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A MULTIPLICITY ESTIMATE FOR PROJECTIONS OF SURFACES

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Let $0 < \alpha < 1$ and let p_ν: $\mathbf{R}^3 \to \mathbf{R}^3$ be the orthogonal projection defined by $p_\nu(x) = x - (x \cdot \nu)\nu$ for $\nu \in S^2$. In this paper, we prove the following estimate for a $C^{3,\alpha}$ embedded, orientable bordered surface M in $\mathbf{R}^3$:

$$\int_{S^2} n(\nu)\, d\mathscr{H}^2 \leqslant 2\int_M \left[\|DN(x)\|^2 + \|D^2N(x)\|\right] d\mathscr{H}^2 x$$

$$+ \int_{\partial M} \left[6\|DN(w)\| + 8\|D\tau(w)\|\right] d\mathscr{H}^1 w,$$

where $n(\nu) = \sup_{y \in \operatorname{Im} p_\nu}\{n(\nu, y)\}$, $n(\nu, y) = \mathscr{H}^0\{M \cap p_\nu^{-1}(y)\}$, N: $M \to S^2$ is an orienting unit normal vectorfield for M, τ: $\partial M \to S^2$ is a continuous unit tangent vectorfield for ∂M, $\mathscr{H}^0$, $\mathscr{H}^1$ and $\mathscr{H}^2$ denote Hausdorff measures ($\mathscr{H}^0(A) = \operatorname{card} A$) and D and D^2 denote covariant differentiation. With some notational inconvenience one can easily generalize our work to treat immersed, nonorientable surfaces. The full strength of the $C^{3,\alpha}$ hypothesis will be used twice: in Lemma 5 to prove the condition

$$\text{(A)} \quad \mathscr{H}^2\left\{\nu \in S^2 : \nu \cdot N(x) = 0, \left|\langle \nu, DN(x)\rangle\right| = 0 \text{ for some } x \in M\right\} = 0,$$

and in an argument at the end of Theorem 11. Elsewhere, C^3 will suffice.

We begin by describing the singularities that can occur under generic projections of M and obtaining integral estimates on the curvature of the projections of the singular sets.

Definition. Let $S_\nu^1 = \{x \in M: \ \nu \cdot N(x) = 0, \ \nu \cdot \langle \nu, \ DN(x)\rangle \neq 0\}$ and $S_\nu^2 = \{x \in M: \ \nu \cdot N(x) = \nu \cdot \langle \nu, \ DN(x)\rangle = 0, \ \nu \cdot \langle \nu \otimes \nu, \ D^2N(x)\rangle \neq 0, |\langle \nu, DN(x)\rangle| \neq 0\}$. Also let $C_\nu = \{x \in M: \nu \cdot N(x) = 0\}$.

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Lemma 1. *If M is of class 2, then*

$$\int_{S^2}\int_{S^2\cap \operatorname{Im} p_\nu} \mathscr{H}^0\{y\in p_\nu(S^1_\nu): N\in \operatorname{Nor}(p_\nu(S^1_\nu), y)\}\, d\mathscr{H}^1 N\, d\mathscr{H}^2\nu$$

$$= 2\pi\int_M J_2 N(x)\, d\mathscr{H}^2 x.$$

Proof. If τ is a unit tangent vector to S^1_ν at x, the condition $\nu\cdot\langle\nu, DN(x)\rangle\neq 0$ implies that $\tau\wedge\nu\neq 0$. If $N\in \operatorname{Im} p_\nu\cap \operatorname{Nor}(p_\nu(S^1_\nu), p_\nu(x))$, then $N\cdot\nu = 0$ and $N\cdot\tau = 0$, hence $N\in\operatorname{Nor}(S^1_\nu, x)$. Thus

$$\int_{S^2}\int_{\operatorname{Im} p_\nu\cap S^2} \mathscr{H}^0\{y\in p_\nu(S^1_\nu): N\in \operatorname{Nor}(p_\nu(S^1_\nu), y)\}\, d\mathscr{H}^1 N\, d\mathscr{H}^2\nu$$

$$= \int_{S^2}\int_{N\cdot\nu=0, |N|=1} \mathscr{H}^0\{x\in S^1_\nu: N\in\operatorname{Nor}(M,x)\}\, d\mathscr{H}^1 N\, d\mathscr{H}^2\nu.$$

By applying the area formula to the set

$$\Sigma = \{(\nu, N)\in S^2\times S^2: \nu\cdot N = 0\}$$

we interchange the order of integration, and the integral reduces to

$$2\pi\int_{S^2}\mathscr{H}^0\{x\in M: N\in\operatorname{Nor}(M,x)\}\, d\mathscr{H}^2 = 2\pi\int_M J_2N(x)\, d\mathscr{H}^2 x.$$

Definition. If M is negatively curved at x (i.e. $\det DN(x) < 0$), then there exist two linearly independent unit vectors $e_1, e_2\in\operatorname{Tan}(M,x)$ (unique up to sign and order) such that $e_i\cdot\langle e_i, DN(x)\rangle = 0$ $(i = 1,2)$. These are called *asymptotic vectors* at x. If $\det DN(x) = 0$ but $\|DN(x)\|\neq 0$ (i.e. rank $DN(x) = 1$), then there is a unique (up to sign) asymptotic vector $\nu\in\operatorname{Tan}(M,x)$, and we define $e_1(x) = e_2(x) = \nu$.

To apply the co-area formula with e_1 and e_2 as slicing functions, we need to find $De_1(x)$ at points where this exists. If $e_1^\perp$ is a unit tangent vector perpendicular to e_1, by differentiating the equation $e_1\cdot\langle e_1, DN(x)\rangle = 0$ in the directions e_1 and $e_1^\perp$, we obtain

$$|\langle e_1, De_1(x)\rangle\cdot e_1^\perp| = \tfrac{1}{2}|e_1\cdot\langle e_1\otimes e_1, D^2N(x)\rangle|/|\langle e_1, DN(x)\rangle|$$

and

$$|\langle e_1^\perp, De_1(x)\rangle\cdot e_1^\perp| = \tfrac{1}{2}|e_1^\perp\cdot\langle e_1\otimes e_1, D^2N(x)\rangle|/|\langle e_1, DN(x)\rangle|.$$

Moreover,

$$N(x)\cdot\langle e_1, De_1(x)\rangle = -\langle e_1, DN(x)\rangle\cdot e_1(x) = 0$$

and

$$N(x) \cdot \langle e_1^{\perp}, De_1(x) \rangle = -\langle e_1, DN(x) \rangle \cdot e_1^{\perp}(x) = |\langle e_1, DN(x) \rangle|,$$

hence $J_2 e_1(x) = \frac{1}{2}|e_1 \cdot \langle e_1 \otimes e_1, D^2N(x)\rangle|$. If $J_2N(x') > \varepsilon$ for all x' in a neighborhood U of x, then, since $|\langle e_1, DN(x)\rangle| = (J_2N(x))^{1/2}$, e_1 is a Lipschitzian function in U, with a Lipschitz constant less than or equal to $\varepsilon^{-1/2}(T + K^2)$, where $T = \sup\|D^2N(x')\|$ and $K = \sup\|DN(x')\|$.

