J. DIFFERENTIAL GEOMETRY
20 (1984) 497–521

A FINITENESS THEOREM FOR NEGATIVELY CURVED MANIFOLDS

KENJI FUKAYA

0. Introduction

The purpose of this paper is to investigate the topological structure of negatively curved manifolds. We show the finiteness of the number of diffeomorphism classes containing complete (but not necessarily compact) negatively curved manifolds with bounded volumes and curvatures. This result is related to works of Wang, Cheeger, Gromov, and others. Hence we first give a rough summary of some of their works.

Wang investigated the case of locally symmetric spaces. Let X be a simply connected symmetric space of noncompact type without factors of dimension smaller than 4. Denote by $\mathrm{Iso}(X)$ the group of all selfisometries of X. Wang considered subgroups Γ of $\mathrm{Iso}(X)$ acting on X effectively and properly discontinuously. He proved that, for each positive number V, there exist only a finite number of conjugacy classes of Γ satisfying $\mathrm{Vol}(\Gamma \setminus X) < V$.

On the other hand, in [5], Gromov proved a finiteness theorem for Riemannian manifolds M satisfying

$$(*) \qquad \mathrm{Vol}(M) \leqslant V,$$

$$(**) \qquad 0 > \text{sectional curvature} \geqslant -1.$$

Namely, he proved that, for each positive number V and for each positive integer n greater than 3, there exist only a finite number of diffeomorphism classes containing compact n-dimensional Riemannian manifolds M satisfying $(*)$ and $(**)$.

In the case of locally symmetric spaces, Wang's result is stronger than Gromov's. We reach the problem below.

Consider pairs (X, Γ), where X is a simply connected and complete Riemannian manifold and Γ is a group of isometries of X acting effectively and

Received December 12, 1983 and, in revised form, September 21, 1984.

properly discontinuously. We say (X, Γ) and (X', Γ') are equivariantly diffeomorphic to each other if there exist an isomorphism $\Phi\colon \Gamma \to \Gamma'$ and a diffeomorphism $f\colon X \to X'$ such that $f(\gamma(x)) = \Phi(\gamma)(f(x))$ holds for every $\gamma \in \Gamma$ and $x \in X$.

Problem. Let $n \neq 2, 3$ be a positive integer and V a positive number. Then there exist only a finite number of equivariant diffeomorphism classes containing (X, Γ) such that

$$0 > \text{sectional curvature of } X \geqslant -1,$$
$$\mathrm{Vol}(\Gamma \setminus X) \leqslant V.$$

Gromov's theorem, described above, gives an affirmative answer to the above problem in the case when the following conditions are satisfied:

(a) $\Gamma \setminus X$ is compact.

(b) Γ acts freely on X.

One of the main theorems of this paper is a generalization of Gromov's theorem to the case when (a) is not necessarily satisfied.

Theorem I. *Let V be a positive number and let n be a positive integer with $n \neq 3, 4$ (resp. $n = 4$). Then there exist only a finite number of diffeomorphism classes (resp. homotopy types) containing n-dimensional Riemannian manifolds M satisfying the following conditions*:

(1) *M has negative curvature or satisfies the visibility axiom of Eberlein & O'Neill* [3].

(2) *sectional curvature of $M \geqslant -1$.*

(3) $\mathrm{Vol}(M) \leqslant V$.

We cannot replace Condition (2) by "M has nonpositive curvature". A counterexample is given by §5.

Gromov used Cheeger's finiteness theorem (Theorem 3.1 in [2]) in the proof of his theorem. To apply Cheeger's theorem directly, we have to restrict ourselves to compact manifolds. Hence we need a noncompact version of Cheeger's theorem in order to prove Theorem I. But it seems difficult to obtain a noncompact version of Cheeger's theorem itself. Thus we divide the argument in Cheeger [2] into two parts and generalize each of them. Our generalization of the first part is the following.

Theorem A. *For each positive integer n and positive numbers a and D, there exists a positive number $\varepsilon_1(a, D)$ such that the following holds*:

Suppose that N is a n-dimensional Riemannian manifold whose sectional curvature is not smaller than -1, and that there is a closed geodesic l whose length is smaller than $\varepsilon_1(a, D)$. Then, for each point p on l, we have

$$\mathrm{Vol}(\{x \in N \mid d(x, p) \leqslant D\}) \leqslant a.$$

Theorem A can be proved in exactly the same way as Cheeger [2, Corollary 2-2].

Our generalization of the second part is the following.

Theorem 1-1. *Let a, b, c and V be positive numbers with $a < b < c$, and let n be a positive integer not equal to 3 or 4 (resp. $n = 3$ or 4). Then there exist only a finite number of diffeomorphism classes (resp. homotopy types) containing n-dimensional Riemannian manifolds N satisfying the following conditions*:

(1) $-1 \leqslant$ *sectional curvature* $\leqslant 1$.

(2) $\mathrm{Vol}(N) \leqslant V$.

(3) *There exist an open subset N' of N and a PL-homeomorphism Φ: $N - N' \to \partial N' \times [0, 1)$ which have the following properties*:

(i) *$\partial N'$ is a codimension-1 PL-submanifold of N.*

(ii) *For an element p of $\partial N'$ the injectivity radius of N at p is greater than c.*

(iii) *For an element p of $\Phi^{-1}(\partial N' \times \{1/2\})$, the injectivity radius of N at p is greater than a and smaller than b.*

We prove Theorem 1-1 in §§1 and 2. The argument in §1 is similar to the argument in Cheeger [2] or Peters [15]. In §2, we treat the ends of N, making use of a type of h-cobordism theorem. To deduce Theorem I from Theorem 1-1 we need a description of the topological type of the set

$$\{x \mid \text{the injectivity radius of } N \text{ at } x \text{ is smaller than } a\}.$$

(Here a is a positive small number.) Theorem 3-1 gives such a description. To prove Theorem 3-1, we use results in Margulis [13] and Gromov [5]. We prove Theorem I in §4, making use of Theorems 1-1 and 3-1.

In §6 we give an estimate on the number of homotopy types containing manifolds satisfying the conditions of Theorem I (Theorem 6-6), making use of Theorem 6-1, which gives an estimate on the number of homotopy types containing manifolds satisfying the conditions of Theorem 1-1. When applied to locally symmetric spaces, Theorems 6-6 together with Mostow's rigidity theorem ([14], [16]) implies the following theorem.

Theorem II. *For each positive integer n greater than 3, there exists a positive number C_1 depending only on n such that the following holds*:

Let V be a positive number. Consider the n-dimensional rank one locally symmetric spaces whose volumes are smaller than V. Then, there exist at most $\exp(\exp(C_1 V))$ *isometry classes containing such spaces.*

Remark. Gromov, in [7], stated a similar estimate in the case when M has constant negative curvature. His upper bound is "something like $V \cdot \exp(\exp(\exp(V + n)))$". The author does not know whether his method is similar to ours or not.

The author is very grateful to Professors I. Tamura, T. Tsuboi, M. Ue and K. Yano for helpful advice. He also wishes to thank Professors K. Shiohama and T. Yamaguchi.

Notation. $D^n(r)$ = the ball of radius r centered at 0 in the flat n-dimensional Euclidean space.

For a Riemannian manifold N, we put

$c^+(N)$ = the supremum of the sectional curvature of N,

$c^-(N)$ = the infimum of the sectional curvature of N,

$\mathrm{Vol}(N)$ = the volume of N,

$\mathrm{diam}(N)$ = the diameter of N.

For points p and q of N and for a positive number r, set

$D_p(r) = \{s \in N \mid d(p,s) \leqslant r\}$,

$i_N(p)$ = the injectivity radius of N at p, that is $\sup\{r \mid$ the restriction to $D^n(r)$ of the exponential map $\exp_p \colon T_p(N) \to N$, is injective$\}$,

$N(r) = \{p \in N \mid i_N(p) \leqslant r\}$.

$C_i(\cdots)$ and $\varepsilon_i(\cdots)$ always denote the positive constants depending only on the numbers in the parentheses and on the dimension n.

1. A generalization of Cheeger's finiteness theorem—I

We prove Theorem 1-1 in this and the next sections. Since the proof is long, we first give a rough summary of it. In this section we divide the set of all manifolds satisfying the conditions of Theorem 1-1 into finitely many classes, and we construct a local diffeomorphism $f \colon N_1 - N_1(a/4) \to N_2$ for two manifolds N_1, N_2 belonging to the same class. In §2 we modify f outside the set $N_1 - \Phi^{-1}(\partial N_1' \times (1/2, 1))$ and make it a PL-homeomorphism between N_1 and N_2. We need some conditions on f in order to make the argument in §2 go well. These conditions are listed in Lemma 1-2 below.

Now we start the proof of Theorem 1-1. We may assume that N is connected, since the number of connected components can be estimated in terms of c and V. Set $\alpha = \min(a/125, (c-d)/6, 1/10)$, and $N'' = N' \cup \Phi^{-1}(\partial N' \times (0, 1/2])$. For a positive number d smaller than c, we denote by $N^{(d)}$ the connected component of $N - N(d)$ which intersects with N'.

Lemma 1-2. *The set of manifolds satisfying the conditions of Theorem* 1-1 *is divided into finitely many classes* $\mathscr{C}_1, \cdots, \mathscr{C}_\Sigma$ *such that the following conditions are satisfied*:

(1)′ *For each* $\mathscr{C}_k$ *there exists a finite set* Y_k, *and for each* $N \in \mathscr{C}_k$ *there exists a map* $\Psi_N \colon Y_k \to N^{(a/4)}$ *such that the a-neighborhood of* $\Psi_N(Y_k)$ *contains* $N^{(a/4)}$.

