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MINIMAL SURFACES IN A KÄHLER SURFACE

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We consider a smooth real two-dimensional surface M embedded or immersed in a complex two-dimensional manifold N. In case M is compact and has only isolated complex tangents, we give a formula relating the Euler number of M, that of its normal bundle, and the sum over all complex tangents of a certain index attached to each complex tangent. This formula is a generalization of a special case of a formula derived previously for a real n-manifold in a complex n-manifold. If N is Kähler and M is a branched minimal immersion which is not a holomorphic curve, we show that the complex tangents are isolated and that each has a negative index. If M is also compact, then the above mentioned formula, with the inclusion of a term involving the total branching order, still holds. One consequence of our formula is that a two-sphere minimally *embedded* in the complex projective plane must be a holomorphic curve, hence a straight line or conic.

First we describe the index $\operatorname{ind}(p)$ of an isolated complex tangent at a point p in a more general setting. Let M be a real surface, $\tilde{V}$ a complex vector bundle of rank 2 over M, and $V \subset \tilde{V}$ a real sub-bundle of real rank 2. Let J, $J^2 = -I$, denote the complex multiplication on the underlying real vector bundle of $\tilde{V}$. We have an isolated complex tangent at $p \in M$ if V_q and JV_q coincide for $q = p$ and are transverse for q in a deleted neighborhood of p. Let π_q denote the orthogonal projection of $\tilde{V}_q$ along V_q onto its normal space F_q, relative to some metric on $\tilde{V}$ (or equivalently consider π: $\tilde{V} \to \tilde{V}/V$). Let v be a nonzero section of V near p. Then $\pi J v$ is a local section of F with an isolated zero at p. $\operatorname{ind}(p)$ is the degree (i.e. winding number) associated to the fiber coordinate of the map $q \to \pi J v(q)$. This is defined up to a sign, which is fixed by choices of local orientation on M and in the fiber of F near p.

Now suppose f is a smooth immersion of M into a complex surface N, with tangent bundle TN. Set $\tilde{V} = f^{-1}(TN)$, $V = f_*(TM)$, and F the normal bundle of V in $\tilde{V}$. Here f_* denotes the differential of f thought of as a bundle mapping

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from TM to $f^{-1}(TM)$. A choice of local orientation of M near p induces naturally a local orientation of TM, and hence of V via f_*. This local orientation of V together with the natural orientation of TN induce a local orientation of F by the requirement that $V \oplus F = \tilde{V}$ as oriented bundles. Given an isolated complex tangent at p, we define $\operatorname{ind}(p)$ relative to these orientations. It is well defined, since a change in the local orientation of M changes the orientation of both V and F, and is a local biholomorphic invariant of M in N near p. If M is compact, we take on it a smooth tangent vector field $\mathbf{v}$ which is nonzero at each complex tangent and has only isolated zeros. A comparison of the index sums of $\mathbf{v}$ and of $\pi J f_* \mathbf{v}$, exactly as in [8], yields the following, in which M need not be orientable.

Proposition 1. *Let the compact surface M be immersed in the complex surface N with only isolated complex tangents. Then*

$$\chi(M) + \chi(F) = \sum_p \operatorname{ind}(p), \tag{1}$$

where $\chi(M)$ and $\chi(F)$ are the Euler numbers of M and its normal bundle, respectively, and the sum extends over all complex tangents to M.

Similar considerations apply to a compact surface M with boundary, however, we leave this to a future investigation. If M in Proposition 1 is also oriented and embedded, then $\chi(F)$ is the self-intersection number of M in N.

Next we consider a complex vector bundle $\tilde{V}$ of rank r over a manifold. We suppose that $\tilde{V}$ has a hermitian metric and denote by $g(u, v)$ the real part of the hermitian inner product of two vectors u and v. g is a real inner product on the underlying real vector bundle. We extend g and J complex linearly to $\tilde{V} \otimes \mathbf{C} = V' \oplus V''$, $V'' = \overline{V'}$, $J = iI$ on V'. Thus if u and v are sections of V', then $g(u, v) = 0$ and $g(u, \bar{v})$ is their hermitian inner product. g is compatible with J, $J^*g = g$. We also assume that we have a connection D on $\tilde{V}$ which is compatible with the hermitian metric. It induces a connection, still denoted D, on the real bundle with $Dg = 0$.