Lemma 2. *If M is a manifold of class* 3 *which satisfies assumption* (A), *then*

$$\int_{S^2} \mathcal{H}^0\{x \in M: \nu \cdot N(x) = \nu \cdot \langle \nu, DN(x) \rangle = 0\}\, d\mathcal{H}^2\nu$$

$$\leqslant 2\int_M \|D^2N(x)\| d\mathcal{H}^2 x.$$

Proof. Let $M_\varepsilon = \{x \in M: J_2N(x) > \varepsilon\}$. If $U \subset M_\varepsilon$ is a neighborhood on which e_1 and e_2 are well defined, then

$$\int_{S^2} \mathcal{H}^0\{x \in U: \nu \cdot N(x) = \nu \cdot \langle \nu, DN(x) \rangle = 0\}\, d\mathcal{H}^2\nu$$

$$= 2\int_U \tfrac{1}{2}\Big(\big|e_1 \cdot \langle e_1 \otimes e_1, D^2N(x) \rangle\big| + \big|e_2 \cdot \langle e_2 \otimes e_2, D^2N(x) \rangle\big|\Big)\, d\mathcal{H}^2 x$$

$$\leqslant 2\int_U \|D^2N(x)\| d\mathcal{H}^2 x.$$

Since the integrand in the second integral is independent of the way $e_1(x)$ and $e_2(x)$ are assigned, the above formula is also true globally, with U replaced by M_ε. By applying Lebesgue's increasing convergence theorem, we obtain

$$\int_{S^2} \mathcal{H}^0\{x \in M_0: \nu \cdot N(x) = \nu \cdot \langle \nu, DN(x) \rangle = 0\}\, d\mathcal{H}^2\nu$$

$$= \int_{M_0} \tfrac{1}{2}\Big(\big|e_1 \cdot \langle e_1 \otimes e_1, D^2N(x) \rangle\big| + \big|e_2 \cdot \langle e_2 \otimes e_2, D^2N(x) \rangle\big|\Big)\, d\mathcal{H}^2 x$$

$$\leqslant \int_M \|D^2N(x)\| d\mathcal{H}^2 x.$$

Since $M \setminus M_0 = \{x \in M: J_2N(x) = 0\}$, by assumption (A)

$$\int_{S^2} \mathcal{H}^0\{x \in M \setminus M_0: \nu \cdot N(x) = \nu \cdot \langle \nu, DN(x) \rangle = 0\}\, d\mathcal{H}^2\nu = 0.$$

Adding the two integrals, we obtain the desired bound.

Corollary 3. *If M is a manifold of class* 3 *which satisfies assumption* (A), *then for $\mathcal{H}^2$-almost all directions ν, $C_\nu = S_\nu^1 \cup S_\nu^2$.*

Proof.

$$\begin{aligned} &C_\nu \setminus (S_\nu^1 \cup S_\nu^2) \\ &\quad = \{x \in M: \nu \cdot N(x) = \nu \cdot \langle \nu, DN(x)\rangle = 0, \langle \nu, DN(x)\rangle = 0\} \\ &\qquad \cup \left\{x \in M_0: \nu \cdot N(x) = \nu \cdot \langle \nu, DN(x)\rangle = \nu \cdot \langle \nu \otimes \nu, D^2N(x)\rangle = 0\right\}. \end{aligned}$$

(Here, as before, $M_0 = \{x \in M: J_2N(x) \neq 0\}$.) The first set is empty for almost all ν by assumption (A). Since

$$\begin{aligned} &\left\{\nu: \nu \cdot N(x) = \nu \cdot \langle \nu, DN(x)\rangle = \nu \cdot \langle \nu \otimes \nu, D^2N(x)\rangle = 0 \right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. \text{for some } x \in M_0\right\} \\ &\qquad = e_1(\{x \in M_0: J_2 e_1(x) = 0\}) \cup e_2(\{x \in M_0: J_2 e_2(x) = 0\}), \end{aligned}$$

the co-area formula shows that the second set is also empty for almost all ν.

A partial proof of assumption (A) can be given in the spirit of Lemma 2. For this we introduce the well-known *principal curvature vectors* $E_1(x)$ and $E_2(x)$ (the eigenvectors of the linear map $DN(x)|\mathrm{Tan}(M, x)$) and the *principal curvatures* $\lambda_1 = E_1 \cdot \langle E_1, DN(x)\rangle$ and $\lambda_2 = E_2 \cdot \langle E_2, DN(x)\rangle$.

Lemma 4. *If M is a manifold of class* 3,

$$M_0 = \{x \in M: J_2N(x) = 0, \|DN(x)\| \neq 0\},$$

then $\mathcal{H}^2\{\nu \in S^2: M_0 \cap S_\nu^2 \neq \varnothing\} = 0$.

Proof. Let $M_{0\varepsilon} = \{x \in M: J_2N(x) = 0, \|DN(x)\| > \varepsilon\}$. For any point $x \in M_0$, one of the principal vectors, say $E_1(x)$, satisfies $E_1(x) \cdot \langle E_1(x), DN(x)\rangle = 0$, and hence *coincides* with the degenerate vectors $e_1(x) = e_2(x)$. Moreover

$$J_2E_1(x) = \left|E_1 \cdot \langle E_2 \otimes E_2, D^2N(x)\rangle\right| |\lambda_1/(\lambda_1 - \lambda_2)|.$$

Hence $M_0 \subset \{x \in M: J_2E_1 = 0, \|DN(x)\| \neq 0\}$. Since E_1 is locally Lipschitzian on $M_{0\varepsilon}$, it follows that

$$\mathcal{H}^2\{\nu \in S^2: (M_0 \setminus M_{0\varepsilon}) \cap S_\nu^2 \neq \varnothing\} = 0,$$

from which the lemma follows immediately.

