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THE EQUATION OF PRESCRIBED GAUSS CURVATURE WITHOUT BOUNDARY CONDITIONS

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Abstract

A necessary condition for the equation of prescribed Gauss curvature to have a convex solution defined on a domain $\Omega \subset \mathbf{R}^n$ is that the Gauss curvature K satisfies $\int_\Omega K \leqslant \omega_n$. We prove the existence, uniqueness and regularity, under suitable hypotheses, of a convex solution in the extremal case $\int_\Omega K = \omega_n$. We also discuss the boundedness of convex solutions of the equation.

1. Introduction

In this paper we are concerned with the existence, uniqueness, regularity and boundedness of convex solutions of the equation of prescribed Gauss curvature

$$\det D^2 u = K(x)\left(1 + |Du|^2\right)^{(n+2)/2} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbf{R}^n$ without imposing boundary conditions on the function u. In particular, we are interested in a certain extremal case.

Equation (1.1) is elliptic only for functions $u \in C^2(\Omega)$ which are uniformly convex at each point of Ω. For such solutions to exist we must therefore assume that K is positive in Ω.

Suppose that $u \in C^2(\Omega)$ is a uniformly convex solution of (1.1). Then the gradient mapping Du is one-to-one with Jacobian $\det D^2 u$, so by integrating (1.1) and changing variables, we obtain, as in [4],

$$\begin{aligned} \int_\Omega K &= \int_{Du(\Omega)} \frac{dp}{\left(1 + |p|^2\right)^{(n+2)/2}} \\ &\leqslant \int_{\mathbf{R}^n} \frac{dp}{\left(1 + |p|^2\right)^{(n+2)/2}} = \omega_n, \end{aligned} \tag{1.2}$$

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where ω_n denotes the measure of the unit ball in $\mathbf{R}^n$. Thus the condition

$$\int_\Omega K \leqslant \omega_n \tag{1.3}$$

is necessary for the existence of a C^2 convex solution of (1.1).

Equation (1.1) has been studied by several authors, particularly in connection with the Dirichlet problem

$$\det D^2 u = K\left(1 + |Du|^2\right)^{(n+2)/2} \quad \text{in } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega, \tag{1.4}$$

where ϕ is some prescribed boundary function. Bakelman [2], [3] (see also [9, Theorem 17.4]) has proved that the condition

$$\int_\Omega K < \omega_n \tag{1.5}$$

suffices for a maximum modulus bound for convex solutions of (1.4) in a bounded domain with bounded ϕ. Further conditions for the generalized solvability of the Dirichlet problem (1.4) are treated in the papers [3], [4], [5]. The classical solvability of (1.4) is discussed in [17] (following [11], [12]) where it is proved that if (1.5) holds, Ω is $C^{1,1}$ and uniformly convex, $K \in C^{1,1}(\Omega)$, $\phi \in C^{1,1}(\partial\Omega)$ and also

$$K(x) \leqslant C \operatorname{dist}(x, \partial\Omega) \tag{1.6}$$

in a neighborhood of $\partial\Omega$, then there is a unique convex solution $u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ of the Dirichlet problem (1.4).

It was previously shown in [4] and [5] that the condition (1.6) ensures that the generalized solution of (1.4) satisfies the boundary condition in the classical sense, provided Ω is uniformly convex and φ is continuous.

Here we are concerned primarily with the extremal case

$$\int_\Omega K = \omega_n. \tag{1.7}$$

This condition is analogous to one considered by Giusti [10] for the equation of prescribed mean curvature.

The paper is arranged in the following way. In §2 we summarize the main ideas of the theory of generalized solutions of (1.1). In §3 we prove a comparison principle which we use several times in subsequent sections. §4 contains an a priori oscillation estimate for convex solutions of (1.1). This estimate differs from others of this type (see for example [4], [7], [15]) in that no boundary conditions are imposed on the solution. In §5 we prove the existence of a generalized solution of (1.1) with K satisfying (1.7) using an approximation procedure together with the oscillation estimate, and in §6 we discuss the regularity of the solution under some additional hypotheses. The final section is concerned with the boundedness of solutions of (1.1).

Finally, we mention that with minor changes in the proofs our results also hold for more general Monge-Ampère equations of the form

$$\det D^2 u = f(x, u, Du).$$

These results are discussed in [18].

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2. Preliminaries

In the following sections we shall use the concept of a generalized solution of (1.1). Such concepts were introduced for general Monge-Ampère equations by Aleksandrov [1] and Bakelman [2]. Here we summarize the main ideas.

Suppose u is a convex function defined on a domain $\Omega \subset \mathbf{R}^n$. The *normal mapping*, $\chi_u(y)$, of a point $y \in \Omega$ is given by

(2.1) $$\chi_u(y) = \{ p \in \mathbf{R}^n : u(x) \geqslant u(y) + p \cdot (x - y) \text{ for all } x \in \Omega \}.$$

The normal mapping of a set $E \subset \Omega$ is defined by

(2.2) $$\chi_u(E) = \bigcup_{y \in E} \chi_u(y).$$

If E is a Borel set, then so is $\chi_u(E)$. The normal mapping is one-to-one modulo a set of measure zero in the following sense:

(2.3) $$|\{ p \in \mathbf{R}^n : p \in \chi_u(y_1) \cap \chi_u(y_2) \text{ for } y_1, y_2 \in \Omega, y_1 \neq y_2 \}| = 0,$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbf{R}^n$.

