J. DIFFERENTIAL GEOMETRY
19 (1984) 517-560

SELF-DUAL CONNECTIONS ON 4-MANIFOLDS WITH INDEFINITE INTERSECTION MATRIX

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Abstract

Let M be a compact, connected and oriented Riemannian 4-manifold. Sufficient conditions on M and a principal SU(2) bundle $P \to M$ are established which imply that P admits a smooth, irreducible, self-dual connection.

1. The main results

This article addresses the following question: When does a principal SU(2) bundle, P, over a smooth, compact, oriented, 4-dimensional Riemannian manifold, M, admit an irreducible, self-dual connection? In a previous article [20], the author established that if the intersection matrix (cf. [15])

$$Q\colon H_2(M;\mathbf{Z}) \times H_2(M;\mathbf{Z}) \to \mathbf{Z}$$

of M is positive definite, then a necessary and sufficient condition on P is that the second Chern class of $P \times_{\mathrm{SU}(2)} C^2$ satisfy $c_2(P) < 0$. This article extends the existence results to manifolds with indefinite intersection matrix. The main result here is that a principal SU(2) bundle $P \to M$ admits a smooth, irreducible self-dual connection whenever $-c_2(P)$ is large with respect to $b_- = \frac{1}{2}(\mathrm{rank}(Q) - \mathrm{signature}(Q))$. These results are stated in detail in Theorems 1.1 and 1.2 below.

A number of the results which are stated below were deduced independently by S. K. Donaldson, to whom the author is greatly indebted for many invaluable discussions.

This article should be considered as a sequel to [20], where most of the notation and terminology is introduced. The reader may find the expositions in

Received March 15, 1983 and, in revised form, January 30, 1984. This work was completed while the author was a guest at the Mathmatical Institute, 24-29 St. Gilles, Oxford OX1-3LB, England, and a Junior Fellow of the Harvard University Society of Fellows and was supported in part by the National Science Foundation under grant #PHY-82-03669.

[5], [9], [19] useful as introductions to Yang-Mills theory. More advanced in related papers are [2], [3], [6].

The context in which the self-duality question arises is as follows. Let G be a compact, simple and simply connected Lie group and $P \to M$ a principal G-bundle over the 4-manifold M. Denote by $\mathscr{C}(P)$ the space of smooth connections on P. A connection $A \in \mathscr{C}(P)$ is self-dual if its curvature, F_A, satisfies

$$F_A = *F_A. \tag{1.1}$$

(The anti-self-dual equations, $F_A = -*F_A$ are obtained in this context by reversing the orientation of M.) Here, $*$ is the Hodge dual that is defined by the metric on the cotangent bundle $T^* \to M$.

Principal G-bundles of the type under consideration are classified, up to isomorphism, by the first Pontrjagin number, $p_1(\hat{\mathfrak{g}})$, where $\hat{\mathfrak{g}} = P \times_{\mathrm{Ad}\,\mathfrak{g}} \mathfrak{g}$ and $\mathfrak{g} = \mathrm{Lie\,Alg}\, G$. When $p_1(\hat{\mathfrak{g}}) = 0$, all self-dual connections are flat, hence their orbits under $\mathscr{G} = \mathrm{Aut}\, P$ are in one-to-one correspondence with elements of $\mathrm{Hom}(\pi(M), G)$ [7].

For $G = \mathrm{SU}(2)$, $p_1(\hat{\mathfrak{g}}) = -8c_2(P)$, and the first existence theorem is

Theorem 1.1. *Let M be a smooth, compact, oriented 4-dimensional Riemannian manifold, and let Q be its homology intersection matrix. Let $P \to M$ be a principal* SU(2) *bundle. The following relations between $b_- = \frac{1}{2}(\mathrm{rank}(Q)$-$\mathrm{signature}(Q))$, and $c_2(P)$ imply that P admits smooth, irreducible self-dual connections*: (1) $-c_2(P) \geqslant \max(\frac{4}{3}b_-, 1)$ *when* $b_- \neq 2$, *or* (2) $-c_2(P) \geqslant 4$ *when* $b_- = 2$.

If one assumes that the metric on M is generic, then better estimates exist. These are stated next:

Theorem 1.2. *Let M be as in Theorem* 1.1 *and suppose that* $b_- \notin \{0, 1, 3\}$. *Let $r \geqslant 2$. For a dense open set of metrics on M in the C^r-topology, a principal* SU(2) *bundle $P \to M$ admits a smooth irreducible self-dual connection when* $-c_2(P) \geqslant b_-$.

It should be remarked that the integer b_- is also equal to the dimension of

$$P_- H^2_{\mathrm{DR}} = \left\{ \omega \in H^2_{\mathrm{DeRham}} : *\omega = -\omega \right\},$$

where $H^2_{\mathrm{DeRham}}(M)$ is the DeRham cohomology group in dimension 2. Indeed, as $*\colon \Lambda_2 T^* \to \Lambda_2 T^*$ satisfies $*^2 = 1$, there is the direct sum decomposition $\Lambda_2 T^* \cong P_+ \wedge T^* \oplus P_- \wedge_2 T^*$ and elements in $P_- H^2_{\mathrm{DR}}$ are the sections of $P_- \wedge_2 T^*$ which are harmonic with respect to the Laplace-Beltrami operator that is defined by the exterior derivative and the metric on T^*.

The previous theorems should generalize to principal bundles with higher rank structure groups; though the subject is not discussed here.

The structure of the moduli spaces $\mathcal{M} = \{$orbits of self-dual connections on $P\}$ is also not discussed. The reader is referred to [9], [6] for the case $b_- = 0$ and $-c_2(P) = 1$. The dimensions of these moduli spaces, when M is Kähler, are computed in [12].

The existence of self-dual connections on stable, holomorphic vector bundles over complex algebraic surfaces has been established using different techniques by S. K. Donaldson [7]. The special case where $M =$ 4-torus has been investigated by 't Hooft [11].

The question immediately arises as to whether the conditions for the existence of self-dual connections given by Theorems 1.1 and 1.2 are in any cases necessary as well as sufficient. Let $M = \overline{\mathbf{C}P^2}$ with the Fubini-Study metric. Then $b_-(\overline{\mathbf{C}P^2}) = 1$ and $\overline{\mathbf{C}P^2}$ is complex algebraic. S. K. Donaldson has pointed out [8] that the above stated condition, that a principal SU(2) bundle $P \to M = \overline{\mathbf{C}P^2}$ must have $-c_2(P) \geqslant 2$ in order to admit an irreducible self-dual connection, is also a necessary condition: An SU(2) bundle $P \to M$ with an irreducible self-dual connection would define a stable holomorphic rank 2 vector bundle over $\overline{\mathbf{C}P^2}$ with second Chern class -1 and first Chern class 0. The Riemann-Roch theorem forbids this (cf. [7, p. 160ff]). Therefore, on $\overline{\mathbf{C}P^2}$, Theorem 1.1 gives necessary and sufficient conditions.

On the other hand, Theorem 1.3 and Proposition 10.3 show that there are cases where Theorems 1.1 and 1.2 are not optimal.

As a final comment on this subject, it should be remarked that in Chapter 3 of [9] it is proved that for a generic metric on M, the existence of an irreducible self-dual connection on a principal SU(2) bundle $P \to M$ requires $-c_2(P) \geqslant \frac{3}{8}(b_- + 1 - \dim H^1_{\mathrm{DR}})$.

The strategy for proving these theorems generalizes the approach in [20]. Schematically, the approach is the following one. Let $\mathcal{B} = \mathcal{C}/\mathrm{Aut}\, P$ denote the space of orbits of connections under Aut P. The assignment of $[A] \in \mathcal{B}$ to $P_- F_A$ defines a section, ϕ, of an infinite dimensional vector bundle $\Omega \to \mathcal{B}$. The problem is to determine when ϕ has a zero.

The cut and paste operation in §§7, 8 of [20] constructs a finite-dimensional manifold N with an embedding $i\colon N \to \mathcal{B}$. The manifold has the property that a useful norm of ϕ is small. (This manifold N is described shortly, see (1.4).)

Let $\nabla\phi$ denote the differential of ϕ. It is a linear map from $T_{\mathcal{B}}$ to Ω which at each $b \in \mathcal{B}$ is a first order linear elliptic differential operator. Let $\nabla\phi^\dagger$ denote its adjoint; the section of $\mathrm{Hom}(\Omega, T_{\mathcal{B}})$ obtained by taking the L^2 adjoint of $\nabla\phi|_b$ at each $b \in \mathcal{B}$.

Small eigenvalues of the composition $\nabla\phi\nabla\phi^\dagger|_N$ are the obstruction to using directly the implicit function techniques in [20] to prove Theorems 1.1 and 1.2.

However, the manifold N is shown to have the following additional property: At each $y \in N$, the second order elliptic operator $\nabla\phi\nabla\phi^\dagger|_y$ has precisely $3b_-$ eigenvectors with small eigenvalues and all its other eigenvalues are O(1). The span of these $3b_-$ eigenvectors defines the fibre at each $y \in N$ of a smooth $\mathbf{R}^{3b_-}$-vector bundle $V \to N$ as a subbundle of $i^*\Omega$.

A global version of Kuranishi's ideas on complex structure deformations [14] and the implicit function theorem are used to construct a section $f\colon N \to V$ with the crucial property that $f^{-1}(0) \subseteq \phi^{-1}(0)$.

Indeed, Kuranishi's ideas adapt to the present circumstances as follows: Let $\Pi(\cdot)\colon N \to \mathrm{Hom}(i^*\Omega, i^*\Omega)$ denote the section which is orthogonal projection onto $V \subset i^*\Omega$. Let $\exp\colon T_{\mathscr{B}} \to \mathscr{B}$ denote the exponential map. Because $\|\phi\|$ is small along N one can find a section $\rho\colon N \to (1 - \Pi)i^*\Omega$ which solves, at each $y \in N$,

$$(1.2) \qquad \nabla\phi\nabla\phi^\dagger(\rho) + (1 - \Pi)\left[\phi\left(\exp(\nabla\phi^\dagger(\rho))\right) - \nabla\phi\nabla\phi^\dagger(\rho)\right] = 0.$$

The crucial observation is that for $y \in N$, the point $\exp(\nabla\phi^\dagger(\rho))(y) \in \mathscr{B}$ is self-dual if and only if

$$(1.3) \qquad \Pi\phi\left(\exp(\nabla\phi^\dagger(\rho))\right)(y) = 0.$$

Thus the zero's of the section $f = \Pi\phi(\exp(\nabla\phi^\dagger(\rho)))\colon N \to V$ determine self-dual connections.

This rewriting of the self-dual equations would be completely academic were it not for the fortunate fact that the data above, N, $V \to N$ and even $f\colon N \to V$, are explicitly known in terms of natural geometric objects on the *base* manifold M.

Let $F_- \to M$ denote the principal SO(3) bundle over M whose fiber at $p \in M$ are the positively oriented, orthonormal frames in $P_- \wedge_2 T^*$ at p. Let $\mathbf{R}^* = (0, \infty)$. The manifold N is an open set in

$$(1.4) \qquad C_k'(F_- \times \mathbf{R}^*)/\mathrm{SO}(3).$$

Here, $C_k'(F_- \times \mathbf{R}^*)$ is the set of k-tuples of unordered points $\{y_1, \cdots, y_k\} \subset F_- \times \mathbf{R}$ with distinct basepoints in M. The group SO(3) acts on $C_k'(F_- \times \mathbf{R}^*)$ via the diagonal action of SO(3) on $\times_k F_-$.

Now $C_k'(F_- \times \mathbf{R}^*)/\mathrm{SO}(3)$ is parametrizing connections (mod Aut P) on the principal bundle $P \to M$. Some of the parameters have direct geometric interpretation. Indeed, $C_k'(F_- \times \mathbf{R}^*)/\mathrm{SO}(3)$ fibers over $C_k'(M \times \mathbf{R}^*)$. Let $y \in C_k(F_- \times \mathbf{R}^*)/\mathrm{SO}(3)$ and let $\{(p_1, \lambda_1), \cdots, (p_k, \lambda_k)\}$ be its image in $C_k'(M \times \mathbf{R}^*)$. Then the curvature, $F(y)$, of the connection that corresponds to y has its curvature concentrated in k-balls in M with centers at $\{p_i\}_{i=1}^k$ and radii

$\{\lambda_i\}_{i=1}^k$. Each ball contributes the amount -1 to the integral over M of the 4-form

$$\frac{1}{4\pi^2}\operatorname{trace}_{\mathbf{C}^2}[F(y) \wedge F(y)]$$

which represents $c_2(P)$ in $H^2(M; \mathbf{R})$.

The $\mathbf{R}^{3b}$- vector bundle V over N is precisely the restriction of

$$\bigotimes_{b_-}\left[\begin{array}{c} C_k'(F_- \times \mathbf{R}^*) \underset{\mathrm{SO}(3)}{\times} \mathbf{R}^3 \\ \downarrow \\ C_k'(F_- \times \mathbf{R}^*)/\mathrm{SO}(3) \end{array}\right] \tag{1.5}$$

to N.

Equation (1.5) suggests that one might obtain a proof of Theorems 1.1 and 1.2 by studying the characteristic classes of the vector bundle in (1.5). For example, if one could prove that for a given k and b_-, the bundle above does not split a trivial line bundle, then every section of V over N must have a zero. In particular, (1.3) must have a solution.

The trivial case, here, is when $b_- = 0$. Then, for any $k > 0$, V is just $N \times \{0\} = N$ and so tautologically, every section vanishes. This reproduces the main theorem of [20].

A nontrivial case, which demonstrates the potential of a strictly topological analysis of (1.3) is when $k = 1$ and $b_- = 1$. In this case, $N = M$, $V = P_- \wedge_2 T^* \to M$ and one obtains

Theorem 1.3. *Suppose that $b_-(M) = 1$ and the Euler class of the $\mathbf{R}^3$-bundle $P_- \wedge_2 T^* \to M$ is nonzero in $H^3(M; \mathbf{Z}_2)$ (for M spin, if $\dim H_1(M; \mathbf{R}) = 1 \bmod 2$). Then a principal* SU(2) *bundle $P \to M$ with $-c_2(P) \geqslant 1$ admits a smooth, irreducible self-dual connection.*

The reader should compare Theorem 1.3 with the $b_- = 1$ case of Proposition 10.3.

For general k and b_-, the calculations for the characteristic classes of the vector bundle in (1.5) have stymied the author. For this reason, an explicit investigation of the section f: $N \to V$ given by (1.4) was undertaken. Fortuitously, f can be written as $-h + h'$, where h is explicit and h' is not; but $\|h'\| \ll \|h\|$. This allows a treatment of f with the following strategy: Find $y \in N$ where h vanishes transversely; i.e. where $h(y) = 0$ and $\nabla h(y) \in \operatorname{Hom}(T_y; V_y)$ is surjective. Then use a version of the implicit function theorem to prove that any perturbation of h, and in particular f, must vanish near to y.

To describe h, it is convenient to reinterpret a section of (1.5) as an SO(3) equivariant map from $C'_k(F_- \times \mathbf{R}^*)$ to $\times_{b_-} \mathbf{R}^3$. Here, SO(3) acts diagonally on the vector space $\times_{b_-} \mathbf{R}^3$. Choose a basis for $P_- H^2_{\mathrm{DR}}(M)$, $\{\omega_J\}_{J=1}^{b_-}$. Represent a point $y \in C'_k(F_- \times \mathbf{R}^*)$ by an unordered set

$$y = \left\{\left(p_i, \lambda_i, (x_i^a)_{a=1}^3\right)\right\}_{i=1}^k. \tag{1.6}$$

Here, each $p_i \in M$, each $\lambda_i \in \mathbf{R}^*$ and each $(x_i^a)_{a=1}^3$ is a positively oriented orthonormal frame at p_i for $P_- \wedge_2 T^*$.

The section h sends y in (1.6) to

$$h(y) = \sum_{i=1}^k \lambda_i^2\left((\omega_J(p_i), x_i^a)\right)_{a=1,J=1}^{3\quad b_-} \tag{1.7}$$

in $\times_{b_-} \mathbf{R}^3$. Here, $(\cdot, \cdot)$ denotes the Riemannian inner product on $P_- \wedge_2 T^*$.

This article is composed as two parts. Part 1 comprises §§2–6 where the self-dual equations are rewritten as equations for the zeros of h on $C'_k(F_- \times \mathbf{R}^*)/\mathrm{SO}(3)$, as defined in (1.7). §2 contains a brief introduction to the self-dual equations. There, they are rewritten in a form that is similar to Kuranishi's complex structure deformation equations. In §3, the formal aspects of the strategy for proving Theorems 1.1 and 1.2 are discussed. The natural vector bundles over $\mathscr{B}$ are discussed there. From §4 on, the discussion is strictly for the group SU(2). In §4, the manifold N is defined as a subset of $C'_k(F_- \times \mathbf{R}^*)/\mathrm{SO}(3)$ and the inclusion of N into $\mathscr{B}$ is constructed. In §5, the obstruction bundle V over N is identified as the restriction to N of the bundle in (1.5). Also in §5, the decomposition of f into $-h$ + perturbation is accomplished. In §6, the implicit function theorem is used to establish the conditions under which $h^{-1}(0) \neq \varnothing$ implies that $f^{-1}(0) \neq \varnothing$.

Part 2 of this paper comprises the analysis of the section h of (1.7). §7 analyzes the case $b_- = 1$, §§8 and 9 analyze $b_- = 2$ and 3 respectively. These three sections establish Theorems 1.1 and 1.2 for $b_- \in \{1, 2, 3\}$. §10 establishes Theorem 1.2 for $b_- > 3$ and §11 establishes Theorem 1.1 for $b_- > 3$. There are two technical appendices too.

2 The self-dual equations

Let G be a compact, simple Lie group and $P \to M$ a principal G-bundle. Fix $A_0 \in \mathscr{C}(P)$. As $\mathscr{C}(P)$ is an affine space, any connection $A \in \mathscr{C}(P)$ can be written uniquely as $A = A_0 + a$ with $a \in \Omega^1(\hat{\mathfrak{g}}) \equiv \Gamma(\hat{\mathfrak{g}} \otimes T^*)$. The connection

A has self-dual curvature if and only if the 1-form a satisfies

$$\mathscr{D}_{A_0}a + a\#a + P_-F_{A_0} = 0. \tag{2.1}$$

Here, as in [20], I have introduced the notation

$$\mathscr{D}_{A_0} = P_-D_{A_0}\colon \Omega^1(\hat{\mathfrak{g}}) \to \Omega^2_-(\hat{\mathfrak{g}}) \equiv \Gamma(\hat{\mathfrak{g}} \otimes P_-(\Lambda_2 T^*)), \tag{2.2}$$

and

$$a\#b = \tfrac{1}{2}P_-(a \wedge b + b \wedge a). \tag{2.3}$$

Following the discussion in [20], one attempts to solve (2.1) for a of the form $a = \mathscr{D}^\dagger_{A_0}u$, where $u \in \Omega^2_-(\hat{\mathfrak{g}})$, and

$$\mathscr{D}^\dagger_{A_0} = *D_{A_0}\colon \Omega^2_-(\hat{\mathfrak{g}}) \to \Omega^1(\hat{\mathfrak{g}}). \tag{2.4}$$

Thus, rather than try to solve (2.1) for $a \in \Omega^1(\hat{\mathfrak{g}})$, consider, the problem of finding $A_0 \in \mathscr{C}(P)$ such that the nonlinear, partial differential equation

$$\mathscr{D}_{A_0}\mathscr{D}^\dagger_{A_0}u + \mathscr{D}^\dagger_{A_0}u\#\mathscr{D}^\dagger_{A_0}u + P_-F_{A_0} = 0 \tag{2.5}$$

has a solution, $u \in \Omega^2_-(\hat{\mathfrak{g}})$.

If for a given A_0, a solution u to (2.5) exists, then

$$A = A(A_0, u) = A_0 + \mathscr{D}^\dagger_A u \in \mathscr{C}(P) \tag{2.6}$$

is self-dual.

The group $\mathscr{G} = \operatorname{Aut} P = \Gamma(P \times_{\operatorname{Ad} G} G)$ acts on $\mathscr{C}(P)$ and on $\Omega^2_-(\hat{\mathfrak{g}})$ as follows: For $(g, A, u) \in \mathscr{G} \times \mathscr{C} \times \Omega^2_-(\hat{\mathfrak{g}})$, one has $(gA, gu) = (A + g\nabla_A g^{-1}, gug^{-1}) \in \mathscr{C} \times \Omega^2_-(\hat{\mathfrak{g}})$. Observe that if $(A_0, u) \in \mathscr{C} \times \Omega^2_-(\hat{\mathfrak{g}})$ satisfies (2.5) then for any $g \in \mathscr{G}$, so does (gA_0, gu). In addition, (2.6) is $\mathscr{G}$-equivariant in the sense that $A(gA_0, gu) = gA(A_0, u)$.

