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ON A CONJECTURE OF K. OGIUE

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1. Introduction

In this note we will prove the following result.

Theorem. *Let M^n be a complete Kaehler submanifold of complex dimension $n \geqslant 2$ in a complex projective space $CP^{n+m}(1)$ of complex dimension $n+m$ endowed with the Study-Fubini metric of constant holomorphic sectional curvature* 1. *Then, if the sectional curvature K of M^n satisfies*

$$(*) \qquad K > \tfrac{1}{8},$$

M^n is a totally geodesic submanifold of $CP^{n+m}(1)$.

This solves in the affirmative a conjecture by K. Ogiue formulated in [1]. So far, partial results in this direction were obtained by S. T. Yau [5], K. Ogiue [2] and P. Verheyen and one of the authors [4], giving respectively the following estimates on K instead of $(*)$:

$$K > \frac{n(2m-1)+8m-3}{4n(4m-1)}, \qquad K > \frac{n+3}{8n},$$

$$K > \frac{m(n+4)+1}{4n(2m+1)}.$$

These results all were obtained by applying basically the same technique, namely by using a formula of Simons' type for the Laplacian of the square of the length of the second fundamental form such as computed by S. S. Chern, M. do Carmo and S. Kobayashi, combined with the Lemma of Hopf. We remark that by a result of S. B. Myers, each of the above assumptions on K implies that M^n is compact.

Recently, the first author introduced a new approach to this type of problems in his equally affirmative solution of K. Ogiue's conjecture concerning the holomorphic sectional curvature H: *if $H > \frac{1}{2}$ for a compact Kaehler*

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submanifold N^n *of* $\mathbf{C}P^{n+m}(1)$, *then* N^n *is totally geodesic* [3]. Here we will use the same method to prove our Theorem.

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2. Preliminaries

Let M^n be a Kaehler submanifold of $\mathbf{C}P^{n+m}(1)$. The Study-Fubini metric of constant holomorphic sectional curvature 1 on $\mathbf{C}P^{n+m}$ and the induced metric on M^n will both be denoted by g. The complex structure of $\mathbf{C}P^{n+m}$ and the induced complex structure on M^n will both be denoted by J. Let $\overline{\nabla}$ and ∇ be the Riemannian connections of $\mathbf{C}P^{n+m}(1)$ and M^n, respectively, and let σ be the second fundmental form of the immersion. A and $\nabla^\perp$ are the Weingarten endomorphism and the normal connection. $\tilde{\nabla}\sigma$ is the covariant derivative of van der Waerden-Bortolotti of σ, and we define the second covariant derivative of σ by

$$(1)\qquad \begin{aligned}(\tilde{\nabla}^2\sigma)(X, Y, Z, W) = \nabla_X^\perp((\tilde{\nabla}\sigma)(X, Z, W)) - (\tilde{\nabla}\sigma)(\nabla_X Y, Z, W)\\ -(\tilde{\nabla}\sigma)(Y, \nabla_X Z, W) - (\tilde{\nabla}\sigma)(Y, Z, \nabla_X W)\end{aligned}$$

for arbitrary $X, Y, Z, W \in \mathfrak{X}M^n$. Let $\overline{R}$, R and $R^\perp$ denote the curvature tensors of the connections $\overline{\nabla}$, ∇ and $\nabla^\perp$. Then we have

$$(2)\qquad \begin{aligned}\overline{R}(\overline{X}, \overline{Y})\overline{Z} = \tfrac{1}{4}\{g(\overline{Y}, \overline{Z})\overline{X} - g(\overline{X}, \overline{Z})\overline{Y} + g(J\overline{Y}, \overline{Z})J\overline{X}\\ -g(J\overline{X}, \overline{Z})J\overline{Y} + 2g(\overline{X}, J\overline{Y})J\overline{Z}\},\end{aligned}$$

$$(3)\qquad R(X, Y)Z = \overline{R}(X, Y)Z + A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y,$$

$$(4)\qquad g(R^\perp(X, Y)\xi, \eta) = g(\overline{R}(X, Y)\xi, \eta) + g([A_\xi, A_\eta]X, Y)$$

for all vector fields $\overline{X}, \overline{Y}, \overline{Z}$ tangent to $\mathbf{C}P^{n+m}$, vector fields X, Y, Z tangent to M^n and vector fields ξ, η normal to M^n in $\mathbf{C}P^{n+m}$. Moreover σ and $\tilde{\nabla}\sigma$ are symmetric and

$$(5)\qquad \begin{aligned}(\tilde{\nabla}^2\sigma)(X, Y, Z, W) - (\tilde{\nabla}^2\sigma)(Y, X, Z, W)\\ = R^\perp(X, Y)\sigma(Z, W) - \sigma(R(X, Y)Z, W) - \sigma(Z, R(X, Y)W).\end{aligned}$$