Lemma 5. *Assumption* (A) *holds for any manifold of class* $(3, \alpha)$ *for any* $\alpha > 0$.

Proof. To prove this it remains to show that $\mathcal{H}^2\{\nu \in S^2: \nu \cdot N(x) = 0$ for some $x \in P\} = 0$, where $P = \{x \in M: DN(x) \equiv 0\}$. According to a strengthened version of the Morse-Sard theorem proved recently by Y. Yomdin [7], there exists a constant $C(M, \alpha)$ such that the set $N(P)$ can be covered by at most $C(M, \alpha)\varepsilon^{-2/2+\alpha}$ balls of radius ε for any $\varepsilon > 0$. (In Yomdin's notation,

$C(M, \alpha) = \bar{A}_0(2, 2, 2 + \alpha)C(M)$, where $C(M)$ is a constant depending on $\|N\|_{2,\alpha}$ and on the number of balls of radius $1/2K$ needed to cover M, where $K = \|N\|_1 = \sup\{\|DN(x)\|: x \in M\}$, as before.) Corresponding to each ball $B_\varepsilon \subset S^2$ of radius ε is a strip of width ε, $S_\varepsilon \subset S^2$, such that if $N(x) \in B_\varepsilon$, then $\mathrm{Tan}(M, x) \cap S^2 \subset S_\varepsilon$. Hence $\{\nu: \nu \in \mathrm{Tan}(M, x), x \in P\}$ can be covered by $C(M, \alpha)\varepsilon^{-2/2+\alpha}$ strips of width ε, and therefore

$$\mathscr{H}^2\{\nu \in S^2: \nu \in \mathrm{Tan}(M, x), x \in P\}$$
$$\leqslant C(M, \alpha)\varepsilon^{-2/2+\alpha}(2\pi\varepsilon) < C\varepsilon^{\alpha/2+\alpha} \quad \text{for any } \varepsilon > 0.$$

Lemmas 1 and 2 are the two basic integral estimates used in proving the multiplicity estimate. Next we consider the local behavior of the multiplicity near fold points and cusp points.

Lemma 6. *Let M be a manifold of class 2. Suppose $x \in S_\nu^1$, and let $y = p_\nu(x)$. Then there exists a neighborhood U of x with the following property. For any σ such that $\sigma \cdot \nu = 0$, $\sigma \cdot N(x) \neq 0$, there exists $\varepsilon > 0$ such that*

$$\mathscr{H}^0\{(p_\nu|U)^{-1}(y + t\sigma)\} = 1 \pm \operatorname{sign} t,$$

whenever $|t| < \varepsilon$. (Either the + sign or the − sign holds throughout the interval.)

Proof. Let $e_1 = \nu$, e_2 form an orthonormal basis of $\mathrm{Tan}(M, x)$. In a neighborhood U of x, M can be parametrized as the graph of a function f of class 2:

$$U_\varepsilon = \{x + se_1 + te_2 + f(s, t)N(x): |(s, t)| < \varepsilon\},$$

where $f(0) = 0$, $\partial f/\partial s(0) = 0$, $\partial f/\partial t(0) = 0$, $\partial^2 f/\partial s^2(0) = e_1 \cdot \langle e_1, DN(x)\rangle$, $\partial^2 f/\partial s\partial t(0) = e_1 \cdot \langle e_2, DN(x)\rangle$, $\partial^2 f(0)/\partial t^2 = e_2 \cdot \langle e_2, DN(x)\rangle$. Since $e_1 \cdot \langle e_1, DN(x)\rangle \neq 0$ and the second derivatives of f are continuous, there exist $\varepsilon' > 0$ and $\delta > 0$ such that $|\partial^2 f/\partial s^2(s, t)| \geqslant \delta > 0$ for all (s, t) such that $|(s, t)| < \varepsilon'$. We let $U = U_{\varepsilon'}$.

If $\sigma = \sigma_1 N + \sigma_2 e_2 \in \mathrm{Im}\, p_\nu$ and $y' = t\sigma$, then

$$(p_\nu|U)^{-1}(y) = \{x + se_1 + te_2: f(s, t\sigma_2) = t\sigma_1\}.$$

Since $|\partial^2 f/\partial s^2| > 0$ in $U(0, \varepsilon')$, the equation $f(s, t\sigma_2) = t\sigma_1$ can have at most two roots s such that $|(s, t)| < \varepsilon'$. Suppose, without loss of generality, that $\partial^2 f/\partial s^2 > 0$. By hypothesis, $\sigma_1 \neq 0$; assume, for example, $\sigma_1 > 0$. By the implicit function theorem, there is, for each sufficiently small t, a unique minimum $m(t)$ of the function $f(s, t\sigma_2)$, and $m'(0) = 0$. Thus there exists $\varepsilon'' > 0$ such that:

(a) if $-\varepsilon'' < t < 0$, then $t\sigma_1 < m(t)$, hence $f(s, t\sigma_2) = t\sigma_1$ has no roots;
(b) if $0 < t < \varepsilon''$, then $m(t) < t\sigma_1$, hence $f(x, t\sigma_2) = t\sigma_1$ has two roots;
(c) if $t = 0$, then $m(t) = t\sigma_1$, hence $f(s, t\sigma_2) = t\sigma_1$ has one root.