Let e_i, $1 \leqslant i \leqslant 2r$, be a local real orthonormal frame field in $\tilde{V}$ and set

$$E_j = \tfrac{1}{2}\left(e_{2j-1} - ie_{2j}\right), \qquad 1 \leqslant j \leqslant r.$$

Then $g(e_i, e_j) = \delta_{ij}$ is equivalent to

$$g(E_i, E_j) = 0, \qquad g\left(E_i, \overline{E}_j\right) = \tfrac{1}{2}\delta_{ij}. \tag{2}$$

We introduce matrices of connection 1-forms $\xi = (\xi_{ij})$ and $\eta = (\eta_{ij})$ by (summation convention)

$$DE_i = \xi_{ij}E_j + \eta_{ij}\overline{E}_j, \qquad D\overline{E}_i = \bar{\xi}_{ij}\overline{E}_j + \bar{\eta}_{ij}E_j. \tag{3}$$

Covariant differentiation of (2) gives

$$\xi + \bar{\xi}^t = 0, \qquad \eta + \eta^t = 0. \tag{4}$$

Now let v_i, $1 \leqslant i \leqslant r$, be a local unitary frame field in V', $g(v_i, \bar{v}_j) = \frac{1}{2}\delta_{ij}$. We have

$$Dv_i = \psi_{ij} v_j, \qquad \psi + \bar{\psi}^t = 0. \tag{5}$$

The two frame fields are related by

$$E_i = a_{ij} v_j + b_{ij} \bar{v}_j, \qquad \bar{E}_i = \bar{b}_{ij} v_j + \bar{a}_{ij} \bar{v}_j. \tag{6}$$

From (2) we get (in matrix form)

$$0 = ab^t + ba^t, \qquad I = a\bar{a}^t + b\bar{b}^t. \tag{7}$$

We substitute (6) into the right-hand side of (3), and also take the covariant derivative of (6), using (5). Comparison of the results of these two processes gives

$$da + a\psi = \xi a + \eta \bar{b}, \qquad db + b\bar{\psi} = \xi b + \eta \bar{a}. \tag{8}$$

Now suppose that $r = 2$ and $V \subset \tilde{V}$ is a real two-plane sub-bundle with a complex tangent at p. We choose e_i, $i = 1, 2$, to span V and e_α, $\alpha = 3, 4$, to span the normal bundle F. We may further assume that $e_2(p) = Je_1(p)$, $e_4(p) = Je_3(p)$, and that $v_i(p) = E_i(p)$, so that

$$a_{ij}(p) = \delta_{ij}, \qquad b_{ij}(p) = 0. \tag{9}$$

We adapt the unitary frame v_i to the frame E_i as in Chern and Wolfson [3]. The allowable change is

$$v_i \to G_{ij} v_j, \qquad G\bar{G}^t = I, \qquad G(0) = I,$$

which results in $a \to aG$, and in particular $a_{12} \to a_{11}G_{12} + a_{12}G_{22}$. By (9) we may choose the unit vector field (G_{12}, G_{22}) orthogonal to $(\bar{a}_{11}, \bar{a}_{12})$ and equal to $(0, 1)$ at p. Then we choose (G_{11}, G_{21}) orthogonal to (G_{12}, G_{22}), of unit length, and equal to $(0, 1)$ at p. Thus we may assume $a_{12} = 0$. From $0 = g(E_1, E_1) = 2a_{11}b_{11}$, we conclude $b_{11} = 0$ by (9). Also,

$$0 = g(E_1, E_2) = a_{11}b_{21} + b_{12}a_{22},$$

$$0 = g(E_2, E_2) = 2(a_{21}b_{21} + a_{22}b_{22}).$$

These can be solved for b_{21} and b_{22} and substituted into

$$0 = g(\bar{E}_1, E_2) = \bar{a}_{11}a_{21} + b_{12}b_{22} = a_{21}\left(|a_{11}|^2 + |b_{12}|^2\right)a_{11}^{-1}.$$

