t)$ and such that $C(t_1) \subset C(t_2)$ for $t_1 < t_2$. Assume that for any point P of the plane there is a value t_P of t such that, for all $t \geqslant t_P$, we have $P \in C(t)$. We then say that $C(t)$ expands over the whole plane as $t \to \infty$. If $F(t)$ and $L(t)$ denote respectively the area and length of $C(t)$, prove that*:

$$\lim_{t\to\infty} \frac{L(t)}{F(t)} = (-K)^{1/2},$$

where $K < 0$ is the curvature of the hyperbolic plane".

The quotient area length appears in a natural way in problems of classical geometric probability. For instance, given a compact convex set in the Euclidean plane we can consider the length of the intersection of a random straight line (in the sense of the integral geometry) with this convex set. In this way we obtain a random variable σ whose expected value $E(\sigma)$ is

$$E(\sigma) = \pi F/L.$$

If we expand now this convex over the whole Euclidean plane we have (cf. [6])

$$\lim_{t\to\infty} \frac{L(t)}{F(t)} = 0$$

and so the expected value of σ tends to infinity.

L. A. Santaló and I. Yañez remark in [5] that the situation in the hyperbolic plane is quite different. In fact they prove, using the hyperbolic isoperimetric inequality and the Gauss-Bonnet formula, that for a family of sets convex

Received August 7, 1984 and, in revised form, February 4, 1985.

respect to horocycles (h-convex) expanding over the whole plane we have

$$\lim_{t \to \infty} \frac{L(t)}{F(t)} = 1/k,$$

where $K = -1/k^2$.

In this note we prove that this is not true for arbitrary convex sets. In fact we construct, for each $\lambda \in [1/k, \infty)$ a sequence of polygonal convex sets K_n expanding over the whole hyperbolic plane with

$$\lim_{n \to \infty} \frac{L(n)}{F(n)} = \lambda,$$

The convex sets in the above sequence can be modified to obtain counterexamples with smooth boundary. For this we use Bonnesen's approximation theorem (cf. [1]) and the continuity, with respect to the Hausdorff metric, of the hyperbolic length and area functions. We end this paper giving sufficient conditions for the conjecture to be true.

We wish to thank Professor L. A. Santaló for many helpful comments during the development of this work.

2. Preliminaries

We shall consider the Beltrami's model for the hyperbolic geometry, i.e., the manifold

$$D = \{(x, y) \in \mathbf{R}^2 \colon x^2 + y^2 < 1\},$$

with the Riemannian metric

$$ds^2 = \frac{k^2}{1 - r^2}\left\{\frac{1}{1 - r^2} dr^2 + r^2 d\theta^2\right\},$$

where r, θ are the ordinary polar coordinates. (See for instance [2].)

This Riemannian manifold is complete, simply connected, and of constant curvature $K = -1/k^2$.

The advantage of this model is that the geodesics are Euclidean straight lines, and so we have a convenient characterization of the convex hyperbolic sets. They are in fact the Euclidean convex sets contained in D.

In this model the length between two points A, B is given by

$$d(A, B) = \frac{k}{2} \log(A, B, M, N),$$

where M, N are the extremities of the chord AB, and (A, B, M, N) is the anharmonic ratio.

The angle between two lines a, b is given by

$$\sphericalangle(a, b) = \frac{1}{2i} \log(a, b, m, n),$$

where m, n are the imaginary tangents to the circle from the point of intersection of a and b, and (a, b, m, n) is the anharmonic ratio of this four lines.

We shall assume throughout that the convex sets are bounded, closed and with nonempty interior.

Consider now a convex set C in the Euclidean plane E^2. We say that the line

$$H = \{x \in \mathbf{R}^2 \colon \langle x, a\rangle = b\},$$

with $a \in \mathbf{R}^2$ and $b \in \mathbf{R}$ is a support line of C if

(i) $C \cap H \neq \varnothing$,

(ii) $C \subset \{x \in \mathbf{R}^2 \colon \langle x, a\rangle \geqslant b\}$ or $C \subset \{x \in \mathbf{R}^2 \colon \langle x, a\rangle \leqslant b\}$.

The function $p(x) = \sup_{s \in C}\langle s, x\rangle$ for each $x \in S^1$ is called the support function of C. As it is the restriction to S^1 of a convex function it is continuous (see [3]).