Lemma 7. *If M is a manifold of class* 3, $x \in S_\nu^2$, $y = p_\nu(x)$, *there exists a neighborhood U of x such that for any $\sigma \in \operatorname{Im} p_\nu$ such that $\sigma \cdot N(x) \neq 0$, there exists $\varepsilon' > 0$ such that $|t| < \varepsilon$ implies $\mathscr{H}^0\{(p_\nu|U)^{-1}(y + t\sigma)\} = 1$.*

Proof. When we parametrize M as a graph, as in Lemma 6, we have $\partial^2 f(0)/\partial s^2 = 0$ and $\partial^3 f(0)/\partial s^3 = e_1 \cdot \langle e_1 \otimes e_1, D^2N(x)\rangle$. (Note: this equality does not always hold, but in this case is a consequence of the fact that $\partial^2 f(0)/\partial s^2 = 0$.) Hence $\partial^3 f/\partial s^3 \neq 0$; again, without loss of generality, suppose it is greater than 0. Also, since $|e_2 \cdot \langle e_1, DN(x)\rangle| = |\langle e_1, DN(x)\rangle| \neq 0$, $\partial^2 f(0)/\partial s \partial t \neq 0$. By continuity of the third derivatives, there exists $\varepsilon > 0$ such that $|(s, t)| < \varepsilon$ implies that $\partial^3 f(s, t)/\partial s^3 \neq 0$, $\partial^2 f(s, t)/\partial s \partial t \neq 0$. Let $U = \{se_1 + te_2 + f(s, t)N: |(s, t)| < \varepsilon\}$.

By the mean-value theorem, the equation $f(s, t\sigma_2) = t\sigma_1$ can have at most three roots in s such that $se_1 + te_2 + f(s, t)N \in U$, because $\partial^3 f/\partial s^3 > 0$ for all s in that interval. Setting $g_t(s) = f(x, t\sigma_2)$, we see that the equation $g_t(s) = t\sigma_1$ has only one root under two circumstances:

(a) if $g_t(s)$ has no critical points;

(b) if g_t has two critical points $s_1(t)$ and $s_2(t)$ such that $g_t(s_1) < g_t(s_2) < t\sigma_1$ or $t\sigma_1 < g_t(s_1) < g_t(s_2)$.

Now suppose $\sigma_2 \neq 0$.

By the implicit function theorem, g' has a unique minimum point $s_0(t)$ for small enough t, and $s_0'(t)$ exists. Then

$$\frac{d}{dt}[g_t'(s_0(t))](0) = \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial s}(s_0(t), \sigma_2 t)\right](0)$$
$$= \sigma_2\left[\frac{\partial^2 f}{\partial s \partial t}(0)\right] \neq 0.$$

Hence $\mathscr{H}^0\{s: g_t'(s) = 0\} = 1 \pm \operatorname{sign} t$, as in Lemma 6. (Note that at this point we have used all three hypotheses $\partial^2 f(0)/\partial s^2 = 0$, $\partial^3 f(0)/\partial s^3 \neq 0$ and $\partial^2 f(0)/\partial s \partial t \neq 0$.)

Suppose, for example, the + sign holds in the above equation. For $t < 0$ the lemma is proved by (a) above. For $t > 0$ it is sufficient to show that $|g_t(s_1)| < |\sigma_1 t|$ and $|g_t(s_2)| < |\sigma_1 t|$ for sufficiently small t, where $s_1(t)$ and $s_2(t)$ are as defined in (b). Indeed, from the Taylor expansions of $f(s, t)$ and $\partial f(s, t)/\partial s$ it follows that $s_1(t), s_2(t) = O(t^{1/2})$ and $f(s_1(t), t) = O(t^{3/2})$; hence $|g_t(s_i)| = O(t^{3/2})$ $(i = 1, 2)$.

The remaining case $\sigma_2 = 0$ is easy: the fact that $f(s, 0) = t\sigma_2$ has only one root follows from the mean-value theorem and the fact that $\partial^3 f/\partial s^3 \neq 0$, and does not even require the hypothesis $\partial^2 f(0)/\partial s \partial t \neq 0$.

Remark. The theory of the stability of singularities provides another approach to analyzing the multiplicity behavior of p_ν. Indeed, if ψ is any parametrization of a neighborhood of x, say with $\psi_1(0) \equiv \partial\psi(0)/\partial x_1 = e_1$ and $\psi_2(0) = e_2$, then:

(a) if x is a fold point and ψ is of class 2, then $(p_\nu \circ \psi)_{11}(0) \neq 0$ and so there exist one-to-one functions g_1, g_2 (of class 1) such that $p_\nu \circ \psi = g_1 \circ F \circ g_2$ on a small enough neighborhood of x, where $F(x_1, x_2) = (x_1^2, x_2)$;

(b) if x is a cusp point and ψ is of class 4, then $(p_\nu \circ \psi)_{111}(0) \neq 0$ and $(p_\nu \circ \psi)_{12}(0) \neq 0$, and there exist one-to-one functions g_1, g_2 (of class 1) such that $p_\nu \circ \psi = g_1 \circ G \circ g_2$ on a small enough neighborhood of x, where $G(x_1, x_2) = (x_1^3 - x_1 x_2, x_2)$.

Statement (a) is not too hard to prove, though decidedly nontrivial. Statement (b) is very subtle, and evidence of this fact is that it was first proved by Whitney under the assumption that ψ is of class $\geqslant 12$ [**6**].

A key reason for the simplicity of Lemmas 6 and 7 as opposed to this approach is the fact that we only need to obtain such estimates "in one direction at a time", instead of having to obtain estimates that hold uniformly in all directions. (For example, ε' in Lemmas 6 and 7 depends on the direction σ.) Also, in Lemmas 6 and 7 we simply ignore the "trickiest" direction, $\sigma = e_2$ (or $-e_2$).

Lemma 8. *If* $\nu \in S^2$ *satisfies the following conditions*:

(1) $C_\nu = S_\nu^1 \cup S_\nu^2$,

(2) $\mathscr{H}^0(S_\nu^2 \cap \partial M) = 0$,

(3) $\operatorname{Tan}(\partial M, x) \neq \operatorname{Tan}(C_\nu, x)$ *when* $x \in C_\nu \cap \partial M$,

(4) $\mathscr{H}^0\{x \in \partial M: \nu \in \operatorname{Tan}(\partial M, x)\} = 0$,

(5) *all self-intersections of* $p_\nu(C_\nu \cup \partial M)$ *are transverse*,

(6) $\sup\{\mathscr{H}^0[(p_\nu|\partial M)^{-1}(y)]: y \in \operatorname{Im} p_\nu\} = 2$,

and if $n(\nu) < \infty$, *then there exists* $y \in \operatorname{Im} p_\nu \setminus p_\nu(C_\nu \cup \partial M)$ *such that* $\mathscr{H}^0\{(p_\nu|M)^{-1}(y)\} = n(\nu)$. *Hence* $\{y: n(\nu, y) = n(\nu)\}$ *has nonempty interior. If* (1)–(5) *hold and* $n(\nu) = \infty$, *then* $\{y: n(\nu, y) \geqslant N\}$ *has nonempty interior for any* $N > 0$.