(2)′ *For two manifolds N_1, N_2 belonging to the same class $\mathscr{C}_k$, there exists a C^∞-map f: $N_1^{(a/4)} \to N_2$ which satisfies the following conditions*:

(i) *For any $p \in N_1^{(a/4)}$, there is a neighborhood U of p in $N_1^{(a/4)}$ such that the restriction of f to U is a diffeomorphism to its image.* (*Hereafter we call a map a local diffeomorphism if this condition is satisfied.*)

(ii) *For each $p, q \in N_1^{(a/4)}$, we have $d(f(p), f(q)) \leqslant d(p,q) + \alpha$.*

(iii) *For each $i \in Y_k$, we have $d(f\Psi_{N_1}(i), \Psi_{N_2}(i)) \leqslant \alpha$.*

(iv) *For each $i \in Y_k$, the inequality $i_{N_1}\Psi_{N_1}(i) \geqslant 3a/5$ holds if and only if $i_{N_2}\Psi_{N_2}(i) \geqslant 3a/5$.*

(v) *For each $i \in Y_k$, the inequality $i_{N_1}\Psi_{N_1}(i) \leqslant (b+c)/2$ holds if and only if $i_{N_2}\Psi_{N_2}(i) \leqslant (b+c)/2$.*

Proof. The proof of Lemma 1-2 is divided into three steps. In Step 1 we construct the map Ψ_N. In Step 2 we divide the set of manifolds satisfying the conditions in Theorem 1-1 into finitely many classes $\mathscr{C}_1, \cdots, \mathscr{C}_\Sigma$. In Step 3 we construct the map f. The method used to construct f is similar to the argument in Peters [15].

Step 1.

Assertion 1-3. *There exists a positive number C_2 such that*

$$\operatorname{diam}(N^{(a/4)}) \leqslant C_2 V \alpha^{1-n}$$

holds for every manifold N satisfying the conditions of Theorem 1-1.

Proof of Assertion 1-3. Let $Z_N = \{p_1, \cdots, p_i, \cdots\}$ be a maximal subset of $N^{(a/4)}$ such that $d(p,q) \geqslant \alpha/20$ holds for every $p, q \in Z_N$ with $p \neq q$. Then $D_{p_i}(\alpha/20)$ $(p_i \in Z_N)$ are disjoint to each other. On the other hand, since $i_{N_1}(p_i) \geqslant \alpha/20$ and $1 \geqslant c^+(N)$, there exists a positive constant δ which depends only on n and which satisfies $\operatorname{Vol}(D_{p_1}(\alpha/20)) \geqslant \delta\alpha^n$.

The above two facts imply

$$\#Z_N \leqslant \operatorname{Vol}(N)/(\delta\alpha^n) \leqslant FV/(\delta\alpha^n). \tag{1-1}$$

Now let p and q be elements of $N^{(a/4)}$. Since Z_N is maximal, the $\alpha/10$-neighborhood of Z_N contains $N^{(a/4)}$. Hence there exist elements $p_{i(1)}$, $p_{i(2)}, \cdots, p_{i(m)}$ of Z_N such that the following conditions hold:

(a) $D_{p_{i(j)}}(\alpha/10) \cap D_{p_{i(j+1)}}(\alpha/10) \neq \varnothing$.

(b) $p \in D_{p_{i(1)}}(\alpha/10)$ and $q \in D_{p_{i(m)}}(\alpha/10)$.

(c) $p_{i(j)} \neq p_{i(j')}$ if $j \neq j'$.

Condition (c) implies that

$$m \leqslant \#Z_N. \tag{1-2}$$

For each j $(\leqslant m)$, fix an element q_j of $D_{p_{i(j)}}(\alpha/10) \cap D_{p_{i(j+1)}}(\alpha/10)$. Then we have

$$d(q_j, q_{j+1}) \leqslant \operatorname{diam}\Big(D_{p_{i(j+1)}}(\alpha/10)\Big) \leqslant \alpha/5. \tag{1-3}$$

On the other hand, condition (b) implies

$$(1\text{-}4) \qquad d(q_1, p) \leqslant \alpha/10 \quad \text{and} \quad d(q_m, q) \leqslant \alpha/10.$$

By equations (1-1)–(1-4), we obtain

$$d(p,q) \leqslant d(p,q_1) + \sum_{j=1}^{m-1} d(q_j, q_{j+1}) + d(q_m q)$$

$$\leqslant \{2V/(\delta\alpha^n) + 2\} \cdot \alpha/10,$$

as desired. The proof of Assertion 1-3 is completed.

We construct the map Ψ_N in the following assertion.

Assertion 1-4. *The set of manifolds satisfying the conditions in Theorem* 1-1 *is divided into finitely many classes* $\mathscr{C}_1^{(1)}, \cdots, \mathscr{C}_{\Sigma'}^{(1)}$ *such that the following holds*:

(1)″ *For each* $\mathscr{C}_k^{(1)}$ *there exists a finite set* γ_k^*, *and for each* $N \in \mathscr{C}_k^{(1)}$ *there exists a map* Ψ_N: $\gamma_k^* \to N^{(a/4)}$ *such that the* $\alpha/10$ *neighborhood of* $\Psi_N(Y_k')$ *contains* $N^{(a/4)}$.

(2)″ *For two manifolds* N_1, N_2 *belonging to* $C_k^{(1)}$ *and for two elements* i, j *of* Y_k^*, *the following holds*:

(i) $d(\Psi_{N_1}(i), \Psi_{N_1}(j)) \leqslant d(\Psi_{N_2}(i), \Psi_{N_2}(j)) + \alpha/10.$

(ii) $i_{N_1}(\Psi_{N_1}(i)) \geqslant 3\alpha/5$ *holds if and only if* $i_{N_2}(\Psi_{N_2}(i)) \geqslant 3\alpha/5.$

(iii) $i_{N_1}(\Psi_{N_1}(i)) \leqslant (b+c)/2$ *holds if and only if* $i_{N_2}(\Psi_{N_2}(i)) \leqslant (b+c)/2.$

Remark. Conditions (1)″ and (2)″ correspond to the part of conditions (1)′ and (2)′ in Lemma 1-2 concerning to the image of Ψ_N.

Proof of Assertion 1-4. For a positive number m, let $\dot{m}$ denote the set of all positive integers smaller than $m+1$. For each manifold N satisfying the conditions of Theorem 1-1, fix a subset Z_N used in the proof of Assertion 1-3, and fix a bijection Ψ_N: $\#\dot{Z}_N \to Z_N$. Let $\mathscr{C}_k^{(2)}$ be the set of all manifolds N which satisfy the conditions of Theorem 1-1 and $\#Z_N = k$. Equation (1-1) implies that the sets $\mathscr{C}_i^{(2)}$ $(i = 1, 2, \cdots)$ are empty except finitely many ones.

Now (1)″ holds because Z_N is maximal.

On the other hand, Assertion 1-3 implies that the set $\{d(\Psi_N(i), \Psi_N(j)) | i, j \leqslant k, N \in \mathscr{C}_k^{(2)}\}$ is bounded. It follows that we can subdivide the classes $\mathscr{C}_i^{(2)}$ $(i = 1, 2, \cdots)$ into finitely many classes $\mathscr{C}_1^{(1)}, \mathscr{C}_2^{(1)}, \cdots, \mathscr{C}_k^{(1)}, \cdots$ satisfying (2)″. The proof of Assertion 1-4 is completed.

Step 2. In this step, we divide each of the classes $\mathscr{C}_k^{(1)}$ given in Assertion 1-4 into finitely many classes. Also, we do not change the set Y_k and the map Ψ_N.

For $i \in Y_k$ and $N \in \mathscr{C}_k^{(1)}$, we denote by $\varphi_{N,i}$ the composition of the exponential map, $\exp_p$: $T_{\Psi_N(i)}(N) \to N$ and an origin preserving isometric embedding of $D^n(6\alpha/10)$ into $T_{\Psi_N(i)}(N)$. If

$$(1\text{-}5)_{i,j,N} \qquad D_{\Psi_N(i)}(2\alpha/10) \cap D_{\Psi_N(j)}(2\alpha/10) \neq \varnothing,$$

then

$$D_{\Psi_N(i)}(2\alpha/10) \subset D_{\Psi_N(j)}(6\alpha/10)$$

and the map $\varphi_{N_2,j}^{-1}\varphi_{N_1,i}$: $D^n(2\alpha/10) \to D^n(6\alpha/10)$ is well defined. We put $g_{N;i,j} = \varphi_{N,j}^{-1}\varphi_{N,i}$, when $(1\text{-}5)_{i,j,N}$ holds.

Since Y_k is a finite set, we can subdivide the classes $\mathscr{C}_k^{(1)}$ ($k = 1, 2, \cdots$) into finitely many classes $\mathscr{C}_1^{(3)}$, $\mathscr{C}_2^{(3)}$, $\cdots$ such that the following holds: if N_1 and N_2 are contained in the same class $\mathscr{C}_k^{(3)}$, then, for each elements i, j of Y_k, $(1\text{-}5)_{i,j,N_1}$ holds if and only if $(1\text{-}5)_{i,j,N_2}$ holds, in other words $g_{N_1;i,j}$ is well defined if and only if $g_{N_2;i,j}$ is well defined.

On the other hand, since the sectional curvatures of our manifolds are uniformly bounded, the maps $g_{N;i,j}$ are equicontinuous when N moves in $\mathscr{C}_k^{(3)}$. (See [2, Lemma 3.4]).

Therefore, using Ascoli-Arzela's theorem, we obtain the following. For an arbitrary number θ we can subdivide the classes $\mathscr{C}_k^{(3)}$ ($k = 1, 2, \cdots$) into finitely many classes $\mathscr{C}_1^{(4)}$, $\mathscr{C}_2^{(4)}$, $\cdots$ such that

$$d\big(g_{N_1;i,j}(p), g_{N_2;i,j}(p)\big) \leqslant \theta$$

holds for each $p \in D^n(2\alpha/10)$ and i, $j \in Y_k$, if N_1 and N_2 are contained in the same class $\mathscr{C}_k^{(4)}$ and if $g_{N_1;i,j}$ is well defined. We will fix the number θ later.