The parallel convex body C_δ is defined to be

$$C_\delta = \bigcup_{a \in C} B(a, \delta),$$

where $B(a, \delta)$ is the Euclidean closed ball with center a and radius δ.

From this we define the following distance (see [3])

$$d(A, B) = \inf\{\delta \colon A \subset B_\delta \text{ and } B \subset A_\delta\}.$$

Let $\mathscr{E}$ denote the set of all compact convex sets with nonempty interior. Then $(\mathscr{E}, d)$ is a metric space, and this enables us to consider the convergence of sequences of convex sets. This metric is called Hausdorff metric. The restriction to $\mathscr{E}$ of the convex sets contained in $H(K)$ (Hyperbolic space of curvature K) will be denoted by $\mathscr{E}_K$.

We shall use the following theorem due to Bonnesen and Fenchel [1].

Theorem. *Let $C \in \mathscr{E}$, and assume the origin of coordinates belongs to C. Then there exists a sequence C_n in $\mathscr{E}$ such that the following hold*:

(i) $\lim_{n\to\infty} C_n = C$ *with* $C_n \subset C$.

(ii) $C_n = \{(x, y) \in E^2,\ F_n(x, y) \leqslant 1\}$ *where F_n is a real analytic function.*

(iii) *For each point in the boundary of C_n there is a unique tangent line.*

(iv) *The support function of C_n is real analytic.*

3. A counterexample

As the conjecture is true for sequences of h-convex sets, to obtain a counterexample we must consider convex sets which are not h-convex. For instance, regular polygons centered at the origin of coordinates of the Beltrami's model.

Let $\{K_n\}$ be a sequence of centered regular polygons. We can triangulate each K_n in such a way that each triangle has one vertex in the origin and the opposite side is a side of the polygon.

Then, the length and area of K_n are given by $L_n = m \cdot l_n$ and $F_n = m \cdot f_n$ where m is the number of sides of K_n, l_n is the length of each one of this sides, and f_n is the area of each triangle in the above triangulation. Thus, we have

$$\lim_{n\to\infty} (L_n/F_n) = \lim_{n\to\infty} (l_n/f_n).$$

To construct a counterexample we take concretely the sequence $\{K_n\}$ of centered regular polygons in such a way that K_n has $3 \cdot 2^{n-1}$ sides and is inscribed in the circle of Euclidean radius $r_n = 1 - a_n$, being $\{a_n\}$ a sequence of real numbers converging to zero.

As a rotation is an hyperbolic isometry, each triangle in the above triangulation is isosceles with hyperbolic angle at the origin $\alpha_n = 2\pi/3 \cdot 2^{n-1}$.

Then we have (cf. Figure 1)

$$l_n = \frac{k}{2}\log(ABMN) = \frac{k}{2}\log\frac{AM}{BM}\cdot\frac{BN}{AN}$$

$$= k\log\frac{a+b}{a-b} = k\log\frac{a^2+b^2+2ab}{a^2-b^2},$$

where

$$a = MQ = \left(1 - r_n^2\cos^2(\alpha_n/2)\right)^{1/2}, \qquad b = AQ = r_n\sin(\alpha_n/2).$$

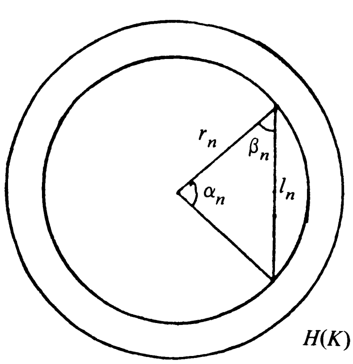

FIGURE 1

That is

$$(1)\quad l_n = k\log\left[1 + \frac{r^n}{a_n(1+r_n)}\left(r_n(1-\cos\alpha_n) + 2\sin\frac{\alpha_n}{2}\sqrt{1-r_n^2\cos^2\frac{\alpha_n}{2}}\right)\right].$$

Now we compute the angles β_n (cf. Figure 1):

$$\beta_n = \frac{1}{2i}\log(m_1, m_2, m_3, m_4),$$

where m_1, m_2 are the angular coefficients of the imaginary tangents to the circle of radius one from B. Computing this anharmonic ratio we obtain

$$\beta_n = \frac{1}{2}\arctan\left(\frac{2\tan(\alpha_n/2)\,\omega_n}{\tan^2(\alpha_n/2)\,\omega_n^2 - 1}\right),$$

where $\omega_n = (1 - r_n^2)^{-1/2}$ and $0 \leqslant \beta_n < \pi/2$.