It follows that $a_{21} = 0$, and hence $b_{22} = 0$ by (9). Thus we may write

$$
\begin{aligned}
(10) \qquad E_1 &= qv_1 + r\bar{v}_2, \qquad |q|^2 + |r|^2 = |s|^2 + |t|^2 = 1, \\
E_2 &= sv_2 + t\bar{v}_1, \ qt + rs = 0, \ q(0) = s(0) = 1, \ r(0) = t(0) = 0.
\end{aligned}
$$

With this normalization (8) gives

$$
\begin{aligned}
(11) \qquad dq + q\psi_{11} &= q\xi_{11} + \bar{t}\eta_{12}, \qquad q\psi_{12} = s\xi_{12}, \\
dr - r\psi_{22} &= r\xi_{11} + \bar{s}\eta_{12}, \qquad t\xi_{12} = -r\psi_{12}.
\end{aligned}
$$

If we set $Q = r/q$, then (11) gives

$$
(12) \qquad dQ = Q(\psi_{11} + \psi_{22}) + (q\bar{s} - r\bar{t})q^{-2}\eta_{12}.
$$

In the special case where V is the tangent bundle of a surface immersed in a complex manifold, an essentially equivalent formula to (12) was derived in [3].

Note that there is a complex tangent precisely where $JE_1 = iE_1$, i.e. where r, or Q, vanishes. If p is an isolated complex tangent, we may approximate F near p by the plane spanned by $\operatorname{Re} v_2$, $\operatorname{Im} v_2$, without altering the index. Since

$$
\begin{aligned}
e_1 = 2\operatorname{Re}(qv_1 + r\bar{v}_2), \quad Je_1 &= 2\operatorname{Re}(iqv_1 - ir\bar{v}_2), \\
\pi Je_1 &\approx 2\operatorname{Re}(iqv_1 - ir\bar{v}_2 + cE_1), \ c = -i,
\end{aligned}
$$

it follows that $\operatorname{ind}(p)$ is the winding number associated to the map

$$
(13) \qquad p_1 \to 4i\bar{r}(p_1).
$$

Now we consider a smooth branched minimal immersion f of M into the Kähler surface (N, g). The theory of branch points of such maps is developed by Gulliver, Osserman and Royden [6]. (I also want to thank Bob Gulliver for several helpful conversations.) In particular there are well-defined tangent and normal planes varying smoothly with $p \in M$. We denote by V and F the corresponding bundles as before. Then, even at a branch point, we may define the index of a complex tangent as above. We may assume that there is a smooth metric g^0 on M for which f is a weakly conformal, branched harmonic immersion. $f^*g = \tilde{c}g^0$, where $\tilde{c} \geqslant 0$ and has only isolated zeros corresponding to the branched points.

Proposition 2. *Let M, N and f be as just described. If $f(M)$ is not a holomorphic curve, then it has only isolated complex tangents, each with negative index.*

Proof. We study f near a point p, possibly a branch point, at which the tangent plane V_p is complex. Let

$$
E^0 = \tfrac{1}{2}(e_1^0 - ie_2^0), \qquad \varphi^0 = \theta_1^0 + i\theta_2^0 = \mu\, dz
$$

be local frame and coframe fields on M relative to g^0. $z = x + iy$, $z(p) = 0$, is a local isothermal parameter and $\mu \neq 0$. If D^0 denotes the Levi-Civita convection of g^0, then

$$D^0\varphi^0 = -\xi^0 \otimes \varphi^0, \qquad \xi^0 + \bar{\xi}^0 = 0,$$
$$d\varphi^0 = \varphi^0 \wedge \xi^0.$$