If $R \in L^1_{\text{loc}}(\mathbf{R}^n)$ is a positive function, we define a set function $\omega(u, R)$ on the Borel subsets of Ω by

(2.4) $$\omega(u, R)(E) = \int_{\chi_u(E)} R(p)\, dp$$

for each Borel set $E \subset \Omega$. Using (2.3) it can be shown that $\omega(u, R)$ is a nonnegative countably additive measure on the σ-algebra of Borel subsets of Ω. Furthermore, $\omega(u, R)$ is finite on compact subsets of Ω and therefore has the following regularity properties:

(2.5) $$\omega(u, R)(A) = \inf\{ \omega(u, R)(U) : U \text{ is open}, A \subset U \subset \Omega \}$$

for each Borel set $A \subset \Omega$, and

(2.6) $$\omega(u, R)(U) = \sup\{ \omega(u, R)(K) : K \text{ is compact}, K \subset U \}$$

for each open set $U \subset \Omega$.

Suppose that $\{u_i\}$ is a sequence of convex functions defined on Ω which converges uniformly on compact subsets of Ω to a convex function u. Then for each compact set $K \subset \Omega$ and each open set $U \subset \Omega$, we have

$$\limsup_{i \to \infty} \omega(u_i, R)(K) \leqslant \omega(u, R)(K) \tag{2.7}$$

and

$$\liminf_{i \to \infty} \omega(u_i, R)(U) \geqslant \omega(u, R)(U). \tag{2.8}$$

It follows from (2.7) and (2.8) that $\omega(u_i, R)$ converges weakly to $\omega(u, R)$, i.e.,

$$\int \phi \, d\omega(u_i, R) \to \int \phi \, d\omega(u, R)$$

for each continuous function with compact support in Ω.

A convex function u defined on Ω is said to be a *generalized solution* of the equation

$$R(Du) \det D^2 u = K \tag{2.9}$$

in Ω if for each Borel set $E \subset \Omega$ we have

$$\int_{\chi_u(E)} R(p)\, dp = \int_E K. \tag{2.10}$$

A convex C^2 solution of (2.10) is also a generalized solution. In the remainder of the paper we write $\omega(u)$ instead of $\omega(u, R)$ if

$$R(p) = \frac{1}{\left(1 + |p|^2\right)^{(n+2)/2}}.$$

It is clear that if u is a generalized solution of (1.1), then (1.3) holds. If (1.7) holds, then $\chi_u(\Omega) = \mathbf{R}^n - E$, where $|E| = 0$.

We write $\omega(u, R) \geqslant \omega(v, R)$ in Ω if for each Borel set $E \subset \Omega$ we have $\omega(u, R)(E) \geqslant \omega(v, R)(E)$. In the remainder of the paper we write $(\det D^2 u)(1 + |Du|^2)^{-(n+2)/2}$ as $F[u]$. We denote the k-dimensional Hausdorff measure by $\mathscr{H}^k$.

Finally, unless otherwise stated our notation, is standard, as, for example, in [9].

3. A comparison principle

In this section we prove a comparison principle which will be used several times in the remainder of the paper. We first introduce some terminology and prove two lemmas.

Suppose that Ω is a bounded domain in $\mathbf{R}^n$ and u is a convex function defined on Ω. For each $x \in \partial\Omega$, let

$$L(x) = \overline{\operatorname{graph} u} \cap (\{x\} \times \mathbf{R}).$$

If $L(x) \neq \varnothing$, we define the value of u at x to be $u(x) = \inf\{t\colon (x, t) \in L(x)\}$. We note that u is then a lower semicontinuous function on the set of points of $\overline{\Omega}$ at which it is defined. We say that the graph of u is vertical on $\Gamma \subset \partial\Omega$ if for every affine function f such that $f \leqslant u$ in Ω, we have $f < u$ on $\Gamma \cap \{x \in \partial\Omega\colon L(x) \neq \varnothing\}$.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbf{R}^n$ and $\Gamma \subset \partial\Omega$. Suppose that u, $v \in C^0(\Omega \cup (\partial\Omega - \Gamma))$ are convex functions satisfying $u = v$ on $\partial\Omega - \Gamma$, $u \leqslant v$ in Ω, and the graph of u is vertical on Γ. Then $\chi_v(\Omega) \subset \chi_u(\Omega)$.*

Proof. Every supporting hyperplane of graph v can be translated to give a parallel supporting hyperplane of graph u, from which the result follows.

Lemma 3.2. *Let f and g be convex functions defined on a bounded open set $I \subset \mathbf{R}$, and suppose that $f < g$ in I. For each $\alpha \geqslant 0$ let $I_\alpha = \{t \in I\colon f(t) + \alpha < g(t)\}$. Suppose that for each rational $\alpha \geqslant 0$ we have*

$$\chi_f(I_\alpha) = \chi_g(I_\alpha) \quad a.e., \tag{3.1}$$

and for each $\alpha \geqslant 0$, we have for $t \in \partial I_\alpha$ either $f(t) + \alpha = g(t)$, or the graph of f is vertical at t. Then $f - g$ is constant.

Proof. Let $\alpha > 0$ be irrational and $\{\alpha_i\}_{i=1}^\infty$ an enumeration of the rationals greater than α. Then from (3.1) we obtain

$$\chi_f(I_\alpha) = \bigcup_{i=1}^\infty \chi_f(I_{\alpha_i}) = \bigcup_{i=1}^\infty \chi_g(I_{\alpha_i}) \quad \text{a.e.}$$
$$= \chi_g(I_\alpha).$$

Thus (3.1) holds for all $\alpha \geqslant 0$.

Now suppose that the conclusion of the lemma is false. Then there exists a point $t_0 \in I$ such that f and g are differentiable at t_0, and

$$f'(t_0) \neq g'(t_0). \tag{3.2}$$

Choose $\beta > 0$ so that $t_0 \in \partial I_\beta$ and let $J_1 = \{t \in I_\beta\colon t > t_0\}$ and $J_2 = \{t \in I_\beta\colon t < t_0\}$. Then t_0 is in either ∂J_1 and ∂J_2; assume the latter. Using the convexity of f and g and (3.2), we deduce that $\chi_f(J_2) - \chi_g(J_2)$ contains a nonempty open interval. However, by Lemma 3.1, we have $\chi_g(J_1) \subset \chi_f(J_1)$, and clearly $\chi_f(J_1) \cap \chi_f(J_2)$ contains at most one point, so that $\chi_f(I_\beta) - \chi_g(I_\beta)$ contains a nonempty open set, contradicting (3.1).

