J. DIFFERENTIAL GEOMETRY
20 (1984) 471–478

MODULI OF CURVES WITH TWO EXCEPTIONAL WEIERSTRASS POINTS

STEVEN DIAZ

1. Introduction

As usual let $\mathcal{M}_g$ be the moduli space of curves of genus g. Many interesting subloci of $\mathcal{M}_g$ are defined in terms of Weierstrass points.

(1.1) Definition. (a) For a curve C of genus g, a point $p \in C$ and an integer k with $2 \leqslant k \leqslant g$ we say p is a Weierstrass point of type k if $h^0(C, kp) \geqslant 2$.

(b) $D_{k,k} = \{[C] \in \mathcal{M}_g \colon C$ possesses a Weierstrass point of type $k\}$.

It is known that $D_{k,k}$ is an irreducible variety of dimension $2g - 3 + k$ (see [1], [2], [7], [8]). Also for fixed k with $2 < k < g$ it is known that a generic point in $D_{k,k}$ corresponds to a curve possessing only one Weierstrass point of type k (see [4]). In this article we study the locus of curves with two Weierstrass points of type k and more generally curves with a Weierstrass point of type k and a Weierstrass point of type l.

A curve with two distinct points, one of which is a Weierstrass point of type k and the other a Weierstrass point of type l, corresponds locally to a point where $D_{k,k}$ and $D_{l,l}$ meet each other. By simply counting codimensions one might then say that the expected dimension of the locus of points in $\mathcal{M}_g$ corresponding to curves possessing both a Weierstrass point of type k and a distinct Weierstrass point of type l is $3g - 3 - (g - k) - (g - l) = g - 3 + k + l$. We show that if g is even and $k, l \geqslant \frac{1}{2}(g + 2)$ or if g is odd and $k, l \geqslant \frac{1}{2}(g + 3)$, then at least one component of this locus has this expected dimension. In many cases when k and l are not in this range we will find examples whose dimension is larger than this expected dimension.

Along the way we construct examples of reducible Hurwitz schemes in which it can be seen that different components correspond to curves with significantly different geometry. We also prove a lemma about the dimension of the image of a Hurwitz scheme in $\mathcal{M}_g$.

We work over the complex numbers.

Received August 21, 1984.

The author would like to thank Joe Harris and Will Kazez for helpful discussions in the course of the investigations which led to this paper.

2. Deformation theory

An important technique in this paper will be the use of deformation theory of pointed curves to calculate the tangent spaces to subloci of $\mathscr{M}_g$ defined in terms of Weierstrass points. We use the theory developed in [4]. Here we briefly review this and state the results we will need.

Consider a smooth curve C and n marked points $p_1, \cdots, p_n$ on C. Let Θ be the tangent bundle, K the canonical bundle, and $\mathcal{O}$ the structure sheaf of C. First order deformations of C are classified by $H^1(C, \Theta)$ and first order deformations of C together with $p_1, \cdots, p_n$ are classified by $H^1(C, \Theta(-p_1 - \cdots - p_n))$. Given a rational function f on C whose divisor is supported set theoretically on $p_1, \cdots, p_n$ and an element ϕ of $H^1(C, \Theta(-p_1 - \cdots - p_n))$, then it is possible to deform f along the deformation of C, $p_1, \cdots, p_n$ given by ϕ so that the divisor of f remains supported on the deformations of the points if and only if the cup product $\phi \cdot d\log f = 0$ in $H^1(C, \mathcal{O})$.

One is usually interested in sets of the form $\{\phi \in H^1(C, \Theta(-p_1 - \cdots - p_n)): \phi \cdot d\log f = 0\}$. By Kodaira-Serre duality this is the same as the annihilator of the image of the multiplication map.

$$(2.1) \qquad H^0(C, K) \otimes \{d\log f\} \to H^0(C, 2K(p_1 + \cdots + p_n)).$$

Now let C be a curve corresponding to a point in $D_{k,k}$, p a Weierstrass point of type k on C, f a rational function on C with divisor $(f) = -kp + q_1 + \cdots + q_k$ and B the branch divisor of f away from p. Let π: $\mathscr{C}_g \to \Delta$ be the universal curve over a small neighborhood of $[C]$ in $\mathscr{M}_g$ (or if C has automorphisms the universal deformation of C). Denote by $VD_{k,k}$ the locus of Weierstrass points of type k near p in $\mathscr{C}_g$. Notice that near $[C]$ the branch of $D_{k,k}$ corresponding to p is the locus of all deformations of C for which it is also possible to deform p, $q_1, \cdots, q_k$ and f so that the divisor of f remains supported on the deformations of p, $q_1, \cdots, q_k$. $VD_{k,k}$ has a similar description.