Proof. Clearly if $n(\nu) < \infty$ then there does exist $y \in \operatorname{Im} p_\nu$ such that $\mathscr{H}^0\{(p_\nu|M)^{-1}(y)\} = n(\nu)$. Let $x_1, \cdots, x_n$ be the preimages of y under $(p_\nu|M)$. If $x_i \notin C_\nu$, let U_i be a neighborhood of x_i such that $p_\nu|U_i$ is a diffeomorphism; otherwise choose U_i as in Lemma 6 (if $x_i \in S_\nu^1$) or Lemma 7 (if $x_i \in S_\nu^2$). (If $x_i \in \partial M$, U_i is to be interpreted as a neighborhood diffeomorphic to an open disk such that $U_i \cap M$ is a half-disk.) By Lemmas 6 and 7 and hypotheses (2)–(4), to each x_i is associated an open half-plane $W_i \subset \operatorname{Im} p_\nu$ such that for any $\sigma \in W_i$ there exists $\varepsilon_i > 0$ such that whenever $0 < t < \varepsilon_i$,

$$\mathscr{H}^0\left\{(p_\nu|U_i)^{-1}(y + t\sigma)\right\} \geqslant c\mathscr{H}^0\left\{(p_\nu|U_i)^{-1}(y)\right\},$$

where $c = 1$ if $x_i \in \partial M \cup S_\nu^2$, $c = 2$ if $x_i \in S_\nu^1$. If $x_i \in S_\nu^2$, we may replace W_i with $W_i \cup (-W_i)$.

Let $r = \mathscr{H}^0[p_\nu^{-1}(y) \cap S_\nu^1]$ and $s = \mathscr{H}^0[p_\nu^{-1}(y) \cap \partial M] \leqslant 2$. It is simple to show (using hypothesis (5)) that there exists $\sigma \neq 0$ such that

$$\mathscr{H}^0\{x_i \in p_\nu^{-1}(y) \cap S_\nu^1 \colon \sigma \in W_i\} \geqslant [r/2] + 1,$$
$$\mathscr{H}^0\{x_i \in p_\nu^{-1}(y) \cap \partial M \colon \sigma \in W_i\} \geqslant 2(r/2 - [r/2])$$

and $\sigma \in \bigcap_{x_i \in S_\nu^2} W_i \cup (-W_i)$. Then for small enough t,

$$\begin{aligned} n(\nu) &\geqslant \mathscr{H}^0\{(p_\nu|M)^{-1}(y + t\sigma)\} \\ &\geqslant (n - r - s) + 2([r/2] + 1) + 2(r/2 - [r/2]) \geqslant n(\nu). \end{aligned}$$

Since the line through y parallel to σ is transverse to $p_\nu(C_\nu \cup \partial M)$, $y + t\sigma \notin p_\nu(C_\nu \cup \partial M)$ for small enough t, which proves the first conclusion. Replacing y by $y + t\sigma$ and choosing new neighborhoods U_i accordingly (which now *all* project onto open neighborhoods of $y + t\sigma$), we easily show that the multiplicity of $p_\nu|M$ is locally constant near such a point, establishing the second claim. The final conclusion also follows by a trivial modification of the above argument.

This provides us with sufficient local information on the multiplicity of $p_\nu|M$. It remains to demonstrate how to use a bound on the curvature of $p_\nu(C_\nu)$ and the number of cusp points to obtain a global bound on the multiplicity.

To begin with, we prove the analog of our theorem for 1-dimensional curves in 2-space.

Theorem 9. *If $C \subset \mathbf{R}^2$ is a finite union of compact immersed curves of class 2 with disjoint boundaries and $n(\sigma) = \sup\{\mathscr{H}^0[(p_\sigma|C)^{-1}(y)] \colon y \in \operatorname{Im} p_\sigma\}$, then*

$$\int_{S^1} n(\sigma)\, d\mathscr{H}^1\sigma \leqslant 2\int_C \|DN(x)\| d\mathscr{H}^1 x + \pi\mathscr{H}^0(\partial C).$$

Proof. If $S_\sigma^1 = \{x \in C \colon \sigma \cdot N(x) = 0,\ \sigma \cdot \langle \sigma, DN(x)\rangle \neq 0\}$ and $C_\sigma = \{x \in C \colon \sigma \cdot N(x) = 0\}$, then

$$\begin{aligned} \int_C \|DN(x)\| d\mathscr{H}^1 x &= \int_C J_1 N(x)\, d\mathscr{H}^1 x = \int_{S^1} \mathscr{H}^0(N^{-1}\{\sigma\})\, d\mathscr{H}^1\sigma \\ &= \tfrac{1}{2}\int_{S^1} \mathscr{H}^0\{x \in C \colon \sigma \cdot N(x) = 0\}\, d\mathscr{H}^1\sigma \\ &= \tfrac{1}{2}\int_{S^1} \mathscr{H}^0(S_\sigma^1) + \mathscr{H}^0(C_\sigma \setminus S_\sigma^1)\, d\mathscr{H}^1\sigma. \end{aligned}$$

If we define τ to be a unit tangent vectorfield to C, we find that $J_1\tau(x) = \tau \cdot \langle \tau, DN(x)\rangle$, hence by the co-area formula and setting $A = \{x \in C\colon J_1\tau(x) = 0\}$,

$$\int_{S^1} \mathscr{H}^0(C_\sigma \setminus S_\sigma^1)\, d\mathscr{H}^1\sigma = \int_A J_1\tau(x)\, d\mathscr{H}^1 x = 0.$$

Hence for almost all directions σ, $C_\sigma = S_\sigma^1$. In such a direction σ, the multiplicity function $n(\sigma, y)$ is easy to describe: it is locally constant on $\operatorname{Im} p_\sigma \setminus p_\sigma(C_\sigma)$, and has a discontinuity of magnitude at most $2n$ if $\mathscr{H}^0[(p_\sigma|C)^{-1}(y) \cap S_\sigma^1] = n$. Since the multiplicity is 0 for large values of y, we conclude that (for almost all $\sigma \in S^1$) $n(\sigma) \leqslant \mathscr{H}^0(C_\sigma) = \mathscr{H}^0(S_\sigma^1)$, proving the theorem in the case where C has no boundary. More generally, if C has m boundary points and σ is not tangent to C at any boundary points (again, this condition holds for almost all σ), then each boundary point x_i introduces a discontinuity of magnitude 1 in the multiplicity function $n(\sigma, y)$ at $y_i = p_\sigma(x_i)$. Thus $n(\sigma) \leqslant \mathscr{H}^0(C_\sigma) + m/2$, and we obtain the desired result by integration.