We are now ready to prove the comparison principle.

Theorem 3.3. *Let Ω be a bounded domain in $\mathbf{R}^n$ and $\Gamma \subset \partial\Omega$, $\Gamma \neq \partial\Omega$. Suppose that $u, v \in C^0(\Omega \cup (\partial\Omega - \Gamma))$ are convex functions satisfying $u \leqslant v$ on $\partial\Omega - \Gamma$, and $\omega(u, R) \geqslant \omega(v, R)$ in Ω, where $R \in L^1(\mathbf{R}^n)$ is a positive function. Suppose also that the graph of v is vertical on Γ and for each Borel set $E \subset \Omega$ with $|E| > 0$, we have $|\chi_v(E)| > 0$. Then $u \leqslant v$ in Ω.*

Proof. Suppose not. For each $\alpha \geqslant 0$ let $U_\alpha = \{x \in \Omega: v(x) + \alpha < u(x)\}$. By Lemma 3.1 we have

$$\chi_u(U_\alpha) \subset \chi_v(U_\alpha), \tag{3.3}$$

from which we obtain

$$\omega(u, R)(U_\alpha) = \omega(v, R)(U_\alpha). \tag{3.4}$$

Since $\omega(u, R)(U_0) < \infty$, we infer that for each $\alpha \geqslant 0$, we have

$$\chi_u(U_\alpha) = \chi_v(U_\alpha) \quad \text{a.e.} \tag{3.5}$$

By the last hypothesis of the theorem we have $|\chi_u(U_0) \cap \chi_v(U_0)| > 0$, so by adding an affine function to u and v, we may assume that for some $x_0 \in U_0$ we have

$$0 \in \chi_u(x_0) \cap \chi_v(x_0). \tag{3.6}$$

Observing that

$$\chi_u(U_\alpha) = \bigcup_{\eta \in S^{n-1}} \chi_u(U_\alpha) \cap \operatorname{span}\{\eta\}, \tag{3.7}$$

and a similar expression for $\chi_v(U_\alpha)$, we obtain from (3.5) that for each fixed $\alpha \geqslant 0$ we have for $\mathscr{H}^{n-1}$ almost all $\eta \in S^{n-1}$,

$$\chi_u(U_\alpha) \cap \operatorname{span}\{\eta\} = \chi_v(U_\alpha) \cap \operatorname{span}\{\eta\} \mathscr{H}^1 \quad \text{a.e.} \tag{3.8}$$

Hence for $\mathscr{H}^{n-1}$ almost all $\eta \in S^{n-1}$ we have (3.8) holding for all rational $\alpha \geqslant 0$.

For each $\eta \in S^{n-1}$ let P_η denote the orthogonal projection of $U_0 \times \mathbf{R}$ onto $\operatorname{span}\{\eta, e_{n+1}\}$, where e_{n+1} is a unit vector pointing along the axis of the cylinder $U_0 \times \mathbf{R}$. Then for each $\eta \in S^{n-1}$, the sets

$$P_\eta(\{(x, u(x)): x \in U_0, \chi_u(x) \cap \operatorname{span}\{\eta\} \neq \varnothing\}),$$
$$P_\eta(\{(x, v(x)): x \in U_0, \chi_v(x) \cap \operatorname{span}\{\eta\} \neq \varnothing\})$$

are the graphs of two convex functions f_η and g_η defined on a bounded relatively open subset of $\operatorname{span}\{\eta\}$. For $\mathscr{H}^{n-1}$ almost all $\eta \in S^{n-1}$, f_η and g_η satisfy the hypotheses of Lemma 3.2, and hence differ by a constant. By (3.6), f_η and g_η must differ by the same constant for $\mathscr{H}^{n-1}$ almost all $\eta \in S^{n-1}$, and hence it follows that $u - v$ is constant in U_0. However, $u = v$ at some point of $\partial U_0 - \Gamma$, so we have $u = v$ in U_0, which is a contradiction.

Theorem 3.3 yields the following uniqueness result.

Corollary 3.4. *Let Ω be a bounded convex domain in $\mathbf{R}^n$. Let u and v be convex functions on Ω satisfying $\omega(u, R) = \omega(v, R)$ in Ω, where $R \in L^1(\mathbf{R}^n)$ is a positive function. Suppose that the graphs of u and v are vertical on $\partial\Omega$, and for each Borel set $E \subset \Omega$ with $|E| > 0$ we have $|\chi_u(E)|, |\chi_v(E)| > 0$. Then $u - v$ is constant.*

Proof. If not, then by adding a constant to v we can assume that $U = \{x \in \Omega \colon u(x) < v(x)\} \neq \varnothing$ or Ω, and $\partial U = \Gamma_1 \cup \Gamma_2$, where $u = v$ on $\Gamma_1 \subset \Omega$ and the graphs of u and v are vertical on $\Gamma_2 \subset \partial\Omega$. Then from Theorem 3.3 we obtain $u = v$ in U, a contradiction.

Remarks. (1) If $\Gamma = \varnothing$, we can allow $R \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ in the proof of Theorem 3.3 by replacing v by $v + \varepsilon$ for $\varepsilon > 0$ and letting $\varepsilon \to 0$ at the end.

(2) The last hypothesis of Theorem 3.3 is not necessary and is made only to ensure that $\chi_u(U_0) \cap \chi_v(U_0) \neq \varnothing$. The case $|\chi_u(U_0)| = |\chi_v(U_0)| = 0$ can easily be treated separately.