The analysis of (2.5) begins by considering the operator $\mathscr{D}_{A_0}\mathscr{D}^\dagger_{A_0}\colon \Omega^2_-(\hat{\mathfrak{g}}) \to \Omega^2_-(\hat{\mathfrak{g}})$. Recall that the natural $\mathscr{G}$ invariant inner product on $\hat{\mathfrak{g}}$, and the given Riemannian metric on T^* induce a $\mathscr{G}$ invariant metric $(\cdot,\cdot)$ and norm $|\cdot|$ on all vector bundles $\hat{\mathfrak{g}} \otimes V$, where V is associated to the SO(4)-frame bundle of M. In the usual way, one obtains $L_p, p \geqslant 1$, inner products on $\Gamma(\hat{\mathfrak{g}} \otimes V)$.

With respect to the L_2-inner products on $\Omega^2_-(\hat{\mathfrak{g}})$ and $\Omega^1(\hat{\mathfrak{g}})$, the operators $\mathscr{D}^\dagger_A$ and $\mathscr{D}_A$ are formal adjoints. The operator $\mathscr{D}_{A_0}\mathscr{D}^\dagger_{A_0}$ is an essentially self-adjoint, unbounded operator on $L_2(\hat{\mathfrak{g}} \otimes P_-\wedge_2 T^*)$ with dense domain Ω^2_- (cf. Appendix A). One property of such operators is that they have discrete spectrum with finite multiplicity.

For any $E \geqslant 0$, define the projection operators $\Pi_E(A_0)\colon L_2(\hat{\mathfrak{g}} \otimes P_-\wedge_2 T^*) \to L_2(\hat{\mathfrak{g}} \otimes P_-\wedge_2 T^*)$ to be the finite rank spectral projections onto the subspace of L_2 spanned by the eigenvectors of $\mathscr{D}_{A_0}\mathscr{D}^\dagger_{A_0}$ with eigenvalues less than or equal to E. Define the projection operator $\Pi_E^{\perp}(A_0)$ to be the L_2-orthogonal complement.

Follow the teachings of Kuranishi [14] and write (2.5) as two equations by using the operator identity $1 = \Pi_E(A_0) + \Pi_E^{\perp}(A_0)$. In addition, consider only those $u \in \Omega_-^2(\mathfrak{g})$ which satisfy $\Pi_E(A_0)u = 0$. Thus, one is now considering the problem of finding $A_0 \in \mathscr{C}(P)$ and $E \geqslant 0$ such that the two equations

$$\mathscr{D}_{A_0}\mathscr{D}_{A_0}^{\dagger}u + \Pi_E^{\perp}(A_0)\left(\mathscr{D}_{A_0}^{\dagger}u\#\mathscr{D}_{A_0}^{\dagger}u + P_-F_{A_0}\right) = 0 \tag{2.7}$$

and

$$\Pi_E(A_0)\left(\mathscr{D}_{A_0}^{\dagger}u\#\mathscr{D}_{A_0}^{\dagger}u + P_-F_{A_0}\right) = 0 \tag{2.8}$$

have simultaneously a solution $u \in \Pi_E^{\perp}(A_0)\Omega_-^2(\mathfrak{g})$. The properties of (2.7) and (2.8) are the subject of the next section.

3. The global setting

The purpose of this section is to place (2.7) and (2.8) in a more geometric setting. This requires first understanding (2.7) and then understanding and reinterpreting (2.8).

For fixed $E \geqslant 0$, one is to consider (2.7) as an equation for $u = u_E(A_0)$. Then (2.8) becomes a finite set of equations for A_0. The treatment of (2.8) is greatly facilitated by establishing under what conditions $u_E(A)$ changes smoothly with respect to smooth changes of A. For this reason, it is necessary to define what is meant by a smooth map of a smooth manifold N into $\mathscr{C}$, $\mathscr{C}/\mathscr{G}$ and $\Omega_-^2(\mathfrak{g})$. This is usually done by considering $\mathscr{C}$, for example, as a dense subset of an underlying smooth Banach manifold. One can do this using a Sobolev $L_{k,p}$ topology (locally on M, derivatives through order k are in L_p, cf. [18]) as is done in [9], [6]. The details are now standard and to avoid a dense, technical discussion here the details are relegated to Appendix A. Suffice it to say that differentiation is defined here with respect to the $L_{4,2}$ structure on $\mathscr{C}$, the $L_{5,2}$ structure on $\mathscr{G}$ and the $L_{3,2}$ structure on $\Omega_-^2(\mathfrak{g})$. Introduce, as notation, $\|\cdot\|_p$ for the L_p norm on sections of the various bundles involved.

The discussion of (2.7) is facilitated by making the following definitions (compare Definition 3.1 of [20]).

Definition. For $A \in \mathscr{C}$ and $E > 0$, define

$$\begin{aligned} \zeta_E(A) &= E^{-1/2}\left(1 + E + \|P_-F_A\|_3^3\right)^{1/2}, \\ \delta_E(A) &= \|P_-F_A\|_2 + \zeta_E(A)\|P_-F_A\|_{4/3}\left(1 + \|F_A\|_4\right). \end{aligned} \tag{3.1}$$

It is an exercise in the Sobolev inequalities to verify that $\zeta_E(\cdot)$ and $\delta_E(\cdot)$ are continuous functions on $\mathscr{C}$. This is left to the reader.

The basic existence theorem for (2.7) follows as a straightforward generalization of Theorem 3.2 of [20]. The following theorem is proved in Appendix A.

Theorem 3.2. *Let M be a compact, oriented, 4-dimensional Riemannian manifold. Let $P \to M$ be a principal G-bundle, with G a compact, semi-simple Lie group. There exists $\varepsilon_0 > 0$, which is independent of P and $A \in \mathscr{C}(P)$, with the following significance: for fixed $E \geqslant 0$, suppose that*

$$(1 + \zeta_E(A))\delta_E(A) < \varepsilon_0.$$

Then there exists a unique solution $u_E \in \Omega^2_-(\hat{\mathfrak{g}})$ to (2.7) with the properties $\Pi_E(A)u_E = 0$, and

$$\left\| \nabla_A \mathscr{D}_A^\dagger u_E \right\|_2 + \left\| \mathscr{D}_A^\dagger u_E \right\|_2 \leqslant c \cdot \delta_E(A), \tag{3.2}$$

$$\left\| \mathscr{D}_A^\dagger u_E \right\|_2 \leqslant c \cdot \zeta_E(A) \| P_- F_A \|_{4/3}. \tag{3.3}$$

Here $c < \infty$ is independent of A, P, and ε_0.

Theorem 3.2 establishes a mapping $u_E(\cdot)$ from $\mathscr{C}_E = \{A \in \mathscr{C}(P)\colon (1 + \zeta_E(A))\delta_E(A) < \varepsilon_0\}$ to $\Omega^2_-(\hat{\mathfrak{g}})$ which is $\mathscr{G}$-equivariant: $u_E(gA) = gu_E(A)$.

The existence of this map $u_E(\cdot)$ allows one to consider (2.8) as an equation for $A \in \mathscr{C}$. The left-hand side of (2.8) defines a mapping which associates to each $A \in \mathscr{C}(P)$ a point

$$f_E(A) = \Pi_E(A)\big(\mathscr{D}_A^\dagger u_E(A) \# \mathscr{D}_A^\dagger u_E(A) + P_- F_A\big) \tag{3.4}$$

in $\Omega^2_-(\hat{\mathfrak{g}})$. To describe the map $f_E(\cdot)$, introduce the assignment $n_E(\cdot)$ of $A \in \mathscr{C}$ to the integer $\operatorname{rank}(\Pi_E(A))$.

Proposition 3.3. *Let N be a smooth manifold and let $\Psi\colon N \to \mathscr{C}_E$ be a smooth map with the property $\Psi^* n_E$ is constant. Then $\Psi^*\Pi_E$ varies smoothly as a projection operator on $\Omega^2_-(\hat{\mathfrak{g}})$, the map $\Psi^*\mathscr{D}^\dagger u_E\colon N \to \Omega^1(\hat{\mathfrak{g}})$ is smooth and also the map $\Psi^* f_E\colon N \to \Omega^2_-(\hat{\mathfrak{g}})$ is smooth.*

This proposition is proved in Appendix A.

Let N be a smooth manifold, and let $\Psi\colon N \to \mathscr{C}_E$ be a smooth map. One defines a new map $\Psi_1\colon N \to \mathscr{C}_E$ by exploiting the affine structure of $\mathscr{C}$ to assign $y \in N$ to

$$\Psi_1(y) = \Psi(y) + \big(\Psi^*\mathscr{D}^\dagger u_E\big)(y). \tag{3.5}$$

This map Ψ_1 is smooth when $\Psi^* n_E$ is constant on N. In addition, $\Psi_1(x)$ is self-dual if $\Psi^* f_E(x) = 0$ in $\Omega^2_-(\hat{\mathfrak{g}})$.

The map Ψ_1 is also $\mathscr{G}$-equivariant in the following sense: Let $l\colon N \to \mathscr{G}$ be a smooth map. Define $\Psi' = (l\Psi)(\cdot)\colon N \to \mathscr{C}_E$, and define Ψ_1' by (3.5) with Ψ'

replacing Ψ. Then $\Psi_1'(y) = l(y)\Psi_1(y)$ for all $y \in N$. It is this $\mathcal{G}$-equivariance which makes it useful to continue the discussion in terms of structures on the quotient space $\mathcal{B} = \mathcal{C}/\mathcal{G}$.

Let $\mathcal{C}^* \subseteq \mathcal{C}$ denote the dense open subspace of irreducible connections. It is now a standard argument that the quotient $\mathcal{C}^* \to \mathcal{C}^*/\mathcal{G} = \mathcal{B}^*$ defines a C^∞ principal $\mathcal{G}$-bundle [9]. For $E \geqslant 0$, define $\mathcal{B}_E^* = (\mathcal{C}_E \cap \mathcal{C}^*)/\mathcal{G}$.

As $\mathcal{G}$ acts naturally on $\Omega_-^2(\hat{\mathfrak{g}})$, one can form the associated vector bundle

$$\Omega \equiv \mathcal{C}^* \times_{\mathcal{G}} \Omega_-^2(\hat{\mathfrak{g}}). \tag{3.6}$$

Since the assignment of $A \in \mathcal{C}$ to $P_-F_A \in \Omega_-^2(\hat{\mathfrak{g}})$ is $\mathcal{G}$-equivariant, it defines a section Φ of Ω over $\mathcal{B}^*$.

The assignment of $A \in \mathcal{C}$ to $n_E(A) = \operatorname{rank} \Pi_E(A) \in \{0, 1, \cdots\}$ is $\mathcal{G}$-equivariant, and so defines a (discontinuous) map n_E: $\mathcal{B}^* \to \{0, 1, \cdots\}$. Also, the assignment of $A \in \mathcal{C}$ to $f_E(A) \in \Omega_-^2(\hat{\mathfrak{g}})$ is $\mathcal{G}$-equivariant and so defines a (discontinuous) section f_E of Ω over $\mathcal{B}_E^*$.

Now, if Ψ: $N \to \mathcal{B}_E^*$ is a smooth map of smooth manifold N into $\mathcal{B}_E^*$ such that $\Psi^* n_E$ is constant, then Proposition 3.3 implies that $\Psi^* f_E$ is a smooth section of $\Psi^*\Omega$ over N. What is more, the subset $V = (\Psi^*\Pi_E)\Psi^*\Omega$ of $\Psi^*\Omega$ defines a smooth vector bundle over N with fiber dimension $\Psi^* n_E$ and $\Psi^* f_E$: $N \to V$ is a smooth section of V over N.

The construction of useful manifolds N with maps Ψ: $N \to \mathcal{B}_E^*$ starts in the next section.

4. Nearly self-dual connections

In this section and the remaining ones, $P \to M$ is a principal SU(2) bundle with $k = -c_2(P) > 0$. Theorem 8.2 of [20] states that for any $E > 0$, $\mathcal{C}_E(P) \neq \varnothing$. In order to construct some useful manifolds with maps into $\mathcal{B}_E^*$, it is necessary to consider certain aspects of the proof of Theorem 8.2 of [20] in more detail. In partcular, it is important to keep careful track of the parameters that are required to specify the connections provided by Definitions 8.3 and 8.4 of [20]; these parameters provide the manifold N. The manifold N and the map Ψ: $N \to \mathcal{B}_E^*$ are defined at the end of this section.

A brief review of the construction of the connections in [20] is the first topic of §4. The recipe that appears in §8 of [20] requires the ingredients that are listed below.

Identify $\mathbf{R}^4 \cong \mathbf{H} =$ quaterions, SU(2) = unit quaternions, and $\mathfrak{su}(2) = \operatorname{Im} \mathbf{H}$. On $\mathbf{R}^4$, define

$$U_1 = \{x \in \mathbf{R}^4 : |x| < 1\} \quad \text{and} \quad U_2 = \mathbf{R}^4 \setminus \{0\}. \tag{4.1}$$

The bundle $\hat{P} \to \mathbf{R}^4$ is specified by giving the transition function

$$(4.2) \qquad g_{12}\colon U_1 \cap U_2 \to \mathrm{SU}(2), \qquad g_{12}(x) = x/|x|.$$

A connection, $W \in \mathscr{C}(\hat{P})$, is specified by data consisting of a pair of $\mathfrak{su}(2)$ valued 1-forms, W^i on U_i $(i = 1, 2)$ which are restricted on $U_1 \cap U_2$ to obey the cocycle condition

$$(4.3) \qquad W_1(x) = g_{12}(x) W_2(x) g_{12}^{-1}(x) + g_{12}(x)\, dg_{12}^{-1}(x).$$

For each $\lambda \in (0,1)$ define the connection

$$(4.4) \qquad W_\lambda = \left(W_\lambda^1, W_\lambda^2\right) = \left(\mathrm{Im}\left(\frac{x\, d\bar{x}}{\lambda^2 + |x|^2}\right), \mathrm{Im}\left(\frac{\lambda^2\, \bar{x}\, dx}{|x|^2(\lambda^2 + |x|^2)}\right)\right).$$

In fact, the connection W_λ is self-dual [4]. This connection is discussed in detail in [2], [10]. The curvature of W_λ is given in U_1 by

$$(4.5) \qquad F_l^1 = \frac{\lambda^2\, dx \wedge d\bar{x}}{\left(\lambda^2 + |x|^2\right)^2}.$$

As in [20, §8], connections over M which are approximately self-dual will be constructed by finding the "good" maps $\phi\colon M \to \mathbf{R}^4$ by which the pair $(\hat{P}, W_\lambda)$ of bundle and connection can be pulled back to M. The good maps come from Gaussian coordinate charts on M.

The set of all Gaussian coordinate systems on M is parametrized by the oriented orthonormal frame bundle $\pi\colon F_M \to M$ in the following way: A point $f \in F_M$ consists of a point $p = \pi(f) \in M$, and an orthonormal frame $e \in \pi^{-1}(p) = F_M|_p$. Let $\exp_p\colon T_M|_p \to M$ denote the exponential map at p. One obtains a unique identification of $T_M|_p \simeq \mathbf{R}^4$ from the frame e. Thus a point $f \in F_M$ yields a unique map, $\exp_f\colon \mathbf{R}^4 \to M$. Because M is compact, there is a ball $B_\rho \subset \mathbf{R}^4$ of fixed radius, $\rho > 0$, centered at $0 \in \mathbf{R}^4$, such that $\exp_f|_{B_\rho}$ is a diffeomorphism onto its image, U. Let ϕ_f denote the inverse map. The map ϕ_f is a Gaussian coordinate system centered at p, and has the following properties: Let $(\cdot,\cdot)$ denote the metric on T^*. Then

$$(4.6) \qquad \begin{aligned} &(1) \quad \phi_f(p) = 0 \quad \text{and} \quad d\phi_f|_p e = \left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^4}\right\}, \\ &(2) \quad \left(\phi_f^*\, dx^\alpha, \phi_f^* dx^\beta\right) = \delta^{\alpha\beta} + \mathcal{O}\left(|x|^2\right), \qquad \alpha, \beta \in (1, \cdots, 4), \\ &(3) \quad \left|d\left(\phi_f^*\, dx^\alpha, \phi_f^*\, dx^\beta\right)\right| = \mathcal{O}(|x|), \qquad \alpha, \beta \in (1, \cdots, 4). \end{aligned}$$

At this point, all of the data necessary for the definition of the manifold N and the map $\Psi\colon N \to \mathscr{C}_E$ have been introduced.

Definition 4.1. For each integer $l > 0$, $t \in [0,1)$ and $0 < \lambda < \lambda_0 = 16^{-1}\rho^2$, define $N_1(l, t, \lambda)$ to be the following subset of $\times_l(F_M \times (0, \lambda))$: A point $y = ((f_i, \lambda_i)_{i=1}^l) \in N_1$ if:

(1) $d(y) = \min_{i \neq j} \operatorname{dist}(\pi(f_i), \pi(f_j)) > 0$.

(2) For each $i \in (1, \cdots, l)$, $\lambda_i \leqslant 64^{-1} d^2(y)$.

(3) Define $\lambda(y) \equiv \max_{i \in (1,\cdots,l)} \lambda_i$, $\bar{\lambda}(y) = \min_{i \in (1,\cdots,l)} \lambda_i$ and $t(y) = \bar{\lambda}(y)/\lambda(y)$. Require that $t(y) > t$.

The set N_1 is a smooth manifold, $\dim N_1 = 11 \cdot l$.

For each point $y \in N_1$, a pair $(P(y), A(y))$ consisting of a principal SU(2) bundle $P(y) \to M$ and a connection $A(y) \in \mathscr{C}(P(Y))$ will be defined. In the following definition, $\beta(t) \in C_0^\infty(\mathbf{R})$ is a smooth bump function satisfying $0 \leqslant \beta \leqslant 1$, and $\beta(t) \equiv 1$ ($\equiv 0$) when $t < 1$ ($t > \frac{3}{2}$). For $x \in \mathbf{R}^4$ and $r > 0$, set $\beta_r(x) \equiv \beta(r^{-1}|x|)$. Let $P_0 = M \times \mathrm{SU}(2)$ and denote by $\theta \in \mathscr{C}(P_0)$ the flat, product connection.

Definition 4.2. Define the family of bundles $(P(y), A(y))_{y \in N_1}$ by the following data:

(1) For $y = ((f_i, \lambda_i)_{i=1}^l) \in N_1$, the cover $\mathscr{V}(y) = \{U_0, U_1, \cdots, U_l\}$ of M is

$$U_i = \phi_{f_i}^{-1}\left(B_{\sqrt{\lambda_i}}\right), \quad i \in (1, \cdots, l), \qquad U_0 = M \backslash \{\pi(f_i)\}_{i=1}^l. \tag{4.7}$$

(2) The transition functions $\{g_{i0}: U_i \cap U_0 \to \mathrm{SU}(2)\}_{i=1}^l$ are

$$g_{i0} = \phi_{f_i}^*\left(x/|x|\right). \tag{4.8}$$

(3) The connection $A(y) = \{a_\alpha \in C^\infty(U_\alpha; T^*) \times \mathfrak{su}(2)\}$ is

$$\begin{aligned} a_i &= \phi_{f_i}^*\left(W_{\lambda_i}^1\right) \quad \text{for } i \in (1, \cdots, l), \\ a_0 &= \theta + \sum_{i=1}^{l} \phi_{f_i}^*\left(\beta_{\sqrt{\lambda_i}} W_{\lambda_i}^2\right). \end{aligned} \tag{4.9}$$

At this point, some remarks are in order. First, the pair $(P(y), A(y))$ is smooth; this follows because of the restrictions $\lambda < 16^{-1}\rho^2$ and $\lambda_i < 64^{-1}d(y)^2$. Second, each $A(y)$ is irreducible. Third, by appealing to Proposition 8.5 of [20], for fixed l, the bundles $P(y)$, $y \in N_1(l, 0, \lambda_0)$, are mutually isomorphic with $-c_2(P(y)) = l$.