Not only is this theorem of great interest in itself, but it also comes in handy in proving the next lemma.

Lemma 10. *If $M \subset \mathbf{R}^3$ is a manifold of class* 3 *and ν is a direction such that*:

(1) $C_\nu = S_\nu^1 \cup S_\nu^2$,

(2) $\int_{p_\nu(C_\nu)} J_1 N(y)\, d\mathscr{H}^1 y < \infty$,

(3) $\int_{p_\nu(\partial M)} J_1 N(y)\, d\mathscr{H}^1 y < \infty$,

(4) $\mathscr{H}^0(S_\nu^2) < \infty$,

(5) $\mathscr{H}^0(C_\nu \cap \partial M) < \infty$,

and conditions (2)–(6) *of Lemma* 8 *hold, then*

$$2\pi n(\nu) \leqslant 2\int_{p_\nu(C_\nu)} J_1 N\, d\mathscr{H}^1 + 2\pi\mathscr{H}^0(S_\nu^2)$$

$$+ 3\pi\mathscr{H}^0(C_\nu \cap \partial M) + 2\int_{p_\nu(\partial M)} J_1 N\, d\mathscr{H}^1.$$

(*In the above equations N refers either to $N_{p_\nu(\partial M)}$ or $N_{p_\nu(C_\nu)}$, depending on whether the integral is taken over $p_\nu(\partial M)$ or $p_\nu(C_\nu)$. These should not be confused with N_M, the normal map to the surface M.*)

Proof. The idea is quite simple: apply Theorem 9 to the curve $p_\nu(C_\nu)$, treating the points of $p_\nu(S_\nu^2)$ and $p_\nu(C_\nu \cap \partial M)$ as boundary points. One complication arises, though: $p_\nu(C_\nu)$ is *not* a class two manifold in general. Indeed, if ϕ is a parametrization of C_ν in a neighborhood of a cusp point x such that $0 < |D\phi(t)| < \infty$ for all t, then

$$|D(p_\nu \circ \phi)(0)| = |\langle \phi'(0), p_\nu\rangle| = |\phi'(0)|\, |\langle \nu, p_\nu\rangle| = 0.$$

However, using the fact that ϕ is of class 2, we can show that the parametrization $p_\nu \circ \phi(t^{1/2})$ parametrizes $p_\nu(C_\nu)$ as a *class* 1 manifold-with-boundary: indeed, it is readily verified that

$$\frac{d}{dt}\left[p_\nu \circ \phi(t^{1/2})\right](0) = \tfrac{1}{2}\frac{d^2}{dt^2}\left[p_\nu \circ \phi(t)\right](0) \neq 0$$

if $\phi(0)$ is a cusp point.

Next we observe that the proof of Theorem 9 really required C to be only of class one at the boundary points (so that, using the implicit function theorem, we could assert that the multiplicity of $(p_\sigma|U)^{-1}(y)$ is 0 on one side of $p_\sigma(x)$ and 1 on the other side for some neighborhood U of x). Thus, if we assume that M is of class two at the boundary, that $C_\nu = S_\nu^1 \cup S_\nu^2$, that $S_\nu^2 \cap \partial M = \varnothing$ and that $\operatorname{Tan}(\partial M, x) \neq \operatorname{Tan}(C_\nu, x)$ at all points $x \in C_\nu \cap \partial M$, then $p_\nu(C_\nu) \cup p_\nu(\partial M)$ satisfies the (weakened) hypotheses of Theorem 9, with each point of $p_\nu(S_\nu^2)$ counted as a boundary point of two different arcs and each point of $p_\nu(S_\nu^1 \cap \partial M)$ as a boundary point of three. (If y is the image of more than one point of $(S_\nu^1 \cap \partial M) \cup S_\nu^2$, we count the multiplicity.)

Hence there exists $\sigma \in \operatorname{Im} p_\nu$ such that for any $z \in \operatorname{Im}(p_\sigma \circ p_\nu)$,

$$2\pi\mathscr{H}^0\left[(p_\sigma|p_\nu(C_\nu))^{-1}(z)\right] \leqslant 2\int_{p_\nu(C_\nu)} J_1 N \, d\mathscr{H}^1 + 2\int_{p_\nu(\partial M)} J_1 N \, d\mathscr{H}^1 + 2\pi\mathscr{H}^0\left(S_\nu^2\right) + 3\pi\mathscr{H}^0(C_\nu \cap \partial M) \equiv k.$$

To put it another way, any line L_σ parallel to σ intersects $p_\nu(C_\nu) \cup p_\nu(\partial M)$ in at most k points. Moreover, σ can be chosen so that for all but a measure zero set of lines $\{L_{\sigma,i}\}$, L_σ intersects $p_\nu(\partial M)$ and $p_\nu(C_\nu)$ transversely at all points of intersection and $L_\sigma \cap p_\nu(S_\nu^2) = \varnothing$, $L_\sigma \cap p_\nu(C_\nu \cap \partial M) = \varnothing$. On each such L_σ, by Lemma 6 and a counting argument,

$$\sup\{n(\nu, y) : y \in L_\sigma\} \leqslant \mathscr{H}^0\{L_\sigma \cap p_\nu(C_\nu)\} + \tfrac{1}{2}\mathscr{H}^0\{L_\sigma \cap p_\nu(\partial M)\} \leqslant k.$$

Thus $n(\nu, y) \leqslant k$ on a dense subset of the plane, and by Lemma 8 we conclude that $n(\nu) \leqslant k$. This proves the lemma.