Therefore the area of each triangle is

$$(2)\quad f_n = k^2(\pi - \alpha_n - 2\beta_n) = k^2\left(\pi - \alpha_n - \arctan\left(\frac{2\tan(\alpha_n/2)\,\omega_n}{\tan^2(\alpha_n/2)\,\omega_n^2 - 1}\right)\right).$$

In the particular case where $a_n = \sigma \cdot 2^{-2n}$ with $\sigma \in (0,\infty)$ we obtain from (1) and (2):

$$\lim_{n\to\infty} l_n = k\cdot\log\left[1 + (2\pi/9\sigma)\left(2\pi + (18\sigma + 4\pi^2)^{1/2}\right)\right],$$

$$\lim_{n\to\infty} f_n = k^2\left(\pi - \arctan\left[6\pi(2\sigma)^{1/2}/(2\pi^2 - 9\sigma)\right]\right).$$

Thus we have

$$\lim_{n\to\infty}\frac{L_n}{F_n} = \frac{\log\left[1 + (2\pi/9\sigma)\left(2\pi + (18\sigma + 4\pi^2)^{1/2}\right)\right]}{k\left(\pi - \arctan\left[6\pi(2\sigma)^{1/2}/(2\pi^2 - 9\sigma)\right]\right)}.$$

The two functions in this ratio are continuous, and the denominator vanishes only when σ goes to infinity.

If we study this quotient when $\sigma \to \infty$ and $\sigma \to 0$, we see that this value goes to $1/k$ and infinity respectively.

Thus we have shown that depending on the choice of σ,

$$\lim_{n\to\infty}\frac{L_n}{F_n} = \lambda$$

for all values of λ between $1/k$ and infinity.

Remark. If we want to obtain $\lim_{n\to\infty}(L_n/F_n)=\infty$, we need only to take $a_n = a^{-2n}$ with $a > 2$.

4. Continuity of length and area functions of convex sets in the hyperbolic plane

Let $C \in \mathscr{E}$. We define the length of the boundary of C by

$$L(C) = \sup_{P\subset C} L(P),$$

where $L(P)$ is the hyperbolic length of a convex polygon P contained in C. Note that if C is a piecewise differentiable curve (with finite number of points without continuous derivative) this definition coincides with the standard one

$$L(C) = \int_{\partial C} \|\dot{\gamma}\|,$$

where γ is a parametrization of ∂C.

That the above supremum exists is an easy consequence of the Cauchy-Crofton formula

$$\int_{G\cap S=\varnothing} dG = L(S),$$

where dG is the density for geodesics, and the integral is extended over the geodesics which cut a piecewise differentiable convex curve S (cf. [6]). In fact, if R is a circle such that $C \subset R$, we have

$$L(P) = \int_{G\cap P=\varnothing} dG \leqslant \int_{G\cap =\varnothing} dG = L(R),$$

and so the supremum exists.

We define area in a similar way

$$F(C) = \sup_{P\subset C} F(P),$$

where $F(P)$ is the hyperbolic area of a convex polygon P contained in C. Note that for all convex set C

$$F(C) = \int_C dv,$$

where dv is the volume element.

When the convex set C is defined by a $\mathscr{C}^2$ support function p we have

$$L(C) = k\int_0^{2\pi} \frac{(p+p'')(1-p^2)^{1/2}}{1-p^2-p'^2}\, d\theta,$$

$$F(C) = k^2\int_0^{2\pi} \frac{d\theta}{(1-p^2-p'^2)^{1/2}} - 2\pi k^2.$$

Length and area are real functions defined on $\mathscr{E}_k$, and we want to prove that they are continuous in the following sense. If $\{S_i\}$ is a sequence which converges to a convex set S, then

$$\lim_i L(S_i) = L(S), \qquad \lim_i F(S_i) = F(S).$$

To prove that the real functions length and area defined on $\mathscr{E}_k$ are continuous we shall use the following lemmas [4].