Choose a fixed basepoint $y_0 \in N_1(l, 0, \lambda_0)$ and write $P = P(y_0)$. For $y \in N_1(l, 0, \lambda_0)$, two isomorphisms $\eta_1, \eta_2 \in \Gamma(\mathrm{Iso}(P, P(y)))$ differ by an element in $\mathscr{G}(P)$. Thus, one has

Definition 4.3. Given $1 \in \mathbf{Z}_+$, define a map Ψ: $N_1(l, 0, \lambda_0) \to \mathscr{B}^*(P)$ as follows: For each $y \in N_1$, choose $\eta(y) \in \Gamma(\mathrm{Iso}(P; P(y)))$ and set

$$\Psi(y) \equiv [\eta^*(y) A(y)] \in \mathscr{B}^*(P),$$

where $A(y)$ is given in Definition 4.2.

Whether or not Ψ is a smooth map is a local question with respect to N_1, and a simple calculation, which shall be omitted, yields

Proposition 4.4. *The map* $\Psi\colon N_1(l, 0, \lambda_0) \to \mathscr{B}^*$ *of Definition* 4.3 *is smooth.*

Further relevant properties of Ψ are related to the fact that associated to an oriented, 4-dimensional Riemannian manifold are two canonical, principal SO(3) bundles, F_+, $F_- \to M$, which are respectively the bundles of orthonormal frames in $P_\pm \wedge {}_2 T_M$. There are bundle maps $\hat{\rho}_\pm\colon F_M \to F_\pm$ which are obtained by taking an orthonormal frame $e \in T_M$, and constructing the induced orthonormal basis of $P_\pm \wedge {}_2 T_M$. There are also two homomorphism $\rho_\pm\colon \mathrm{SO}(4) \to \mathrm{SO}(3)$ which come from the isomorphism $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times_{\{\pm 1\}} \mathrm{SU}(2)$. The maps $\hat{\rho}_\pm$ have the property that

$$\begin{array}{ccc} F_M \times \mathrm{SO}(4) & \xrightarrow{\hat{\rho}_\pm \times \rho_\pm} & F_\pm \times \mathrm{SO}(3) \\ \downarrow & & \downarrow \\ F_M & \xrightarrow{\hat{\rho}_\pm} & F_\pm \end{array}$$

is a commutative diagram. Here, the verticle arrows give the standard group actions.

The bundle $F_- \to M$ is important. The map $\hat{\rho}_-$ extends, for each $l > 0$, to a bundle map

$$\hat{\rho}_-\colon \times_l(F_M \times (0, \lambda_0)) \to \times_l(F_- \times (0, \lambda_0)). \tag{4.10}$$

Let $N_2(l, 0, \lambda_0) \equiv \hat{\rho} N_1(l, 0, \lambda_0)$. The group Σ_l of permutations of l objects acts naturally on N_2. In Addition, SO(3) acts on N_2 by multiplication from the right simultaneously on each factor of F_-. This SO(3) action commutes with the Σ_l action. The quotients, $\overline{N}(l, 0, \lambda_0) \equiv N_2(l, 0, \lambda_2)/\Sigma_l$ and $N(l, 0, \lambda_0) = \overline{N}/\mathrm{SO}(3)$ are smooth manifolds of dimension $8l$ and $8l - 3$, respectively. In fact, the projection from $\overline{N} \to N$ defines a principal SO(3) bundle.

Proposition 4.5. *The map* $\Psi\colon N_1(l, 0, \lambda_0) \to \mathscr{B}^*$ *of Definition* 4.3 *factors through* $\overline{N}(l, 0, \lambda_0)$, *and* $N(l, 0, \lambda_0)$.

Denote the induced maps by Ψ also. It is a fact that Ψ embeds N in $\mathscr{B}^*$. This will not be proved here as it is not directly relevant to the proof of the theorems in §1. The proof of Proposition 4.5 is deferred to the end of this section.

Concerning the map $\Psi^* n_E\colon N \to \mathbf{Z}$, one has

Proposition 4.6. *Let* $E_0(M) = \frac{1}{2} \cdot$ (*the lowest, nonzero eigenvalue of* $P_- dd^\dagger$: $\Omega^2_-(M) \to \Omega^2_-(M)$). *Given* $l > 0$ *and* $0 < E \leqslant E_0(M)$, *there exists* $0 < \lambda_1(E, l) \leqslant \lambda_0$ *such that* $\Psi^* n_E\colon N(l, 0, \lambda_1) \to \mathbf{Z}$ *has the constant value* $3b_-(M)$.

Proposition 4.6 is also proved at the end of this section.

The utility of the manifold N to the strategy of §3 is due to Proposition 4.6 and

Proposition 4.7. *Given $l > 0$, $E > 0$ and $t \in (0,1)$, there exists $0 < \lambda_2(E, t, l) < \lambda_0$ such that for all $\lambda < \lambda_2$ and all $y \in N(l, t, \lambda)$, $\Psi(y) \in \mathscr{B}_E^*$. In fact*

(1) $\delta_E(A(y)) \leqslant c(l)(1 + E^{-1/2})(1 + t^{-1})\lambda^2(y)$,

(2) $\|P_- F_{A(y)}\|_p \leqslant c(l)\lambda(y)^{2/p}$,

(3) $\|F_{A(y)}\|_p \leqslant c(l)(\bar{\lambda}(y))^{4/p-2}$.

Proposition 4.7 is Proposition 8.6 of [20].

The remainder of this section contains the proofs of Propositions 4.5 and 4.6.

Proof of Proposition 4.5. Let $y = ((f_i, \lambda_i)_{i=1}^l) \in N_1$. Restrict attention to a ball $B_{3\sqrt{\lambda_1}}$ centered at $p = \pi(f_1)$. Let $(x^\nu)_{\nu=1}^4$ denote the Gaussian coordinate system defined by f_1. Represent $e \in \mathrm{SO}(4)$ by $(e_+, e_-) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$. Let $(x_e^\nu)_{\nu=1}^4$ denote the Gaussian coordinate system defined by $f_1 e$. Think of $\mathbf{R}^4 \simeq \mathbf{H}$ and $\mathrm{SU}(2) = S^3 \subset \mathbf{H}$. Then

$$x_e = e_+ x\, e_-^{-1}. \tag{4.11}$$

Hence, the transition function in $U_i \cap U_0$ defined by f_1 and $f_1 e$ are related by

$$g_{10}[f_1 e] = \phi_{f_1 e}^*(x/|x|) = e_+ g_{10}[f_1] e_-^{-1}. \tag{4.12}$$

The connection 1-forms in U_1 are related by

$$a_1[f_1 e] = \phi_{f_1 e}^*\left(W_{\lambda_1}^1\right) = e_+ a_1[f_1] e_+^{-1}. \tag{4.13}$$

One concludes from (4.12) and (4.13) that the image of $y = ((f_1, \lambda_1), (f_j, \lambda_j)_{j=2}^l)$ and $y' = ((f_1 e, \lambda_1), (f_j, \lambda_j)_{j=2}^l)$ in $\mathscr{B}^*$ coincide when $e = [(e_+, 1)] \in \mathrm{SO}(4)$. Therefore the map Ψ factors through N_2.

Next, let $y = ((f_i, \lambda_i)_{i=1}^l)$ and $y' = ((f_i e, \lambda_i)_{i=1}^l)$ for $e = [(e_-, e_+)] \in \mathrm{SO}(4)$. Using (4.12), one observes that for each $i \in (1, \cdots, l)$, the transition functions in $U_i \cap U_0$ defined by y and y' are related by $g_{i0}[y'] = e_+ g_{i0}[y] e_-^{-1}$, while the connection 1-forms in U_0 are related by

$$a_0[y'] = e_- a_0[y] e_-^{-1}. \tag{4.14}$$

Therefore, y and y' have the same image in $\mathscr{B}_E^*$, and Ψ factors through $N_2/\mathrm{SO}(3)$. Since a permutation of the factors of $y = ((f_i, \lambda_i)_{i=1}^l)$ changes nothing, Ψ factors through N as claimed.

Proof of Proposition 4.6. The fact that for $E \leqslant E_0$, there exists $\lambda_1(E, l) > 0$ such that $\Psi^* n_E$: $N(l, 0, \lambda_1) \to \mathbf{Z}_+$ has value less than or equal to $3b_-$ is proved in the same way that Proposition 8.8 of [20] was proved; the reader is referred there. To show that λ_1 can be chosen so that $\Psi^* n_E = 3b_-$, the following construction is required: Choose an orthonormal basis $\{\sigma^a\}_{a=1}^3$ for $M \times \mathfrak{su}(2)$ such that $\nabla_\Theta \sigma^a = 0$, $a = 1,2,3$. Let $y = ((f_i, \lambda_i)) \in \overline{N}(l, 0, \lambda)$, for $\lambda \leqslant \lambda_0$. Define $M(y) = M \setminus \{\pi(f_i)\}_{i=1}^l$. The construction of $P(y) \to M$ gives a natural identification $P(y)|_{M(y)} \simeq M(y) \times \mathrm{SU}(2)$, and through this identification,

$$\{\sigma^a\}_{a=1}^3 \in C^\infty\big(M(y); P(y) \times_{\mathrm{SU}(2)} \mathfrak{su}(2)\big).$$

Choose an L_2-orthornormal basis $\{\omega_J\}_{J=1}^{b_-}$ of $P_- H^2_{\mathrm{DR}}(M)$. Finally, for $a \in (1,2,3)$ and $J \in (1,\cdots,b_-)$, set

$$\omega_{J,a}[y] = \prod_{i=1}^{l}\big(1 - \beta_{\lambda_i}[i]\big) \cdot \omega_J \cdot \sigma^a, \tag{4.15}$$

where $\beta_{\lambda_i}[i] = \phi_{f_i}^* \beta_{\lambda_i}$. Observe that $\omega_{J,a}[y] \in \Omega^2_-(\mathfrak{g}(y))$. A straightforward calculation reveals that for λ sufficiently small,

$$\text{(4.16)}\quad \begin{aligned} &(1)\quad \big\|\mathscr{D}^\dagger_{A(y)}\omega_{J,a}(y)\big\|_2 \leqslant z \cdot \lambda(y), \quad a \in (1,2,3), J \in (1,\cdots,b_-),\\ &(2)\quad \big\langle \omega_{J,a}(y), \omega_{J',a'}(y)\big\rangle_2 = \delta_{J,J'}\delta_{a,a'} + O(\lambda^4(y))\\ &\qquad\qquad \text{for } a, a' \in (1,2,3) \text{ and } J, J' \in (1,\cdots,b_-). \end{aligned}$$

Here, z is a numerical constant which is independent of λ. Thus, if $\lambda_3 \leqslant z^{-1}E_0(M)$ with z defined by (4.16), then for each $y \in N(l, 0, \lambda_3)$, the rank of $\Pi_{z\cdot\lambda_3}(A(y))$ is $3b_-$. Then the Graham-Schmidt procedure provides exactly $3b_-$ eigenvectors of $\mathscr{D}_{A(y)}\mathscr{D}^\dagger_{A(y)}$ with eigenvalue less than or equal to $z \cdot \lambda_3$. In fact, (4.16) implies that the projection $\Pi_{z\cdot\lambda_3}(A(y))$ on $\operatorname{span}\{\omega_{j,a}\}_{j=1,a=1}^{b_-,3}$ is an isomorphism onto. The following lemma summarizes.

Lemma 4.8. *Given $l > 0$, there exist $c(M, l) < \infty$ and $0 < \lambda_3(M, l) \leqslant \lambda_0$ such that for each $y \in N(l, 0, \lambda_3)$ the operator $\mathscr{D}_{A(y)}\mathscr{D}^\dagger_{A(y)}$ has $3b_-(M)$ eigenvectors with eigenvalues less than $c \cdot \lambda(y)^2$ and all other eigenvalues are greater than $E_0(M)$.*

Proposition 4.6 follows immediately.

5. The obstruction

For each $l > 0$, the principal SO(3) bundles $\overline{N}(l, t, \lambda_2) \to N(l, t, \lambda_2)$, have been constructed for values of $t \in (0,1)$ and for $\lambda_2 = \lambda_2(E_0, t, l)$ as specified by Propositions 4.6 and 4.7. Definition 4.3 provides a smooth map Ψ: $N \to \mathscr{B}^*_{E_0}$ with the property that $\Psi^* n_E = 3b_-$. According to Proposition 3.5, the induced section $\Psi^* f_{E_0} \in \Gamma(\Psi^*\Omega)$ is smooth. The task of determining the conditions which imply that $\Psi^* f_{E_0}$ has a zero is simplified because the vector bundle $\Pi_{E_0}(\Psi)\Psi^*\Omega \to N$ is isomorphic to an associated vector bundle to the SO(3) principal bundle $\overline{N} \to N$. This observation is stated in the proposition below.

Proposition 5.1. *Let $l > 0$, and let $\lambda_1(E_0, l)$ be as specified in Proposition 4.6. There exists $0 < \lambda_4(l) < \lambda_1(E_0, l)$ such that the pulled-back $\mathbf{R}^{3b_-}$-vector bundle $\Pi_{E_0}(\Psi)\Psi^*\Omega \to N(l, 0, \lambda_4)$ is isomorphic to the associated vector bundle*

$$V \equiv \overline{N} \times_{SO(3)} \mathbf{R}^{3b_-} \to N,$$

where SO(3) *acts diagonally on* $\mathbf{R}^{3b_-} = \times_{b_-} \mathbf{R}^3$.

Proposition 5.1 follows from the next result. Below, $\langle \cdot, \cdot \rangle_2$ denotes the L_2 inner product on $\Omega^2_-(\hat{\mathfrak{g}})$.

Proposition 5.2. *Let $l > 0$ and let $\lambda_1(E_0, l)$ be as specified in Proposition* 4.6. *There exists* $0 < \lambda_4(l) \leqslant \lambda_1(E_0, l)$ *such that the pulled-back* $\mathbf{R}^{3b_-}$- *vector bundle*

$$\hat{V} \equiv \Pi_{E_0}(\Psi)\Psi^*\Omega \to \overline{N}(l, 0, \lambda_4(l))$$

is isomorphic to $\overline{N} \times \mathbf{R}^{3b_-}$. *Let* $\{\omega_{J,a}(y)\}$ *be defined by* (4.15). *Then a trivialization of this bundle is given by the map* ϕ: $\hat{V} \to \overline{N}(l, 0, \lambda_4) \times \mathbf{R}^{3b_-}$ *which takes* $u \in \hat{V}|_y$ *to*

$$\phi(y, u) = \left(y, \langle \omega_{J,a}[y], u\rangle_2\right)_{J=1,a=1}^{b_-\ 3}\Big). \tag{5.1}$$

Proof of Proposition 5.1, *assuming Proposition* 5.2. Let $y = ((f_i, \lambda_i))_{i=1}^{l} \in \overline{N}$, let $u \in \hat{V}|_y$ and let $\Lambda \in$ SO(3). Represent Λ by the 3×3 matrix Λ^{ab}, $a, b \in (1, 2, 3)$. From (4.14) and (4.15), one obtains that

$$\phi(y\Lambda, u) = \left(y\Lambda, \left(\sum_{b=1}^{3} \langle \omega_{J,b}[y], u\rangle_2 \Lambda^{ab}\right)_{J=1,a=1}^{b_-\ 3}\right), \tag{5.2}$$

where $y\Lambda = ((f_i\Lambda, \lambda_i)_{i=1}^{l})$ and ϕ is the map in (5.1). Proposition 5.1 follows readily from (5.2).

Proof of Proposition 5.2. A consequence of Proposition 4.6 and Proposition 3.5 is that the bundle in question is smooth. The map ϕ is a smooth, bundle map, so it is necessary to establish that it is an isomorphism on the fibres. This fact follows readily from Lemma 4.8 and (4.16) when λ_4 is taken sufficiently small.

The applications of Propositions 5.1 and 5.2 require one to define for $l > 0$ and $t \in (0, 1)$, the number

$$\overline{\lambda}(l, t) = \min(\lambda_4(l), \lambda_2(E_0, t, l)), \tag{5.3}$$

where $\lambda_2(E_0, t, l)$ is specified in Proposition 4.7. It is convenient to use the trivialization of V that Proposition 5.2 provides to reinterpret sections of the vector bundle

$$\overline{N}(l, t, \overline{\lambda}(l, t)) \times_{\mathrm{SO}(3)} \mathbf{R}^{3b_-} \to N(l, t, \overline{\lambda}(l, t))$$

as SO(3)-equivariant maps from $\overline{N}(l, t, \overline{\lambda})$ to $\mathbf{R}^{3b_-}$. The problem at hand is to determine under what circumstances the SO(3)-equivariant map $\Psi^* f_{E_0}$ has a zero. Lacking a more powerful technique, the strategy for solving this problem is to split $\Psi^* f_{E_0}$ into a sum $-h + h'$ of SO(3) equivariant maps from $\overline{N}(l, t, \overline{\lambda})$ to $\mathbf{R}^{3b_-}$. The map h will contain the leading order term in $\lambda(y)$, $y \in N$; the map h' will contain terms of lower order.

The definition of the splitting of $\Psi^* f_{E_0}$ into $-h + h'$ is the next order of business.

Definition 5.3. For $l > 0$, define the space

$$F(l) = C_l'(F_- \times \mathbf{R}^*) = \Big\{(f_i, \lambda_i)_{i=1}^l \in \times_l (F_- \times (0, \infty)):$$
$$\text{for } i \neq j, \pi(f_i) \neq \pi(f_j)\Big\}/\Sigma_l.$$

Write $y \in F(l)$ as $y = \{(f_i, \lambda_i)_{i=1}^l\}$ and represent each $f_i \in F_-$ by a pair $(p_i, (x_i^a)_{a=1}^3)$, where $p_i = \pi(f_i) \in M$ and $(x_i^a)_{a=1}^3$ is an orthonormal frame for $P_- \wedge_2 T^*$ at p_i. Define $h(y)$ in $\mathbf{R}^{3b_-}$ by (1.7). For each $l > 0$ and $t \in (0, 1)$, restrict h to $\overline{N}(l, t, \overline{\lambda}(l, t))$ and define h': $\overline{N}(l, t, \overline{\lambda}(l, t)) \to \mathbf{R}^{3b_-}$ by $h' = (1/\sqrt{2}\pi^2)\Psi^* f_{E_0} + h$.

The utility of this splitting of $\Psi^* f_{E_0}$ is in part due to the following proposition.

Proposition 5.4. *Let h': $\overline{N}(l, t, \overline{\lambda}(l, t)) \to \mathbf{R}^{3b_-}$ be as in the previous definition. Then for $y \in \overline{N}$, $|h'(y)| \sim \mathcal{O}(\lambda(y)^{5/2})$.*

The remainder of this section is occupied with the proof of this proposition.

Proof of Proposition 5.4. Consider (2.8) with $A_0 = A(y)$. This equation defines $\Psi^* f_{E_0}$. It is simplest to prove Proposition 5.4 by eliminating terms in (2.8) which are $\mathcal{O}(\lambda(y)^{5/2})$. What remains will be the map h of Definition 5.3. First, observe that if τ is an eigenvector of $\mathscr{D}_A \mathscr{D}_A^\dagger$ of eigenvalue $E < 1$ and $A = A(y)$ then $\|\tau\|_\infty$ is uniformly bounded, independent of y. This is established in §8 of [20], cf. (8.48). Therefore,

$$\left|\Pi_{E_0}(A)\left(\mathscr{D}_A^\dagger u_{E_0} \wedge \mathscr{D}_A^\dagger u_{E_0}\right)\right| \leqslant c \cdot \left\|\mathscr{D}_A^\dagger u_{E_0}\right\|_2^2 \leqslant c \cdot E_0^{-1} \lambda(y)^3.$$

The last inequality above is obtained with the aid of Proposition 4.7 and (3.4).

Next, consider the term $\Pi_{E_0}(A) P_- F_A$ in (2.8). The following identity holds: Let $\omega = \omega_{J,A}(y)$ and $A = A(y)$. Then

$$(5.5) \qquad \left\langle \omega, \Pi_{E_0}(A) P_- F_A \right\rangle_2 = \langle \omega, P_- F_A \rangle_2 - \left\langle \Pi_{E_0}^\perp \omega, P_- F_A \right\rangle_2.$$

Lemma 5.5. *Under the conditions of Proposition* 5.4, *the second term in* (5.5) *is* $O(\lambda(y)^{5/2})$.