Suppose i and j are elements of Y_k such that $g_{N;i,j}$ is well defined. Then $d(\Psi_N(i), \Psi_N(j)) \leqslant i_N(\Psi_N(i))$ holds for every $N \in C_k^{(4)}$. Therefore there exists a unique geodesic segment joining $\Psi_N(i)$ with $\Psi_N(j)$. Let $P_{N;i,j}$: $T_{\Psi_N(i)}(N) \to T_{\Psi_N(j)}(N)$ be the parallel displacement along this geodesic.

Since the maps

$$(\Phi_{N,j})_*^{-1}P_{N;i,j}(\Phi_{N,i})_*\colon T_0(D^n(2\alpha/10)) \to T_0(D^n(2\alpha/10))$$

are elements of $SO(n)$, that is the group of linear isometries of $T_0(D^n(2\alpha/10))$, and since $SO(n)$ is compact, we can subdivide the classes $\mathscr{C}_k^{(4)}$ ($k = 1, 2, \cdots$) into finitely many classes $\mathscr{C}_1, \mathscr{C}_2, \cdots$ such that

$$d\Big(\big(\Phi_{N_1,j}\big)^{-1}P_{N_1;i,j}\Phi_{N_1,i}, \big(\Phi_{N_2,j}\big)^{-1}P_{N_2;i,j}\Phi_{N_2,i}\Big) \leqslant \theta$$

holds for each N_1 and N_2 belonging to the same class $\mathscr{C}_k$. Here d is a distance function on $SO(n)$.

Step 3. We prove that the classes $\mathscr{C}_k$ divided in Step 2 satisfy conditions (1)′, (2)′ and (3)′ in Lemma 1-2. We have already constructed the set Y_k and the map Ψ_N satisfying conditions (1)′ and (2)′. We will construct the map f for each pair (N_1, N_2) of elements of $\mathscr{C}_k$.

Following Peters, we use center of mass technique here. Let $\eta_1\colon N_1^{(a/4)} \to [0,1]$ be C^∞-functions such that $\sum_{i\in Y_k} \eta_i = 1$ and that the support of η_i is contained in $D_{\Psi_N(i)}(6\alpha/10)$.

For $p \in N_1^{(a/4)}$ and $q \in N_2$, set

$$\omega(p,q) = \sum_{i\in Y_k} \left\{\eta_i(p) \times d\left(\Phi_{N_2,i}\Phi_{N_1,i}^{-1}(p), q\right)\right\}.$$

(Remark that if p is not contained in the image of $\Phi_{N_1,i}$, then $\eta_i(p) = 0$. Hence the above function is well defined.)

Let $f(p)$ be the point of N_2 such that

$$d(p, f(p)) = \min_{q\in N_2} \omega(p,q).$$

The unique existence of such a point $f(p)$ is known and can be proved by making use of the convexity of the function $q \to \omega(p,q)$. For these facts, see Buser & Karcher [1].

On the other hand, Peters showed that if $\theta \leqslant \min(6^{2-n}, \beta/700)$, then f is a local diffeomorphism ([15, Lemma], where he treated the case when N was compact, but his proof can be applied without any change to our case). Put $\theta = \min(6^{2-n}, \beta/700)$. Then f satisfies condition (2)′(i).

By the definition of f, the inequality

$$d\left(f(p), \Psi_{N_2}(i)\right) \leqslant 6\alpha/10$$

holds if

$$d\left(p, \Psi_{N_1}(i)\right) \leqslant 2\alpha/10.$$

By using this fact and Assertion 1-3, we can easily prove that f satisfies conditions (2)′(ii), (iii) and (3)′. Thus Lemma 1-2 is proved.

2. A generalization of Cheeger's finiteness theorem—II

In this section we complete the proof of Theorem 1-1. Take one of the classes $\mathscr{C}_k$ given in Lemma 1-2. Suppose N_1 and N_2 belong to $\mathscr{C}_k$. If the dimension is not 4, there exist only a finite number of diffeomorphism classes in a given PL-homeomorphism class. Therefore it suffices to show that N_1 is PL-homeomorphic to N_2. (In the case when $n = 3$ or 4, it suffices to show that N_1 is homotopy equivalent to N_2.)

Let $f\colon N_1^{(a/4)} \to N_2$ and $f'\colon N_2^{(a/4)} \to N_1$ be the maps given in Lemma 1-2(2).

Assertion 2-1. (1) *$f(N_1^{(3a/4)}) \subset N_2^{(a/2)}$. In particular, $f'f|_{N_1^{(3a/4)}}$ is well defined.*

(2) *For every point p of* $N_1^{(3a/4)}$, *we have* $d(f'f(p), p) \leqslant 7\alpha$.

Proof. Using the fact |sectional curvature| $\leqslant 1$, we see easily that

$$|i_N(p) - i_N(q)| \leqslant d(p, q) \tag{2-1}$$

holds for each $p, q \in N$ with $d(p, q) \leqslant \pi/4$.

Now let p be an arbitrary point of $N_1^{(3a/4)}$. By Lemma 1-2, there exists an element j of Y_k such that $d(p, \Psi_{N_1}(j)) \leqslant \alpha$. Therefore, (2-1) implies that

$$i_{N_1}\big(\Psi_{N_1}(j)\big) \geqslant i_{N_1}(p) - \alpha \geqslant 3a/5.$$

It follows that

$$\begin{aligned} i_{N_2}\big(f\big(\Psi_{N_1}(j)\big)\big) &\geqslant i_{N_2}\big(\Psi_{N_2}(j)\big) - d\big(f\big(\Psi_{N_1}(j)\big), \Psi_{N_2}(j)\big) \\ &\geqslant 3a/5 - \alpha \geqslant 11a/20. \end{aligned} \tag{2-2}$$

The first inequality follows from (2-1). The second inequality follows from Lemma 1-2(iii), (iv) and the fact $i_{N_1}(\Psi_{N_1}(j)) \geqslant 3a/5$. The third inequality follows from the fact $\alpha \leqslant a/20$.

Equation (2-1), together with Lemma 1-2(2)(ii) and the fact $d(p, \Psi_{N_1}(j)) \leqslant \alpha$, imply that

$$i_{N_2}(f(p)) \geqslant 11a/20 - d\big(f(p), f\big(\Psi_{N_1}(q)\big)\big) \geqslant 11a/20 - 2\alpha \geqslant a/2.$$

This proves (1) of Assertion 2-1.

By Lemma 1-2(2)(ii), (iii), we have

$$d\big(f'f\big(\Psi_{N_1}(j)\big), \Psi_{N_1}(j)\big) \leqslant d\big(f'\big(\Psi_{N_2}(j)\big), \Psi_{N_1}(j)\big) + 2\alpha \leqslant 3\alpha.$$

Using Lemma 1-2(2)(ii), (iii) again, we obtain

$$d(f'f(p), p) \leqslant d\big(f'f\big(\Psi_{N_1}(j), f'f(p)\big)\big) + d\big(\Psi_{N_1}(j), p\big) + 3\alpha \leqslant 7\alpha.$$

The proof of Assertion 2-1 is completed.

Assertion 2-2. *The restriction of f to* N_1' *is a homotopy equivalence between* N_1' *and* N_2.

Proof. Condition (4) in Theorem 1-1 implies that N_1' is a deformation retract of N_1. Let $j_1\colon N_1 \to N_1'$ and $j_2\colon N_2 \to N_2'$ be retractions. Set

$$\bar{f} = j_2 f|_{N_1'}, \qquad \bar{f}' = j_1 f'|_{N_2'}.$$

We will show that $\bar{f}'\bar{f}$ is homotopic to the identity map of N_2'. It can be shown in a similar way that $\bar{f}\bar{f}'$ is homotopic to the identity map of N_1'. Since the maps j_1 and j_2 are homotopy equivalences, Assertion 2-2 follows from the above two facts.

To prove that $\bar{f}'\bar{f}$ is homotopic to the identity map, it suffices to show that $f'f|_{N_1'}$ is homotopic to the inclusion map i of N_1' into N_1 (because j_1 and j_2 are homotopy equivalences).

By Assertion 2-1, we have

$$d(f'f(p), p) \leqslant 7\alpha \leqslant i_{N_1}(p).$$

Therefore there exists uniquely a minimum geodesic segment l: $[0,1] \to N_1$ joining p with $f'f(p)$. For each element t of $[0,1]$ and for each $p \in N_2$, let $g_t(p)$ denote the point on l such that $d(p, g_t(p)) = t \cdot d(p,q)$. Clearly g is continuous, $g_0 = i$ and $g_1 = f'f$. Namely g is a homotopy from i to $f'f$. The proof of Assertion 2-2 is completed.

Assertion 2-3. *The restriction of f to N_1'' is injective.*

Proof. Let p, q be points of N_1'' such that $f(p) = f(q)$. (Recall that $N_1'' = N_1' \cup \Phi^{-1}(\partial N_1' \times [0, 1/2])$.) We will prove $p = q$.

By Assertion 2-1(2), we have

$$d(p,q) \leqslant d(p, f'f(p)) + d(f'f(q), q) \leqslant 14\alpha.$$

On the other hand, $i_{N_1}(p)$ is greater than 14α.

Hence there exists a geodesic l: $[0,1] \to N_1$ such that l is parametrized proportionally to arc length and that $l(0) = p$, $l(1) = q$ and that the length of l is smaller than 14α.

By Lemma 1-2(ii), we have, for each $s \in [0,1]$,

$$d(f(p), f(l(s))) \leqslant d(p, l(s)) + \alpha \leqslant 15\alpha.$$

It follows that

$$fl([0,1]) \subset D_{f(p)}(15\alpha).$$

Therefore, since $i_{N_2}(f(p)) \geqslant 15\alpha$, there exists a continuous map g: $[0,1] \times [0,1] \to D_{f(p)}(15\alpha)$ such that $g(t,0) = fl(t)$ and $g(0,t) = g(1,t) = g(t,1) = f(p)$.

Claim. *There exists a continuous map $\bar{g}$: $[0,1] \times [0,1] \to N_1^{(a/2)}$ such that $f\bar{g} = g$ and $\bar{g}(t,0) = l(t)$.*

Before proving the claim, we prove Assertion 2-4 making use of the claim.