Combining these descriptions of $D_{k,k}$ and $VD_{k,k}$ with the previously mentioned deformation theory one can prove the following results (see [4] for more details). Remember when we say "annihilator of" we are using Kodaira-Serre duality.

(2.2) If k is the first nongap of p, then the tangent space to $VD_{k,k}$ at p has the same dimension as $VD_{k,k}$ and is given by $TVD_{k,k}$ = annihilator of

$\{H^0(C, K - q_1 - \cdots - q_k) \otimes d\log f = H^0(C, 2K + p - B)\}$ in $H^1(C, \Theta(-p))$. If in addition $k + 1$ is a gap for p, then the tangent space to the branch of $D_{k,k}$ corresponding to p has the same dimension as $D_{k,k}$ and is given by $TD_{k,k} =$ annihilator of $\{H^0(C, K - p - q_1 \cdots - q_k) \otimes d\log f = H^0(C, 2K - B)\}$ in $H^1(C, \Theta)$.

3. Curves with two exceptional Weierstrass points

First we construct some curves with two Weierstrass points of type k.

(3.1) Lemma. *Fix integers $g \geqslant 2$ and k with $2 \leqslant k \leqslant g$. Then there exists a smooth curve C of genus g with two points $p, q \in C$ satisfying the following conditions.*

(1) *There exists a rational function f on C with divisor $(f) = kp - kq$.*

(2) *In the Weierstrass gap sequence for p the only nongaps smaller than the largest gap are multiples of k.*

(3) *In the Weierstrass gap sequence for q the first nongap is k.*

Proof. A curve satisfying (1) can be obtained by constructing a k sheeted cover of $\mathbf{P}^1$ whose branching consists of 2 total ramification points and $2g$ other simple branch points over distinct points. It is a simple matter to show such covers exist, see for instance [5]. To show that (2) can also be satisfied we will first construct a singular curve and then smooth it.

Start with a smooth rational curve D expressed as a k sheeted cover of $\mathbf{P}^1$ with two points of total ramification. Let $\pi: D \to \mathbf{P}^1$ be the covering map and call p and q the two total ramification points on D. Choose g pairs of points $x_1, y_1, \cdots, x_g, y_g$ on D with $\pi(x_i) = \pi(y_i)$. If we identify each x_i with y_i to make g simple nodes, then we obtain a singular curve of genus g. Call it D'. Regular functions on D' are regular functions f on D with $f(x_i) = f(y_i)$, all i. Thus by choosing the x_i and y_i appropriately we can make p on D' have a Weierstrass gap sequence as described in condition (2) of the lemma. We may smooth the g nodes to obtain a smooth curve C mapping to $\mathbf{P}^1$ as described in the first paragraph of the proof. (It is not hard to construct such a smoothing—see for instance [6, §4].) By semicontinuity, p will continue to have the desired gap sequence. Condition (3) follows because kp is linearly equivalent to kq.

Remark. Using this lemma it is easy to construct many examples of reducible Hurwitz schemes. Choose integers k and g as in the lemma. Suppose k is composite, say $k = mn$, $m, n > 1$. From the lemma we know that the Hurwitz scheme H_k of $\{k$ sheeted covers of $\mathbf{P}^1$ with two points of total ramification and $2g$ other simple branch points over distinct points$\}$ has at least one component a general point of which corresponds to a curve for which the first nongap for each of the two total ramification points is k. The same statement is true with k replaced by m or n throughout. In all cases we may assume that the two points of total ramification lie over 0 and ∞. By raising the function giving an m sheeted cover to the nth power or the other way around we construct a k sheeted cover. Doing this for all points in H_m and H_n we get at least two (one if $n = m$) other components of H_k. In one the first nongap for the total ramification points is generically m and in the other it is generically n. These are different components because they all have dimension $2g - 1$. Notice that this shows that curves corresponding to points in different components of a Hurwitz scheme can have significantly different geometry.