Theorem 11 (*Multiplicity Estimate for 2-Manifolds in* $\mathbf{R}^3$). *If* $M \subset \mathbf{R}^3$ *is a manifold-with-boundary of class* $(3, \alpha)$ *and dimension* 2, *and if* $n(\nu) = \sup\{\mathscr{H}^0[(p_\nu|M)^{-1}(y)] : y \in \operatorname{Im} p_\nu\}$, *then*

$$\int_{S^2} n(\nu)\, d\mathscr{H}^2 \leqslant 2\int_M \|DN\|^2 \, d\mathscr{H}^2 + 2\int_M \|D^2 N\| d\mathscr{H}^2 + 6\int_{\partial M} \|DN\| d\mathscr{H}^1 + 8\int_{\partial M} \|D\tau\| d\mathscr{H}^1,$$

where $N\colon M \to S^2$ is the Gauss map and $\tau\colon \partial M \to S^2$ is a unit tangent vector field to ∂M.

Proof. We have already seen that

$$\int_{S^2}\int_{p_\nu(C_\nu)} J_1 N \, d\mathscr{H}^1 \, d\mathscr{H}^2\nu \leqslant 2\pi \int_M \|DN(x)\|^2 \, d\mathscr{H}^2 x \quad \text{(Lemma 1)},$$

$$\int_{S^2} \mathscr{H}^0(S_\nu^2) \, d\mathscr{H}^2\nu \leqslant \int_M \|D^2N(x)\| d\mathscr{H}^2 x \quad \text{(Lemma 2)}.$$

To prove the theorem, we need integral estimates on the two boundary terms $\mathscr{H}^0(C_\nu \cap \partial M)$ and $\int_{p_\nu(\partial M)} J_1(N_{p_\nu(\partial M)}) \, d\mathscr{H}^1$, and we need to prove that conditions (2)–(6) of Lemma 8 hold for $\mathscr{H}^2$-almost all ν. (Condition 1 is just assumption (A), and conditions (2)–(5) of Lemma 10 are guaranteed by the integral estimates we will obtain.)

Define $B = \{(x, \nu) \in \partial M \times S^2$ such that $\nu \cdot N(x) = 0\}$, and let $Q(x, \nu) = \nu$, $P(x, \nu) = x$ for any $(x, \nu) \in B$. By computing a basis for $\operatorname{Tan}(B, (x, \nu))$, one easily computes the Jacobians J_2Q and J_1P, and obtains

$$\begin{aligned}
\int_{S^2} \mathscr{H}^0\{x \in \partial M\colon \nu \cdot N(x) = 0\} \, d\mathscr{H}^2 &= \int_B J_2Q(x, \nu) \, d\mathscr{H}^2(x, \nu) \\
&= \int_{\partial M}\int_{\nu \cdot N(x) = 0} J_2Q(x, \nu)/J_1P(x, \nu) \, d\mathscr{H}^1\nu \, d\mathscr{H}^1 x \\
&= \int_{\partial M}\int_{\nu \cdot N(x) = 0} |\nu \cdot \langle \tau, DN_M\rangle| d\mathscr{H}^1\nu \, d\mathscr{H}^1 x \\
&= 4\int_{\partial M} |\langle \tau, DN_M\rangle| d\mathscr{H}^1 x \quad \left(\text{since } \int_0^{2\pi} |\cos\theta| d\theta = 4\right) \\
&\leqslant 4\int_{\partial M} \|DN_M\| d\mathscr{H}^1 x,
\end{aligned}$$

which gives us the desired estimate on $\mathscr{H}^0(\partial M \cap C_\nu)$.

By a similar computation,

$$\int_{S^2}\int_{p_\nu(\partial M)} J_1\Big(N_{p_\nu(\partial M)}(y)\Big) \, d\mathscr{H}^1 y \, d\mathscr{H}^2\nu = 8\pi \int_{\partial M} \|D\tau(x)\| d\mathscr{H}^1 x.$$

In this calculation, we are justified in assuming $J_1(N_{p_\nu(\partial M)})$ is defined for all $y \in p_\nu(\partial M)$ and $\mathscr{H}^2$-almost all ν because the only singular points of $p_\nu(\partial M)$ are points x where $\nu \in \operatorname{Tan}(\partial M, x)$. By the co-area formula, $\mathscr{H}^1\{\nu \in S^2\colon \nu \in \operatorname{Tan}(\partial M, x)$ for some $x \in \partial M\} < \infty$, hence the $\mathscr{H}^2$ measure of that set is 0.

Next we verify that $\mathscr{H}^0(S_\nu^2 \cap \partial M) = 0$ for almost all ν. By definition, $S_\nu^2 \cap \partial M \subset \partial M_0 \equiv \partial M \cap \{x\colon \det DN(x) \neq 0\}$, and the functions $e_1(x)$ and

$e_2(x)$ are defined locally on ∂M. By hypothesis, M is of class 3 at the boundary, so e_1 and e_2 are locally Lipschitzian and $e_1(\partial M_0)$ and $e_2(\partial M_0)$ are countably $\mathcal{H}^1$-rectifiable; hence $\mathcal{H}^2(e_1(\partial M) \cup e_2(\partial M)) = 0$. This is equivalent to the desired assertion.

Next we verify condition (3) of Lemma 8. If $\tau(x)$ is, as before, a unit tangent vectorfield to ∂M, another way of saying $\operatorname{Tan}(\partial M, x) = \operatorname{Tan}(C_\nu, x)$ is $\nu \cdot \langle \tau(x), DN(x) \rangle = 0$ or $J_2 Q(x, \nu) = 0$, where $Q\colon B \to S^2$ is as defined before; hence

$$\int_{S^2} \mathcal{H}^0\{x \in M\colon \nu \cdot N(x) = 0, \nu \cdot \langle \tau(x), DN(x) \rangle = 0\}\, d\mathcal{H}^2\nu = 0,$$

as desired.

It is trivial that condition (4) of Lemma 8 holds for $\mathcal{H}^2$-almost all ν.