Since $f\bar{g}(t,1)$ does not depend on t and since f is a local diffeomorphism, it follows that $\bar{g}(t,1)$ does not depend on t. In particular $\bar{g}(0,1) = \bar{g}(1,1)$. Similarly we obtain $\bar{g}(1,0) = \bar{g}(1,1)$ and $\bar{g}(0,1) = \bar{g}(0,0)$.

Therefore

$$p = l(0) = \bar{g}(0,0) = \bar{g}(1,0) = l(1) = q,$$

as required.

Proof of the claim. We will prove by contradiction. Suppose the claim is false. Then the following is valid, since f is a local diffeomorphism.

(∗) There exist $s_0, t_0 \in$,[0, 1] and a continuous map $\bar{g}(t_0, \cdot)$ which maps $s \in [\,0, s_0)$ to $\bar{g}(t, s) \in N_1^{(a/2)}$ and which satisfies $f(\bar{g}(t_0, s)) = g(t_0, s)$ and $\bar{g}(t_0, 0) = l(t_0)$. Furthermore s_0 is the maximum among the numbers which have the above property.

We will deduce a contradiction from (∗). For each $s \in [0, s_0)$, we have

$$d(p, \bar{g}(t_0, s)) < d(p, f'f(p)) + d(f'f(p), f'g(t_0, s)) + d(f'f(\bar{g}(t_0, s)), \bar{g}(t_0, s)).$$

This formula, together with Assertion 2-1(2), Lemma 1-2(2)(ii) and the fact $g(t_0, s) = D_{f(p)}(15\alpha)$, implies that

$$(2\text{-}3)\qquad d(p, \bar{g}(t_0, s)) \leqslant 7\alpha + \big(d(f(p), g(t_0, s)) + \alpha\big) + 7\alpha \leqslant 30\alpha.$$

It follows from (2-3), (2-1) and the fact $i_{N_1}(p) \geqslant 3a/5$, that

$$(2\text{-}4)\qquad i_{N_1}(g(t_0, s)) \geqslant 3a/5 - 30\alpha > a/2.$$

Therefore, "all accumulation points of $\lim_{s \to s_0} \bar{g}(t_0, s)$ are contained in $N_1^{(a/2)}$".

On the other hand,

(2-5) "f is defined on $N_1^{(a/2)}$",

(2-6) "f is a local diffeomorphism",

(2-7) "$\lim_{s \to s_0} fg(t_0, s)$ converges".

The facts (2-4)–(2-7) imply that $\lim_{s \to s_0} g(t_0, s)$ converges to a point of $N_1^{(a/2)}$. Hence $g(t_0, \cdot)$ can be extended to $[0, s_0 + \delta)$ for sufficiently small δ. This contradicts (∗). Thus the proof of the claim is completed.

Assertion 2-4. $f(\partial N_1'') \subset N_2 - N_2'$.

(**Remark.** Condition (4)(ii) in Theorem 1-1 is added to make this assertion valid.)

Proof of Assertion 2.4. Let p be a point of $\partial N_1''$. By Lemma 1-2(1)$'$, there exists an element j of Y_k such that

$$(2\text{-}8)\qquad d(p, \Psi_{N_1}(j)) \leqslant \alpha.$$

Since $p \in \partial N_1''$, condition (4)(ii) in Theorem 1-1 implies

$$(2\text{-}9)\qquad i_{N_1}(p) \leqslant b.$$

By (2-8), (2-9) and (2-1), we have

$$i_{N_1}\Psi_{N_1}(j) \leqslant b + \alpha \leqslant (b + c)/2.$$

Therefore, using Lemma 1-2(2)′, we have

$$i_{N_2}\Psi_{N_2}(j) \leqslant (b+c)/2. \tag{2-10}$$

On the other hand, by Lemma 1-2(2)(ii), we have

$$\begin{aligned} d(f(p), \Psi_{N_2}(j)) &\leqslant d(f(p), f\Psi_{N_1}(j)) + d(f\Psi_{N_1}(j), \Psi_{N_2}(j)) \\ &\leqslant 3\alpha. \end{aligned} \tag{2-11}$$

Inequalities (2-10), (2-11) and (2-1) imply

$$i_{N_2}(f(p)) \leqslant (b+c)/2 + 3\alpha \leqslant c.$$

This inequality, together with condition (4)(ii), implies

$$f(p) \in N_2(c) \subset N_2 - N_2',$$

as required.

Assertion 2-5. *The restriction of f to $\partial N_1''$ is a homotopy equivalence between $\partial N_1''$ and $N_2 - N_2'$.*

The proof of Assertion 2-5 is similar to the proof of Assertion 2-2, and hence is omitted.

Set $U = \overline{f(N_1'') - N_2'}$ and $V = \overline{N_2 - f(N_1'')}$.

Assertion 2-6. *The embedding i_1 of $f(\partial N_1'')$ into U and the embedding i_2 of $f(\partial N_1'')$ into V are homotopy equivalences.*

Proof. In order to avoid complicated notations, we assume that $N_2 - N_2'$ is connected.

By Van-Kampen's theorem, we have

$$\pi_1(N_2 - N_2') \simeq \pi_1(U)\pi_1 \underset{\pi_1(f(\partial N_1''))}{*} \pi_1(V).$$

Assertion 2-5 implies that the inclusion map: $f(\partial N_1'') \to N_2$ induces an isomorphism on fundamental groups.

Using these facts, we can prove easily that i_1 and i_2 induce isomorphisms on fundamental groups.

On the other hand, by using Assertion 2-6 and the Mayer-Vietoris exact sequence

$$\cdots \to H_*(f(\partial N_1'')) \to H_*(U) \oplus H_*(V) \to H_*(N_2 - N_2') \to \cdots$$

we can prove easily that i_1 and i_2 induce isomorphisms on homotopy groups of any local coefficient system.

Assertion 2-6 follows easily from these two facts.

Assertion 2-7. *If $n \neq 3, 4$, then V is PL-homeomorphic to $f(\partial N_1'') \times [0,1)$.*

Proof. By condition (4) in Theorem 1-1, we can attach to N_2 a boundary $\partial \overline{N}_2$, which is PL-homeomorphic to $\partial N_2''$, and can make N_2 a compact PL-manifold $\overline{N}_2$. Set $\overline{V} = V \cup \partial \overline{N}_2$.

Since the restriction of f to N_1' is a PL-homeomorphism to its image and since $\partial N_1''$ is contained in N_1', it follows that $f(N_1'')$ is a PL-submanifold of $\overline{N}_2$.

Now Assertion 2-7 implies that $(\overline{V}, f(\partial N_1''))$ and $(\overline{V}, \partial \overline{N}_2)$ are ∞-connected. Therefore, Theorem 7-11 in [13] implies that $\overline{V} - \partial \overline{N}_2$ $(= V)$ is PL-homeomorphic to $f(\partial N_1'') \times [0, 1)$, as required.

Assertion 2-7 implies that $f|_{N_1''}$ is extended to a PL-homeomorphism from N_1 to N_2. This completes the proof.

3. Negatively curved manifolds

In this section we review negatively curved manifolds. First we need some notations. Let M be a complete Riemannian nonpositively curved manifold. Let X be the universal covering space of M, and let π be the natural projection $\pi: X \to M$, and let Γ be the group $\pi_1(M)$ acting on X as the group of covering transformations.

In this section and the next, we assume either that X satisfies the visibility axiom of Eberlein & O'Neill [3] or that M has negative curvature (namely, for each point p of X and for each plane $\pi \subset T_p(X)$, the sectional curvature of X at π is strictly negative).

Let γ be a selfisometry of X. We call γ an elliptic isometry if γ has a fixed point in X, a hyperbolic isometry if γ has a unique invariant geodesic and has no fixed point in X, and a parabolic isometry if γ is neither elliptic nor hyperbolic. For two points p and q of X, we denote by $\overline{p\,q}$ the geodesic joining p with q.

For a point p of X and subset A of Γ, we set $\delta_A(p) = \inf_{\gamma \in A - \{1\}} d(p, \gamma(p))$. If A is invariant under the inner automorphisms of Γ, then δ_A is invariant by the action of Γ. Hence δ_A induces a function on M $(= \Gamma \setminus X)$. We denote this function also by δ_A. It is easy to see that

$$\delta_\Gamma/4 \leqslant i_M \leqslant \delta_\Gamma.$$

We set

$$\begin{aligned} X_{a,A} &= \{ p \in \Gamma | \delta_A(p) \leqslant a \}, \\ M_{a,A} &= \{ p \in M | \delta_A(p) \leqslant a \}, \\ X_a &= X_{a,\Gamma}, \qquad M_a = M_{a,\Gamma}, \\ \Gamma_{\varepsilon,p} &= \text{subgroup of } \Gamma \text{ generated by } \{ \gamma | d(\gamma(p), p) < \varepsilon \}. \end{aligned}$$

Now we give a description of the set M_a. Assume M satisfies $c^-(M) \geqslant -1$, and $\mathrm{Vol}(M) < \infty$. Let ε_2 be the Margulis' constant (see Gromov [5, 3-2] or Buser & Karcher [1, 2-5]). Let ε be a positive number smaller than ε_2. We denote by $S_1, S_2, \cdots$ all connected components of M_ε. Choose, for each i, one of the connected components of $\pi^{-1}(S_i)$, and denote it by $\tilde{S}_i$. In the case when X satisfies the visibility axiom, Eberlein [4, Corollary 3-3] implies that there uniquely exists a maximal almost nilpotent subgroup containing $\Gamma_{\varepsilon,p}$. (Here we say a group is almost nilpotent if it has a nilpotent subgroup with finite index.) It is easy to see that this group depends only on i and does not depend on $p \in \tilde{S}_i$. We denote this subgroup by Γ_i.