Proof of Lemma 5.5. Observe that because of (4.16) and because $\pi_{E_0}(A)$ commutes with $\mathscr{D}_A\mathscr{D}_A^\dagger$,

$$\|\mathscr{D}_A^\dagger \Pi_{E_0}^\perp \omega\|_2 \leqslant c \cdot \lambda(y). \tag{5.6}$$

By applying a result from Appendix A, (A.3), one obtains from (5.6) that

$$\|\Pi_{E_0}^\perp \omega\|_4 \leqslant c \cdot \zeta_E(A)\lambda(y). \tag{5.7}$$

From this last inequality and Proposition 4.7, one has

$$\left|\left\langle \Pi_{E_0}^\perp \omega, P_- F_A\right\rangle_2\right| \leqslant c \cdot \zeta_{E_0}(A)\lambda(y)\|P_- F_A\|_{4/3} \leqslant c \cdot \zeta_{E_0}(A)\lambda^{5/2}(y). \tag{5.8}$$

The first term in (5.5) has contributions only from the balls $B_{2\sqrt{\lambda_i}}(p_i)$, $i = 1,\cdots,l$. Focus on one such ball and for convenience, write $\lambda = \lambda_i$, $f = f_i$ and $p = p_i$. In $U \equiv B_{2\sqrt{\lambda}}(p)\backslash p$, F_A is

$$\begin{aligned} F_A = d\beta_{\sqrt{\lambda}} \wedge \operatorname{Im}\left(\frac{\lambda^2 \bar{x}\, Dx}{|x|^2\left(|x|^2 + \lambda^2\right)}\right) + \beta_{\sqrt{\lambda}} \frac{\bar{x}}{|x|} F_\lambda^1(x) \frac{x}{|x|} \\ + \beta_{\sqrt{\lambda}}(\beta_{\sqrt{\lambda}} - 1) W_\lambda^2(x) \wedge W_\lambda^2(x). \end{aligned} \tag{5.9}$$

Here, $\{x^\nu\}_{\nu=1}^4$ are the Gaussian normal coordinates defined by some $f' \in \hat{\rho}_-^{-1} f \subset F_M|_p$, and F_λ^1 is given in (4.5) while W_λ^2 is given in (4.4).

Lemma 5.6. *Under the conditions of Proposition* 5.4,

$$\left|\left\langle \omega, P_-\left(\beta_{\sqrt{\lambda}} \frac{\bar{x}}{|x|} F_\lambda^1 \frac{x}{|x|} + \beta_{\sqrt{\lambda}}(\beta_{\sqrt{\lambda}} - 1) W_\lambda^2 \wedge W_\lambda^2\right)\right\rangle_2\right|$$

is $O(\lambda^3(y))$.

Proof of Lemma 5.6. The lemma follows from the following observations:

(1) $\|\omega\|_\infty$ is uniformly bounded.

(2) The metric norms and measure differ in B from the Euclidean ones by $O(|x|^2 \sim \lambda)$.

(3)

$$\left|P_- \beta_{\sqrt{\lambda}} \frac{\bar{x}}{|x|} F_\lambda^1 \frac{x}{|x|}\right| \leqslant c \cdot \lambda^2 (x^2 + \lambda^2)^{-1},$$

$$|\beta_{\sqrt{\lambda}}(1 - \beta_{\sqrt{\lambda}}) W_\lambda^2 \wedge W_\lambda^2| \leqslant c \cdot \beta_{\sqrt{\lambda}}(1 - \beta_{\sqrt{\lambda}}) \cdot \lambda.$$

The details are left to the reader.

Lemma 5.7. *Under the conditions of Proposition* 5.4,

$$\left\langle \omega_{J,a}, P_-\left(d\beta_{\sqrt{\lambda}} \wedge \operatorname{Im}\left(\frac{\lambda^2 \bar{x}\, dx}{|x|^2\left(|x|^2+\lambda^2\right)}\right)\right)\right\rangle_2 = -\lambda^2\sqrt{2}\,\pi^2(\omega_J(p), x^a) + O(\lambda^{5/2}).$$

Here, the point $f = (p, \{x^a\}_{a=1}^3) \in F_-$.

Proof of Lemma 5.7. One observes that due to observation (2) in the previous paragraph,

$$(5.10)\quad P_-\left(d\beta_{\sqrt{\lambda}} \operatorname{Im} \frac{\lambda^2 \bar{x}\, dx}{|x|^2\left(|x|^2+\lambda^2\right)}\right) = \frac{1}{\sqrt{2}} \frac{\partial\beta_{\sqrt{\lambda}}}{\partial|x|} \frac{\lambda^2}{|x|\left(|x|^2+\lambda^2\right)} \sigma^a x^a + r.$$

Here, $|r| \leqslant c \cdot \lambda\beta_{2\sqrt{\lambda}}(1-\beta_{1/2\sqrt{\lambda}})$. In addition, the 2-form ω_J can be expanded in a Taylor's expansion about p, whence

$$(5.11)\qquad \omega_{J,a}(x) = \omega_J(p)\sigma^a + O(\lambda^{1/2}).$$

Thus, except for terms of $O(\lambda^{5/2})$,

$$(5.12)\quad \begin{aligned} &\left\langle \omega_{J,a}, P_-\left(d\beta_{\sqrt{\lambda}} \wedge \operatorname{Im} \frac{\lambda^2 \bar{x}\, dx}{|x|^2(x^2+\lambda^2)}\right)\right\rangle_2 \\ &\quad = \frac{2\pi^2}{\sqrt{2}}(\omega_J(p), x^a) \int_0^\infty \frac{\lambda^2 x^2\, dx}{(x^2+\lambda^2)} \frac{\partial\beta_{\sqrt{\lambda}}}{\partial|x|}, \\ &\quad = -\lambda^2\sqrt{2}\,\pi^2(\omega_J(p), x^a) + O(\lambda^{5/2}), \end{aligned}$$

as claimed.

One concludes from Lemmas 5.5–5.7 that $\Psi^* f_{E_0}$ is $O(\lambda(y)^{5/2})$ except for a sum, $\sum_i -\lambda_i^2\sqrt{2}\,\pi^2(\omega_J(p_i), x_i^a)$.

But this is precisely $-\sqrt{2}\,\pi^2$ times h of Definition 5.3. Hence $|\Psi^* f_{E_0}/\sqrt{2}\pi^2 + h| \sim O(\lambda(y)^{5/2})$ as claimed.

6. Bundles with $-c_2(P)$ large

In order to utilize the decomposition of $\Psi^* f_{E_0}$ into $-h + h'$ of Definition 5.3, one additional result is necessary; this is a proposition which relates the vanishing of the section h to the vanishing of the section of interest, $\Psi^* f_{E_0}$. Proposition 6.1, below, provides the necessary relation.

In order to efficiently use the decomposition of $\Psi^* f_{E_0}$ into $-h + h'$, it is helpful to have a result which relates the zero's of $\Psi^* f_{E_0}$ in $N(l)$ to those of $\Psi^* f_{E_0}$ in $N(l')$ for $l' > l$. This relation is provided by Proposition 6.2.

Proposition 6.1. *Let $k \in (1, 2, \cdots)$ and $n \in (0, 1, \cdots)$. Let v be a C^2 map of the ball of radius $\delta > 0$, $B_\delta \subset \mathbf{R}^{n+k}$ into $\mathbf{R}^k$ with the following properties*:

(1) $v(0) = 0$.

(2) $H = dv|_0$ *is surjective. Let* $\mu = |HH^\dagger|^{1/2}$.

(3) $|v(x) - Hx| < \mu \cdot \delta/2$ *if* $x \in B_\delta$.

Let v': $B_\delta \to \mathbf{R}^k$ *be continuous with* $|v'| < \mu\delta/2$. *Then there exists* $x \in B_\delta$ *such that* $v(x) + v'(x) = 0$.

Proposition 6.2. *Let* $l > 0$. *Suppose that there exists* $y \in F(l)$ *where the map* h *of Definition* 5.3 *satisfies*

(1) $h(y) = 0$, *and*

(2) dh: $T_F|_y \to \mathbf{R}^{3b_-}$ *is surjective.*

Then there exist smooth, irreducible self-dual SU(2) *connections on all principal* SU(2) *bundles* $P \to M$ *satisfying* $-c_2(P) \geqslant l$.

In practice, Proposition 6.2 is used in conjunction with the following observation: if $Q \in \mathrm{GL}(3b_-, \mathbf{R})$ is a constant matrix, and $Qh(\cdot)$ satisfies conditions (1) and (2) above, then $h(\cdot)$ does also.

Because of this fact, $h(\cdot)$ and $Qh(\cdot)$ for $Q \in \mathrm{GL}(3b_-, \mathbf{R})$ will not be explicitly distinguished. (Matrices Q arise by considering linear combinations of the orthonormal basis $\{\omega_J\}_{J=1}^{b_-}$ of $P_- H^2_{\mathrm{DR}}$.)

The proof of Propositions 6.1 and 6.2 occupies the rest of this section.

Proof of Proposition 6.1. Define a one-parameter family of maps,

$$l(\cdot): S^{k-1} \times [0,1] \to S^{k-1},$$

by identifying the domain S^{k-1} with the δ-sphere in $(\mathrm{Ker}\, H)^\perp$ and writing

$$l(z,t) = \frac{z + tH^\dagger(HH^\dagger)^{-1}(v(z) - Hz + v'(z))}{|z + tH^\dagger(HH^\dagger)^{-1}(v(z) - Hz + v'(z))|}.$$

The map $l(\cdot,\cdot)$ is continuous. Because $l(\cdot, 0)$: $S^{k-1} \to S^{k-1}$ has degree 1, so does $l(\cdot, 1)$. Therefore, no continuous extension of $l(\cdot, 1)$ to a map from $B^k \to S^{k-1}$ exists, so necessarily, there exists $x \in \mathrm{int}\, B_\delta \cap (\mathrm{Ker}\, H)^\perp$ which satisfies

$$x + H^\dagger(HH^\dagger)^{-1}(v(x) - Hx + v'(x)) = 0.$$

Proof of Proposition 6.2. The crucial observation to make is that if $y = ((f_i, \lambda_i)_{i=1}^l) \in F(l)$, then there exists $r(y) \in (0,1)$ such that for all $r > r(y)$, the point

$$ry = \left((f_i, r\lambda_i)_{i=1}^l\right) \tag{6.1}$$

lies in $\overline{N}(l, t(y), \overline{\lambda}(l, t(y)))$. Indeed, $\lambda(ry) = r\lambda(y)$ while $t(ry) = t(y)$. The map h: $F(l) \to \mathbf{R}^{3b_-}$ if Definition 5.3 transforms homogeneously with respect to the scaling $y \to ry$:

$$h(ry) = r^2 h(y). \tag{6.2}$$

This scaling $y \to ry$ maps $\overline{N}(l, t, \overline{\lambda}(l, t))$ into itself, but the map h': $\overline{N}(l, t, \overline{\lambda}(l, t)) \to \mathbf{R}^{3b_-}$ of Definition 5.3 is not necessarily homogeneous with respect to it. However, by Proposition 5.4, if $y \in \overline{N}(l, t, \overline{\lambda}(l, t))$ then

$$|h'(ry)| \leqslant r^{5/2}|h(y)|. \tag{6.3}$$

Now suppose that $y \in F(l)$ has $h(y) = 0$ and $dh|_y$ surjective. Then this is true for ry as well. For $r \in (0,1)$ sufficiently small, $ry \in \overline{N}(l, t(y), \overline{\lambda}(l, t(y)))$. For $r \in (0,1)$ smaller still, (6.2) and (6.3) insure that Proposition 6.1 is applicable to the decomposition $\Psi^* f_{E_0} = -h + h'$ by setting $v = -h$, $v' = h'$. Thus $(\Psi^* f_{E_0})^{-1}(0) \neq \varnothing$ and the SU(2) bundle $P \to M$ with $-c_2(P) = l$ admits a smooth, irreducible, self-dual connection.

Next, consider SU(2) bundles $P' \to M$ with $-c_2(P') = k > l$. Let $y = ((f_i, \lambda_i)_{i=1}^l) \in F(l)$ be the given point. Choose, arbitrarily, $k - l$ points $\{g_j\}_{j=1}^{k-l} \in F_-$, except require that $\{\pi(g_j)\}_{j=1}^{k-l} \in M$ are distinct, and distinct from the points in $\{\pi(f_j)\}_{j=1}^l$.

For all $s \in (0,1)$, the point

$$y_k = ((f_1, \lambda_1), \cdots, (f_l, \lambda_l), (g_1, s\hat{\lambda}(y)), \cdots, (g_{k-l}, s\hat{\lambda}(y))) \tag{6.4}$$

lies in $F(k)$, and satisfies $t(y_k) = st(y)$.

Therefore, if $r = r(s) \in (0,1)$ is sufficiently small, $ry_k \in \overline{N}(k, st(y), \overline{\lambda}(k, st(y)))$.

Observe that there is a natural splitting of $h(ry_k) = h_1 + h_2$ where

$$h_1(ry_k) = r^2 h(y), \tag{6.5}$$

$$h_2(ry_k) = -r^2 s^2 \sum_{j=1}^{k-l} \omega_J(\pi(g_j), x_j^a). \tag{6.6}$$

Here, $g_j = (\pi(g_j), \{x_j^a\}_{a=1}^3)$. Meanwhile, one still has

$$|h'(ry_k)| \leqslant \text{const}(r\lambda(y))^{5/2}. \tag{6.7}$$

Since h_2 is homogeneous of degree 2 in s and r, it follows that there exists $s \in (0,1)$ and $r = r(s) \in (0,1)$ such that Proposition 6.1 is applicable when one sets $v = -h_1$ and $v' = -h_2 + h'$. Therefore, $(\Psi^* f_{E_0})^{-1}(0) \cap N(k, st(y), \overline{\lambda}(k, st(y)))$ is nonempty, and $P' \to M$ admits a smooth, irreducible self-dual connection as well.

7. The case $b_- = 1$

It is relatively easy to establish that the conditions of Proposition 6.1 are satisfied for $l = 2$ when $b_- = 1$. This is done here. The result is stated as Proposition 7.1.

Proposition 7.1. *Let $b_- = 1$. Smooth, irreducible self-dual connections exist on principal* SU(2) *bundles $P \to M$ with $-c_2(P) \geqslant 2$.*

Preliminary to proving Proposition 7.1, it is useful to describe the map h of Definition 5.3 in greater detail. It is convenient to use the exponential function $t \to \exp(-t/2)$ to identify $\mathbf{R}$ with $(0, \infty)$. Via this identification, $F(l) \simeq (\times(F_- \times \mathbf{R}))/\Sigma_l$. As no confusion will arise, let $(\times_l(F_- \times \mathbf{R}))/\Sigma_l$ be denoted by $F(l)$ and for $y = ((p_i, \{x_i^a\}_{a=1}^3, t_i)_{i=1}^l) \in F(l)$, the map h of Definition 5.3 is given by

$$h(y) = \sum_{i=1}^{l} e^{-t_i}((\omega_J(p_i)), x_i^a)_{J=1,a=1}^{b_-\ 3}. \tag{7.1}$$

The vector $h(y) = \sum_{i=1}^l \hat{h}(i) \in \mathbf{R}^{3b_-}$ is the sum of contributions which depend only on $(p_i, \{x_i^a\}_{a=1}^3, t_i) \in F_- \times \mathbf{R}$. Here, $\hat{h}\colon F_- \times \mathbf{R} \to \mathbf{R}^{3b_-}$ is given by

$$\hat{h}\big(p, \{x^a\}_{a=1}^3, t\big) = e^{-t}((\omega_J(p)), x^a)_{J=1,a=1}^{b_-\ 3}.$$

The differential of $\hat{h}$ at $(f, t) \in F_- \times \mathbf{R}$ is a linear map from $T_{F_-}|_f \times \mathbf{R}$ to $\mathbf{R}^{3b_-}$. Before describing this differential, it should be pointed out that the fiber $T_{F_-}|_f$ is spanned by the vertical vector fields at f, and the horizontal (with respect to the Riemannian metric) vector fields at f. The space of vertical vector fields at f is naturally isomorphic, via pull back, to the fiber at $p = \pi(f)$ of the bundle $A\,\mathrm{End}(P_- \wedge_2 T^*)$ of skew-symmetric endomorphisms of $P_- \wedge_2 T^*$. The space of horizontal vector fields at f is isomorphic to $T_M|_p$. The differential of h at $(p, \{x^a\}_{a=1}^3, t)$ is

$$\begin{aligned} d\hat{h}(\zeta, v, s) = e^{-t}(&-s(\omega_J(p)), x^a) + \big(\nabla_\zeta \omega_J(p), x^a\big) \\ &+ \big(v\omega_J(p), x^a\big)_{J=1,a=1}^{b_-\ 3}. \end{aligned} \tag{7.2}$$

In this notation, $\zeta \in T_M|_p$, ∇_ζ is the covariant derivative and $v \in A\,\mathrm{End}(P_- \wedge_2 T^*)|_p$.

At times, it will be convenient to consider a notation where, for fixed $p \in M$, $(\omega_J(p), x^a)_{a=1}^3$ is a vector $\vec{\omega}_J(p) \in \mathbf{R}^3$ whose direction is determined by the frame $\{x^a\}$. Changing the frame to $\{x'^a\}$ corresponds to rotating $\vec{\omega}(p)$ to $\Lambda(x')\vec{\omega}(p)$, where $\Lambda(x') \in \mathrm{SO}(3)$ is a 3×3 orthogonal matrix. Equivalently, the choice of a fixed frame $\{x^a\}$ in $F_-|_p$ allows one to write $y \in F_-|_p \times \mathbf{R}$

as $y = (p, \Lambda, t)$ with $\Lambda \in SO(3)$. This representation for y will be used here, and in later sections.

Proof of Proposition 7.1. It is enough to verify the conditions of Proposition 6.1 for $l = 2$. Choose distinct points, $p_1, p_2 \in M$ where the 2-form $\omega \in P_-H^2_{DR}$ does not vanish. Choose fixed frames in F_- at p_1 and p_2. Let $\Lambda \in SO(3)$ and $t \in \mathbf{R}$. A point in $F(2)$ is defined by $y = ((p_1, \Lambda, t), (p_2, 1, 0))$. As SO(3) acts transitively on S^2, it is no loss of generality to assume that $\vec{\omega}(p_1) = -\alpha\vec{\omega}(p_2)$ with $\alpha > 0$, and by rescaling t, one can assume that $\alpha = 1$.

The map h of (7.1), when restricted to y of the above form, defines a map h: $SO(3) \times \mathbf{R} \to \mathbf{R}^3$ that is given by $h(\Lambda, t) = (e^{-t}\Lambda\vec{\omega} - \vec{\omega})$, where $\vec{\omega} = \vec{\omega}(p_1)$. Hence $h(1, 0) = 0$.

According to (7.2), a vector $\vec{v} \in \mathbf{R}^3$ is in Coker dh at $(\Lambda = 1, t = 0)$ iff (1) $\vec{\omega} \cdot \vec{v} = 0$, and (2) $\vec{\omega} \wedge \vec{v} = 0$. Here, "$\cdot$" is the scalar product on $\mathbf{R}^3$ and "$\wedge$" is the exterior (vector) product. These conditions imply that $\vec{v} = 0$. Therefore $y = ((p_1, 1, 0), (p_2, 1, 0))$ satisfies the conditions of Proposition 6.2.

8. The case $b_- = 2$

The case $b_- = 2$ is quite different than the case $b_- = 1$, as evidenced by

Proposition 8.1. *Let $b_- = 2$. (1) Smooth, irreducible, self-dual connections exist on all principal* SU(2) *bundles $P \to M$ with $-c_2(P) \geqslant 4$. (2) Suppose that no L_2-orthonormal $\omega_1, \omega_2 \in P_-H^2_{DR}$ are pointwise orthonormal and pointwise the same length. Then smooth irreducible self-dual connections exist on all principal* SU(2) *bundles $P \to M$ if $-c_2(P) \geqslant 3$.*

Proposition 8.2. *Let $b_- = 2$ and $k \geqslant 2$. For an open, dense set of metrics on T_M in the C^k-topology, smooth, irreducible self-dual connections exist on all principal* SU(2) *bundles $P \to M$ with $-c_2(P) > 2$.*

Both propositions, and the results for $b_- > 2$ in later sections, require the next lemma about 2-forms in H^2_{DR}.

Lemma 8.3. *Let $\omega_1, \omega_2 \in P_-H^2_{DR}$ be linearly independent over* **R**. *Then the set of points $p \in M$, where $\omega_1(p) = \alpha(p)\omega_2(p)$, $\alpha(p) \in \mathbf{R}$, has measure zero.*

Proof of Lemma 8.3. Indeed, if $\omega_1 = \alpha\omega_2$ in an open set $U \subseteq M$, then the fact that ω_1 and ω_2 are closed forms implies that α is constant in U. The lemma now follows from the principle of unique continuation of solutions to elliptic PDEs [18].