We must define precisely what we mean by the phrase "a component of the locus of points corresponding to curves possessing both a Weierstrass point of type k and a Weierstrass point of type l". Let C be a smooth curve of genus $g \geqslant 3$, k and l integers with $2 \leqslant k, l \leqslant g - 1$, and p and q two distinct points on C which are Weierstrass points of type k and l respectively. Let $\pi\colon \mathscr{C} \to \Delta$ be the universal deformation of C. Let W_k (W_l) be the locus of Weierstrass points of type k (l) near p (q). Choose an irreducible component X of $\pi(W_k) \cap \pi(W_l)$. If X' is the image of X in $\mathscr{M}_g$, then there is a unique irreducible subvariety of $\mathscr{M}_g$ of dimension equal to the dimension of X' containing X'. Call it Y. Any point of Y will correspond to a curve possessing a Weierstrass point of type k and a Weierstrass point of type l and, for a generic point of Y, these two Weierstrass points will be distinct. We will call Y "a component of the locus of points corresponding to curves possessing both a Weierstrass point of type k and a Weierstrass point of type l". Such subvarieties of $\mathscr{M}_g$ will be denoted by $W(k, l)$.

In a sense this is incorrect terminology. Suppose that besides p and q there are other Weierstrass points of type k or l on C. Based on current knowledge it might be that Y is contained as a proper subvariety of an irreducible subvariety Z of $\mathscr{M}_g$ such that any point in Z corresponds to a curve possessing a Weierstrass point of type k and a Weierstrass point of type l.

(3.2) Theorem. *Fix a genus $g \geqslant 4$. Choose two integers k and l with $k, l \leqslant g - 1$ and $k, l \geqslant \frac{1}{2}(g + 2)$ if g is even, or $k, l \geqslant \frac{1}{2}(g + 3)$ if g is odd. Then there exists a component $W(k, l)$ of the locus of points in $\mathscr{M}_g$ corresponding to curves possessing both a Weierstrass point of type k and a Weierstrass point of type l which has dimension $g - 3 + k + l$.*

Proof. From the description of these subvarieties in the preceding paragraphs we see that locally $W(k, l)$ is the intersection of two subvarieties of dimensions $2g - 3 + k$ and $2g - 3 + l$ inside $\mathscr{M}_g$ which has dimension $3g - 3$. Since $\mathscr{M}_g$ is locally the quotient of a smooth variety by a finite group, we conclude that $W(k, l)$ has dimension at least $g - 3 + k + l$. Next, by looking at tangent spaces we show that this lower bound is the actual dimension when $k = l$.

A curve C constructed in (3.1) represents a point in some $W(k, k)$. The tangent space to the branch of $W(k, k)$ represented by p and q at $[C]$ will be the intersection of the tangent spaces to the two branches $D_{k,k}(p)$ and $D_{k,k}(q)$ of $D_{k,k}$ represented by p and q at $[C]$. In the map of C to $\mathbf{P}^1$ constructed in (3.1) let B be the divisor consisting of the $2g$ simple branch points. Notice that K, the canonical divisor of C, is linearly equivalent to $B - p - q$. Using (2.2) we obtain

$$\begin{aligned} T_pD_{k,k} &= \text{annihilator of } H^0(C, 2K - B - (k-1)q), \\ T_qVD_{k,k} &= \text{annihilator of } H^0(C, 2K + q - B - (k-1)p). \end{aligned} \tag{3.3}$$

Inside $T_q\mathscr{C}_g$, intersect $T_qVD_{k,k}$ and $\pi^*T_pD_{k,k}$. The dimension of this intersection gives an upper bound on the dimension of $W(k, k)$.

$$\begin{aligned} \dim W(k, k) \leqslant g - 3 + 2k + \dim\{&H^0(C, 2K - B - (k-1)q) \\ &\cap H^0(C, 2K + q - B - (k-1)p)\}. \end{aligned} \tag{3.4}$$

The vector space inside the braces is $H^0(C, 2K - (k-1)p - (k-1)q) \cong H^0(C, K - 2kp) \cong 0$.

Now we do the case $k \neq l$. We may assume $k < l$. Construct $W(k, k)$ as before. Near $[C]$, $D_{k,k}(p)$ must be contained in a branch of $D_{l,l}$. Call it $D_{l,l}(p)$. By construction $h^0(C, lp) = 2$. Theorem 2 in [7] then assures us that $VD_{l,l}$ is smooth near p, which means that the singularities of $D_{l,l}(p)$, if any, are mild enough so that on it intersections work out dimensionally as on a smooth variety. On $D_{l,l}(p)$ the intersection $D_{k,k}(p) \cap \{D_{l,l}(p) \cap D_{k,k}(q)\}$ has at least one component, $W(k, k)$, of the minimum possible dimension. Therefore, $D_{l,l}(p) \cap D_{k,k}(q)$ must have at least one component of the minimum possible dimension. Take this as $W(k, l)$.