Theorem 3-1. *Suppose that X satisfies the visibility axiom. Then, for each i, one of the following statements holds*:

(1) (a) *S_i is diffeomorphic to an $\mathbf{R}^{n-1}$-bundle over S^1.*

(b) *Γ_i is isometric to $\mathbf{Z}$ and all nontrivial elements of Γ_i are hyperbolic.*

(c) *There exists a geodesic l such that l is invariant by all elements of Γ_i and that $\pi(l)$ is a closed geodesic contained in S_i.*

(2) (a) *Γ_i acts on $\mathbf{R}^{n-1}$ freely such that*

(i) *$\Gamma_i \setminus \mathbf{R}^{n-1}$ is compact.*

(ii) *S_i is homeomorphic to* $[0,1) \times (\Gamma_i \setminus \mathbf{R}^{n-1})$.

(b) *There exists a unique point on ∂X which is invariant by Γ_i.*

In the case when we do not assume the visibility axiom for X and when we assume that X has negative curvature, an analogue of Theorem 3-1 holds. But in this case, we do not know whether S_i is homeomorphic to $[0,1) \times \Gamma_i \setminus \mathbf{R}^{n-1}$ in case (2).

Theorem 3-2. *Suppose X has negative curvature. Then, for each i, one of the following statements holds*:

(1) *For each $p \in S_i$, there exists uniquely a maximal almost nilpotent subgroup $\Gamma_{p,i}$ containing $\Gamma_{\varepsilon,p}$. The group $\Gamma_{p,i}$ does not depend on p and depends only on i. Put $\Gamma_i = \Gamma_{p,i}$. Then conditions* (a), (b) *and* (c) *in Theorem* 3-1(1) *hold.*

(2) *There exist a compact manifold L and a homeomorphism Φ between S_i and $L \times [0,1)$ such that conditions* (a) *and* (b) *below hold for each $p \in L$.*

(a) *If $t_1, t_2 \in [0,1)$ and if $t_1 < t_2$, then we have*

$$\delta_\Gamma\big(\Phi^{-1}(p, t_1)\big) > \delta_\Gamma\big(\Phi^{-1}(p, t_2)\big).$$

(b) $\lim_{t \to 1} \delta_\Gamma(\Phi^{-1}(p, t)) = 0$.

Proofs of Theorems 3-1 *and* 3-2. First we need a lemma.

Lemma 3-3. *If $p_0 \in S_i$ and if Γ_{ε,p_0} contains a hyperbolic isometry, then, for every $p \in S_i$, the group $\Gamma_{\varepsilon,p}$ contains a hyperbolic isometry.*

Proof of Lemma 3-3. Set $U = \{\, p \in S_i | \Gamma_{\varepsilon,p}$ contains a hyperbolic isometry.$\}$

Claim 1. U is open.

Proof. Let p be an element of U. By the definition of U, there exists a hyperbolic element in $\Gamma_{\varepsilon,p}$. Therefore, Gromov [6, 2-5] implies that all nontrivial elements of $\Gamma_{\varepsilon,p}$ are hyperbolic. Hence there exists a hyperbolic isometry

γ such that $\delta_{\{\gamma\}}(p) < \varepsilon$. Then there exists a neighborhood W of p such that $\delta_{\{\gamma\}}$ is smaller than ε in W. Hence $W \subset U$, as required.

Claim 2. U is closed in S_i.

Proof. Let p be an element of $S_i \cap \overline{U}$. Since p is contained in S_i, there exists a nontrivial element γ of Γ such that $\delta_{\{\gamma\}}(p) < \varepsilon$. Hence there exists a neighborhood W of p such that $\delta_{\{\gamma\}}$ is smaller than ε on W. Since $p \in \overline{U}$, it follows that $W \cap U \neq \varnothing$. Let $q \in W \cap U$. Since $\Gamma_{\varepsilon,q}$ contains a hyperbolic isometry, all elements of $\Gamma_{\varepsilon,q}$ are hyperbolic [5, 2-5]. Hence γ is hyperbolic. It follows that $p \in U$, as desired.

Since S_i is connected, Lemma 3-3 follows from Claims 1 and 2.

We return to the proofs of Theorems 3-1 and 3-2. We show that (1) holds if $\Gamma_{\varepsilon,p}$ contains a hyperbolic element and (2) holds if $\Gamma_{\varepsilon,p}$ does not contain a hyperbolic element.

Case 1. The case when Γ_p contains a hyperbolic element.

In this case, Lemma 3-3 and [5, 2-5] imply that there exists uniquely a maximal almost nilpotent subgroup $\Gamma_{p,i}$ containing $\Gamma_{\varepsilon,p}$, and that $\Gamma_{p,i}$ does not depend on p. Then, Margulis' lemma ([1, 2-5], [5, 3-2]) and [5, 2-5] imply that $\Gamma_i \simeq \mathbf{Z}$ and that all nontrivial elements of Γ_i are hyperbolic with the same invariant geodesic l. It is easy to see that $\tilde{S}_i$ contains l. Let p be an element of $\tilde{S}_i$ and q be the element of l such that $d(p, q) = d(p, l)$. Then, it is easy to see that $\overline{p\,q} \subset \tilde{S}_i$. On the other hand, [5, 3-4] implies that $S_i = \Gamma_i \setminus \tilde{S}_i$. Therefore S_i is diffeomorphic to an $\mathbf{R}^{n-1}$-bundle over S^1. Thus we have proved that (1) holds in this case.

Case 2. The case when $\Gamma_{\varepsilon,p}$ does not contain a hyperbolic element.

In this case, we must prove Theorems 3-1 and 3-2 separately.

Proof of Theorem 3-1. Since all elements of Γ_i are parabolic, Eberlein [4, Corollary 3-3] implies that there exists $p_0 \in \partial X$ such that $\Gamma_{p_0} = \Gamma_i$. Since a parabolic isometry has only one fixed point, it follows that p_0 is uniquely determined.

We need some facts on horosphere here.

Let p_0 be a point on ∂X. We define the Buseman function β_{p_0} as follows. Take $q \in X$, and let l: $[0, \infty) \to X$ be the half geodesic satisfying $l[0, \infty)) = \overline{p_0 q} - \{p_0\}$. For an element p of X, set $\beta_{p_0}(p) = \lim_{t\to\infty}(d(l(t), p) - t)$. Then, it is proved in Eberlein-O'Neill [3, p. 56, Propositions 3-1 and 3-5], that the above functions converge to a C^1-function of p, and the resulting function does not depend on q modulo a constant number. Let us denote the limit by β_{p_0}. For a point p_0 on ∂X, a horosphere of p_0 is a set which is $\{p \in X | \beta_{p_0}(p) = a\}$ for some positive number a.

Lemma 3-4. *Any horosphere is diffeomorphic to Euclidean space.*

Proof. Set $V = \{ p \in X | \beta_{p_0}(p) = a \}$. We prove that V is diffeomorphic to $\mathbf{R}^{n-1}$.

Take $q \in X - V$ such that $\overline{q\, p_0} \cap V \neq \emptyset$. It is easy to see that $\overline{q\, p_0} \cap V$ consists of one point. Let o denote this point. Put $\vec{x} = \mathrm{grad}(\beta_{p_0})$. Then we see

$$\vec{x}(p) = \frac{d}{dt}\left(\overline{q\, p_0}(t)\right)|_{t=0} \qquad \left(t \in T_p(X)\right).$$

Hence β_{p_0} has no singular point. It follows that V is a C^1-submanifold of X. Denote by $\vec{y}$ the vector field on V such that $\vec{y}(p)$ is the orthogonal projection to $T_p(V)$ of $d/dt\overline{pq}(t)|_{t=0}$ $(\in T_p(X))$. The vector field $\vec{y}$ is continuous but not necessarily differentiable.

Assertion 3-5. *If $p \neq 0$, then $\vec{y}(p) \neq 0$.*

Proof. If $\vec{y}(p) = 0$, then $d/dt\overline{pq}(t)|_{t=0}$ is parallel to $\vec{x}(p)$. Hence q, p and p_0 lie on one geodesic. It follows that $q \in \overline{qp_0}$. Therefore $p = o$, as desired.

We return to the proof of Lemma 3-4. For a point p of V, set $g(p) = d(p, q)$. It is easy to see that $\mathrm{grad}(g) = \vec{y}$. Hence, Assertion 3-5 implies that g has only one critical point o in V.

If o is a nondegenerate critical point, Morse theory would imply the lemma. But we do not know this fact. Hence we proceed as follows.

Choose a C^∞-structure on V which is compatible with the C^1-structure as a submanifold of X. Let D be a neighborhood of o in V such that D is diffeomorphic to D^{n-1}. Let U and U' be open subsets of V such that $o \in U \subset \overline{U} \subset U' \subset \overline{U'} \subset \mathrm{Int}(D)$.

Since $\vec{y}(p)$ is not equal to 0 for each p not equal to o, it follows that there exists a C^1-function g' on V such that the following conditions are satisfied:

(1) g' is of C^∞ class on $V - U$.

(2) g' coincides with g in a neighborhood of o.

(3) g' is nonsingular on $V - \{0\}$.

(4) For every positive number b, the set $\{ p \in V | g'(b) \leqslant b \}$ is compact.

Let φ be a C^∞-function on V satisfying the following conditions:

(1) $\varphi(p) = 0$, for $p \in U'$.

(2) $\varphi(p) = 1$, for $p \in X - D$.

(3) $0 \leqslant \varphi(p) \leqslant 1$.

Choose a (C^∞-) Riemannian metric on V and set $\vec{z} = \varphi\, \mathrm{grad}\, g'$.

By condition (1) on g' and condition (1) on φ, the vector field $\vec{z}$ is of C^∞ class. Hence there exists a one-parameter family of transformations Φ_t associated with $\vec{z}$.

Condition (4) on g' and condition (2) on φ imply the following: for each compact subset K of V there exists a positive number $t(K)$ such that $\Phi_t(D)$ contains K for each $t \geqslant t(K)$.