We also choose local adapted frames in $\tilde{V} = f^{-1}TN$ as in (10) and denoted by φ_i the one-forms dual to E_i. It follows that

$$f_* E^0 = \lambda E_1, \quad \text{or}$$
$$f^*\varphi_1 \equiv \varphi_1 = \lambda\varphi^0, \qquad f^*\varphi_2 \equiv \varphi_2 = 0 \tag{14}$$

for a complex factor λ, $\lambda\bar{\lambda} = \tilde{c}$, after changing the local orientation of M if necessary. Since N is Kähler its Levi-Civita connection D coincides with the hermitian connection, so the previous computations apply. We also have

$$\begin{aligned} D\varphi_i &= -\xi_{ji} \otimes \varphi_j - \bar{\eta}_{ji} \otimes \bar{\varphi}_j, \\ d\varphi_i &= \varphi_j \wedge \xi_{ji} + \bar{\varphi}_j \wedge \bar{\eta}_{ji}. \end{aligned} \tag{15}$$

We write the Jacobian of f as

$$f_* = \varphi^0 f_* E^0 + \bar{\varphi}^0 f_* \bar{E}^0 = \lambda\varphi^0 E_1 + \lambda\bar{\varphi}^0 \bar{E}_1.$$

The second fundamental form of f, a symmetric $\tilde{V}$-valued 2 form, is defined by

$$\mathrm{II}_f = \tilde{D} f_*, \qquad \tilde{D} = D^0 \otimes I + I \otimes D.$$

If we take the exterior derivative of the last two equations in (14) and use (15), (14), and Cartan's lemma, we get

$$d\lambda + \lambda\left(\xi_{11} - \xi^0\right) = \lambda'\varphi^0, \tag{16}$$

and

$$\lambda\xi_{12} = a\varphi^0 + b\bar{\varphi}^0, \qquad \lambda\eta_{12} = c\varphi^0 + \bar{b}\bar{\varphi}^0 \tag{17}$$

for certain smooth functions a, b, c, λ'. From (16) it follows that λ satisfies an equation of the form $\lambda_{\bar{z}} = K\lambda$. By a well-known theorem [1], since $\lambda \not\equiv 0$, it has the form

$$\lambda = z^l \lambda_0, \qquad l > 0, \lambda_0 \neq 0, \tag{18}$$

at a branch point p. Also (16) and (17) give

$$\begin{aligned} \mathrm{II}_f = \lambda'(\varphi^0)^2 E_1 + \bar{\lambda}'(\bar{\varphi}^0)^2 \bar{E}_1 &+ (\xi_{12} \otimes \varphi_1 + \bar{\eta}_{12} \otimes \bar{\varphi}_1) E_2 \\ &+ \left(\bar{\xi}_{12} \otimes \bar{\varphi}_1 + \eta_{12} \otimes \varphi_1\right) \bar{E}_2, \end{aligned} \tag{19}$$

where

$$\xi_{12} \otimes \varphi_1 + \bar{\eta}_{12} \otimes \bar{\varphi}_1 = a(\varphi^0)^2 + 2b\varphi^0\bar{\varphi}^0 + \bar{c}(\bar{\varphi}^0)^2.$$

The tension field of f is

$$\begin{aligned}\tau_f = \operatorname{tr} \mathrm{II}_f &= \mathrm{II}_f(e_1^0, e_1^0) + \mathrm{II}_f(e_2^0, e_2^0) \\ &= 4\mathrm{II}_f(E^0, \bar{E}^0) = 8(bE_2 + \bar{b}\bar{E}_2).\end{aligned} \tag{20}$$

Thus, f is harmonic if and only if $b = 0$. In that case (12), (17) imply, again by [1], that either $Q \equiv 0$, or Q has the form $Q = z^k Q_0$, $k \geqslant 1$, $Q_0 \neq 0$. If $Q \equiv 0$, then $f(M)$ is a holomorphic curve near $f(p)$. Otherwise p is an isolated complex tangent, and the map (13) has the form $z \to 4i\bar{r}(z) = 4i\bar{z}^k\bar{Q}_0\bar{q}$. It follows that $\operatorname{ind}(p) = -k < 0$. q.e.d.

