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A NEW PROOF OF THE SYMMETRIZER LEMMA AND A STRONGER WEAK TORELLI THEOREM FOR PROJECTIVE HYPERSURFACES

R. DONAGI & M. L. GREEN

Let V be a vector space of dimension $n + 1$ and $F \in S^d V$ a homogeneous polynomial of degree d representing a nonsingular hypersurface in $\mathbf{P}(V^*)$. For $k \geqslant 0$, let $J_k \subset S^k V$ be the Jacobi ideal of F generated by the first partials of F. Let $R^k = S^k V / J_k$. In the first author's proof [1] of the generic Torelli theorem for projective hypersurfaces, a key step is the Symmetrizer Lemma:

Theorem 1. *The Koszul complex*

$$R^{b-a} \to (R^a)^* \otimes R^b \to \Lambda^2(R^a)^* \otimes R^{a+b} \tag{1}$$

is exact at the middle term, provided that

$$\max(a + b + 1, d - 1 + b) \leqslant (n + 1)(d - 2). \tag{2}$$

Note. In [1], this is stated in the form

$$\{P \in \operatorname{Hom}(R^a, R^b) \mid P(u)v = P(v)u \text{ in } R^{a+b} \text{ for all } u, v \in R^a\} \simeq R^{b-a},$$

where $R^{b-a} \to \operatorname{Hom}(R^a, R^b)$ by the map induced by multiplication.

In [1], Theorem 1 is shown only for F generic. The interest of having Theorem 1 in the stronger version given here is that it offers an improvement of the generic Torelli theorem given there. In particular, it shows that the prolonged period map from projective hypersurfaces to first order infinitesimal variations of Hodge structure is one-to-one.

Theorem 2. *A smooth hypersurface F of degree d in $\mathbf{P}^{n+1}$ can be reconstructed up to projective equivalence from the derivative of the period map at F, provided $d \nmid n + 2$, and we are not in the cases $n = 2$, $d = 3$; $n \equiv 0 \pmod 4$, $d = 4$; $n \equiv 1 \pmod 6$, $d = 6$.*

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The proof that Theorem 2 implies Theorem 1 follows exactly the argument of [1]. We now prove Theorem 1. Let $\sigma = (n+1)(d-2)$. To start, recall:

Macaulay's Theorem. *$R^\sigma \approx \mathbf{C}$ and the multiplication map $R^a \otimes R^{\sigma-a} \to R^\sigma \approx \mathbf{C}$ is a perfect pairing, provided $0 \leqslant a \leqslant \sigma$.*

By Macaulay's Theorem, the sequence (1) is dual to the sequence

$$\Lambda^2 R^a \otimes R^{\sigma-a-b} \to R^a \otimes R^{\sigma-b} \to R^{\sigma+a-b}. \tag{3}$$

We now have the commutative diagram:

$$\begin{array}{ccccc} \Lambda^2(S^aV) \otimes S^{\sigma-a-b}V & \xrightarrow{G} & S^aV \otimes S^{\sigma-b}V & \xrightarrow{H} & S^{\sigma+a-b} \\ \downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_3 \\ \Lambda^2 R^a \otimes R^{\sigma-a-b} & \xrightarrow{g} & R^a \otimes R^{\sigma-b} & \xrightarrow{h} & R^{\sigma+a-b} \end{array} \tag{4}$$

By diagram chasing, we see that the statements

(5) the top row of (4) is exact

and

$$H(\ker \pi_2) = \ker \pi_3 \tag{6}$$

imply the desired exactness of the bottom row. Now, (6) holds whenever $\sigma - b \geqslant d - 1$.

We are thus done modulo the following lemma.

Lemma. *The Koszul sequence*

$$\Lambda^2(S^aV) \otimes S^{b-a}V \xrightarrow{G} S^aV \otimes S^bV \xrightarrow{H} S^{a+b}V \tag{7}$$

is exact at the middle term, provided $b > a$.

This lemma appears to be a well-known property of polynomials; a generalization of it is proved in [2]. The standard argument uses the $\mathrm{GL}(V)$-invariance of G, H to reduce to checking the behavior under G and H of the irreducible components of the $\mathrm{GL}(V)$-modules $\Lambda^2(S^aV) \otimes S^{b-a}V$ and $S^aV \otimes S^bV$, using the Littlewood-Richardson rule. We include here an expecially nice proof suggested to us by R. Steinberg:

Using standard multi-index notation, let

$$\sum_{\substack{|I|=a \\ |J|=b}} C_{I,J} Z^I \otimes Z^J \in \ker H, \qquad C_{I,J} \in \mathbf{C}. \tag{8}$$

Then

$$\sum_{I+J=K} C_{I,J} = 0 \quad \text{for all } K \text{ with } |K| = a+b. \tag{9}$$

Elements of $\operatorname{im} G$ are spanned by

$$Z^I \otimes Z^{I'+L} - Z^{I'} \otimes Z^{I+L}, \qquad |I| = |I'| = a,\ |L| = b - a. \tag{10}$$

Thus

$$Z^I \otimes Z^J \equiv Z^{I'} \otimes Z^{J'} \mod \operatorname{im} G$$

if $I + J = I' + J'$ and $J' - I \geqslant 0$. By concatenating two such relations, we have that

$$Z^I \otimes Z^{K-I} \equiv Z^{I'} \otimes Z^{K-I'} \mod \operatorname{im} G$$

if there exists a J with $|J| = b$, $J \leqslant K$, and $J - I' \geqslant 0$, $J - I \geqslant 0$.

Since, by hypothesis, $b > a$, we have

$$Z^I \otimes Z^{K-I} \equiv Z^{I'} \otimes Z^{K-I'} \mod \operatorname{im} G$$

if I, I' differ by one change of index, i.e., $I - I' = (0,0,\cdots,1,0,\cdots,-1,0,\cdots,0)$. By transitivity,

$$Z^I \otimes Z^{K-I} \equiv Z^{I'} \otimes Z^{K-I'} \mod \operatorname{im} G$$

whenever $K \geqslant I$ and $K \geqslant I'$. Thus

$$\sum_{\substack{I+J=K \\ |I|=a \\ |J|=b}} C_{I,J} Z^I \otimes Z^J \equiv 0 \mod \operatorname{im} G$$

whenever (9) holds. Since (8) $\to$ (9), we are done.

References

[1] R. Donagi, *Generic Torelli for projective hypersurfaces*, Compositio Math. **50** (1983) 325–353.

[2] M. L. Green, *Koszul cohomology and the geometry of projective varieties*. II, J. Differential Geometry **20** (1984) 279–289.

NORTHEASTERN UNIVERSITY
UNIVERSITY OF CALIFORNIA, LOS ANGELES