Take a sequence of compact sets $K_1, K_2, \cdots$ such that $K_i \subset K_{i+1}$, and $\bigcup_{i=1} K_i = V$. Define positive numbers $t_1, t_2, \cdots$ inductively as follows: $t_1 = t(D \cup K_1), \cdots, t_{i+1} = t(\Phi_{t_i}(D) \cup K_i)$. Set $D_i = \Phi_{t_i}(D)$. Then $D_i \subset D_{i+1}$, $\bigcup_{i=1} D_i = V$ and each D_i is diffeomorphic to $D^{n-1}(1)$.

Therefore, the annulus theorem (see for example [12, Corollary 2-16-1]) implies that $D_i - D_{i-1}$ is PL-homeomorphic to $S^{n-2} \times [0,1)$. Therefore V is diffeomorphic to $\mathbf{R}^{n-1}$ (see [11]). The proof of Lemma 3-4 is completed.

We return to the proof of Theorem 3-1.

We have shown that there exists uniquely an element p_0 of X such that $\Gamma_i = \Gamma_{p_0}$. It follows easily that

$$\tilde{S}_i = \left\{ p \in X | \delta_{\Gamma_i}(p) < \varepsilon \right\}.$$

By a method similar to the proof in [5, 3-4], we can prove that

$$S_i \simeq \Gamma_i \setminus \tilde{S}_i. \tag{3-1}$$

We see easily that

$$\overline{p\, p_0} \cap X \subset \tilde{S}_i \quad \text{for each } p \in \tilde{S}_i. \tag{3-2}$$

On the other hand, Eberlein [4, Lemma 3-1(e)] implies that there exists a positive number a such that

$$\left\{ p \in X | \beta_{p_0}(p) \leqslant a \right\} \subset \tilde{S}_i. \tag{3-3}$$

Now set $\operatorname{grad}(\beta_{p_0}) = \vec{x}$. Since $\vec{x}$ is invariant by the action of Γ_i, it follows that $\vec{x}$ induces a vector field on $S_i = \Gamma_i \setminus \tilde{S}_i$. We denote this vector field also by $\vec{x}$. Take the number a given in (3-3), and put $S_i' = \{ p \in M | \beta_{p_0}(p) \leqslant a \}$.

By (3-2), we obtain a homeomorphism h: $S_i \to S_i'$ such that, for $p \in \tilde{S}_i$, two points p and $h(p)$ are contained in the same orbit of $\vec{x}$.

On the other hand, if we let V denote $\{ p \in X | \beta_{p_0}(p) = a \}$, then Lemma 3-4 implies that V is diffeomorphic to $\mathbf{R}^{n-1}$. On the other hand, $S_i' \simeq \Gamma_i \setminus V \times [0,1)$. Therefore $S_i \simeq \Gamma_i \setminus V \times [0,1)$.

Since ∂S_i is closed in $M - \bigcup_i \operatorname{Int} S_i$, and since $M - \bigcup_i \operatorname{Int} S_i$ is compact, it follows that $\partial S_i \simeq \Gamma_i \setminus V$ is compact. Thus the proof of Theorem 3-1 is completed.

Proof of Theorem 3-2 *in the case when* $\Gamma_{\varepsilon,p}$ *contains no hyperbolic elements.*

Assertion 3-6. *Suppose* $\Gamma_{\varepsilon,q}$ *contains no hyperbolic elements for each* $q \in \tilde{S}_i$. *Then there exists a* C^∞*-vector field* $\vec{x}$ *on* S_i *such that the following holds*:

Let p *be an arbitrary point of* S_i, *and let* $\gamma_1, \gamma_2, \cdots$, *be all elements of* Γ *such that* $\delta_{\{\gamma_j\}}(p) = \delta_\Gamma(p)$. *Then, for each* j, *we have*

$$(\vec{x}(p))\left(\delta_{\{\gamma_j\}}\right) > 0. \tag{3-4}$$

Proof. Margulis' lemma and [5, 2-7], imply that, for each $p \in S_i$, there exists a vector $\vec{x}(p) \in T_p(X)$ such that (3-4) holds for each j. Then Assertion 3-6 follows by using a partition of unity.

Now we can complete the proof of Theorem 3-2. Assertion 3-6 immediately implies that $L = \{p \in S_i | \delta_\Gamma(p) = \varepsilon/2\}$ is a topological submanifold of S_i. It is easy to see that L is compact.

Let Ψ_t be the 1-parameter group of transformations associated to $\vec{x}$. For each point p of L, the intersection of ∂S_i and $\{\Psi_t(p) | t \in \mathbf{R}\}$ consists of one point, which we denote by $\Psi_{g(t)}(p)$. For $(p, t) \in L \times [0, 1)$, set

$$F(p, t) = \Psi_{g(t) + \tan(\pi t/2)}(p).$$

Then, clearly, $F\colon L \times [0, 1) \to S_i$ is a homeomorphism. Set $\Phi = F^{-1}$. Condition (2)(a) follows from Assertion 3-6.

Proof of condition (2)(b). If $\limsup_{t \to 1} \delta_\Gamma(\Phi^{-1}(p, t)) \neq 0$, then there exists a sequence of elements, $t_1, t_2, \cdots$ of $[0, 1)$ and a positive number θ such that

$$\tan(\pi t_{i+1}/2) - \tan(\pi t_i/2) > 1, \tag{3-5}$$

$$\delta_\Gamma\big(\Psi^{-1}(p, t_i)\big) > \theta. \tag{3-6}$$

On the other hand, since the set $K = \{q \in S_i | \delta_\Gamma(p) > \theta\}$ is compact, there exists a positive number λ such that

$$\delta_\Gamma(q) - \delta_\Gamma\big(\Phi_t(q)\big) < -\lambda \tag{3-7}$$

for every $q \in K$ and $t \geqslant 1$.

Equations (3-5)–(3-7) imply that

$$\delta_\Gamma\big(\Phi^{-1}(p, t_i)\big) < \delta_\Gamma\big(\Phi^{-1}(p, t_{i-1})\big) - \lambda.$$

This contradicts (3-6). The proof of Theorem 3-2 is now completed.

We call S_i an ε-tube if (1) is satisfied and we call S_i an ε-cusp if (2) is satisfied.

Remark 3-7. For each point p of X, there exist a finite number of elements $\gamma_1, \gamma_2, \cdots, \gamma_k$ of Γ such that $\delta_\Gamma = \min_{j=1}^{k}(\delta_{\gamma_j})$ on some neighborhood of p. Using this fact, we see that L is a PL-submanifold and that Φ is a PL-homeomorphism.

4. Proof of Theorem I

In this section, we prove Theorem I. We need the following lemma of Gromov, which played the key role in the proof by Gromov of his finiteness theorem.

In this section, we assume either that M has negative curvature or that the universal covering space of M satisfies the visibility axiom.

Let $n \geqslant 4$ be an integer. Let ε be a number smaller than $\varepsilon_2/2$. Suppose M satisfies $\mathrm{Vol}(M) < \infty$, $c^-(M) \geqslant -1$ and $\dim(M) = n$. Let S be an ε-tube of M and S' be the 2ε-tube containing S, and let $\pi(l)$ be the closed geodesic which is contained in S and whose length is smaller than ε. Assume $d(\partial S, \pi(l)) > 2\varepsilon$ and $d(M - S, \pi(l)) > 3$.

Lemma B (*Gromov* [5, 4.4]). *There exists a positive number* C_3 *such that*

$$\mathrm{Vol}(S') \geqslant C_3 \cdot \mathrm{diam}(S)^{p_n} \varepsilon^n,$$

where $p_n = 1$ *if* $n \geqslant 8$, $p_n = 3/2$ *if* $n = 6$ *or* 7, $p_n = 3$ *if* $n = 4$ *or* 5.

Gromov, in [5], also remarked the following: for each $\delta > 0$ we have a constant $C_4(\delta)$ such that,

$$\mathrm{Vol}(S') \geqslant C_4(\delta) \cdot \mathrm{diam}(S) \cdot \varepsilon^n \tag{4-1}$$

holds for M satisfying $c^+(M) < -\delta$ in addition.

Gromov deduced, from Lemma B, the inequality

$$\mathrm{diam}(M) \leqslant \mathrm{const} \cdot \mathrm{Vol}(M)^{p_n}$$

in the case when M is compact. This formula does not hold in the case when M is noncompact. But we can prove a similar formula.

Theorem 4-1. *For each positive integer* n *with* $n \geqslant 4$ *and for each positive number* ε *with* $\varepsilon \leqslant \varepsilon_2/2$, *there exists a positive number* $C_5(\varepsilon)$ *such that the following holds. If* M *is connected and satisfies* $\dim(M) = n$ *and* $c^-(M) \geqslant -1$, *then we have*

$$\mathrm{diam}(M - (\textit{the union of all } \varepsilon\textit{-cusps})) \leqslant C_5(\varepsilon) \cdot \mathrm{Vol}(M)^{p_n}.$$

Theorem 4-1′. *For each positive number* δ *and* $\varepsilon \leqslant \varepsilon_2/2$, *there exists a positive number* $C_6(\varepsilon, \delta)$ *such that*

$$\mathrm{diam}(M - (\textit{the union of all } \varepsilon\textit{-cusps})) \leqslant C_6(\varepsilon, \delta) \cdot \mathrm{Vol}(M)$$

holds if M *satisfies* $c^+(M) < -\delta$ *in addition.*

Proof of Theorems 4-1 *and* 4-1′. By $i_M \leqslant \delta_\Gamma$, we have

$$M - M_\varepsilon \subset M - M(\varepsilon). \tag{4-2}$$

Hence Assertion 1-3 and (4-2) imply

$$\mathrm{diam}(M - M_\varepsilon) \leqslant C_2 \cdot \mathrm{Vol}(M) \cdot \varepsilon^{1-n}.$$

On the other hand, Lemma B implies

$$\mathrm{diam}(\text{an } \varepsilon\text{-cusp of } M) \leqslant \left(C_3^{-1} \cdot \mathrm{Vol}(M) \cdot \varepsilon^{-n}\right)^{p_n}. \tag{4-4}$$

Theorem 4-1 immediately follows from (4-3) and (4-4). The proof of Theorem 4-1′ is similar.