We also consider the relations

$$(6)\qquad \sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y),$$

$$(7)\qquad A_{J\xi} = JA_\xi = -A_\xi J,$$

$$(8)\qquad \nabla_X^\perp J\xi = J\nabla_X^\perp \xi,$$

$$(9)\qquad \begin{aligned}(\tilde{\nabla}\sigma)(JX, Y, Z) &= (\tilde{\nabla}\sigma)(X, JY, Z)\\ &= (\tilde{\nabla}\sigma)(X, Y, JZ) = J(\tilde{\nabla}\sigma)(X, Y, Z).\end{aligned}$$

If u is a unit tangent vector to M^n, then the holomorphic sectional curvature $H(u)$ of M^n determined by u is given by

$$(10)\qquad H(u) = 1 - 2\|\sigma(u, u)\|^2.$$

Let π: $UM \to M$ and UM_p be the unit tangent bundle of M and its fiber over $p \in M$, respectively. Then we consider the function f: $UM \to \mathbf{R}$ defined by

$$(11)\qquad f(u) = \|\sigma(u, u)\|^2$$

for $u \in UM_p$.

3. Proof of the theorem

By the assumption $(*)$ and its completeness, we know that M^n is compact. Hence UM is compact, such that the above function f attains its maximum at some vector v in UM_p for some $p \in M$. Then from [3] we have

$$(1)\qquad f(v)\cdot(1 - 4f(v)) \leqslant 0,$$

$$(2)\qquad A_{\sigma(v,v)}v = \|\sigma(v, v)\|^2 v.$$

We will prove the Theorem by showing that under its assumptions the hypothesis that M^n is not totally geodesic leads to a contradiction.

From (1) it follows that, by the hypothesis $\sigma \neq 0$,

$$(3)\qquad \|\sigma(v, v)\|^2 \geqslant \tfrac{1}{4}.$$

For any $u \in UM_p$, let $\gamma_u(t)$ be the geodesic in M^n determined by the initial conditions $\gamma_u(0) = p$ and $\gamma_u'(0) = u$. Parallel translation of v along $\gamma_u(t)$ yields a vector field $V_u(t)$. By the special choice of v we know that the function f_u defined by

$$(4)\qquad f_u(t) = f(V_u(t))$$

attains a maximum at $t = 0$. This implies that

$$(5)\qquad \frac{d^2}{dt^2}f_u(0) + \frac{d^2}{dt^2}f_{Ju}(0) \leqslant 0$$

for all $u \in UM_p$. By direct computations we obtain

$$\frac{d}{dt} f_u(t) = 2g((\tilde{\nabla}\sigma)(\gamma_u', V_u, V_u), \sigma(V_u, V_u))(t), \tag{6}$$

$$\frac{d^2}{dt^2} f_u(0) = 2g((\tilde{\nabla}^2\sigma)(u, u, v, v), \sigma(v, v)) + 2\|(\tilde{\nabla}\sigma)(u, v, v)\|^2. \tag{7}$$

From (7) we derive that

$$\begin{aligned} \frac{d^2}{dt^2} f_u(0) + \frac{d^2}{dt^2} f_{Ju}(0) &= 2g((\tilde{\nabla}\sigma)(u, u, v, v) \\ &\quad + (\tilde{\nabla}^2\sigma)(Ju, Ju, v, v), \sigma(v, v)) + 4\|(\tilde{\nabla}\sigma)(u, v, v)\|^2. \end{aligned} \tag{8}$$

By similar arguments as in [3] we also obtain that

$$\begin{aligned} &g((\tilde{\nabla}^2\sigma)(Ju, Ju, v, v), \sigma(v, v)) \\ &= g((\tilde{\nabla}^2\sigma)(Ju, u, Jv, v), \sigma(v, v)) \\ &= -g((\tilde{\nabla}^2\sigma)(u, u, v, v), \sigma(v, v)) + g(R^\perp(Ju, u)J\sigma(v, v), \sigma(v, v)) \\ &\quad -2g(R(Ju, u)Jv, A_{\sigma(v, v)}v). \end{aligned} \tag{9}$$