(3) An alternative proof of Theorem 3.3, based on a method of Aleksandrov, is presented in [18]. The proof given here, which is the earlier one, is included because the method is of interest. We also note that in both proofs the requirement that graph v be vertical on $\partial\Omega$ can be replaced by a suitable measure theoretic statement (see [18]).

In §6 we shall impose conditions on K and Ω which ensure that the graph of a generalized solution of (1.1) in Ω is vertical on $\partial\Omega$. This is important for proving regularity, as well as uniqueness.

4. An oscillation estimate

The oscillation estimate proved in this section is the key idea used to prove the existence of a generalized solution of (1.1) with K satisfying (1.7). We first prove two lemmas, the first of which is taken from [4].

Lemma 4.1. *Suppose u is a generalized solution of* (1.1) *in a domain $\Omega \subset \mathbf{R}^n$ with K satisfying $K \geqslant \lambda > 0$ in Ω for some constant λ. Then*

$$\int_\Omega \sqrt{1 + |Du|^2} \leqslant \frac{C(n)}{\lambda}. \tag{4.1}$$

Proof. If $u \in C^2(\Omega)$, then we have

$$\int_\Omega \sqrt{1 + |Du|^2} \leqslant \frac{1}{\lambda}\int_\Omega \frac{\det D^2 u}{\left(1 + |Du|^2\right)^{(n+1)/2}}. \tag{4.2}$$

Changing variables and estimating as in (1.2) gives the result. The general case follows by approximation.

Remark. Using Hölder's inequality we can prove a sharper form of (4.1), namely,

$$\int_\Omega \sqrt{1+|Du|^2} \leqslant C(n,r)\left(\int_\Omega K^{-1/(r-1)}\right)^{1-1/r} \tag{4.3}$$

for $r \in (1,2)$. For our purposes (4.1) will suffice.

Lemma 4.2. *Let u be a convex function defined on a bounded domain $\Omega \subset \mathbf{R}^n$, and suppose that*

$$\int_\Omega |Du| \leqslant M. \tag{4.4}$$

Then for each $\varepsilon > 0$ there is a t_0, depending only on n, ε, M, $d = \operatorname{diam}\Omega$ and $m = \inf_\Omega u$, such that if $t \geqslant t_0$, then

$$L_t = \{x \in \Omega\colon u(x) = t\} \subset \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \varepsilon\}. \tag{4.5}$$

Proof. We may assume that $m = 0$. For $t > 0$ let $A_t = \{x \in \Omega\colon u(x) > t\}$ and $B_t = \{x \in \Omega\colon |Du(x)| > t\}$. Then

$$|B_t| \leqslant M/t, \tag{4.6}$$

and by the convexity of u, since $m = 0$, we have

$$\inf_{A_t} |Du| \geqslant t/d. \tag{4.7}$$

Thus $A_t \subset B_{t/d}$, and for $t > t_0 = 2Md/\omega_n\varepsilon^n$ we have

$$|A_t| \leqslant \tfrac{1}{2}\omega_n\varepsilon^n. \tag{4.8}$$

For $t > t_0$, let $y \in L_t$ and T be an $n-1$ dimensional supporting plane of L_t at y. Let T^+ and T^- be the associated half spaces. L_t is an $n-1$ dimensional convex submanifold of Ω, and is therefore contained in one of the half spaces, say T^-. Then by the convexity of u, $T^+ \cap \Omega \subset A_t$. If $B_\rho(y) \subset \Omega$, then $B_\rho(y) \cap T^+ \subset A_t$ and hence

$$\tfrac{1}{2}\omega_n\rho^n \leqslant |B_\rho(y) \cap T^+| \leqslant \tfrac{1}{2}\omega_n\varepsilon_n,$$

from which the result follows.

Corollary 4.3. *Let u be a generalized solution of* (1.1) *in a domain $\Omega \subset \mathbf{R}^n$. Then if $\Omega'' \subset\Subset \Omega' \Subset \Omega$, we have*

$$\operatorname*{osc}_{\Omega''} u \leqslant C, \tag{4.9}$$

where C depends only on n, $\operatorname{diam}\Omega'$, $\operatorname{dist}(\Omega'', \partial\Omega')$ *and* $\inf_{\Omega'} K > 0$.

The local oscillation estimate (4.9) is in fact sufficient to prove the existence theorem of the following section. We now prove the global analogue of (4.9). We say that a domain $\Omega \subset \mathbf{R}^n$ satisfies a *uniform interior sphere condition with* *radius* R if for each point $x \in \partial\Omega$ there is a ball $B = B_R(y) \subset \Omega$ such that $\partial\Omega \cap \partial B = \{x\}$.

Theorem 4.4. *Let* Ω *be a bounded domain in* $\mathbf{R}^n$ *satisfying a uniform interior sphere condition with radius* R, *and* u *a generalized solution of* (1.1) *in* Ω. *If*

$$K(x) \geqslant M \operatorname{dist}(x, \partial\Omega)^{\delta} \tag{4.10}$$

for some constants $M > 0$ *and* $\delta \in [0, 1)$, *then we have*

$$\operatorname*{osc}_{\Omega} u \leqslant C, \tag{4.11}$$

where C *depends only on* n, δ, M, R *and* $\operatorname{diam}\Omega$.

Proof. For convenience we assume that $\inf_\Omega u = 0$ and $R \leqslant 1$. Applying Lemmas 4.1 and 4.2 on $U = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > R/4\}$ we obtain

$$L_t \subset \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < R/2\} \tag{4.12}$$

for all $t \geqslant t_0(n, \delta, M, R, \operatorname{diam}\Omega)$.