The remainder of this section contains the proofs of Propositions 8.1 and 8.2. Although Proposition 8.2 implies Proposition 8.1 except for special metrics, the proof of Proposition 8.1 will be given in detail because it establishes results that are necessary in later sections. The proofs are presented in opposite order. In both proofs, the strategy is to verify that the respective assumptions imply that the conditions of Proposition 6.2 are satisfied.

Proof of Proposition 8.2. The key observation is provided by the following lemma.

Lemma 8.4. *Let $r \geqslant 2$. For an open, dense set of metrics on T_M in the C^r topology, there exists $p \in M$ where $P_-H^2_{\mathrm{DR}}$ spans a plane in $\Lambda_2 T^*|_p$ and where $\{d|\omega|^2\colon \omega \in P_-H^2_{\mathrm{DR}}\}$ spans a plane in $T^*|_p$.*

This lemma is proved in Appendix B.

Assume that the given metric on T_M satisfies the conclusions of Lemma 8.4. With this the case, there is a point $p \in M$ and a basis $\{\omega_1, \omega_2\}$ of $P_-H^2_{\mathrm{DR}}$ which is normalized so that $|\omega_1|(p) = |\omega_2|(p) = 1$ and $(\omega_1, \omega_2)(p) = 0$. In addition, the functions

$$f_1 = \tfrac{1}{2}\left(|\omega_1|^2 - |\omega_2|^2\right) \quad \text{and} \quad f_2 = (\omega_1, \omega_2) \tag{8.1}$$

have linearly independent gradients at p. Then the set $Z = \{x \in M\colon f_1(x) = f_2(x) = 0\}$ is locally a smooth 2-surface through p. Choose a second point $q \in Z$ different from p. Fix frames in F_- at the points p and q. Let B centered at p be a ball of radius $\frac{1}{2}\operatorname{dist}(p, q)$. For $p' \in B$ and $\Lambda \in \mathrm{SO}(3)$, a point $y \in F(2)$ is given by

$$y = ((p', \Lambda, t), (q, 1, 0)). \tag{8.2}$$

By considering the point q fixed, the map h of (7.1), when restricted to y of the form above, gives a map $h(p', \Lambda, t)$ from $B \times \mathrm{SO}(3) \times \mathbf{R} \to \times_2 \mathbf{R}^3$.

Let $\vec{\sigma}_1 = \vec{\omega}_1(p)$ and $\vec{\sigma}_2 = \vec{\omega}_2(p)$. By rescaling t and redefining Λ, one obtains

$$h((p, \Lambda, t)) = e^{-t}\Lambda\begin{pmatrix} -\vec{\sigma}_1 \\ -\vec{\sigma}_2 \end{pmatrix} + \begin{pmatrix} \vec{\sigma}_1 \\ \vec{\sigma}_2 \end{pmatrix}. \tag{8.3}$$

Clearly, $h((p, 1, 0)) = 0$. Now suppose that $v = (\vec{v}_1, \vec{v}_2) \in \times_2 \mathbf{R}^3$ is in Coker dh at $(p, 1, 0)$. By considering dh on tangent vectors to the factor $\mathrm{SO}(3) \times \mathbf{R}$ (cf. (7.2)), one concludes that v must be of the form

$$\vec{v}_1 = \alpha\vec{\sigma}_1 + \beta\vec{\sigma}_2, \quad \vec{v}_2 = -\alpha\vec{\sigma}_2 + \beta\vec{\sigma}_1. \tag{8.4}$$

By considering dh on tangent vectors to the factor B, one concludes that

$$0 = \alpha[(\nabla\omega_1, \omega_1)(p) - (\nabla\omega_2, \omega_2)(p)] + \beta[(\nabla\omega_1, \omega_2)(p) + (\omega_1, \nabla\omega_2)(p)].$$

Rewriting this last expression gives the following condition on α and β:

$$\alpha\, df_1|_p + \beta\, df_2|_p = 0. \tag{8.5}$$

Since $df_1|_p$ and $df_2|_p$ are by assumption linearly independent, one concludes that Coker $dh = (0)$ at $(p, 1, 0)$. Thus the point $y = ((p, 1, 0), (q, 1, 0))$ satisfies the conditiions of Proposition 6.2.

Proof of Propositioin 8.1. There are two cases which can arise. Case 1 occurs when there exists $q \in M$, where either $|\omega_1|(q) \neq |\omega_2|(q)$ or $(\omega_1, \omega_2)(q) \neq 0$. Case 2 occurs when no such q exists.

Let $\{\omega_1, \omega_2\} \subset P_- H^2_{\mathrm{DR}}$ be linearly independent and such that, at $p \in M$, $|\omega_1|(p) = |\omega_2|(p) = 1$ and $(\omega_1, \omega_2)(p) = 0$. With no loss of generality, one can take p so that there exists $p' \neq p$, where $|\omega_1|(p') = |\omega_2|(p') \neq 0$ and $(\omega_1, \omega_2)(p') = 0$.

Proof for Case 1. Let $q \in M$ be the point described above. Choose a frame in F_- at p, p' and q so that a point $y \in F(3)$ is given by

$$y = ((q, 1, 0), (p, \Lambda_1, t_1), (p', \Lambda_2, t_2)), \tag{8.6}$$

where each $(\Lambda_i, t_i) \in \mathrm{SO}(3) \times \mathbf{R}$. For fixed $\{q, p, p'\}$ the map h of (7.1) when restricted to points y as in (8.6) and after a possible redefinition of $(\Lambda_i, t_i)_{i=1}^2$, has the form $h(y) = h_1((\Lambda_i, t_i)_{i=1}^2) + (\vec{\omega}_1(q), \vec{\omega}_2(q))$, where

$$h_1\big((\Lambda_i, t_i)_{i=1}^2\big) = e^{-t_1}\Lambda_1\begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\end{pmatrix} + e^{-t_2}\Lambda_2\begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\end{pmatrix}. \tag{8.7}$$

Here, $\vec{\sigma}_{1,2} = \vec{\omega}_{1,2}(p_1)$.

For the present purposes, the key fact about the map h of (8.7) is provided by the next lemma.

Lemma 8.5. *Let* h_1: $\times_2(\mathrm{SO}(3) \times \mathbf{R}) \to \times_2 \mathbf{R}^3$ *be the map in* (8.7). *Let* $\mathscr{I} = \{v = (\vec{v}_1, \vec{v}_2) \in \times_2 \mathbf{R}^3$: *Either* $|\vec{v}_1| \neq |\vec{v}_2|$ *or* $\vec{v}_1 \cdot \vec{v}_2 \neq 0\}$. *For each* $v \in \mathscr{I}$, *there exists* $\zeta \in \times_2(\mathrm{SO}(3) \times \mathbf{R})$ *satisfying* $h_1(\zeta) = v$ *and* dh_1 *at* ζ *is surjective.*

The lemma provides a point $y \in F(3)$ which satisfies the conditions of Proposition 6.2. Thus, proving Lemma 8.5 completes the proof for Case 1 of Proposition 8.1.

Proof of Lemma 8.5. Let $v = (\vec{v}^1, \vec{v}^2) \in \mathscr{I}$. There is no loss of generality by assuming that $\vec{v}_1$ and $\vec{v}_2$ are linear combinations of $(\vec{\sigma}_1, \vec{\sigma}_2)$. Write $\vec{v}_i = \beta_{ik}\vec{\sigma}_k$ for $i \in (1, 2)$, where $\beta = (\beta_{ik})$ is a 2×2 matrix. The transformation $\beta \to S\beta$ where $S \in \mathrm{O}(2)$ can be compensated by a redefinition of Λ_1 and Λ_2 in (9.10). Thus, there is no loss of generality in assuming that β is positive semidefinite. Let $\Lambda_1 = 1$. The matrix Λ_2 will be a rotation in the $(\vec{\sigma}_1, \vec{\sigma}_2)$ plane and it can be represented by

$$\Lambda_2\begin{pmatrix}\vec{\sigma}^1\\ \vec{\sigma}^2\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\vec{\sigma}^1\\ \vec{\sigma}^2\end{pmatrix} \tag{8.8}$$

with $\theta \in [0, 2\pi]$. The values

$$t_1 = -\ln\left(\tfrac{1}{2}\operatorname{trace}\beta\right),$$
$$t_2 = -\tfrac{1}{2}\ln\left(\tfrac{1}{4}(\operatorname{trace}\beta)^2 - \det\beta\right),$$
$$\Theta = \tan^{-1}[2\beta_{12}/(\beta_{11} - \beta_{22})]$$

yield $h_1((1, t_1), (\Lambda_2, t_2)) = (\vec{v}_1, \vec{v}_2)$. Notice that because $v \in \mathscr{J}$, both t_1 and t_2 are finite. The proof that dh_1 is surjective at $\zeta = ((1, t_1), (\Lambda_2, t_2))$ is left as an exercise.

Proof of Case 2. In this case, $(\omega_1, \omega_2) \equiv 0$ and $|\omega_1|^2 - |\omega_2|^2 \equiv 0$. Choose distinct points $\{p_i\}_{i=1}^4$, where $\omega_1 \neq 0$. By choosing a frame at each p_i, one defines a point $y \in F(4)$ by $y = ((p_i, \Lambda_i, t_i)_{i=1}^4)$, with each $(\Lambda_i, t_i) \in \mathrm{SO}(3) \times \mathbf{R}$. By restricting the map h of (7.1) to y of the above form, one obtains, after possible redefinitions of $(\Lambda_i, t_i)_{i=1}^4$, that $h(y) = h_4((\Lambda_i, t_i)_{i=1}^4)$ with

$$h_4\left((\Lambda_i, t_i)_{i=1}^4\right) = \sum_{i=1}^4 e^{-t_i}\Lambda_i\begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\end{pmatrix} + \sum_{i=3}^4 e^{-t_i}\Lambda_i\begin{pmatrix}\vec{\sigma}_1\\ -\vec{\sigma}_2\end{pmatrix}.$$

Restrict h_4 further by setting $t_2 = t_4 = 1$ and $\Lambda_2 = \Lambda_4 = 1$. Then Lemma 8.5 is applicable to this restricted map, and the lemma provides $y \in F(4)$ which satisfies the conditions of Proposition 6.2.

9. The case $b_- = 3$

The case when $b_- = 3$ is similar in many respects to the case $b_- = 1$. The situation is described by the following proposition.

Proposition 9.1. *Let* $b_- = 3$. *There exist smooth, irreducible self-dual connections on principapl* SU(2) *bundles* $P \to M$ *if* $-c_2(p) \geqslant 4$.

In the proof of Proposition 9.1, there are two cases which will be considered separately. The first case is the generic one, when there exists $p \in M$, where $P_-H^2_{\mathrm{DR}}$ spans $P_-\wedge_2 T^*|_p$. This is Case 1. Case 2 is when there is no $p \in M$, where $P_-H^2_{\mathrm{DR}}$ spans $P_-\wedge_2 T^*|_p$. Both cases will be done in detail as they are necessary for analyzing the situation when $b_- > 3$. As always, the strategy is to verify that under the assumptions above, the conditions of Proposition 6.2 are satisfied.

Proof of Proposition 9.1 *for Case* 1. When Case 1 is true, choose $p_1 \in M$, where $P_-H^2_{\mathrm{DR}}$ spans $P_-\wedge_2 T^*$ at p_1. This is true for all x in a neighborhood $U \ni p_1$. Let $\{\omega_i\}_{i=1}^3$ be a linearly independent basis for $P_-H^2_{\mathrm{DR}}$ which satisfies $(\omega_i(p_1), \omega_j(p_1)) = \delta_{ij}$ for $i, j \in (1,2,3)$. Next choose a point $p_0 \in U$, distinct from p_1. Then choose points $p_2, p_3 \neq p_0$ which are a distance $\delta > 0$ from each other and from p_1. The distance δ will be determined shortly; essentially the requirement that $\delta\|\nabla\omega_i\|_\infty \ll 1$ determines this number. Choose frames in F_- at each $\{p_i\}_{i=1}^4$. A point $y \in F(4)$ is given by $y = ((p_0, 1, 0), (p_i, \Lambda_i, t_i)_{i=1}^3)$ with each $\Lambda_i \in \mathrm{SO}(3)$ and $t_i \in \mathbf{R}$.

For fixed $\{p_i\}_{i=0}^4$ the map h of (7.1) when restricted to points y of the above form is

$$h(y) = \sum_{i=1}^{3} e^{-t_i}\Lambda_i \begin{pmatrix} \vec{\omega}_1(p_i) \\ \vec{\omega}_2(p_i) \\ \vec{\omega}_3(p_i) \end{pmatrix} + \begin{pmatrix} \vec{\omega}_1(p_0) \\ \vec{\omega}_2(p_0) \\ \vec{\omega}_3(p_0) \end{pmatrix}. \tag{9.1}$$

Let $\{\vec{\sigma}_i = \vec{\omega}_i(p_1)\}_{i=1}^3$. For δ small, the map h of (9.1) has the following decomposition.

$$h = h_1 + R + \begin{pmatrix} \vec{\omega}_1(p_0) \\ \vec{\omega}_2(p_0) \\ \vec{\omega}_3(p_0) \end{pmatrix}, \tag{9.2}$$

where $h_1 = h_1((\Lambda_i, t_i)_{i=1}^3)$ is given by

$$h_1\big((\Lambda_i, t_i)_{i=1}^3\big) = \sum_{i=1}^{3} e^{-t_i}\Lambda_i \begin{pmatrix} \vec{\sigma}_1 \\ \vec{\sigma}_2 \\ \vec{\sigma}_3 \end{pmatrix}, \tag{9.3}$$

and $R = R((\Lambda_i, t_i)_{i=1}^3)$ is the remainder.

This remainder satisfies

$$|R| + |dR| \leqslant c \cdot \delta \sum_{i=1}^{3} e^{-t_i}. \tag{9.4}$$

Here, c is a constant which depends only on the metric for T_M. The R term will be treated as a perturbation to the map h_1 of (9.3).

Case 1 of Proposition 9.1 is proved by analyzing the map h_1. The relevant question is this: For which $v = (\vec{v}_1, \vec{v}_2, \vec{v}_3) \in \times_3 \mathbf{R}^3$ does there exist $\zeta \in \times_3(\mathrm{SO}(3) \times \mathbf{R})$ with $h_1(\zeta) = v$ and $dh_1|_\zeta$ is surjective? To answer this question, write $\vec{v}_i = \beta_{ik}\vec{\sigma}_k$, where β_{ik} is a 3×3 matrix. The matrix β has a decomposition $\beta = T \cdot U$, where T is a symmetric, negative semidefinite matrix, and $U \in \mathrm{O}(3)$. Whether or not $v \in \times_3 \mathbf{R}^3$ is the image of a noncritical point of h_1 depends on the form of T and U. The dependence is given in the next lemma.

Lemma 9.2. *Define a set* $\mathscr{F} = \{v = (\vec{v}_1, \vec{v}_2, \vec{v}_3) \in \times_3 \mathbf{R}^3$: (1) *The span of* $\{\vec{v}^i\}$ *has dimension* 2 *or more.* (2) *In the representation* $v = T \cdot U\sigma$, *as described above, either* $U \in \mathrm{SO}(3)$, *or the least negative eigenvalue of* T *is simple*$\}$. *Each element of* $\mathscr{F}$ *is the image by* h_1 *of a point in* $\times_3(\mathrm{SO}(3) \times \mathbf{R})$, *where* dh_1 *is surjective.*

Assume for the moment that Lemma 9.2 is true. The implications of this assumption for the map h of (9.1) are summarized next:

Lemma 9.3. *For fixed $p_0 \in U$, there exists $\delta > 0$ such that when $\{p_i\}_{i=1}^3$ are a distance δ apart, the following is true: There exists a point $((\Lambda_i, t_i)_{i=1}^3 \in \times_3(\mathrm{SO}(3) \times \mathbf{R}))$ such that $y = ((p_0, 1, 0), (p_i, \Lambda_i, t_i)_{i=1}^3)$ satisfies $h(y) = 0$ and dh at y is surjective onto $\times_3 \mathbf{R}^3$, where h is the map of* (9.1).

Proof of Lemma 9.3. Because $p_0 \in U$, the element $\omega \equiv (\vec{\omega}_i(p_0))_{i=1}^3 \in \mathscr{F}$. By Lemma 9.2, there exists $\zeta' = (\Lambda_i', t_i')_{i=1}^3 \in \times_3(\mathrm{SO}(3) \times \mathbf{R})$ with $h_1(\zeta') = \omega$, and where dh_1 is surjective. For δ sufficiently small the remainder, R, of (9.3) and (9.4) is a perturbation to h_1 and Proposition 6.2 provides $(\Lambda_i, t_i)_{i=1}^3$ which satisfy the requirements of the lemma.

The proof of Case 1 is completed by proving Lemma 9.2.

Proof of Lemma 9.2. Write $v = \beta\sigma$ (that is, $\vec{v}^i = \beta^i_k \vec{\sigma}_k$), where β is a 3×3 matrix. Let $\beta = T \cdot U$ be the decomposition of β into (symmetric, negative semidefinite) $\times$ (orthogonal). The matrix T has the representation $T = VDV^T$, where $V \in \mathrm{SO}(3)$ and D is a diagonal matrix, $D^i_j = d_{(i)}\delta^i_j$. Two possibllities arise. The first is when $U \in \mathrm{SO}(3)$. Then the matrix D has the decomposition $D = \sum_{k=1}^3 r_k L_k$, where $r_k = -\frac{1}{2}(\operatorname{trace} D - d_{(k)})$ and $L_k \in \mathrm{SO}(3)$ are the matrices $L_1 = \operatorname{diag}(1, -1, -1)$; $L_2 = \operatorname{diag}(-1, 1, -1)$; $L_3 = \operatorname{diag}(-1, -1, 1)$. One concludes that β can be written as

$$\beta = \sum_{k=1}^{3} r_k V L_k V^T U. \tag{9.5}$$

Observe that each $S_k = VL_kV^TU \in \mathrm{SO}(3)$. Since the basis $\{\vec{\sigma}_i\}_{i=1}^3$ of $\mathbf{R}^3$ is orthonormal, there exists a unique $\Lambda_k \in \mathrm{SO}(3)$ such that

$$\Lambda_k \vec{\sigma}_i = (S_k)^i_j \vec{\sigma}_j. \tag{9.6}$$

Since T has at least two nonzero eigenvalues, each $r_k > 0$. Thus $(\Lambda_k, t_k = -\ln r_k)_{k=1}^3 \in \times_3(\mathrm{SO}(3) \times \mathbf{R})$ is mapped onto v by h_1.

The second possibility which occurs is when $\det U = -1$, and the largest eigenvalue of T is simple. Assume that this eigenvalue is $d_{(3)}(|d_{(3)}| < |d_{(1)}|, |d_{(2)}|)$.

Write $D = \hat{D}\hat{U}$, where $\hat{U} = \operatorname{diag}(1, 1, -1)$. The matrix $\hat{D}$ has a decomposition $\hat{D} = \sum_{k=1}^3 \rho_k L_k$, where $\rho_1 = -(d_{(2)} - d_{(3)})$, $\rho_2 = -(d_{(1)} - d_{(3)})$ and $\rho_3 = -(d_{(1)} + d_{(2)})$. Each $\rho_k > 0$, and β has the decomposition

$$\beta = \sum_{k=1}^{3} \rho_k V L_k \hat{U} V^T U.$$

Note that each $\hat{S}_k = VL_k\hat{U}V^TU \in \mathrm{SO}(3)$. Let $\Lambda_k \in \mathrm{SO}(3)$ be such that $\Lambda_k\vec{\sigma}_i = (\hat{S}_k)^i_j\vec{\sigma}_j$. Thus $(\Lambda_k, t_k = -\ln \rho_k)_{k=1}^3 \in \times_3(\mathrm{SO}(3) \times \mathbf{R})$ is mapped onto v by h_1.

It is an exercise that is left to the reader to show that for both cases, dh_1 is surjective at $(\Lambda_k, t_k)_{k=1}^3$.