From (5), (8) and (9), for all $u \in UM_p$, we have

$$g(R^\perp(Ju, u)J\sigma(v, v), \sigma(v, v)) - 2g(R(Ju, u)Jv, A_{\sigma(v, v)}v) \leqslant 0. \tag{10}$$

By (2),

$$g(R(Ju, u)Jv, A_{\sigma(v, v)}v) = -\|\sigma(v, v)\|^2 g(R(u, Ju)Jv, v), \tag{11}$$

and by the Ricci equation,

$$g(R^\perp(Ju, u)J\sigma(v, v), \sigma(v, v)) = -\tfrac{1}{2}\|\sigma(v, v)\|^2 - 2\|A_{\sigma(v, v)}u\|^2. \tag{12}$$

Hence (5) yields that

$$2\|\sigma(v, v)\|^2 g(R(u, Ju)Jv, v) - \tfrac{1}{2}\|\sigma(v, v)\|^2 - 2\|A_{\sigma(v, v)}u\|^2 \leqslant 0 \tag{13}$$

for all $u \in UM_p$. Now, since $n \geqslant 2$, we can always choose a unit eigenvector u of $A_{\sigma(v, v)}$ such that $g(u, v) = g(u, Jv) = 0$. Using the equation of Gauss which implies that

$$R(u, v)v = \tfrac{1}{4}u + A_{\sigma(v, v)}u - A_{\sigma(u, v)}v, \tag{14}$$

$$R(u, Jv)Jv = \tfrac{1}{4}u - A_{\sigma(v, v)}u - A_{\sigma(u, v)}v, \tag{15}$$

we have

$$\tag{16} \begin{aligned} A_{\sigma(v,v)}u &= \tfrac{1}{2}(R(u,v)v - R(u,Jv)Jv) \\ &= \tfrac{1}{2}(K(u,v) - K(u,Jv))u, \end{aligned}$$

where $K(r, s)$ is the sectional curvature of M at p for the plane spanned by $r, s \in T_pM$. The Bianchi identity shows that

$$\tag{17} g(R(u,Ju)Jv, v) = K(u,v) + K(u,Jv).$$

From (13), (16) and (17) we obtain

$$\tag{18} \begin{aligned} &2\|\sigma(v,v)\|^2(K(u,v) + K(u,Jv)) - \tfrac{1}{2}\|\sigma(v,v)\|^2 \\ &\quad - \tfrac{1}{2}\big(K(u,v)^2 + K(u,Jv)^2 - 2K(u,v)K(u,Jv)\big) \leqslant 0, \end{aligned}$$

or equivalently

$$\tag{19} aK(u,v) + bK(u,Jv) - \tfrac{1}{2}\|\sigma(v,v)\|^2 \leqslant 0,$$

where

$$\tag{20} a = 2\|\sigma(v,v)\|^2 - \tfrac{1}{2}K(u,v) + \tfrac{1}{2}K(u,Jv),$$

$$\tag{21} b = 2\|\sigma(v,v)\|^2 - \tfrac{1}{2}K(u,Jv) + \tfrac{1}{2}K(u,v).$$

Now we prove that $a, b > 0$. From the equation of Gauss it follows that

$$\tag{22} K(u,v) + K(u,Jv) = \tfrac{1}{2} - 2\|\sigma(u,v)\|^2 \leqslant \tfrac{1}{2}.$$

By (3) and (20) we have

$$\tag{23} 1 - K(u,v) + K(u,Jv) \leqslant 2a.$$

By (22) and (23),

$$\tag{24} \tfrac{1}{2} + 2K(u,Jv) \leqslant 2a,$$

which by the assumption (*) implies that $a > 0$. In the same way it follows that also $b > 0$. Since a and b are strictly positive and since $K > \frac{1}{8}$, by (19) we obtain the strict inequality

$$\tag{25} \tfrac{1}{8}(a + b) - \tfrac{1}{2}\|\sigma(v,v)\|^2 < 0.$$

But from (20) and (21) it follows that

$$\tag{26} a + b = 4\|\sigma(v,v)\|^2,$$

which inserted in (25) yields the desired contradiction.

Hence M^n is totally geodesic in $\mathbf{C}P^{n+m}(1)$.

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