Theorem 4-2. *For each integer n with $n \geqslant 4$, and for each positive number V, there exists a positive number $\varepsilon_3(V)$ such that the following holds. If an n-dimensional manifold M satisfies* $\mathrm{Vol}(M) < V$ *and* $c^-(M) \geqslant -1$, *then M contains no $\varepsilon_3(V)$-tubes. In other words, $M_{\varepsilon_3(V)}$ consists only on cusps.*

Proof. We may assume that M is connected. Since $M - M_{\varepsilon_2} \neq \varnothing$, we have

$$\mathrm{Vol}(M - (\varepsilon_2/2\text{-cusps})) \geqslant \mathrm{Vol}(D^n(1)) \cdot (\varepsilon_2/4)^n. \tag{4-5}$$

Set

$$\varepsilon_3(V) = \varepsilon_1\big((\varepsilon_2/4)^n \cdot \mathrm{Vol}(D^n(1)), C_5(\varepsilon_2/2) \cdot V^{p_n}\big).$$

We will prove by contradiction that M has no $\varepsilon_3(V)$-tubes.

Suppose that S is an $\varepsilon_3(V)$-tube of M. Take a point p_0 on l. We see

$$\text{length of } l \leqslant \varepsilon. \tag{4-6}$$

Since p_0 is contained in a tube, it is not contained in any cusps. On the other hand, Theorem 4-1 implies that

$$\mathrm{diam}(M - (\varepsilon_2/2\text{-cusps})) \leqslant C_5(\varepsilon_2/2) \cdot V^{p_n}. \tag{4-7}$$

Therefore, Theorem A, the definition of $\varepsilon_3(V)$ and equations (4-6) and (4-7) imply

$$\mathrm{Vol}(M - (\varepsilon_2/2\text{-cusps})) < (\varepsilon_2/4)^n \cdot \mathrm{Vol}(D^n(1)).$$

This contradicts (4-5). The proof of Theorem 4-2 is completed.

Proof of Theorem I. Now we can prove Theorem I. First we assume $n \geqslant 4$. Set $a = \varepsilon_3(V)/64$, $b = \varepsilon_3(V)/16$ and $c = \varepsilon_3(V)/8$. Suppose M satisfies the conditions of Theorem I. It suffices to show that M satisfies the conditions of Theorem 1-1 when we take the numbers a, b and c as above.

It is clear that M satisfies Conditions (1), (2) and (3) in Theorem 1-1. We will verify Condition (4). We treat only the case when the universal covering space of M satisfies the visibility axiom. The proof in the case when M has negative curvature is similar (use Theorem 3-2 instead of Theorem 3-1).

Set $M' = M_{\varepsilon_3(V)}$. Theorem 4-2 implies that $M_{\varepsilon_3(V)}$ consists of cusps. Take one of the cusps, and let us denote it by S_i. Let p be the point on ∂X such that $\Gamma_i = \Gamma_{p_0}$. (We are using the notation used in Theorem 3-1.) Denote by Φ_i the PL-homeomorphism between S_i and $[0,1) \times (\Gamma_i \setminus \mathbf{R}^{n-1})$ given in Theorem 3-2. (As was remarked in 3-7, the homomorphism given in Theorem 3-2 is a PL-homeomorphism.)

Then $\Phi_i^{-1}([0,1) \times \{\text{a point}\})$ is an orbit of $\vec{x}$. (Here $\vec{x}$ is the gradient vector field of Buseman function.) Hence there exists a PL-homeomorphism Φ_i': $S_i \to \partial S_i \times [0,1)$ such that

$$\Phi_i'^{-1}(\partial S_i \times \{1/2\}) = \{\, p \in S_i | \delta_\Gamma(p) = \varepsilon_3(V)/16 \,\}.$$

Let Φ: $M - M' \to (0,1) \times \partial M'$ be the PL-homeomorphism satisfying $\Phi|_{S_i} = \Phi_i'$.

Now we will show that condition (3) is satisfied if we take this function Φ and this open subset M'. (i) is clear.

Proof of (ii). We have

$$M' \cap M(c) = \{ p \in M | \delta_\Gamma(p) \geqslant \varepsilon_3(V), i_M(p) \leqslant \varepsilon_3(V)/8 \}.$$

On the other hand, we see $\delta_\Gamma/4 \leqslant i_M$. Therefore we have $M' \cap M(c) = \varnothing$, as desired.

Proof of (iii). Suppose p is an element of $\Phi^{-1}(\partial M' \times \{1/2\})$. By the definition of Φ, we have $\delta_\Gamma(p) = \varepsilon_3(V)/16$. Hence

$$a = \varepsilon_3(V)/64 \leqslant i_M(p) \leqslant \varepsilon_3(V)/16 = b.$$

Therefore we have $p \in M(b) - M(a)$, as desired.

Thus we have proved Theorem I in the case when $n \geqslant 4$.

In the case when $n = 2$, Theorem I can be easily deduced from Gromov's Betti number estimate $b_i(M) \leqslant \text{const} \cdot \text{Vol}(M)$ [8, p. 12].

5. A counterexample

In this section, we given an example which shows that we cannot replace Condition (2) in Theorem I by "M has nonpositive curvature".

Proposition 5-1. *For each positive integer n greater than* 2, *there exists a sequence of Riemannian manifolds M_1, $M_2, \cdots$ which has the following properties*:

(1) *The volumes of M_i ($i = 1, 2, \cdots$) are uniformly bounded.*

(2) *There exists a positive number C_7 such that*

$$0 \geqslant \textit{sectional curvature of } M_i \geqslant -C_7$$

holds for each i.

(3) *If $i \neq j$, then the Betti numbers of M_i and M_j are distinct.*

Since our construction is quite similar to the one in Gromov [5], we give only an outline of the construction. (See also Eberlein [4, p. 459].) First we study the case when $n = 3$. In [5], Gromov took infinitely many manifolds X_i, which are diffeomorphic to (Torus–two open disks) $\times$ circle. And he made a manifold M_∞ by identifying one of the boundaries of X_i to that of X_{i-1} and by identifying the other boundary of X_i to that of X_{i+1}. Then M_∞ is a nonpositively curved manifold which has finite volume and bounded curvature but whose Betti number is infinite. Now, we take finitely many manifolds $X_1, \cdots, X_k$ from X_i ($i = 1, 2, \cdots$). Then we get a manifold M_k' by identifying their boundaries in a similar way. The boundary of M_k' is one of the boundaries of

X_k. We modify M_k' and obtain a closed manifold M_k. These manifolds have required properties.

In the case when $n \neq 3$, we take $M_k \times (S^1)^{n-3}$.

6. An estimate on the number of homotopy types

In this section, we give an estimate on the number of homotopy types containing manifolds satisfying the conditions of Theorem 1-1.

Theorem 6-1. *In each dimension, there exists a positive number C_8 such that the following holds. For each positive number a, b, c and V satisfying $a < b < c$ and $V \leqslant 1$, there exist at most $(d^{-n}V)^{C_8 d^{-n}V}$ homotopy types containing manifolds N satisfying the conditions of Theorem* 1-1. *Here we put $d = \min\{a/3, (b-c)/2, 1\}$.*

Proof of Theorem 6-1.

Lemma 6-2. *There exist positive numbers C_9 and C_{10} such that the following holds. For each N satisfying the conditions of Theorem* I, *there exist open subsets $D_1, \cdots, D_L$ of N such that following holds*:

(1) *$L \leqslant C_9 V d^{-n}$.*

(2) *There exists $L' \leqslant L$ such that*

$$\text{(i)} \quad N^{(b)} \supset \bigcup_{i=1}^{L'} D_i \supset N^{(c)},$$

$$\text{(ii)} \quad \bigcup_{i=1}^{L} D_i \supset N^{(a)}$$

(3) *For each $\{i_1, \cdots, i_k\} \subset \{1, 2, \cdots, L\}$, the set $\cap_{j=1}^{k} D_{i_j}$ is contractible if it is nonempty.*

(4) *For each $i \leqslant L$, we have*

$$\#\left\{ j \leqslant L \mid D_i \cap D_j \neq \varnothing \right\} \leqslant C_{10}.$$

Proof. Let $Z = \{p_1, p_2, \cdots\}$ be a maximal subset of $N^{(a/4)}$ such that $d(p, q) \geqslant d/2$ holds for the two elements p and q of Z. A method similar to the proof of (1-1) in §1 shows that $D_i = D_{p_i}(d)$ has the desired property.

Now we construct a simplicial complex $\mathscr{S}_M$ as follows:

(a) The vertex set of $\mathscr{S}_M$ is $\{1, 2, \cdots, L\}$.

(b) For $0 < i_1 < i_2 < \cdots < i_k$, the set $\{i_1, \cdots, i_k\}$ is the vertex set of some simplex of $\mathscr{S}_M$ if and only if

$$D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k} \neq \varnothing .$$

Denote by $\mathscr{S}_M'$ the full subcomplex of $\mathscr{S}_M$ whose vertex set is $\{1, \cdots, L'\}$. The suffix M is omitted when no confusion arises. Let $|\mathscr{S}|$ and $|\mathscr{S}'|$ be geometric realizations of $\mathscr{S}$ and $\mathscr{S}'$ respectively. It is well known that (3) in Lemma 6-2 implies that there exist homotopy equivalences

$$\Psi\colon |\mathscr{S}| \to \bigcup_{i=1}^{L} D_i \quad \text{and} \quad \Psi'\colon |\mathscr{S}'| \to \bigcup_{i=1}^{L'} D_i$$

which commute with natural inclusions $|\mathscr{S}'| \to |\mathscr{S}|$ and $\bigcup_{i=1}^{L'} D_i \to \bigcup_{i=1}^{L} D_i$.

Now, by (1) and (4) in Lemma 6-2, we see easily that $\mathscr{S}$ satisfies the following conditions:

(i) The number of vertices is smaller than $C_9 V d^{-n}$.