To obtain (4.11) we use a barrier argument which was used in [17] to prove a nonexistence result for the Dirichlet problem for a class of Monge-Ampère equations including (1.1). Let $B = B_R(y)$ be an interior ball at $x_0 \in \partial\Omega$ and $r(x) = \operatorname{dist}(x, \partial B) = R - |x - y|$. Let $w = \psi(r) = A - ar^{\beta}$, where $\beta \in (0, 1)$, a and A are constants to be chosen. For $x \in B$ satisfying $|x - y| \geqslant R/2$ we obtain

$$\begin{aligned} F[w] &= \frac{\psi''(-\psi'/|x-y|)^{n-1}}{(1 + |\psi'|^2)^{(n+2)/2}} \\ &\leqslant 2^{n-1}(a\beta)^{-2} R^{1-n} r^{2(1-\beta)-1} \leqslant K, \end{aligned}$$

provided $2(1 - \beta) - 1 > \delta$ and $2^{n-1}(a\beta)^{-2}R^{1-n} \leqslant M$. Thus we have $\omega(w) \leqslant \omega(u)$ in $B_R(y) \cap \{x \in \Omega\colon u(x) > t_0\}$. By choosing a suitable value for A we may ensure that $w \geqslant t_0 = u$ on $B_R(y) \cap L_{t_0}$, so by Theorem 3.3 we obtain $u \leqslant w$ in $B_R(y) \cap \{x \in \Omega\colon u(x) > t_0\}$, from which (4.11) follows.

5. Existence

This section contains a principal result of this paper. We state it for quite general functions K and domains Ω, although to prove the regularity result of the next section some restrictions will be necessary. Our proof is based on an approximation procedure using the following existence result for the Dirichlet problem for (1.1). A proof is given in [17] (see also [11], [12]).

Lemma 5.1. *Let* Ω *be a* $C^{1,1}$ *uniformly convex domain in* $\mathbf{R}^n$ *and* $K \in C^{1,1}(\Omega)$ *a positive function satisfying* (1.5) *and* (1.6). *Let* $\phi \in C^{1,1}(\bar{\Omega})$. *Then the Dirichlet problem*

$$F[u] = K \quad \text{in } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega$$

has a unique convex solution u in $C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$.

Theorem 5.2. *Let* Ω *be a bounded convex domain in* $\mathbf{R}^n$ *and* $K \in L^1(\Omega)$ *a positive function satisfying* (1.7) *and* $\inf_{\Omega'} K > 0$ *for all* $\Omega' \Subset \Omega$. *Then there is a generalized solution of* (1.1) *in* Ω.

Proof. Let $\{\Omega_m\}$ be an increasing sequence of $C^{1,1}$ uniformly convex subdomains of Ω satisfying $\bigcup \Omega_m = \Omega$ and $\{K_m\}$ a sequence of functions in $C^{1,1}(\Omega)$ converging to K in $L^1(\Omega)$ and satisfying $K_m > 0$ in Ω_m,

$$\int_{\Omega_m} K_m < \omega_n \tag{5.1}$$

and

$$K_m(x) \leqslant C(m)\,\mathrm{dist}(x, \partial\Omega_m). \tag{5.2}$$

Let u_m be the unique solution of the Dirichlet problem $F[u_m] = K_m$ in Ω_m, $u_m = c_m$ on $\partial\Omega_m$, where c_m are constants to be chosen.

Choose $x_0 \in \Omega$; we may assume that $x_0 \in \Omega_m$ for each m. Fix the constants c_m so that $u_m(x_0) = 0$ for all m. Using the oscillation estimate (4.9) we see that for each subdomain $\Omega' \Subset \Omega$ we have a uniform bound for $\sup_{\Omega'}|u_m|$ for all sufficiently large m, and hence there is a subsequence converging uniformly on compact subsets of Ω to a convex function u which is a generalized solution of (1.1) in Ω.

Remarks. (1) The existence assertion of Theorem 5.2 clearly holds if we assume (1.5) in place of (1.7). This is also proved in [4]. In this case we can also obtain a uniformly Lipschitz solution by choosing a $C^{1,1}$ uniformly convex domain $\tilde{\Omega} \Supset \Omega$ and a sequence of positive functions $\{K_m\} \subset C^{1,1}(\tilde{\Omega})$ converging in $L^1(\Omega)$ to K and satisfying for each m the inequalities

$$\int_{\tilde{\Omega}} K_m \leqslant \omega_n - \varepsilon \tag{5.3}$$

for some $\varepsilon > 0$ and

$$K_m(x) \leqslant C(m)\,\mathrm{dist}(x, \partial\tilde{\Omega}). \tag{5.4}$$

We can then obtain uniform bounds for $|u_m|_{0;\tilde{\Omega}}$ (see [4], or [9, Theorem 17.4]), where u_m is the convex solution of the Dirichlet problem

$$F[u_m] = K_m \quad \text{in } \tilde{\Omega}, \qquad u_m = 0 \quad \text{on } \partial\tilde{\Omega},$$

and hence uniform bounds for $|Du_m|_{0;\Omega}$, using the obvious estimate

$$\sup_{\Omega} |Du_m| \leqslant \operatorname*{osc}_{\tilde{\Omega}} u_m \operatorname{dist}(\Omega, \partial\tilde{\Omega})^{-1}, \tag{5.5}$$

from which the required assertion follows.

(2) From the proof it is clear that Theorem 5.2 also holds for unbounded convex domains.

(3) It has been pointed out by the referee that Theorem 5.2 can also be proved by solving the dual boundary value problem obtained by taking the Legendre transform of the equation (1.1).

6. Regularity

In this section we prove that the generalized solution obtained in Theorem 5.2 is a regular solution provided Ω and K satisfy some additional conditions.

We shall require interior second derivative estimates for smooth solutions of (1.1).