Lemma 1. *If $\{S_i\}$ is a sequence in $\mathscr{E}$ convergent to a compact set S, then S is convex.*

Lemma 2. *Let C be a convex set in $H(K)$ with support function p. Then for each pair of real numbers ε and δ such that*

$$0 < \varepsilon < 1 - \max p, \qquad 0 < \delta < 1/\max p,$$

there exist convex polygons P and Q such that

$$C \subset P \subset C_\varepsilon \subset H(K), \qquad Q \subset C \subset \delta Q \subset H(K),$$

where δQ means the δ-homothetic polygon of Q, and $\partial Q \cap \partial C = \partial\delta Q \cap \partial C = \varnothing$.

Now we shall prove the following:

Proposition 1. *If P is a convex polygon and δP is δ-homothetic, then*

$$\text{(i)} \qquad F(\delta P) \leqslant \left(\frac{\delta^2 - M^2}{1 - M^2}\right)^{1/2} \frac{1}{\delta}\left(F(P) + 2\pi k^2\right) - 2\pi k^2,$$

$$\text{(ii)} \qquad L(\delta P) \leqslant \frac{\delta^2 - M^2}{1 - M^2}\frac{1}{\delta}L(P),$$

where $M = \max\|\delta x_i\| < 1$, $P = \operatorname{conv}(x_1, x_2, \cdots, x_k)$ the convex hull of the set of points $(x_1, x_2, \cdots, x_k)$.

Proof. P is the union of the triangles V_i of vertices Ox_ix_{i+1}, $i = 1, \cdots, k-1$ and V_k of vertices Ox_kx_1, and δP is the union of the triangles V_i' of vertices $O\delta x_i\delta x_{i+1}$, $i = 1, \cdots, k-1$ and V_k' of vertices $O\delta x_k\delta x_1$.

An easy computation shows that

$$F(V_i) = k^2\int_{\theta_1}^{\theta_2} (1 - r^2)^{-1/2}\, d\theta - k^2\Delta\theta,$$

$$F(V_i') = k^2\int_{\theta_1}^{\theta_2} \left(1 - (\delta r)^2\right)^{-1/2}\, d\theta - k^2\Delta\theta,$$

where $\Delta\theta$ is the Euclidean angle between Ox_i and Ox_{i+1}, and $r = r(\theta)$ the polar parametrization of x_ix_{i+1}.

As $f(x) = ((1 - x^2)/(1 - \delta^2 x^2))^{1/2}$ for $x \in (0, M/\delta)$ is an increasing function, it attains his maximum at $x = M/\delta$ and then

$$f(M/\delta) = \left(\frac{\delta^2 - M^2}{1 - M^2}\right)^{1/2} > 1.$$

Thus,

$$F(V_i') \leqslant f(M/\delta) k \int_0^{2\pi} (1 - r^2)^{-1/2} \, d\theta - k^2 \Delta\theta$$

and hence (i) follows.

We also have

$$L(V_i) = k \int_{\theta_1}^{\theta_2} (\dot{r}^2 - r^4 + r^2)^{1/2} / (1 - r^2) \, d\theta,$$

$$L(V_i') = k \int_{\theta_1}^{\theta_2} (\delta^2 \dot{r}^2 - \delta^4 r^4 + \delta^2 r^2)^{1/2} / (1 - \delta^2 r^2) \, d\theta.$$

Where $L(V)$ and $L(V')$ denotes the length of a side of P and δP respectively.

As before

$$\max(1 - r^2)/(1 - \delta^2 r^2) = (1/\delta^2)(\delta^2 - M^2)/(1 - M^2)$$

and then

$$\frac{(\dot{r}^2 - \delta^2 r^4 + r^2)^{1/2}}{(1 - \delta^2 r^2)} \leqslant (1/\delta) \frac{\delta^2 - M^2}{1 - M^2} \frac{(\dot{r}^2 + r^2 - r^4)^{1/2}}{1 - r^2}$$

and the second inequality follows.

Now we have

Theorem. *The functions length and area defined on $\mathscr{E}_K$ are continuous.*

Proof. Let $S = \lim_i S_i$. From Lemma 2 for each $1 < \delta < 1/\max p$ there exists a convex polygon P such that $P \subset S \subset \delta P \subset H(K)$.