Proof of Proposition 9.1 *for Case* 2. Choose a point $p_1 \in M$, where $P_-H^2_{\mathrm{DR}}$ spans a 2-dimensional subspace of $P_-\wedge_2 T^*$. Up to a scale, there exists a unique $\omega_3 \in P_-H^2_{\mathrm{DR}}$ which satisfies $\omega_3(p_1) = 0$. Choose $p_0 \in M$, where $P_-H^2_{\mathrm{DR}}$ spans a 2-dimensional subspace of $P_-\wedge_2 T^*$ and where $\omega_3(p_0) \neq 0$. The 2-form ω_3 is specified completely by requiring that $|\omega_3|(p_0) = 1$. The remaining elements $\{\omega_1, \omega_2\}$ of a basis for $P_-H^2_{\mathrm{DR}}$ are specified by the requirements (1) $\{\omega_1(p_1), \omega_2(p_1)\}$ are orthonormal and (2) $\omega_1(p_0)$ and $\omega_2(p_0)$ are orthogonal to $\omega_3(p_0)$.

Select $p_2 \in M$ a distance δ from p_1 and select $p_3 \in M$ a distance δ from p_0. The number $\delta > 0$ will be specified shortly; essentially by the requirement that $\delta\|\nabla\omega_i\|_\infty \ll 1$. To order δ, $\omega_{1,2}(p_2) = \omega_{1,2}(p_1)$ and $\omega_{1,2}(p_3) = \omega_{1,2}(p_0)$.

Choose a frame in F_- at each p_i so that a point $y \in F(4)$ is given by

$$y = \big((p_0, 1, 0), (p_i, \Lambda_i, t_i)_{i=1}^3\big), \tag{9.7}$$

where each $(\Lambda_i, t_i) \in \mathrm{SO}(3) \times \mathbf{R}$. For fixed $\{p_i\}_{i=0}^3$, the map h of (7.1) when restricted to points y as in (9.7) has the form $h = h' + R$, where $h' = h'((\Lambda_i, t_i)_{i=1}^3)$ and $R = R((\Lambda_i, t_i)_{i=1}^3)$. Here, h', after a possible redefinition of the Λ_i's, is given by

$$\begin{aligned} h'\big((\Lambda_i, t_i)_{i=1}^3\big) &= e^{-t_1}\Lambda_1\begin{pmatrix}\vec\sigma_1\\ \vec\sigma_2\\ 0\end{pmatrix} + e^{-t_2}\Lambda_2\begin{pmatrix}\vec\sigma_1\\ -\vec\sigma_2\\ 0\end{pmatrix}\\ &\quad + e^{-t_3}\Lambda_3\begin{pmatrix}\alpha_1\vec\sigma_1\\ \alpha_2\vec\sigma_1\\ -\vec\sigma_2\end{pmatrix} + \begin{pmatrix}\alpha_1\vec\sigma_1\\ \alpha_2\vec\sigma_1\\ \vec\sigma_2\end{pmatrix}, \end{aligned} \tag{9.8}$$

where $\vec\sigma_{1,2} = \vec\omega_{1,2}(p_1)$ and at least one of (α_1, α_2) is nonzero. The remainder, R, satisfies

$$|R| + |dR| \leqslant c \cdot \delta \sum_{i=1}^{3} e^{-t_i}, \tag{9.9}$$

where c is a constant which depends only on the metric on T_M. This R will be treated as a perturbation to the map h' of (9.8).

The map h' is upper triangular in the sense that when $(\Lambda_3 = 1, t_3 = 0)$, h' defines a map $h_1\colon \times_2(\mathrm{SO}(3) \times \mathbf{R}) \to \times_2 \mathbf{R}^2$ given by the relation

$$h'\big((\Lambda_i, t_i)_{i=1}^2, (1, 0)\big) = \begin{pmatrix} h_1\big((\Lambda_i, t_i)_{i=1}^2\big)\\ 0\end{pmatrix} + 2\begin{pmatrix}\alpha_1\vec\sigma_1\\ \alpha_2\vec\sigma_1\\ 0\end{pmatrix}.$$

Thus,

$$h_1\left((\Lambda_i, t_i)_{i=1}^2\right) = e^{-t_1}\Lambda_1\begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\end{pmatrix} + e^{-t_2}\Lambda_2\begin{pmatrix}\vec{\sigma}^1\\ -\vec{\sigma}^2\end{pmatrix}. \tag{9.10}$$

This is the same map h_1 that appears in (8.7). Hence, Lemma 8.5 is applicable. Let $\mathcal{J}$ be the set defined therein. Observe that $2(\alpha_1\vec{\sigma}_1, \alpha_2\vec{\sigma}_1) \in \mathcal{J}$ if at least one of (α_1, α_2) is nonzero. Hence Lemma 8.5 provides $(\Lambda_i, t_i)_{i=1}^2 \in \times_2(\mathrm{SO}(3) \times \mathbf{R})$ such that $h'((\Lambda_i, t_i)_{i=1}^2, (1,0)) = 0$ and (cf. the proof of Proposition 7.1) dh' at $((\Lambda_i, t_i)_{i=1}^2, (1,0))$ is surjective. For δ sufficiently small, Proposition 6.1 is applicable to the decomposition $h = h' + R$. This follows from (9.9). Therefore, there is a point $y \in F(4)$ of the form given in (9.7) which satisfies the conditions of Proposition 6.2.

10. When $b_- > 3$

For a given value of $b_- > 3$, it is straightforward to show that the conditions of Proposition 6.2 are satisfied when l is sufficiently large. One has

Proposition 10.1. *Let $b_- > 3$. There exists a smooth irreducible self-dual connection on a principal* SU(2) *bundle $P \to M$ if $-c_2(P) \geqslant 4/3b_-$.*

Proposition 10.1 does not necessarily give the minimum value of $-c_2(P)$ for which $P \to M$ can admit a self-dual connection. One obtains a much smaller value of $-c_2(P)$ as being sufficient if a generic assumption about the metric on T_M is made. A conjecture is that this generic condition is actually not necessary. A suspicion is that an argument of a more topological bent will eliminate it. (Perhaps an argument along the lines of the proof of Theorem 1.3 will succeed.)

Proposition 10.2. *Let $b_- > 3$ and let $r \geqslant 2$. For an open, dense set of metrics on T_M in the C^r-topology, there exists a smooth, irreducible self-dual connection on a principal* SU(2) *bundle $P \to M$ if $-c_2(P) \geqslant b_-$.*

This proposition is similar in its proof to that of the following result for the cases $b_- \leqslant 3$. S. K. Donaldson [8] suggested the proposition below to the author.

Proposition 10.3. *Suppose that there exists $\omega \in P_-H^2_{\mathrm{DR}}$ and $q \in M$ such that $\omega(q) = 0$ and $\nabla\omega$: $T_M|_q \to P_-\wedge_2 T^*|_q$ has maximal rank. Let $r \geqslant 2$. When $b_- = 1$, or for an open, dense set of metrics on T_M in the C^r-topology when $b_- = 3$, there exists a smooth, irreducible self-dual connection on a principal G-bundle $P \to M$ when $-c_2(P) \geqslant b_-$.*

The reader should compare Theorem 1.3 (and its proof) with Proposition 10.3 when $b_- = 1$.

The proofs of Propositions 10.2 and 10.3 are tedious and at the same time unenlightening. For this reason, only the $b_- = 1$ case of Proposition 10.3 will be proved in detail. The proofs of Propositions 10.2 and the remaining case in Proposition 10.3 will only be sketched and the reader is referred to [21]. Proposition 10.1 is proved in the next section.

Proof of Proposition 10.3 *when* $b_- = 1$. Use equations (7.1), (7.2) and Proposition 6.2.

Proof of Propositions 10.2 *and* 10.3, *outline*. In the proof, the three cases $b_- = 0, 1, 2 \bmod(3)$ are considered. In each of these cases, a specific basis of linearly independent forms in $P_- H^2_{\mathrm{DR}}$ will be chosen. Certain properties of this basis are required and for a C^r generic metric ($r \geqslant 2$), a basis with these properties can be found. Write $b_- = 3n + m$ with $m = (0, 1, 2)$ and $n \geqslant 1$. The first such requirement is that there exist a linearly independent basis of $P_- H^2_{\mathrm{DR}}$, indexed as $\{\omega_j, \omega_{\alpha i}\}$ for $j \in (1, \cdots, m)$, $\alpha \in (1, \cdots, n)$ and $i \in (1, 2, 3)$ with the properties:

(1) There are distinct points $\{p_\alpha\}_{\alpha=1}^n \subset M$ at which $\{\omega_{\alpha i}(p_\alpha)\}_{i=1}^3$ is an orthonormal basis for $P_- \wedge_2 T^*$ at p_α.

(2) For $\beta \neq \alpha$, $\{\omega_{\alpha i}(p_\beta)\}_{i=1}^3 = 0$, and for all α, $\{\omega_j(p_\alpha)\}_{j=1}^m = 0$.

(3) There is a point $q \notin \{p_\alpha\}$ in M at which $\omega_1(q) = 0$, and the linear map $\nabla\omega_1 : T_M|_q \to P_- \wedge_2 T^*|_q$ has rank 3.

When $b_- = 3$, denote the 2-form ω of Proposition 10.3 by ω_1 and label the remaining elements of an L_2-basis for $P_- H^2_{\mathrm{DR}}$ by $\{\omega_j\}_{j=2}^3$.

For $b_- > 3$, each of the cases $b_- = 0, 1, 2 \bmod(3)$ requires that one fix $\delta > 0$ so that

$$\delta \ll \min\Big\{ \min_{\alpha \neq \beta} \big(\mathrm{dist}(p_\alpha, p_\beta)\big),\ \min_{\alpha} \big(\mathrm{dist}(q, p_\alpha)\big) \Big\}.$$

The number δ will be determined by considerations similar to those that occurred in the proof of Proposition 9.1. For each $\alpha \in (1, \cdots, n)$ choose points $\{p_{\alpha 1}, p_{\alpha 2}\}$ to be a distance δ from each other and from $p_\alpha = p_{\alpha 3}$. Choose an orthonormal frame $\{x^a\}_{a=1}^3$ for $P_- \wedge_2 T_M|_q$ and by choosing paths, parallel transport $\{x^a\}$ to each $p_{\alpha i}$.

In order to prove Propositions 10.2 and 10.3 simultaneously, the convention is made that when $b_- = 3$, the set $\{p_{\alpha i}\}$ is empty, and remarks concerning them should be ignored.

The strategy for the proof of Propositions 10.2 and 10.3 is to appeal to Proposition 6.2 to obtain the existence under the relevant assumptions.

Let $\{q_\nu\}_{\nu=1}^4$ be Gaussian coordinates on a ball B, centered at $q = \{q_\nu = 0\}$. Since $\nabla\omega_1$ at q has maximal rank, ω_1 vanishes along a curve through q, and one can choose distinct points $\{q_i\}_{i=1}^{2} \notin \{q, p_{\alpha i}\}_{\alpha=1,i=1}^{n\ \ 3}$ on this curve, where $\omega_1(q_i) = 0$, for $i \in \{1,2\}$. Require that $\operatorname{dist}(q_2, q_3) < \delta$.

A point $y \in F(b_-)$ is defined by

$$y = \Big((q_\nu, 1, 0), (q_j, \Lambda_j, t_j)_{j=1-m}^{2-m}, (p_{\alpha i}, \Lambda_{\alpha i}, t_{\alpha i})_{\alpha=1,i=1}^{n\ \ 3}\Big),$$

with each (Λ_j, t_j), $(\Lambda_{\alpha i}, t_{\alpha i}) \in \mathrm{SO}(3) \times \mathbf{R}$. The convention here is to ignore index subscripts in $\{0, -1, -2, \cdots\}$.

The map h of (7.1), when restricted to y of the above form, defines a map

$$h\colon B \times (\mathrm{SO}(3) \times \mathbf{R})^{2-m} \times \Big(\bigtimes_{3} (\mathrm{SO}(3) \times \mathbf{R})\Big)^n \to \mathbf{R}^3 \times (\mathbf{R}^3)^{2-m} \times \Big(\bigtimes_{3} \mathbf{R}^3\Big)^n.$$

This map has the form $h = \tilde{h} + R$, where $\tilde{h} = (h_0'(q_\nu), h_0, (h_\alpha)_{\alpha=1}^n)$. Here

$$h_\alpha = \sum_{i=1}^{3} e^{-t_{\alpha i}} \Lambda_{\alpha i} \begin{pmatrix} \vec{\omega}_{\alpha 1}(p_\alpha) \\ \vec{\omega}_{\alpha 2}(p_\alpha) \\ \vec{\omega}_{\alpha 3}(p_\alpha) \end{pmatrix} + \begin{pmatrix} \vec{\omega}_{\alpha 1}(0) \\ \vec{\omega}_{\alpha 2}(0) \\ \vec{\omega}_{\alpha 3}(0) \end{pmatrix} + \sum_{j=1-m}^{2-m} e^{-t_j} \Lambda_j \begin{pmatrix} \vec{\omega}_{\alpha 1}(q_j) \\ \vec{\omega}_{\alpha 2}(q_j) \\ \vec{\omega}_{\alpha 3}(q_j) \end{pmatrix},$$

$$h_0 = \begin{pmatrix} \vec{\omega}_2(0) \\ \vec{\omega}_3(0) \end{pmatrix} + \sum_{j=1-m}^{2-m} e^{-t_j} \Lambda_j \begin{pmatrix} \vec{\omega}_2(q_j) \\ \vec{\omega}_3(q_j) \end{pmatrix},$$

and $h_0' = q_\nu(\nabla_\nu \vec{\omega}_1)(0)$.

The remainder, $R = (R_0', R_0, (R_\alpha)_{\alpha=1}^n)$, has $|R_\alpha|$, $|R_0|$ small ($\mathscr{O}(\delta)$) with respect to $|h_\alpha|$, $|h_0|$ and $|R_0'|$ is $\mathscr{O}(\sum_\nu q_\nu q_\nu)$. The remainder R is treated with Proposition 6.2.

The main term $\tilde{h}$ is "upper triangular", and the conditions of Proposition 6.2 are verified for $\tilde{h}$ by treating the blocks $(h_0', h_0, (h_\alpha)_{\alpha=1}^n)$ independently with Lemmas 9.2 and 8.5. But here, further generic assumptions are required; they are listed below: Let $\mathscr{F}$, $\mathscr{J}$ be as in Lemmas 9.2 and 8.5 respectively. Then (1) $\{\omega_j\}_{j=1-m}^{2-m}$ spans a 2-plane in $P_- \wedge_2 T^*$ at q; (2) the pair $(\vec{\omega}_2(0), \vec{\omega}_3(0)) \in \mathscr{J}$; (3) there exists $(t_j, \Lambda_j)_{j=1-m}^{2-m} \in \bigtimes_m (\mathrm{SO}(3) \times \mathbf{R})$ with $h_0(t_j, \Lambda_j) = 0$, $dh_0(t_j, \Lambda_j)$ is surjective, and (4) for each α, $(\vec{\omega}_{\alpha i}(0) + \sum_{j=1-m}^{2-m} e^{-t_j} \Lambda_j \vec{\omega}_{\alpha i}(q_j))_{i=1}^3$ is in $\mathscr{F}$.

11. When $b_- > 3$, continued

The purpose of this section is to prove Proposition 10.1. The proof is obtained by demonstrating that the conditions of Proposition 6.2 are satisfied for $P \to M$ when $-c_2(P) \geqslant \frac{4}{3} b_-$ if $b_- > 3$.

To begin, write $b_- = 3n + m$, with either $n \geqslant 2$ and $m \in (0,1,2)$ or $n = 1$ and $m \in (1,2)$. The proof requires a basis of $P_- H^2_{\mathrm{DR}}$ with some specific qualities which are spelled out in the first lemma:

Lemma 11.1. *There exists an integer $l \in (0, \cdots, n)$ and distinct points $\{p_\alpha\}_{\alpha=1}^{l}$ with the following properties*:

(1) *The subspace $Z_l = \{\omega \in P_- H^2_{\mathrm{DR}}: \omega(p_\alpha) = 0\}$ spans no more than a two-plane in $P_- \wedge_2 T^*$ at any point $p \in M$.*

(2) *There is a basis $\{\omega_{\alpha i}\}_{\alpha=1,i=1}^{l\ \ 3}$ of $P_- H^2_{\mathrm{DR}}/Z_l$ which is uniquely defined up to elements in Z_l by the requirements that for each α, $\{\omega_{\alpha i}(p_\alpha)\}_{i=1}^{3}$ is orthonormal and for each $\beta \neq \alpha$, $\{\omega_{\alpha i}(p_\beta) = 0\}_{i=1}^{3}$.*

(3) *Let l' be the greatest integer which is less than or equal to $\frac{1}{2}(b_- - 3l)$. There exist distinct points $\{q_\beta\}_{\beta=1}^{l'}$ and a linearly independent set $\{\omega_{\beta A}\}_{\beta=1,A=1}^{l'\ \ 2}$ $\subset Z_l$ with the property that for each β, $\{\omega_{\beta A}(q_\beta)\}_{A=1}^{2}$ is orthonormal and for each $\beta \neq \alpha$, $\{\omega_{\alpha A}(q_\beta) = 0\}_{A=1}^{2}$.*

Proof of Lemma 11.1. Use the Graham-Schmidt procedure and Lemma 8.3 to choose this basis.

For each $\alpha \in (1, \cdots, l)$, choose points $\{p_{\alpha i}\}_{i=1}^{3}$ to be a distance $\delta > 0$ from each other, and from $p_{\alpha 0} = p_\alpha$. For each $\beta \in (1, \cdots, l')$ choose a point $q_{\beta 1}$ to be a distance δ from $q_{\beta 0} \equiv q_\beta$. The number δ will be specified shortly; for now require that δ is much less than the minimum distance separating the points in $\{p_\alpha, q_\beta\}_{\alpha=1,\beta=1}^{l\ \ l'}$. Require also that

$$\delta \ll \left[\sup_{P_- H^2_{\mathrm{DR}}} \|\nabla \omega\|_\infty / \|\omega\|_\infty \right].$$

There are three cases which arise and are considered separately. Case 1 is when $b_- - 3l$ is odd; Case 2 is when $b_- - 3l$ is even, but not 2; and Case 3 is when $b_- - 3l = 2$.

Proof for Case 1. When $b_- - 3l$ is odd, there exists a 2-form $\omega \in P_- H^2_{\mathrm{DR}}$ which vanishes at each p_α and q_β. This form ω is unique up to scale. A consequence of Lemma 8.3 is that if $Z_l \neq \mathrm{Span}\{\omega\}$, there exists $q \in M$ where $\{\omega(q), \omega_{\beta A}(q)\}$ spans a 2-plane in $P_- \wedge_2 T^*|_q$ for each $\beta \in (1, \cdots, l')$ and $A \in (1,2)$. If $Z_l = \mathrm{Span}\{\omega\}$ choose $q \in M$ where $\omega(q) \neq 0$. Let $q' \notin \{p_\alpha, q_\beta\}$ be a distance δ from q.

By choosing frames for F_- at each $p_{\alpha i}$, $q_{\beta A}$, q, q', one can write $y \in F(4l + 2(l' + 1))$ as

$$\begin{aligned} y = \Big(&\big((p_{\alpha 0}, 1, 0), (p_{\alpha i}, \Lambda_{\alpha i}, t_{\alpha i})_{i=1}^{3}\big)_{\alpha=1}^{l}; \\ &\big(q_{\beta A}, \Lambda_{\beta A}, r + t_{\beta A}\big)_{\beta=1,A=1}^{l'\ \ 2}; (q, \Lambda_1, r + t_1); (q', 1, r)\Big), \end{aligned} \tag{11.1}$$

where the (Λ, t)'s are in $SO(3) \times \mathbf{R}$ and $r \gg 1$. By redefining the (Λ, t)'s if necessary one discovers that the map h of (7.1), when restricted to y as in (11.1), has the form $h = h' + R$, where $h', R\colon \times_{3l+2l'+1}(SO(3) \times \mathbf{R}) \to \mathbf{R}^{3b_-}$ are functions of the (Λ, t)'s. The remainder, R, satisfies

$$|R| + |dR| \leqslant c \cdot \delta \cdot \left(\sum_{\alpha i} e^{-t_{\alpha i}} + e^{-r}\left(\sum_{\beta A} e^{-t_{\beta A}} + e^{-t_1} + 1 \right) \right). \tag{11.2}$$

Here, c is a constant which depends only on the metric on T_M. As in the previous sections, R is treated as a perturbation to h'.