Proposition 2 was originally motivated by the following observation. If M is immersed in N and has an elliptic point p [7], then as proven in [7] there is a smooth one-parameter family of analytic discs bounding on M and shrinking down to p. It can be used to construct a local variation of M which strictly decreases the area, by Wirtinger's inequality. Hence, M cannot be minimal. An elliptic point has index $+1$ [8].

At each branch point p the exponent $l = l(p)$ in (18) is the branching order. The total branching order is

$$B = \sum_p l(p), \tag{21}$$

the sum extending over all branch points. If M does not have a complex tangent at p we set $\operatorname{ind}(p) = 0$.

Theorem 3. *Let M be a compact surface, N a Kähler surface, and f a branched minimal immersion of M into N, with normal bundle F. If $f(M)$ is not a holomorphic curve, then*

$$\chi(M) + \chi(F) + B = \sum_p \operatorname{ind}(p) \leqslant 0, \tag{22}$$

with equality holding if and only if $f(M)$ is totally real.

Proof. By Proposition 2 and the argument preceding Proposition 1

$$\chi(V) + \chi(F) = \sum_p \operatorname{ind}(p) \leqslant 0,$$

and equality holds if and only if V is totally real. We relate $\chi(V)$ to $\chi(M)$ as follows. Let v be a smooth vector field on M, not vanishing at any branch point, and having only isolated zeros. $f_*(v)$ is a section of V. To each zero of v corresponds a zero of $f_*(v)$ of the same index. At a branch point p, $f_*(v)$ picks up an additional zero of index $l(p)$. Thus $\chi(V) = \chi(M) + B$. q.e.d.

If M is also oriented and smoothly embedded in N, then $B = 0$ and the sum of its Euler number and self-intersection number is nonpositive.

Suppose, for example, that $N = \mathbf{P}_2(\mathbf{C})$ (with *any* Kähler metric!), and that M is oriented and embedded. Then M has a homological degree $k \in \mathbf{Z}$, $M = k \cdot \mathbf{P}_1$ in $H_2(\mathbf{P}_2, \mathbf{Z})$, and $\chi(F) = k^2$.

Corollary 4. *Let M be a compact, orientable surface of genus g minimally embedded in $\mathbf{P}_2(\mathbf{C})$ with degree k. If M is not a complex algebraic curve, then*

$$k^2 \leqslant 2g - 2, \tag{23}$$

with equality holding if and only if M is totally real.

It follows that an embedded minimal two-sphere ($g = 0$) in $\mathbf{P}_2$ is an algebraic curve of degree k, $(k-1)(k-2) = 2g$, hence is a complex line or conic. There is an extensive literature with many examples of *immersed* minimal spheres starting with the work of Calabi [2]. For the case of $\mathbf{P}_2(\mathbf{C})$ see Din and Zakrzewski [4], Eells and Wood [5], and [3]. If M is an embedded minimal torus, then (23) implies that it is either a nonsingular complex cubic curve or has degree $k = 0$ and is totally real. For the Fubini-Study metric an example of the latter alternative is provided by the Clifford torus, $\{|z| = |w| = 1\} \subset \mathbf{C}^2 \subset \mathbf{P}_2$, where (z, w) are standard nonhomogeneous coordinates.

Finally, we remark that Theorem 3 can be easily generalized to the case $\dim N > 2$ under the same hypotheses. We set $\tilde{V}_p = V_p + J_p V_p$ for p a totally real point. $\tilde{V}$ is a complex 2-plane sub-bundle of $f^{-1}(TN)$ which can be shown to extend smoothly to the points where M has a complex tangent. F is taken to be the normal bundle of V in $\tilde{V}$. Proposition 2 and (22) follow by essentially the same arguments.

Note. After circulating this work in preprint form in August, 1984, we were informed of two works containing results related to ours and completed independently at nearly the same time. These are Gauduchon-Lawson [11] and Eells-Salamon [10]. We should also mention the work of D. Burns [9].

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