4. Hurwitz schemes

When k and l are both smaller than is allowed in (3.2), then $g - 3 + k + l$ is at most $2g - 2$. The locus of points corresponding to hyperelliptic curves which has dimension $2g - 1$ is therefore a variety $W(k, l)$ of dimension greater than $g - 3 + k + l$. The Hurwitz scheme of $\{k$ sheeted covers of $\mathbf{P}^1$ of genus g with two points of total ramification and $2g$ other simple branch points over distinct points$\}$ also has dimension $2g - 1$. One might therefore expect that the image of this Hurwitz scheme in $\mathscr{M}_g$ would also give rise to varieties $W(k, l)$ of dimension greater than $g - 3 + k + l$. We shall prove that this is at least sometimes true by proving a lemma about the dimensions of the images of certain Hurwitz schemes in $\mathcal{M}_g$.

Let k be a positive integer and B a sequence $b_1, \cdots, b_b$, where each b_i is itself a partition of k, that is, each b_i is a set of integers $b_{i,1}, \cdots, b_{i,m(i)}$ such that $\sum_{j=1}^{m(i)} b_{i,j} = k$. By branching as defined by b_i we mean that over p_i there are $m(i)$ points, one with ramification index equal to each of the $b_{i,j}$.

(4.1)

$$H_{k,B} = \begin{bmatrix} \text{moduli space of the data } \pi\colon C \to \mathbf{P}^1 \text{ of degree } k, p, \cdots, p_b \in \mathbf{P}^1 \\ \text{distinct points, } C \text{ smooth irreducible curve; } \pi \text{ has branching} \\ \text{over each } p_i \text{ as described by } b_i, \text{ otherwise unbranched} \end{bmatrix}.$$

There is a morphism $\sigma\colon H_{k,B} \to \mathcal{M}_g$, where g can be determined by the Riemann-Hurwitz formula. Let $\overline{\mathcal{M}}_g$ be the Deligne-Mumford compactification of $\mathcal{M}_g$. There exists a compactification $\overline{H}_{k,B}$ of $H_{k,B}$ such that σ extends to a morphism $\sigma\colon \overline{H}_{k,B} \to \overline{\mathcal{M}}_g$. This compactification was developed for the case of simple branching in [6]. The same proof works for arbitrary preassigned branching (see [3]). The points which are added to $H_{k,B}$ to get $\overline{H}_{k,B}$ correspond to admissible covers of reducible stable b pointed curves of arithmetic genus 0.

A stable b pointed curve is a reduced, connected curve C with at most ordinary double points, plus b smooth points, $p_1, \cdots, p_b \in C$ such that every smooth rational component of C contains at least 3 points which are either p_i's or double points of C. A morphism $\pi\colon C \to B$, where B is a stable b pointed curve of genus 0, is called an admissible cover if (1) C is reduced connected and of arithmetic genus g, (2) π has degree k, (3) over p_i the branching is as described by b_i, and (4) any branching not over a p_i must be over a double point of B.

In general it is not known whether $H_{k,B}$ is nonempty. We will need the following sufficient condition found in [5, Remark, p. 785].

(4.2) If there exists a permutation τ of $\{1, \cdots, b\}$ and an integer s with $1 \leqslant s \leqslant b$ such that

$$\sum_{i=1}^{s} k - m(\tau(i)) \geqslant k - 1 \quad \text{and} \quad \sum_{i=s+1}^{b} k - m(\tau(i)) \geqslant k - 1,$$

then $H_{k,B}$ is nonempty. Branching data which satisfies this condition is said to be splittable.

When $H_{k,B}$ is nonempty it clearly has dimension max$(0, b - 3)$. We wish to know the dimension of $\sigma(H_{k,B})$, the image of $H_{k,B}$ in $\mathcal{M}_g$.

(4.3) Lemma. *Assume $H_{k,B}$ is nonempty and that all except possibly two of the b_i are of the form $2, 1, \cdots, 1$. Then at least one component of $\sigma(H_{k,B})$ has dimension* $\min\{\dim H_{k,B}, \dim \mathcal{M}_g\}$.