Lemma 6.1. *Let $u \in C^{0,1}(\bar{\Omega}) \cap W^{4,n}_{\mathrm{loc}}(\Omega)$ be a convex solution of* (1.1) *in a bounded convex domain $\Omega \subset \mathbf{R}^n$ where $K \in C^{1,1}(\bar{\Omega})$ satisfies $K \geqslant \lambda > 0$ in Ω for some constant λ. If u is equal to an affine function on $\partial\Omega$, then for any $\Omega' \Subset \Omega$, we have*

$$\sup_{\Omega'} |D^2 u| \leqslant C, \tag{6.1}$$

where C depends only on n, λ, $|u|_{1;\Omega}$, $|K|_{1,1;\Omega}$, diam Ω *and* dist$(\Omega', \partial\Omega)$.

Lemma 6.2. *Let $u \in C^{1,1}(\bar{\Omega}) \cap W^{4,n}_{\mathrm{loc}}(\Omega)$ be a convex solution of* (1.1) *in a bounded domain $\Omega \subset \mathbf{R}^n$ where $K \in C^{1,1}(\bar{\Omega})$ satisfies $K \geqslant \lambda > 0$ in Ω for some constant λ. Then for any $\Omega' \Subset \Omega$, we have*

$$[D^2 u]_{\alpha;\Omega'} \leqslant C, \tag{6.2}$$

where $\alpha \in (0,1)$ depends only on n, λ, and $|D^2 u|_{0;\Omega}$ and C depends in addition on $|u|_{1;\Omega}$, $|K|_{1,1;\Omega}$, diam Ω *and* dist$(\Omega', \partial\Omega)$.

Lemma 6.1 is proved in [9] and [12] following Pogorelov [13], and Lemma 6.2 is proved in [16]. Before stating the next lemma we recall the definition of the *generalized Gauss map* of a convex hypersurface. If M is a convex hypersurface in $\mathbf{R}^n$, the generalized Gauss image of a set $E \subset M$ is given by

$$G(E) = \bigcup_{y \in E} \{\eta \in S^{n-1} \colon \eta \text{ is the outer unit normal to a supporting hyperplane of } M \text{ at } y\}. \tag{6.3}$$

Thus G is a set function.

We say that a domain $\Omega \subset \mathbf{R}^n$ satisfies an *enclosing sphere condition* at $x_0 \in \partial\Omega$ if there is a ball $B = B_R(y) \supset \Omega$ such that $\partial\Omega \cap \partial B = \{x_0\}$.

Lemma 6.3. *Let Ω be a bounded convex domain in $\mathbf{R}^n$ satisfying an enclosing sphere condition at each point of $\partial\Omega$. Let u be a generalized solution of* (1.1) *in Ω with K satisfying* (1.7). *Suppose that for each $y \in \partial\Omega$ we have*

$$K(x) \leqslant M \operatorname{dist}(x, \partial\Omega)^{-\delta} \tag{6.4}$$

in a neighborhood of y, where $M > 0$ and $\delta \in (0,1)$ are constants depending only on y. Then the graph of u is vertical on $\partial\Omega$.

Proof. If not, then there exists an affine function f such that $f(x_0) = u(x_0)$ for some $x_0 \in \partial\Omega$ and $f < u$ in Ω. By replacing u by $u - f$ we can assume that u satisfies the equation

$$\frac{\det D^2 u}{(1 + |Du + Df|^2)^{(n+2)/2}} = K \tag{6.5}$$

in Ω in the generalized sense, $u(x_0) = 0$ and $u > 0$ in Ω. For $t > 0$ let

$$\Gamma_t = \{x \in \Omega : u(x) = t\} \cup \{x \in \partial\Omega : u(x) \leqslant t\}.$$

Then Γ_t is a closed convex $n-1$ dimensional submanifold of $\overline{\Omega}$, and $x_0 \in \Gamma_t$ for all t. Let $B_R(y) \supset B_{R/2}(z)$ be enclosing spheres at x_0. Let G_t and G denote the generalized Gauss maps of Γ_t and $\Gamma = \partial B_R(y)$ respectively. Then for each $t > 0$ we have

$$G_t(\Gamma_t - B_\varepsilon) \subset G(\Gamma - B_\varepsilon) \tag{6.6}$$

for $\varepsilon > 0$, where $B_\varepsilon = B_\varepsilon(x_0)$.

If $x \in \Omega - B_\varepsilon$ for some $\varepsilon > 0$, then $x \in \Gamma_t$ for some $t > 0$. Let g be an affine function whose graph is a supporting hyperplane of graph u at $(x, u(x))$. Then provided $Dg \neq 0$, we have

$$Dg/|Dg| \in G_t(\Gamma_t - B_\varepsilon), \tag{6.7}$$

and hence

$$\chi_u(\Omega - B_\varepsilon) \subset \{p \in \mathbf{R}^n : p/|p| \in G(\Gamma - B_\varepsilon)\} \cup \{0\}. \tag{6.8}$$

Since $\chi_u(\Omega) = \mathbf{R}^n - E$, where $|E| = 0$, we then have

$$\chi_u(\Omega \cap B_\varepsilon) \supset \{p \in \mathbf{R}^n : p/|p| \in G(\Gamma \cap B_\varepsilon)\}, \tag{6.9}$$

except possibly for a set of measure zero. From (6.9) we obtain

$$\begin{aligned}
\int_{\Omega \cap B_\varepsilon} K &= \int_{\chi_u(\Omega \cap B_\varepsilon)} \frac{dp}{(1 + |p + Df|^2)^{(n+2)/2}} \\
&= \int_{Df + \chi_u(\Omega \cap B_\varepsilon)} \frac{dp}{(1 + |p|^2)^{(n+2)/2}} \\
&\geqslant \frac{C(n, |Df|)\varepsilon^{n-1}}{R^{n-1}}
\end{aligned} \tag{6.10}$$

for all $\varepsilon \in (0, \varepsilon_0)$, where ε_0 depends only on R.