The map

$$h' = \left((h_\alpha)_{\alpha=1}^{l}, (\bar{h}_\beta)_{\beta=1}^{l'}, \hat{h} \right) \in \times_{l} \left(\times_{3} \mathbf{R}^3 \right) \times_{l'} \left(\times_{2} \mathbf{R}^3 \right) \times \mathbf{R}$$

has the upper triangular form

$$(1) \quad h_\alpha = \sum_{i=1}^{3} e^{-t_{\alpha i}} \Lambda_{\alpha i} \begin{pmatrix} \vec{\sigma}_1 \\ \vec{\sigma}_2 \\ \vec{\sigma}_3 \end{pmatrix} + \begin{pmatrix} \vec{\sigma}_1 \\ \vec{\sigma}_2 \\ \vec{\sigma}_3 \end{pmatrix}$$

$$+ e^{-r} \left[\sum_{\beta=1}^{l'} \sum_{A=1}^{2} e^{-t_{\beta A}} \Lambda_{\beta A} \begin{pmatrix} \vec{\omega}_{\alpha 1} \\ \vec{\omega}_{\alpha 2} \\ \vec{\omega}_{\alpha 3} \end{pmatrix} (q_\beta) + \left(e^{-t_1}\Lambda_1 + 1 \right) \begin{pmatrix} \vec{\omega}_{\alpha 1} \\ \vec{\omega}_{\alpha 2} \\ \vec{\omega}_{\alpha 3} \end{pmatrix} (q) \right], \tag{11.3}$$

$$(2) \quad \bar{h}_\beta = e^{-r} \left[e^{-t_{\beta 1}} \Lambda_{\beta 1} \begin{pmatrix} \vec{\sigma}_1 \\ \vec{\sigma}_2 \end{pmatrix} + e^{-t_{\beta 2}} \Lambda_{\beta 2} \begin{pmatrix} \vec{\sigma}_1 \\ -\vec{\sigma}_2 \end{pmatrix} + \left(e^{-t_1}\Lambda_1 + 1 \right) \begin{pmatrix} \beta_{\beta 1}\vec{\sigma}_1 \\ \beta_{\beta 2}\vec{\sigma}_1 \end{pmatrix} \right],$$

$$(3) \quad \hat{h} = e^{-r}\left[e^{-t_1}\Lambda_1 \vec{\sigma}_2 + \vec{\sigma}_2 \right].$$

Here, $\{\vec{\sigma}_i\}_{i=1}^3$ is an orthonormal frame for $\mathbf{R}^3$, and by assumption, at least one of $\beta_{\beta 1}, \beta_{\beta 2}$ is not zero for $\beta \in (1, \cdots, l')$.

The map h' has a zero. Indeed, set $t_1 = 0$ and specify Λ_1 by the requirements $\Lambda_1 \vec{\sigma}_2 = -\vec{\sigma}_2$ and $\Lambda_1 \vec{\sigma}_1 = \vec{\sigma}_1$. Lemma 8.5 determines each $(\Lambda_{\beta A}, t_{\beta A})$, $\beta \in (1, \cdots, l')$ and $A \in (1, 2)$. For r sufficiently large, Lemma 9.2 determines each $(\Lambda_{\alpha i}, t_{\alpha i})$, $\alpha \in (1, \cdots, l)$ and $i \in (1, 2, 3)$. For these values of the (Λ, t)'s, dh' is surjective. Thus, if $\delta > 0$ is sufficiently small, Proposition 6.1 provides $y \in F(4l + 2(l' + 1))$ which satisfies the conditions of Proposition 6.2. Because $4l + 2l' + 2 \leqslant \frac{4}{3} b_-$, Case 1 is true.

Proof for Case 2. When $b_- - 3l$ is even, but not equal to 2, then either $b_- = 3l$ or $b_- - 3l \geqslant 4$. When $b_- = 3l$, repeat the argument for Case 1 with $\{q_{\beta A}, q, q'\}_{\beta=1, A=1}^{l', 2} = \varnothing$.

Now assume that $b_- - 3l \geqslant 4$. Choose a point $q \in M$, where none of the 2-forms $\omega_{\beta A}(q)$ vanish for $\beta \in (1, \cdots, l')$ and $A \in (1, 2)$. Choose a point $q' \in M$ to be a distance δ from q and disjoint from each $p_{\alpha i}$, and $q_{\beta A}$. Let y as given in (11.1) define a point in $F(4l + 2l' + 2)$. When h of (7.1) is restricted to y as given in (11.1), it has the form $h = h' + R$, where h', R: $\times_{4l+2l'+2}(\mathrm{SO}(3) \times \mathbf{R}) \to \mathbf{R}^{3b_-}$ are again functions of the (Λ, t)'s. The remainder R satisfies (11.2) again. Now the map h' has the upper triangular form $h' = ((h_\alpha)_{\alpha=1}^{l}, (\bar{h}_\beta)_{\beta=1}^{l'}) \in \times_l(\times_3 \mathbf{R}^3) \times_{l'}(\times_2 \mathbf{R}^3)$. The h_α's are given by (1) of (11.3) while

$$(11.4)\quad \bar{h}_\beta = e^{-r}\left[e^{-t_{\beta 1}}\Lambda_{\beta 1}\begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\end{pmatrix} + e^{-t_{\beta 2}}\Lambda_{\beta 2}\begin{pmatrix}\vec{\sigma}_1\\ -\vec{\sigma}_2\end{pmatrix} + \left(e^{-t_1}\Lambda_1 + 1\right)\begin{pmatrix}\vec{\omega}_{\beta 1}(q)\\ \vec{\omega}_{\beta 2}(q)\end{pmatrix}\right],$$

for $\beta \in (1,\cdots,l')$. By assumption, for each $\beta \in (1,\cdots,l')$ and $A \in (1,2)$, $\vec{\omega}_{\beta 1}(q) \neq 0$. For generic values of $(\Lambda_1, t_1) \in \mathrm{SO}(3) \times \mathbf{R}$, Lemma 8.5 provides for each β, A the $(\Lambda_{\beta A}, t_{\beta A})$ for which $\bar{h}_\beta(\Lambda_{\beta A}, t_{\beta A}) = 0$ and $d\bar{h}_\beta$ is surjective. For r sufficiently large, Lemma 9.2 does the equivalent provisioning of the $(\Lambda_{\alpha i}, t_{\alpha i})$ for each h_α. At this point, the argument follows as in Case 1 to show that $y \in F(4l + 2l' + 2)$ exists satisfying the conditions of Proposition 6.2. Case 2 is true because if $b_- - 3l \geqslant 4$, then $4l + 2l' + 2 \leqslant \frac{4}{3}b_-$.

Proof for Case 3. In this case, $b_- = 3n + 2$. Let $\{p_{\alpha i}\}$, $\{q_{1A} = q_A\}_{A=1}^{2}$ be the points that are specified by Lemma 11.1. Let $q \in M$ be disjoint from $\{p_{\alpha i}, q_A\}$. By choosing frames for F_- at each $\{p_{\alpha i}, q_A, q\}$, a point $y \in F(4n + 3)$ is defined by

$$(11.5)\quad \begin{aligned} y = \Big(&\big((p_{\alpha 0}, 1, 0), (p_{\alpha i}, \Lambda_{\alpha i}, t_{\alpha i})_{i=1}^{3}\big)_{\alpha=1}^{n}, \\ &(q_A, \Lambda_A, t_A + r)_{A=1}^{2}, (q, \Lambda_3, t_3 + r)\Big). \end{aligned}$$

Here, the (Λ, t)'s $\in \mathrm{SO}(3) \times \mathbf{R}$ and $r \gg 1$ will be specified shortly. When h is restricted to y of the type above, it defines a function from

$$\begin{aligned} W \equiv (M \backslash \{p_{\alpha i}, q_A\}) \underset{3}{\times} (\mathrm{SO}(3) \times \mathbf{R}) \underset{l}{\times} \Big(\underset{3}{\times} (\mathrm{SO}(3) \times \mathbf{R})\Big) \\ \to \underset{2}{\times} (\mathbf{R}^3) \underset{l}{\times} \Big(\underset{3}{\times} \mathbf{R}^3\Big). \end{aligned}$$

Here, the points $\{p_{\alpha i}, q_A\}$ are considered fixed and q is variable. As before, $h = h' + R$, where $h' = ((h_\alpha)_{\alpha=1}^{n}, \hat{h})$ is block diagonal and R is the remainder. The map h_α: $\times_3(\mathrm{SO}(3) \times \mathbf{R}) \to \times_3 \mathbf{R}^3$ is

$$(11.6)\quad \begin{aligned} h_\alpha = &\sum_{i=1}^{3} e^{-t_{\alpha i}}\Lambda_{\alpha i}\begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\\ \vec{\sigma}_3\end{pmatrix} + \begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\\ \vec{\sigma}_3\end{pmatrix} \\ &+ e^{-r}\left[\sum_{A=1}^{2} e^{-t_A}\Lambda_A\begin{pmatrix}\vec{\omega}_{\alpha 1}(q_1)\\ \vec{\omega}_{\alpha 2}(q_1)\\ \vec{\omega}_{\alpha 3}(q_1)\end{pmatrix} + e^{-t_3}\Lambda_3\begin{pmatrix}\vec{\omega}_{\alpha 1}(q)\\ \vec{\omega}_{\alpha 2}(q)\\ \vec{\omega}_{\alpha 3}(q)\end{pmatrix}\right]. \end{aligned}$$

The map $\hat{h}$ is

$$(11.7)\qquad \hat{h} = e^{-r}\left[\sum_{A=1}^{2} e^{-t_A}\Lambda_A\begin{pmatrix}\vec{\sigma}^1\\ \vec{\sigma}^2\end{pmatrix} + e^{-t_3}\Lambda_3\begin{pmatrix}\vec{\omega}_1(q)\\ \vec{\omega}_2(q)\end{pmatrix}\right],$$

where $\{\omega_A \equiv \omega_{1A}\}_{A=1}^{2}$. The remainder R satisfies bounds that are similar to those in (11.2).

If there exists $q \in M$, where either $|\omega_1|(q) \neq |\omega_2(q)|$ or $(\omega_1, \omega_2)(q) \neq 0$, then Case 3 is proved by applying the arguments for the proof of Case 1 of Proposition 8.1 (cf. Lemma 8.5) and arguments that are completely analogous to those used to prove Cases 1 and 2 in this section. The details are left to the reader.

The situation where $|\omega_1| - |\omega_2| \equiv (\omega_1, \omega_2) \equiv 0$ is more subtle. Assume that this pathology occurs on M. Let $p \in M$ be distinct from $\{p_{1i}\}_{i=1}^{3}$, $\{p_{\alpha i}\}_{i=0,\alpha=2}^{3\ \ n}$, $\{q_A, q\}_{A=1}^{2}$ and consider a point $y \in F(4n+3)$ of the form given by (11.5) but with p replacing p_{10} in that equation. The points $\{p_{1i}\}_{i=1}^{3}$ are still a distance δ from p_{10} which is fixed; the point p is variable.

As before, $h = h' + R$ with $h' = (h_\alpha, \hat{h})$, where now

$$h_1 = \sum_{i=1}^{3} e^{-t_{1i}}\Lambda_{1i}\begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\\ \vec{\sigma}_3\end{pmatrix} + \begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\\ \vec{\sigma}_3\end{pmatrix} + \begin{pmatrix}\vec{\omega}_{11}(p) - \vec{\sigma}^1\\ \vec{\omega}_{12}(p) - \vec{\sigma}^2\\ \vec{\omega}_{13}(p) - \vec{\sigma}^3\end{pmatrix} + e^{-r}\begin{pmatrix}\vec{q}_{11}\\ \vec{q}_{12}\\ \vec{q}_{13}\end{pmatrix}.$$

For $\alpha > 1$,

$$h_\alpha = \sum_{i=1}^{3} e^{-t_{\alpha i}}\Lambda_{\alpha i}\begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\\ \vec{\sigma}_3\end{pmatrix} + \begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\\ \vec{\sigma}_3\end{pmatrix} + \begin{pmatrix}\vec{\omega}_{\alpha 1}(p)\\ \vec{\omega}_{\alpha 2}(p)\\ \vec{\omega}_{\alpha 3}(p)\end{pmatrix} + e^{-r}\begin{pmatrix}\vec{q}_{\alpha 1}\\ \vec{q}_{\alpha 2}\\ \vec{q}_{\alpha 3}\end{pmatrix}.$$

Here,

$$(11.8)\qquad \vec{q}_{\alpha i} = \sum_{A=1}^{2} e^{-t_A}\Lambda_A\vec{\omega}_{\alpha i}(q_1) + e^{-t_3}\Lambda_3\vec{\omega}_{\alpha i}(q), \quad \vec{\sigma}_i = \vec{\omega}_{1i}(p_{10}).$$

The map $\hat{h}$ is now

$$(11.9)\qquad \hat{h} = e^{-r}\left[\sum_{A=1} e^{-t_A}\Lambda_A\begin{pmatrix}\vec{\sigma}^1\\ \vec{\sigma}^2\end{pmatrix} + e^{-t_3}\Lambda_3\begin{pmatrix}\vec{\sigma}^1\\ \vec{\sigma}^2\end{pmatrix}\right] + \begin{pmatrix}\vec{\omega}_1(p)\\ \vec{\omega}_2(p)\end{pmatrix}.$$

Choose p sufficiently close to p_{10} so that $0 < |\omega_1|(p) \ll 1$ and $|\vec{\omega}_{\alpha i}(p) - \vec{\omega}_{\alpha i}(p_{10})| \ll 1$. Set $r = -\ln|\omega_1|(p)$. By a redefinition of $\{\Lambda_A, \Lambda_3\}_{A=1}^{2}$, $\hat{h}$ can be put in the form

$$\hat{h}\big((\Lambda_A, t_A)_{A=1}^{2}, (\Lambda_3, t_3)\big) = e^{-r}\left[\sum_{A=1}^{2} e^{-t_A}\Lambda_A\begin{pmatrix}\vec{\sigma}_1\\ \vec{\sigma}_2\end{pmatrix} + \big(e^{-t_3}\Lambda_3 + 1\big)\begin{pmatrix}\vec{\sigma}^1\\ \vec{\sigma}^2\end{pmatrix}\right].$$

Lemma 8.5 provides $\zeta = \{(\Lambda_A, t_A), (\Lambda_3, t_3)\}$ for which $\hat{h}(\zeta) = 0$ and $d\hat{h}|_\zeta$ is surjective. As $(|\vec{\omega}_{\alpha i}(p) - \vec{\omega}_{\alpha i}(p_{10})| + e^{-r}) \ll 1$, Lemma 9.2 provides the data $\{(\Lambda_{\alpha i}, t_{\alpha i})\}$, where $h_\alpha = 0$ and dh_α is surjective. At this point, the proof of Case 3 follows in the same manner as the previous cases, the details are omitted.

Appendix A. Proofs of results in §3

This appendix contains the proofs of Theorem 3.2 and Proposition 3.1. To begin, it is useful to introduce some notation. Let $\overline{\mathscr{C}}$ denote the Hilbert manifold of $L_{5,2}$ connections on P [9]. Let $\hat{\mathfrak{g}}^2_-$, $\hat{\mathfrak{g}}^1$ denote $\hat{\mathfrak{g}} \otimes P_- \wedge_2 T^*$ and $\hat{\mathfrak{g}} \otimes T^*$, respectively. Let $\|\cdot\|_{k,p}$ denote the Sobolev $L_{k,p}$ norm (derivatives through order k are in L_p).

Consider the operator $H_A = \mathscr{D}_A^\dagger \mathscr{D}_A$ on sections of $\hat{\mathfrak{g}}^2_-$ when $A \in \overline{\mathscr{C}}$. It is a straightforward exercise with the Sobolev inequalities to verify that H_A is a bounded, closed, linear operator from $L_{5,2}(\hat{\mathfrak{g}}^2_-) \to L_{3,2}(\hat{\mathfrak{g}}^2_-)$ (cf. [9]). In fact, if $A_0 \in \mathscr{C}(P)$ is fixed, then $V_A \equiv H_{A_0} - H_A$ is a compact operator, and the mapping $A \to V_A$ is a smooth map from $\overline{\mathscr{C}}$ into the Banach space of compact operators from $L_{5,2}(\hat{\mathfrak{g}}^2_-)$ to $L_{3,2}(\hat{\mathfrak{g}}^2_-)$ (cf. Chapter II.4 of [13] for definitions).

Equivalently, for $A \in \overline{\mathscr{C}}$, H_A is a closed operator on $L_{3,2}(\hat{\mathfrak{g}}^2_-)$ or $L_2(\hat{\mathfrak{g}}^2_-)$ with dense domain $L_{5,2}(\hat{\mathfrak{g}}^2_-)$. For $A \in \mathscr{C}$, H_A has smooth coefficients, and is a self-adjoint operator on $L_2(\hat{\mathfrak{g}}^2_-)$ with dense domain $L_{2,2}(\hat{\mathfrak{g}}^2_-)$. Generally, one has

Lemma A.1. *For $A \in \overline{\mathscr{C}}$, the operator H_A is self-adjoint on $L_2(\hat{\mathfrak{g}}^2_-)$ with dense domain $L_{2,2}(\hat{\mathfrak{g}}^2_-)$.*

Proof of Lemma A.1. The operator H_A is symmetric. If $A_0 \in \mathscr{C}(P)$, then the difference $V_A = H_A - H_{A_0}$ is H_{A_0}-bounded with a H_A-bound which is quadratic in $\|A - A_0\|_{4,2}$. This is a straightforward application of the Sobolev inequalities and the definitions in [13, Chapter V.4.1]. As $\mathscr{C} \subset \overline{\mathscr{C}}$ is dense and H_A is self-adjoint, the self-adjointness of H_A follows from Theorem V.4.3 of [13]. The same theorem states that the domains of H_A and H_{A_0} coincide.

Now consider the spectral projections. For $A \in \overline{\mathscr{C}}$ and $E \geqslant 0$, the existence of $\Pi_E(A)$ as a self-adjoint, bounded operator on $L_2(\hat{\mathfrak{g}}^2_-)$ follows from the spectral theorem, cf. [13, Chapter VI.5.3]. It is well known that when $A \in \mathscr{C}(P)$, $\times$ H_A has discrete spectrum with finite multiplicity and no accumulation points. These last observations hold for $A \in \overline{\mathscr{C}}$, as one can appeal to Theorem IV.3.18 of [13] and the fact that $\mathscr{C} \subset \overline{\mathscr{C}}$ is dense.

Before turning to the proof of Theorem 3.2, it is appropriate to establish some further properties of the spectral projection $\Pi_E(\cdot)$; in particular

Lemma A.2. *Let $A \in \overline{\mathscr{C}}$. For $E < \infty$, the projection $\Pi_E(A)$ is a bounded linear map from $L_{k,2}(\hat{\mathfrak{g}}^2_-)$ to $L_{k+2,2}(\hat{\mathfrak{g}}^2_-)$ for $k \in (0,\cdots,4)$, and from $L_{5,2}(\hat{\mathfrak{g}}^2_-)$ to itself.*

Proof of Lemma A.2. This follows from the standard elliptic regularity theorems for the eigenfunctions of H_A [16, Chapters 5 and 6].

Lemma A.3. *Let $A \in \overline{\mathscr{C}}$ and suppose that $E < \infty$ is not an eigenvalue of H_A. Then there exists an open neighborhood, $V \supset A$ such that $\Pi_E(\cdot)$ is a smooth map of V into the Banach space of bounded operators on $L_{5,2}(\hat{\mathfrak{g}}^2_-)$.*

Proof of Lemma A.3. Introduce the complexified spaces $L_{4,2}(\hat{\mathfrak{g}}^1 \otimes_{\mathbf{R}} C)$, $L_{5,2}(\hat{\mathfrak{g}}^2_- \otimes_{\mathbf{R}} C)$ with the obvious hermitian inner products. Let $\zeta \in C \setminus \text{Spectrum}\, H_A$, and consider, for fixed $v \in L_{5,2}(\hat{\mathfrak{g}}^2_- \otimes_{\mathbf{R}} C)$, the map

$$h_v(a,u) = \left(a, \mathscr{D}_{A+a}\mathscr{D}^\dagger_{A+a}(u+v) + \zeta(u+v)\right)$$

from $B^0 = L_{4,2}(\hat{\mathfrak{g}}^1 \otimes_{\mathbf{R}} C) \times L_{5,2}(\hat{\mathfrak{g}}^2_- \otimes_{\mathbf{R}} C)$ to $B^1 = L_{4,2}(\hat{\mathfrak{g}}^1 \otimes_{\mathbf{R}} C) \times L_{3,2}(\hat{\mathfrak{g}}^2_- \otimes_{\mathbf{R}} C)$. The previous considerations imply that this is a bounded, smooth map. As $\zeta \notin \text{Spectrum}\, H_A$, $dh_v|_{(0,0)}$ is an invertible operator. This fact, and the implicit function theorem imply that for all $\|a\|_{4,2}$ sufficiently small (depending on ζ), the operator $H_{A+a} + \zeta$, from $L_{5,2}(\hat{\mathfrak{g}}^2_- \otimes_{\mathbf{R}} C)$ to $L_{3,2}(\hat{\mathfrak{g}}^2_- \otimes_{\mathbf{R}} C)$, is invertible when $\zeta \notin \text{Spectrum}\, H_A$. In addition, the inverse depends analytically on $a \in L_{4,2}(\hat{\mathfrak{g}}^1 \otimes_{\mathbf{R}} C)$. By Theorem VII.1.3 of [13], the family of operators H_{A+a} that is indexed by $a \in L_{4,2}(\hat{\mathfrak{g}}^2_- \otimes_{\mathbf{R}} C)$ is holomorphic at $a = 0$. Lemma A.3 is now seen to be an immediate consequence of Theorem VII.1.7 of [13].