Proof. We prove the equivalent statement for $\overline{H}_{k,B}$. The proof is by induction on both g and b. The cases $g = 0$ or $b = 2$ or 3 are obvious. Assume the theorem is true whenever the genus is less than g or the number of marked points is less than b. Let p_1 and p_2 be the two marked points over which the branching might not be simple. As usual let Δ equal $\overline{\mathcal{M}}_g - \mathcal{M}_g$. Since Δ is a divisor on $\overline{\mathcal{M}}_g$, the lemma will be proven if we show that $\dim \sigma(\overline{H}_{k,B}) \cap \Delta = \min(b - 4, \dim \mathcal{M}_g - 1)$.

We construct an admissible cover $\pi\colon C \to B$ representing a point in $\sigma(\overline{H}_{k,B}) \cap \Delta$. B consists of two smooth rational curves D_1 and D_2 meeting transversely at one pont q. D_1 contains $p_1, \cdots, p_{b-2}$. D_2 contains p_{b-1}, p_b. Over D_1, C consists of a smooth curve C_1 of genus $g - 1$ filling all k sheets. Such a cover exists by (4.2) because one can calculate that if the data $b_1, \cdots, b_{b-2}$ was not splittable, then g would equal 0 and we would be done. Over D_2, C consists of $k - 2$ rational curves étale over D_2 and one rational curve C_2 two sheeted over D_2. Over q there is no branching. Let s_1 and s_2 be the two points of $C_1 \cap C_2$.

Now we will show that the dimension of the image in $\mathcal{M}_g$ of the family of all admissible covers like the one we have just constructed is $\min(b - 4, \dim \mathcal{M}_g - 1)$.

Case 1. $(b - 2) - 3 \leqslant 3(g - 1) - 3$.

By the induction hypothesis the choice of isomorphism class of C_1 varies in a $b - 5$ dimensional family. For each choice of C_1 the choice of s_1 and s_2 varies in a one-dimensional family—by moving q. This gives a total of $b - 4$.

Case 2. $(b - 2) - 3 > 3(g - 1) - 3$.

In this case $b - 3 \geqslant 3g - 3$, so we want to show that our covers vary in a $3g - 4$-dimensional family. By the induction hypothesis the choice of isomorphism class of C_1 varies in a $3(g - 1) - 3$-family and for each choice of C_1 there is at least a one-dimensional choice of different maps of C_1 to D_1. As we vary the choice of map of C_1 to D_1 the general fiber of the map must also vary. This gives a two-dimensional family of choices of s_1 and s_2. This gives a total of $3g - 4$. q.e.d.

Remark. One of the points on which attempts to generalize this lemma seem to get stuck is the unsolved problem of when $H_{k,B}$ is nonemtpy. This information is needed in order to know which admissible covers can be constructed.

This lemma shows that at least one component of the Hurwitz scheme of $\{k$ sheeted covers of $\mathbf{P}^1$ of genus g with two points of total ramification and $2g$ other simple branch points over distinct points$\}$ has an image in $\mathcal{M}_g$ of dimension $2g - 1$. If k and l are both outside the range of (3.2) and $l \geqslant k$, then this gives a component of $W(k, l)$ of dimension larger than $g - 3 + k + l$.

References

[1] E. Arbarello, *Weierstrass points and moduli of curves*, Compositio Math. **29** (1974) 325–342.
[2] ______, *On subvarieties of the moduli space of curves of genus g defined in terms of Weierstrass points*, Lincei-Memorie Sc. Fisiche, **15** (1978) 3–21.
[3] S. Diaz, *Exceptional Weierstrass points and the divisor on moduli space that they define*, Ph.D. thesis, Brown University, Providence, RI, 1982.
[4] ______, *Tangent spaces in moduli via deformations with applications to Weierstrass points*, Duke Math. J., to appear.
[5] A. Edmonds, R. Kulkarni & R. Strong, *Realizability of branched coverings of surfaces*, Trans. Amer. Math. Soc. **282** (1984) 773–790.
[6] J. Harris & D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982) 23–86.
[7] R. Lax, *Weierstrass points on the universal curve*, Math. Ann. **216** (1975) 35–42.
[8] H. Rauch, *Weierstrass points, branch points, and moduli of Riemann surfaces*, Comm. Pure Appl. Math. **12** (1959) 543–560.

UNIVERSITY OF PENNSYLVANIA