Finally, we wish to show that conditions (5) and (6) of Lemma 8 hold for $\mathcal{H}^2$-almost all ν. We define the functions $\nu_1\colon \partial M \times \partial M \to S^2$, $\nu_2\colon \partial M \times M \to S^2$ and $\nu_3\colon M \times M \to S^2$ by the identical formulas $\nu_i(x, y) = (x - y)/|x - y|$. It is then easy to check that (assuming $x \neq y$)

(1) $J_2\nu_1 = 0 \leftrightarrow \det[x - y, \tau(x), \tau(y)] = 0 \leftrightarrow$ self-intersection of $p_{\nu_1}(\partial M)$ at $p_{\nu_1}(x) = p_{\nu_1}(y)$ is not transverse;

(2) $J_2\nu_2 = 0 \leftrightarrow y \in C_{\nu_2}$ and $\det[x - y, \tau(x), \langle \nu, DN(y) \rangle] = 0 \leftrightarrow$ the intersection of $p_{\nu_2}(\partial M)$ and $p_{\nu_2}(C_{\nu_2})$ at $p_{\nu_2}(x) = p_{\nu_2}(y)$ is not transverse;

(3) $J_2\nu_3 = 0 \leftrightarrow x, y \in C_{\nu_3}$ and $N(x) = N(y) \leftrightarrow$ the self-intersection of $p_{\nu_3}(C_{\nu_3})$ at $p_{\nu_3}(x) = p_{\nu_3}(y)$ is not transverse.

Thus the set of $\nu \in S^2$ for which condition (5) fails is $\bigcup \nu_i\{(x, y)\colon J_2\nu_i(x, y) = 0\})$, and has measure 0 by the Morse-Sard theorem. (Note that $C^{3,\alpha}$ is precisely the degree of smoothness needed to achieve this result for the map ν_3.)

To verify condition (6), let $\Pi = \{(x, y, z) \in \partial M \times \partial M \times \partial M\colon x \neq y, y \neq z, x \neq z, (x - y) \wedge (x - z) = 0\}$. Let $\nu_4\colon \Pi \to S^2$ be given (once again!) by $\nu_4(x, y, z) = (x - y)/|x - y|$. Π is readily seen to be a 1-manifold except at points (x, y, z) for which $\det[x - y, \tau(x), \tau(y)] = 0$. Hence, in view of the equivalence (1) above, $\mathcal{H}^2[\nu_4(\Pi)] = 0$.

Thus all the hypotheses of Lemma 10 are satisfied for $\mathcal{H}^2$-almost all ν, and by integrating the estimate of Lemma 10 over S^2 we obtain

$$\int_{S^2} n(\nu)\, d\mathcal{H}^2\nu \leqslant 2\int_M \|DN(x)\|^2\, d\mathcal{H}^2x + 2\int_M \|D^2N(x)\| d\mathcal{H}^2x$$

$$+6\int_{\partial M} \|DN(x)\| d\mathcal{H}^1x + 8\int_{\partial M} \|D\tau(x)\| d\mathcal{H}^1x.$$

Theorem 11 enables us to prove a generalization of a 1-dimensional integral geometric result given in **[5]**.

Proposition 12. *If* $M \subset B(0, R)$, $\partial M = \varnothing$ *and* $\int_M \|DN\|^2 \, d\mathscr{H}^2 \leqslant K_1$, $\int_M \|D^2N\| \, d\mathscr{H}^2 \leqslant K_2$, *then* $\mathscr{H}^2(M) \leqslant (K_1 + K_2)R^2$.

Proof. Let $\mathscr{L}$ denote the set of oriented lines in $\mathbf{R}^3$, and let μ denote the invariant measure of $\mathscr{L}$ under Euclidean motions. Then

$$\mathscr{H}^2(M) = (2\pi)^{-1} \int_{\mathscr{L}} \mathscr{H}^0(l \cap M) \, d\mu$$

$$= (2\pi)^{-1} \int_{S^2} \int_{l \| \nu} \mathscr{H}^0(l \cap M) \, d\mathscr{H}^2 l \, d\mathscr{H}^2 \nu$$

$$= (2\pi)^{-1} \int_{S^2} (\pi R^2) n(\nu) \, d\mathscr{H}^2 \nu \leqslant (K_1 + K_2) R^2.$$

Remark. The corresponding 1-dimensional result is that $\mathscr{H}^1(C) \leqslant KR$ for a closed curve of total absolute curvature K contained in a ball of radius R. I. Fary claimed to prove the result of Proposition 12 *without* the bound K_2 **[2]**. His proof, however, is erroneous, in that he makes use of the 1-dimensional result without taking into account the possible existence of cusp points. This, of course, is precisely what forced us to introduce the third-derivative integral in the first place. While we do not have an example showing that K_2 must be present in Proposition 12, it seems unlikely that the assertion can be proved in general without it.

Conclusion. The preceding Theorem 11 shows that, if M is a sufficiently smooth compact manifold-with-boundary, then for $\mathscr{H}^2$-almost all directions $\nu \in S^2$ it can be viewed as the *graph* of a *multiple-valued function* over the plane Im p_ν. In particular, it is the graph of a $Q(\nu)$-valued function f_ν, where

$$\int_{S^2} Q(\nu) \, d\mathscr{H}^2 \leqslant C\left(\int_M \|DN\|^2 \, d\mathscr{H}^2 + \int_M \|D^2N\| d\mathscr{H}^2 + \int_{\partial M} \|DN\| d\mathscr{H}^1 + \int_{\partial M} \|D\tau\| d\mathscr{H}^1 \right).$$

Such multiple-valued functions have been defined and studied extensively by F. Almgren (see **[1]**), in whose work they have already proved to be a very versatile analytic tool for the study of variational problems.

In the second part of the author's Ph.D. thesis **[4]**, the question of whether one can obtain a global estimate on the modulus of continuity of the functions f_ν was studied. Under more restrictive assumptions on the manifold M, this turned out to be possible:

Theorem. *If* $M \subset \mathbf{R}^3$ *is a nonpositively-curved surface of class* 4 *such that* $\det DN(x) \neq 0$ *for all* $x \in \partial M$, *then the function*

$$C(\nu) = \sup\left\{ |f_\nu(x) - f_\nu(y)| / |x - y|^{1/3} : x, y \in \text{Im } p_\nu \right\}$$

is $\mathscr{H}^2$-measurable and

$$\int_{S^2} |C(\nu)|^{\alpha} \, d\mathscr{H}^2\nu < \infty \quad \textit{for } \alpha < 3/238.$$

(The norm in the expression $|f_\nu(x) - f_\nu(y)|$ is essentially the flat norm on integral currents.) This result requires a more careful analysis of the singular set of the projection p_ν than the present paper; a proof will be published elsewhere.

The role of the multiplicity estimate for 1-dimensional manifolds (Theorem 9) in proving the multiplicity estimate for 2-dimensional manifolds suggests the possibility of an inductive argument to prove a similar estimate for higher-dimensional manifolds. I am presently investigating that question.

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