Denoting $\operatorname{dist}(x, \partial\Omega)$ by $d(x)$ and using the fact that $|Dd| = 1$ almost everywhere in Ω, together with the coarea formula [8, Theorem 3.2.12], inequality (6.4) and the estimate

$$\mathscr{H}^{n-1}\left(d^{-1}(t) \cap \Omega \cap B_\varepsilon\right) \leqslant C(n)\varepsilon^{n-1}, \tag{6.11}$$

we have, for small ε,

$$\begin{aligned} \int_{\Omega \cap B_\varepsilon} K &\leqslant M \int_0^\varepsilon \int_{d^{-1}(t) \cap \Omega \cap B_\varepsilon} d(x)^{-\delta} d\mathscr{H}^{n-1}(x)\, dt \\ &\leqslant \frac{C(n) M \varepsilon^{n-\delta}}{1-\delta}. \end{aligned} \tag{6.12}$$

For ε sufficiently small (6.10) and (6.12) give a contradiction, which proves the lemma.

Remark. It is evident that even if Ω does not satisfy an enclosing sphere condition at some point $x_0 \in \partial\Omega$, we can still obtain the conclusion of Lemma 6.3 provided K satisfies

$$K(x) \leqslant M \operatorname{dist}(x, \partial\Omega)^\alpha \tag{6.13}$$

in a neighborhood of x_0, where bounds on the value of α depend on n and parameters determined by the behavior of $\partial\Omega$ near x_0.

We are now ready to prove the regularity theorem. We use a technique which has already been used for proving the regularity of generalized solutions of Monge-Ampère equations, for example in [6].

Theorem 6.4. *Let Ω be a bounded convex domain in $\mathbf{R}^n$ satisfying an enclosing sphere condition at each point of $\partial\Omega$ and $K \in C^{1,1}(\Omega)$ a positive function satisfying* (1.7). *Suppose that for each point $y \in \partial\Omega$ K satisfies* (6.4) *in a neighborhood of y with $M > 0$ and $\delta \in (0,1)$ depending only on y. Then if u is a generalized solution of* (1.1) *in Ω, we have $u \in C^2(\Omega)$.*

Proof. By Lemma 6.3 and Corollary 3.4, u is unique up to an additive constant. We may therefore assume that u is obtained by the argument of Theorem 5.2. In the proof of Theorem 5.2 we can now take $\{K_m\} \subset C^{1,1}(\Omega)$ converging locally in $C^{1,1}$ to K. Elliptic regularity theory [9, Lemma 17.16] then yields $u_m \in W^{4,n}_{\mathrm{loc}}(\Omega_m)$. We may also assume that $\{u_m\}$ converges uniformly on compact subsets of Ω to u.

Fix a point $x_0 \in \Omega$. We will show that u is C^2 in a neighborhood of x_0. Since x_0 is arbitrary, this implies that u is in $C^2(\Omega)$.

Let f be an affine function whose graph is a supporting hyperplane of graph u at $(x_0, u(x_0))$. By Lemma 6.3, graph f does not intersect the boundary of graph u. Thus for some $\varepsilon > 0$, $U = \{x \in \Omega\colon u(x) < f(x) + 4\varepsilon\} \Subset \Omega$ and $x_0 \in U$. For all sufficiently large m, the sets $\{x \in \Omega\colon u_m(x) < f(x) + 3\varepsilon\}$ are contained in a fixed compact subset of U, and $\{x \in \Omega\colon u_m(x) < f(x) + \varepsilon\}$ contains a fixed compact neighborhood of x_0. Using the estimates (4.9) and (5.5) we obtain uniform estimates for $|u_m|$ and $|Du_m|$ on $\{x \in \Omega\colon u_m(x) < f(x) + 3\varepsilon\}$ for sufficiently large m. Using Lemmas 6.1 and 6.2 we obtain uniform estimates for $|D^2 u_m|$ on $\{x \in \Omega\colon u_m(x) < f(x) + 2\varepsilon\}$, and then for $[D^2 u_m]_\alpha$ on $\{x \in \Omega\colon u_m(x) < f(x) + \varepsilon\}$ for sufficiently large m. We thus have uniform estimates for $|u_m|_{2,\alpha}$ on a neighborhood of x_0, from which it follows that $u \in C^2(\Omega)$.

Remarks. (1) If $K \in C^k(\Omega)$ for $k \geqslant 2$, elliptic regularity theory [9, Lemma 17.16] implies $u \in C^{k+1,\alpha}(\Omega)$ for all $\alpha < 1$. In particular, if $K \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

(2) If $n = 2$, generalized solutions of (1.1) with $K > 0$ are strictly convex (see [14]). In this case, by modifying the proof of Theorem 6.4, we may drop the hypothesis (6.4), and we can also allow Ω to be an arbitrary domain in $\mathbf{R}^2$.

Corollary 6.5. *Suppose that Ω is a bounded convex domain in $\mathbf{R}^n$ satisfying an enclosing sphere condition at each point of $\partial\Omega$ and $K \in C^{1,1}(\Omega \cup \Gamma)$ satisfies* (1.5), (6.4) *and $K > 0$ on $\Omega \cup \Gamma$, where Γ is a relatively open $C^{1,1}$ portion of $\partial\Omega$. Then there is a convex function $u \in C^2(\Omega \cup \tilde{\Gamma})$ satisfying* (1.1) *in Ω, where $\tilde{\Gamma} \Subset \Gamma$ is a relatively open portion of $\partial\Omega$.*

Proof. We can extend K to a positive function $\tilde{K} \in C^{1,1}(\tilde{\Omega})$ satisfying

$$\tilde{K}(x) \leqslant M \operatorname{dist}(x, \partial\tilde{\Omega})^{-\delta} \tag{6.14}$$

in a neighborhood of each point $y \in \partial\tilde{\Omega}$ with $M > 0$ and $\delta < 1$ depending only on y, and

$$\int_{\tilde{\Omega}} \tilde{K} = \omega_n, \tag{6.15}$$

where $\tilde{\Omega} \supset \Omega$ is a bounded convex domain satisfying an enclosing sphere condition at each point of $\partial\tilde{\Omega}$ and $\tilde{\Gamma} = \tilde{\Omega} \cap \partial\Omega \Subset \Gamma$. Theorems 5.2 and 6.4 imply that there is a convex function $\tilde{u} \in C^2(\tilde{\Omega})$ satisfying $F[\tilde{u}] = \tilde{K}$ in $\tilde{\Omega}$, so $u = \tilde{u}_{|\Omega} \in C^2(\Omega \cup \tilde{\Gamma})$ is a convex solution of $F[u] = K$ in Ω.