On the other hand, from Proposition 1, we have

$$\begin{aligned} 0 < F(\delta P) - F(P) &\leqslant \big(f(M/\delta) - 1\big)\big(F(P) + 2\pi k^2\big) \\ &< \big(f(M/\delta) - 1\big)\big(F(S) + 2\pi k^2\big). \end{aligned}$$

Then for each $\varepsilon > 0$, there exist δ and P such that $0 < F(\delta P) - F(P) < \varepsilon$ (because $f(M/\delta) \to 1$ when $\delta \to 1$).

Since $\lim_i S_i = S$ there exists i_0 such that $P \subset S_i \subset \delta P$ for $i \geqslant i_0$. Then

$$|F(S) - F(S_i)| \leqslant F(\delta P) - F(P) < \varepsilon, \qquad i \geqslant i_0,$$

and the Theorem is proved. q.e.d.

This theorem proves, using Bonnesen approximation theorem, that the counterexample of the above paragraph can be modified to obtain counterexamples with smooth boundary (in fact defined by analytic support functions).

5. Remarks

We give some sufficient conditions for the conjecture to be true.

First we recall that the conjecture is true for sequences of h-convex sets (cf. [5]), so we shall deal with arbitrary convex sets.

Note that in Beltrami's projective model a sequence $\{K_n\}$ of convex sets defined by support functions P_n expands over the whole plane if and only if $p_n(\theta)$ converges to 1. With this notation we have

Proposition 2. *Let* $1 - b_n \leqslant p_n \leqslant 1 - a_n$ *with* $\lim_n a_n = \lim_n b_n = 0$. *If* $\lim_n (a_n/b_n) = 1$, *the conjecture is true.*

Proof. We have $C(1 - b_n) \subset K_n \subset C(1 - a_n)$, where $C(r)$ denotes the circle of Euclidean radius r centered at the origin.

From the expressions of length and area in §4 we have

$$L(C(r)) = \frac{2\pi k r}{(1 - r^2)^{1/2}},$$

$$F(C(r)) = \frac{2\pi k^2\left(1 - (1 - r^2)^{1/2}\right)}{(1 - r^2)^{1/2}}.$$

Substituting this expressions in the inequality

$$\frac{L(C(1 - b_n))}{F(C(1 - a_n))} \leqslant \frac{L(K_n)}{F(K_n)} \leqslant \frac{L(C(1 - a_n))}{F(C(1 - b_n))},$$

we obtain

$$\frac{1}{k} \cdot \frac{1 - b_n}{1 - \sqrt{2a_n - a_n^2}} \cdot \left(\frac{(2 - a_n)a_n}{(2 - b_n)b_n}\right)^{1/2}$$

$$\leqslant \frac{L(K_n)}{F(K_n)} \leqslant \frac{1}{k} \cdot \frac{1 - a_n}{1 - \sqrt{2b_n - b_n^2}} \cdot \left(\frac{(2 - b_n)b_n}{(2 - a_n)a_n}\right)^{1/2},$$

that is, $\lim_n (L_n(K_n))/(F_n(K_n)) = 1/k$.

Proposition 3. *Let the notation be as above. If* $p_n''(\theta)$ *converges uniformly to zero, the conjecture is also true.*

Proof. This is a consequence of the expression for the length and area. A more explicit proof is given in [3]. q.e.d.

If we omit the condition $p_n'' \to 0$ uniformly, the conjecture is not necessarily true, as can be shown by computing the quotient length area for the convex sequence

$$p_n(\theta) = 1 - (1/n^2) - (1/4n^2)\cos^2 n\theta$$

in $H(-1)$.

References

[1] T. Bonnesen & W. Fenchel, *Theorie der Könvexen korper*, Springer, Berlin, 1934.

[2] R. Bonola, *Non-Euclidean geometry*, Dover, New York.

[3] E. Gallego & A. Reventós, *A note on the Santaló conjecture about convex sets in the hyperbolic plane*, (to appear).

[4] S. Lay, *Convex sets and their applications*, Wiley-Interscience, New York, 1982.

[5] L. A. Santaló & I. Yañez, *Averages for polygons by random lines in Euclidean and Hyperbolic planes*, J. Appl. Probability **9** (1972) 140–157.

[6] L. A. Santaló, *Integral geometry and geometric probability*, Addison-Wesley, Reading, MA, 1976.

UNIVERSITAT AUTÒNOMA DE BARCELONA, SPAIN