(ii) There exist at most C_{10} vertices which can be joined with a given vertex of $\mathscr{S}$ by some 1-simplex of $\mathscr{S}$.

We put $C =$ the maximum integer smaller than $C_9 V d^{-n}$. Let Y be the set of all simplicial complexes satisfying conditions (i) and (ii) above.

Lemma 6-3. *There exists a constant C_{11} such that the number of isomorphism classes containing elements of Y is smaller than $(C_{11} V d^{-n})^{C_{11} V d^{-n}}$.*

Proof. Let Δ^C denote the simplex whose vertex set is $\{1, 2, \cdots, C\}$. It is easy to see that every element of Y can be represented as a subcomplex of Δ^C.

Let Z be the set consisting of all subcomplexes of Δ^C which have less than C_{10} vertices.

Define a map $\Omega = (\omega_1, \omega_2, \cdots, \omega_C)\colon Y \to Z^C$, as follows.

$$\omega_i(\mathscr{S}) = \text{the full subcomplex of } \mathscr{S} \text{ whose vertex set is}$$
$$\{\, j \mid j \text{ can be joined with } i \text{ by some 1-simplex of } \mathscr{S}.\}$$

Condition (ii) implies that $\omega_i(\mathscr{S}) \in Z$. Hence the map Ω is well defined. (Here we identified an element of Y to a subcomplex of Δ^C which is isomorphic to it.)

Clearly Ω is injective. On the other hand, it can be proved by an easy combinatorial argument that there exists a positive number C_{12} depending only on n, the dimension, and satisfying

$$\# Z \leqslant D_{12} \cdot C^{C_{10}} \leqslant C_{12} \cdot \left(C_9 V d^{-n}\right)^{C_{10}}.$$

Lemma 6-3 follows immediately from these two facts.

Now we return to the proof of Theorem 6-1. Since $\mathscr{S}'$ is a full subcomplex of $\mathscr{S}$ and since the order of the vertex set of $\mathscr{S}$ is smaller than C, it follows that there exist at most $2^C \cdot (C_{11} V d^{-n})^{C_{11} V d^{-n}}$ isomorphism classes containing pairs $(\mathscr{S}_N, \mathscr{S}_N')$ for some N satisfying the conditions of Theorem 1-1.

Therefore Theorem 1-1 follows immediately from the lemma below.

Lemma 6-4. *If $(\mathscr{S}_N, \mathscr{S}_N')$ is isomorphic to $(\mathscr{S}_{N'}, \mathscr{S}_{N'}')$, then N is homotopy equivalent to N'.*

Proof. Let G be an Abelian group and let $\varphi\colon \pi_1(N) \to G$ be a homomorphism. G and φ induce local coefficient system on $|\mathscr{S}|$ and $|\mathscr{S}'|$. Let G^φ denote this local coefficient system.

Assertion 6-5. *$\pi_1(N)$ is isomorphic to the image of the homomorphism: $\pi_1(|\mathscr{S}'|) \to \pi_1(|\mathscr{S}|)$ induced by the inclusion map. Also, for each G and φ, the group $H_*(N; G^\varphi)$ is isomorphic to the image of the homomorphism: $H_*(|\mathscr{S}'|; G^\varphi) \to H_*(|\mathscr{S}|; G^\varphi)$ induced by the inclusion map.*

Proof. Let Φ be the map given in condition (3) in Theorem 1-1. Set $N'' = \Phi^{-1}(\partial N' \times [0, 1/2]) \cup N'$. Then the homomorphism: $\pi_1(N'') \to \pi_1(N)$ is an isomorphism. Hence considering the diagram

$$\pi_1\left(\bigcup_{i=1}^{L'} D_i\right) \to \pi_1(N'') \to \pi_1\left(\bigcup_{i=1}^{L} D_i\right) \to \pi_1(N)$$

we can easily show that the homomorphism: $\pi_1(\bigcup_{i=1}^{L} D_i) \to \pi_1(N)$ is injective on the image of the homomorphism: $\pi_1(\bigcup_{i=1}^{L'} D_i) \to \pi_1(\bigcup_{i=1}^{L} D_i)$.

On the other hand, since the homomorphism: $\pi_1(N') \to \pi_1(N)$ is an isomorphism, considering the diagram

$$\pi_1(N') \to \pi_1\left(\bigcup_{i=1}^{L'} D_i\right) \to \pi_1(N),$$

we can easily show that the homomorphism: $\pi_1(\bigcup_{i=1}^{L'} D_i) \to \pi_1(N)$ is surjective.

The above two facts immediately imply the statement on the fundamental groups. The proof of the statement on the homology groups is similar. The proof of Assertion 6-5 is completed.

Now we return to the proof of Lemma 6-4. Let ψ be the isomorphism between $(\mathscr{S}_{N_1}, \mathscr{S}'_{N_1})$ and $(\mathscr{S}_{N_2}, \mathscr{S}'_{N_2})$. Let φ: $N_1' \to N_2$ be the composition of the following five maps: the injection: $N_1' \to \bigcup_{i=1}^{L} D_i$, the homotopy equivalence: $\bigcup_{i=1}^{L} D_i \to |\mathscr{S}_{N_1}|$, the map $|\psi|$: $|\mathscr{S}_{N_1}| \to |\mathscr{S}_{N_2}|$, the homotopy equivalence: $|\mathscr{S}_{N_2}| \to \bigcup_{i=1}^{L} D_i$ and the injection $\bigcup_{i=1}^{L} D_i \to N_2$. Then, using Assertion 6-5, it is easy to see that φ induces isomorphisms both on fundamental groups and on homology groups of any local coefficient system. Therefore φ is a homotopy equivalence, as desired. Thus the proofs of Lemma 6-4 and that of Theorem 6-1 are completed.

Next, we give an estimate on the number of homotopy types containing manifolds satisfying the conditions of Theorem I.

First we need an estimate on the number $\varepsilon_1(a, D)$ in Theorem A. Heintze & Karcher [10, Corollary 2.3.2] gives

$$\varepsilon_1(a, D) \geqslant (2\pi a/\mathrm{Vol}(S^m)) \cdot \sinh(D)^{-m+1}.$$

Therefore there exists a constant C_{13} such that

$$\varepsilon_1(a, D) \geqslant a \cdot \exp(-C_{13} D).$$

On the other hand, the number $\varepsilon_3(V)$ given in Theorem 4-2 is

$$\varepsilon_1\left((\varepsilon_2/2)^n \mathrm{Vol}(D^n),\ C_5(\varepsilon_2/2) \cdot V^{p_n}\right).$$

Therefore there exists a constant C_{14} such that

$$\varepsilon_3(V) \geqslant \exp(-C_{14}V^{p_n}).$$

On the other hand, we have proved in §4 the following: if we put $a = \varepsilon_3(V)/64$, $b = \varepsilon_3(V)/16$ and $c = \varepsilon_3(V)/8$, and if we assume that M satisfies the conditions of Theorem I, then M satisfies the conditions of Theorem 1-1.

Using these facts and Theorem 6-1, we obtain the following result.

Theorem 6-6. *For each positive integer n greater than* 3, *there exists a positive number C_{15} such that the following holds. For each positive number V, there exists at most* $\exp(\exp(C_{15}V^{p_n}))$ *homotopy types containing manifolds satisfying the conditions of Theorem* I.

The number of homotopy types containing manifolds M satisfying $c^+(M) \leqslant -\delta$ in addition, can be estimated by $\exp(\exp(C_{16}(\delta)V))$. (This fact can be proved by using Theorem 4-1′ instead of Theorem 4-1 in the proof of Theorem 6-6.) Using this fact and Mostow's rigidity theorem ([14], [16]), we can easily prove Theorem II.

References

[1] P. Buser & H. Karcher, *Gromov's almost flat manifolds*, Astérisque **81** (1981) 1–148.
[2] J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math. **92** (1970) 61–74.
[3] P. Eberlein & B. O'Neill, *Visibility manifolds*, Pacific J. Math. **12** (1960) 45–109.
[4] P. Eberlein, *Lattices in space of nonpositive curvature*, Ann. of Math. **111** (1980) 425–476.
[5] M. Gromov, *Manifolds of negative curvature*, J. Differential Geometry **13** (1978) 223–230.
[6] ______, *Almost flat manifolds*, J. Differential Geometry **13** (1978) 231–242.
[7] ______, *Hyperbolic manifolds according to Thurstone and Jørgensen*, Seminar Bourbaki, 1979/80, No. 549.
[8] ______, *Volume and bounded cohomology*, Inst. Hautes Étude Sci. Publ. Math. **56** (1982) 5–100.
[9] E. Heintze, *Mannigfaltigkeiten negatvier Krümmung*, Habilitation-schrift, University of Bonn, 1976.
[10] E. Heintze & H. Karcher, *A general comparison theorem with application to volume estimate for submanifolds*, Ann. Sci. Ecole Norm. Sup. Ser. 4 **11** (1978) 451–470.
[11] M. Hirsch & B. Mazur, *Smoothings of piecewise linear manifolds*, Ann. of Math. Studies No. 80, Princeton University Press, Princeton, 1974.
[12] J. Hudson, *Piecewise linear topology*, Benjamin, 1969.
[13] G. Margulis, *On connections between metric and topological properties of manifolds of nonpositive curvature*, Proceeding of the Sixth Topological Conference (Tibilisi, U.S.S.R., 1972). (Russian)
[14] G. Mostow, *Strong rigidity of locally symmetric spaces*, Annals of Math. Studies No. 78, Princeton University Press, Princeton, 1973.
[15] S. Peters, *Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds*, preprint.
[16] G. Prasad, *Strong rigidity of* **Q**-*rank* 1 *lattices*, Invent. Math. **21** (1973) 255–283.
[17] H. Wang, *Topics in totally discontinuous groups*, in Symmetric Spaces, Boothy Weiss, 1972.
[18] A. Weinstein, *On the homotopy types of positively pinched manifolds*, Arch. Math. (Basel) **8** (1967) 523–529.

UNIVERSITY OF TOKYO