Remark. If $K \in C^{1,1}(\overline{\Omega})$, $K > 0$ on $\overline{\Omega}$, Ω is $C^{1,1}$ and K satisfies (1.5), then in a similar way we can obtain a convex $C^2(\overline{\Omega})$ solution of (1.1).

7. Boundedness

In this section we discuss the boundedness of generalized solutions of (1.1). From the barrier argument used in the proof of Theorem 4.4 we can immediately conclude the following.

Theorem 7.1. *Let Ω be domain in $\mathbf{R}^n$ and u a generalized solution of* (1.1) *in Ω. Let $x_0 \in \partial\Omega$. If*

$$K(x) \geqslant M \operatorname{dist}(x, \partial\Omega)^\delta \tag{7.1}$$

in a neighborhood of x_0, where $M > 0$ and $\delta < 1$ are constants, and Ω satisfies an interior sphere condition at x_0, then for each ball $B \subset \Omega$ such that $\partial B \cap \partial\Omega = \{x_0\}$, we have

$$\sup_B u < \infty. \tag{7.2}$$

Next we consider conditions for the solution to be unbounded. We first prove a simple geometric lemma.

Lemma 7.2. *Let $B_R \supset B_{R/2}$ be two balls in $\mathbf{R}^n$ with $\partial B_R \cap \partial B_{R/2} = \{x_0\}$. Let $y \in \partial B_{R/2}$ be the center of B_R. Then for each $x \in B_{R/2}$ we have*

$$|x - x_0| \leqslant (2R)^{1/2} \operatorname{dist}(x, \partial B_R)^{1/2}. \tag{7.3}$$

Proof. First suppose that $\zeta \in \partial B_{R/2}$. Then we have

$$\begin{aligned} |\zeta - x_0|^2 &= |y - x_0|^2 - |y - \zeta|^2 \\ &= (R + |y - \zeta|)(R - |y - \zeta|) \\ &\leqslant 2R \operatorname{dist}(\zeta, \partial B_R). \end{aligned} \tag{7.4}$$

If $x \in B_{R/2}$, there is a unique $\zeta \in \partial B_{R/2}$ such that $x = (1 - t)x_0 + t\zeta$ for some $t \in [0, 1]$. Hence

$$|x - x_0| = t|\zeta - x_0| \leqslant t(2R)^{1/2} \operatorname{dist}(\zeta, \partial B_R)^{1/2}. \tag{7.5}$$

Let z be the unique point in B_R such that $\operatorname{dist}(x, \partial B_R) = |x - z|$, and η the unique point where the line through ζ parallel to the line segment $[x, z]$ intersects the line through x_0 and z. Then

$$|x - z| = t|\zeta - \eta|, \tag{7.6}$$

so from (7.5) we obtain

$$|x - x_0| \leqslant t(2R)^{1/2}|\zeta - \eta|^{1/2} \leqslant t^{1/2}(2R)^{1/2}|x - z|^{1/2}, \tag{7.7}$$

which gives the estimate (7.3).

Theorem 7.3. *Let Ω be a domain in $\mathbf{R}^n$ and u a generalized solution of* (1.1) *in Ω. Suppose that Γ is a relatively open portion of $\partial\Omega$ and the graph of u is vertical on Γ. Let $x_0 \in \Gamma$ and suppose that either*

$$K(x) \leqslant M \operatorname{dist}(x, \partial\Omega) \tag{7.8}$$

or

$$K(x) \leqslant M|x - x_0|^2 \tag{7.9}$$

in a neighborhood of x_0, where $M > 0$ is a constant, and Ω satisfies an enclosing sphere condition at x_0. Then

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = \infty. \tag{7.10}$$

Proof. Let $B = B_R(y)$ be an enclosing ball at x_0 and $r(x) = \operatorname{dist}(x, \partial\Omega) = R - |x - y|$. We first consider the case (7.8). Let $w = \psi(r) = A - a \log r$, where A and a are constants to be chosen. We may assume that $r \leqslant 1$ and $a^2 r^{-2} \geqslant 1$. Then we have

$$F[w] = \frac{\psi''(-\psi'/|x-y|)^{n-1}}{(1 + |\psi'|^2)^{(n+2)/2}}$$

$$\geqslant 2^{-(n+2)/2} a^{-2} R^{1-n} r \geqslant K,$$

provided $Ma^2 \leqslant 2^{-(n+2)/2} R^{1-n}$. Thus for some $\varepsilon > 0$ we have $\omega(w) \geqslant \omega(u)$ in $B_\varepsilon(x_0) \cap \Omega$. By choosing a suitable value for A we may ensure that $u \geqslant w$ on $\partial B_\varepsilon(x_0) \cap \Omega$, so by Theorem 3.3 we obtain $u \geqslant w$ in $B_\varepsilon(x_0) \cap \Omega$, from which (7.10) follows.

To prove the second case let $B_{R/2}(y) \subset B_R(z)$ be enclosing spheres at x_0. Then from (7.9) and Lemma 7.2 we obtain

$$K(x) \leqslant 2MRr(x) \tag{7.11}$$

in a neighborhood of x_0, where $r(x) = \operatorname{dist}(x, \partial B_R(z))$. The above barrier argument can now be used to obtain the result.

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