Proof of Theorem 3.3. Armed with Lemmas A.1–A.3, the theorem is a straightforward generalization of Theorem 3.2 of [20], so the discussion will be brief. The existence of the constants ε_0 and c, and the solution $u_E \in L_{5,2}(\hat{\mathfrak{g}}^2_-)$ to (2.7) is proved by repeating the proof of Theorem 3.2 of [20], essentially word for word. As in [20], one obtains that ε_0 and c are independent, and independent from the choice of $A \in \overline{\mathscr{C}}(P)$ and P. Equation (3.3) here is (3.7) of [20]. As for (3.4), observe that because $\Pi_E(A)u_E = 0$, one can take the L_2-inner product of both sides of (2.7) with u_E to obtain

$$\|\mathscr{D}^\dagger_A u_E\|_2^2 = -\left\langle u_E, \mathscr{D}^\dagger_A u_E \wedge \mathscr{D}^\dagger_A u_E + P_- F_A\right\rangle_2. \tag{A.1}$$

Using Hölders inequality on the expression above gives

$$\|\mathscr{D}^\dagger_A u_E\|_2^2 \leqslant \|u_E\|_4\left(\|\mathscr{D}^\dagger_A u_E\|_2 \|\mathscr{D}^\dagger_A u_E\|_4 + \|P_- F_A\|_{4/3}\right). \tag{A.2}$$

Lemma 4.6 of [20], with $\zeta_E(A)$ replacing $\zeta(A)$, together with Lemma 5.2 of [20] yield the following inequality which is valid for any $v \in L_{1,2}(\hat{\mathfrak{g}}^2_-)$ satisfying $\Pi_E(A)v = 0$:

$$\|v\|_4 \leqslant c_1 \zeta_E(A) \|\mathscr{D}^\dagger_A v\|_2. \tag{A.3}$$

Here, c_1 is independent from $A \in \overline{\mathscr{C}}(P)$ and P. Since $\Pi_E(A)u_E = 0$, u_E satisfies inequality (A.3).

By applying (A.3) to (A.2), and utilizing (3.3), one obtains that

$$\|\mathscr{D}_A^\dagger u_E\|_2^2(1 - c'\zeta_E(A)\delta_E(A)) \leqslant c'\zeta_E(A)\|\mathscr{D}_A^\dagger u_E\|_2\|P_-F_A\|_{4/3}.$$

Here, c' is independent of A, P. Equation (3.4) follows immediately when $\zeta_E(A)\delta_E(A) < \varepsilon_0 < \frac{1}{2}(c')^{-1}$.

The uniqueness of the solution, u_E, is proved by supposing the contrary and establishing a contradiction: Let u, v be two solutions to (2.7) satisfying (3.3) and (3.4). Then if $w = u - v$

$$\text{(A.4)} \qquad \mathscr{D}_A\mathscr{D}_A^\dagger w = \Pi_E(A)P_-\left(\mathscr{D}_A^\dagger w \wedge \mathscr{D}_A^\dagger(u + v)\right).$$

By taking the inner product of both sides of (A.4) with w, one obtains

$$\text{(A.5)} \qquad \|\mathscr{D}_A^\dagger w\|_2 \leqslant \|w\|_4\|\mathscr{D}_A^\dagger(u + v)\|_4.$$

Since $\Pi_E(A)w = 0$, (A.3), (A.5) and (3.3) yields the inequality

$$\text{(A.6)} \qquad 1 \leqslant c_1 \cdot c\zeta_E(A)\delta_E(A).$$

As the constant c is independent of ε_0, it is no loss of generality to assume that $\varepsilon_0 < (c_1 \cdot c)^{-1}$. Now (A.6) is a contradiction that establishes the uniqueness of u_E.

In order to prove Proposition 3.3, one must examine how $u_E(A)$ of Theorem 3.3 varies with $A \in \overline{\mathscr{C}}_E$. If E is not an eigenvalue of H_A, then by Lemma A.3, there exists an open neighborhood, $A \in V \subset \overline{\mathscr{C}}_\varepsilon$, such that for all $A' \in V$, E is not an eigenvalue of $H_{A'}$. From Lemma A.3 and the Sobolev inequalities, the map $\mathscr{S}_E(\cdot)$, below, is a smooth map from $V \times L_{5,2}(\hat{\mathfrak{g}}^2_-) \to V \times L_{3,2}(\hat{\mathfrak{g}}^2_-)$:

$$\text{(A.7)} \qquad \begin{aligned}&\mathscr{S}_E(A', u)\\ &= \left(A', \mathscr{D}_{A'}\mathscr{D}_{A'}^\dagger u + \Pi_E^\perp(A')P_-\left(\mathscr{D}_{A'}^\dagger u \wedge \mathscr{D}_{A'}^\dagger u + F_{A'}\right) + \Pi_E(A')u\right).\end{aligned}$$

The map $\mathscr{S}_E$ is smooth on $V \times L_{5,2}(\hat{\mathfrak{g}}^2_-)$.

Lemma A.4. *Let $A \in \mathscr{C}_E$ and suppose that $E \in \mathbf{R}$ is not an eigenvalue of $\mathscr{D}_A\mathscr{D}_A^\dagger$. Then the differential $d\mathscr{S}_E$ at $(A, u_E(A))$ is an isomorphism from $L_{4,2}(\hat{\mathfrak{g}}^1) \times L_{5,2}(\hat{\mathfrak{g}}^2_-) \to L_{4,2}(\hat{\mathfrak{g}}^1) \times L_{3,2}(\hat{\mathfrak{g}}^2_-)$.*

Assume Lemma A.4 for the moment.

Proof of Proposition 3.3, *assuming Lemma* A.4. If Ψ^*n_E is constant along N, then for each $x \in N$, there exists $E' > E$ such that E' is not an eigenvalue of $\mathscr{D}_A\mathscr{D}_A^\dagger$ and $\Pi_{E'}(A) = \Pi_E(A)$ for $A = \Psi(x)$. Let $x \in N$ and let $E' > E$ be as above. There is a neighborhood V of $A = \Psi(x)$ such that $\Pi_{E'}(\cdot)$ is smooth on V and $\Pi_{E'}(A') = \Pi_E(A')$ for all $A' \in V \cap \Psi(N)$. This is Lemma A.3. Thus, $\Psi^*\Pi_E(\cdot)$ is smooth on $\Psi^{-1}(V) \cap N$. The implicit function theorem and

Lemma A.4 implies that the assignment $A' \mapsto u_{E'}(A')$ is a smooth map from V into $L_{5,2}(\hat{\mathfrak{g}}^2_-)$. Therefore, the composition $A' \mapsto \mathscr{D}^\dagger_{A'} u_{E'}(A')$ gives a smooth map from $V \to L_{4,2}(\hat{\mathfrak{g}}^1)$. Restricting to $V \cap \Psi(N)$ establishes smoothness for $\Psi^*\mathscr{D}^\dagger u_E$ on $\Psi^* V \cap N$. Similarly, $f_{E'}$: $V \to L_{3,2}(\hat{\mathfrak{g}}^2_-)$ is smooth, so the restriction to $\Psi^*V \cap N$, $\Psi^* f_E$ is smooth.

Proof of Lemma A.4. The linear operator $d\mathscr{S}|_{(A, u_E(A))}$, applied to $(a, v) \in L_{4,2}(\hat{\mathfrak{g}}^1) \times L_{5,2}(\hat{\mathfrak{g}}^2_-)$, has the form $d\mathscr{S}(a, v) = (a, T(a, v))$. The linear operator T: $L_{4,2}(\hat{\mathfrak{g}}^1) \times L_{5,2}(\hat{\mathfrak{g}}^2_-) \to L_{3,2}(\hat{\mathfrak{g}}^2_-)$ is

$$\begin{aligned} T(a, v) = {} & \mathscr{D}_A \mathscr{D}^\dagger_A v + 2\Pi_E(A) P_-\left(\mathscr{D}^\dagger_A u_E \wedge \mathscr{D}^\dagger_A v\right) + \Pi_E(A) v \\ & + 2P_-\left(a \wedge \mathscr{D}^\dagger_A u_E + \mathscr{D}_A\left(a^\dagger u_E\right)\right) \\ & + \Pi^\perp_E(A) P_-\left(4\mathscr{D}^\dagger_A u_E \wedge a^\dagger u_E + \mathscr{D}_A a\right) \\ & - \left\langle \delta\Pi_E(A), a\right\rangle \left[P_-\left(\mathscr{D}^\dagger_A u_E \wedge \mathscr{D}^\dagger_A u_E\right) + P_- F_A - u_E\right]. \end{aligned} \tag{A.8}$$

Here, $a^\dagger(u) = \frac{1}{2} *(a \wedge u - u \wedge a)$, and $\langle \delta\Pi_E(A), a\rangle[\cdot]$ denotes the differential of the map $A' \to \Pi_E(A')$ of V into the Banach space of bounded, linear operators on $L_{3,2}(\hat{\mathfrak{g}}^2_-)$.

It is clear that $d\mathscr{S}$ is an isomorphism if for fixed $a \in L_{4,2}(\hat{\mathfrak{g}}^1)$, $T(a, \cdot)$: $L_{5,2}(\hat{\mathfrak{g}}^2_-) \to L_{3,2}(\hat{\mathfrak{g}}^2_-)$ is invertible. To see that this is the case, it is useful to split each $v \in L_{3,2}(\hat{\mathfrak{g}}^2_-)$ as $v = v_1 + v_2$, where $v_1 = \Pi_E(A) v$ and $v_2 = \Pi^\perp_E(A) v$.

Now T is invertible iff for each $q \in L_{3,2}(\hat{\mathfrak{g}}^2_-)$, there exists $v = (v_1, v_2) \in L_{5,2}(\hat{\mathfrak{g}}^2_-)$ such that

$$\begin{aligned} \mathscr{D}_A \mathscr{D}^\dagger_A v_1 + v_1 = {} & \Pi_E(A)\left[q - 2P_-\left(a \wedge \mathscr{D}^\dagger_A u_E + \mathscr{D}_A\left(a^\dagger u_E\right)\right)\right. \\ & \left. - \left\langle \delta\Pi_E(A), a\right\rangle \left[P_-\left(\mathscr{D}^\dagger_A u_E \wedge \mathscr{D}^\dagger_A u_E\right) + P_- F - u_E\right]\right], \end{aligned} \tag{A.9}$$

$$\begin{aligned} \mathscr{D}_A \mathscr{D}^\dagger_A v_2 + 2\Pi^\perp_E(A) P_-\left(\mathscr{D}^\dagger_A u_E \wedge \mathscr{D}^\dagger_A v_2\right) = {} & \Pi^\perp_E(A)\left[q - P_-\left(2\mathscr{D}^\dagger_A u_E \wedge \mathscr{D}^\dagger_A v_1\right.\right. \\ & \left. + 2a \wedge \mathscr{D}^\dagger_A u_E + \mathscr{D}_A\left(a^\dagger u_E\right) + 4\mathscr{D}^\dagger_A u_E \wedge a^\dagger u_E + \mathscr{D}_A a\right) \\ & \left. + \left\langle \delta\Pi_E(A), a\right\rangle \left[P_- \mathscr{D}^\dagger_A u_E \wedge \mathscr{D}^\dagger_A u_E + P_- F_A - u_E\right]\right]. \end{aligned} \tag{A.10}$$

Equation (A.9) is a linear equation for v_1 in terms of q, a, u_E and A and it has a unique solution $v_1 \in \Pi_E(A) L_{5,2}(\hat{\mathfrak{g}}^2_-)$.

Because v_1 can be solved for, (A.10) is now a linear equation for v_2. Indeed, the right-hand side of (A.10) depends only on q, a, u_E and A. Whether or not one can solve (A.10) for v_2 depends on whether or not the operator

$$\mathscr{D}_A \mathscr{D}^\dagger_A(\cdot) + 2\Pi^\perp_E(A) P_-\left(\mathscr{D}^\dagger_A u_E \wedge \mathscr{D}^\dagger_A(\cdot)\right)$$

is invertible on $\Pi^\perp_E(A) L_{3,2}(\hat{\mathfrak{g}}^2_-)$. The same arguments that were used in the proof of Theorem 3.3 verify that this is so, provided that $A \in \mathscr{C}_E$, and u_E satisfies (3.3) and (3.4).

The conclusion is that T is invertible and that $d\mathscr{S}_{(A,u_E(A))}$ is an isomorphism, as claimed.

Appendix B: Metrics and $P_-H^2_{\mathrm{DR}}$

The vector space $P_-H^2_{\mathrm{DR}}$ is the linear space $\{\omega \in \Gamma(\wedge_2 T^*)\colon d\omega = 0$ and $*\omega = -\omega\}$. As one considers various metrics on T_M, the $*$ operator changes and so $P_-H^2_{\mathrm{DR}}$ changes too. The question of how $P_-H^2_{\mathrm{DR}}$ changes with the metric is the topic of this appendix. The purpose here is to prove Lemma 8.4. In proving Lemma 8.4, the following result is used.

Lemma B.1. *Let E_1, E_2 be Hilbert manifolds and let $f\colon E_1 \to E_2$ be a smooth map. Let $Z \subset E_2$ be a closed, nowhere dense subset. Then $f^{-1}(E_2 \setminus Z)$ is open in E_1. If at every point $x \in f^{-1}(Z)$, the differential $df\colon T_{E_1}|_x \to T_{E_2}|_{f(x)}$ is surjective, then $f^{-1}(E_2 \setminus Z)$ is dense in E_1. Alternatively, if Z is contained in a smooth submanifold $Y \subset E_2$ and if at each $x \in f^{-1}(Z)$, the induced map $df\colon T_{E_1}|_x \to (T_{E_2}/T_Y)|_{f(x)}$ is nonzero, then $f^{-1}(E_2 \setminus Z)$ is dense in E_1.*

Proof of Lemma B.1. The statement that $f^{-1}(E_2 \setminus Z)$ is open is implied by the fact that f is continuous. The last two statements follow using the implicit function theorem.

To apply Lemma B.1 to the problems at hand, it is convenient to introduce some formal ideas. For an integer $k \geqslant 5$, let Θ denote the Hilbert manifold of $L_{k,2}$ metrics on T_M. This is defined as follows: Given a fixed metric m on T_M, one first defines $L_{k,2;m}(\mathrm{Sym}_2 T^*)$ where the subscript m signifies that the norms and covariant derivatives are defined using m. Then

$$\Theta = \{m' \in L_{k,2;m}(\mathrm{Sym}_2 T^*)\colon \text{For all } p \in M \text{ and } 0 \neq \zeta \in T_M|_p,\ m'(\zeta,\zeta)(p) > 0\}.$$

The spaces so defined by two metrics m and m' are isomorphic. The tangent space to $m \in \Theta$ is naturally $L_{k,2;m}(\mathrm{Sym}_2 T^*)$. Henceforth the subscript "m" will be suppressed.

Let $\overline{\Omega}_p$, $p \in (1,\cdots,4)$, denote the completion of $\Gamma(\wedge_p T^*)$ in the $L_{k,2}$-norm. By the Sobolev theorem, there are the compact embeddings

$$\Theta, \overline{\Omega}_p \to C^{k-3}. \tag{B.1}$$

Let $P_\pm = P_\pm(m)$ be the metric dependent projections on $\wedge_2 T^*$. The assignment of $m \in \Theta$ to the b_- dimensional vector subspace $V(m) = P_-H^2_{\mathrm{DR}} \subset \overline{\Omega}_2$ defines a smooth, $\mathbf{R}^{b_-}$ vector bundle

$$\Pi\colon V \to \Theta. \tag{B.2}$$

The arguments to prove smoothness are similar to those of Appendix A.

Each $h \in T_\Theta|_m$ defines a section

$$X = X(h) \in L_{k,2}(\mathrm{Hom}(P_- \wedge_2 T^*; P_+ \wedge_2 T^*))$$

via the natural isomorphism of the vector bundle $\mathrm{Hom}(P_- \wedge_2 T^*; P_+ \wedge_2 T^*)$ with traceless $\mathrm{Sym}_2(T^*)$. Observe that $(h, u) \in T_\Theta|_m \times \overline{\Omega}_2$ is tangent to $(m, \omega) \in V$ iff

(B.3) $$P_+ u + X\omega = 0 \quad \text{and} \quad du = 0.$$

The results in §8 require a knowledge of the linear dependence of harmonic 2-forms and their derivatives at points in M. For $p \in M$, define the evaluation map e_p: $V \to \wedge_2 T^*|_p \cong \mathbf{R}^6$ by

(B.4) $$e_p(m, \omega) = \omega(p).$$

Due to (B.1), e_p is a smooth map of V into $\mathbf{R}^6$.

The application of these constructions is in the proof of Lemma 8.4.

Proof of Lemma 8.4. Let $m \in \Theta$. Then due to Lemma 8.3, there exists $p \in M$ where $V(m)$ spans a 2-plane in $\wedge_2 T^*$ at p. Since e_p is continuous, there is a neighborhood $U \subseteq \Theta$ of m such that $V(m')$ spans a 2-plane in $\wedge_2 T^*$ at p for all $m' \in U$. Define a connection on V by specifying that the horizontal subspace at $(m, \omega) \in V$ be

(B.5) $$H_{(m,\omega)} = \{(h, u) \in T_V|_{(m,\omega)}: \langle u, V(m)\rangle_2 = 0\}.$$

This is a smooth connection, and by parallel transport out from m, a basis $\{\omega_1, \omega_2\}$ of $V(m)$ can be extended in a smooth way to define a basis $\{\omega_1, \omega_2\}$ for V over U. Define a map f: $U \to T^* \oplus T^*|_p \simeq \mathbf{R}^8$ by

(B.6) $$f(m) = \left(d|\omega_1|^2(p), d|\omega_2|^2(p)\right).$$

The function f is smooth, due to (B.1). Therefore, the set {metrics in U for which $d|\omega_1|^2(p) \wedge d|\omega_2|^2(p) \neq 0$} is open in U.

The next observation is

Lemma B.2. *Let $m \in \Theta$, and suppose that $\{\omega_1, \omega_2\} \subset V(m)$ spans a plane in $\wedge_2 T^*$ at $p \in M$. Extend $\{\omega_1, \omega_2\}$ on an open set, U of m in Θ using the connection of* (B.5) *and let f: $U \to \mathbf{R}^8$ be the map in* (B.6). *Then $df|_m$ is surjective.*

Before proving Lemma B.2, note that Lemma 8.4 is an immediate consequence of Lemmas B.1 and B.2.

Proof of Lemma B.2. Let $B \subset M$ be an open neighborhood of p on which $\{\omega_1, \omega_2\}$ are linearly independent. Choose $\omega_3 \in \Gamma(B; P_- \wedge_2 T^*)$ to be pointwise orthonormal to ω_1 and ω_2. Consider a pair $\{a_1, a_2\} \in \Gamma_c(B, T^*)$, where Γ_c

denotes compactly supported sections. A section $h \in \Gamma_c(B; \mathrm{Sym}_2 T^*)$ is uniquely defined by requiring that

$$\text{(B.7)} \quad \begin{array}{ll} (1) & \mathrm{Trace}\, h = 0, \text{ and} \\ (2) & X(h)\omega_i = P_+ da_i \quad \text{for } i \in (1,2) \quad \text{and } X(h)\omega_3 = 0. \end{array}$$

The horizontal lift of h to $T_V|_{(m,\omega_i)}$, $i = 1,2$, gives the pair

$$(h, -da_i) \in H_{(m,\omega_i)}, \qquad i \in (1,2).$$

One now observes that the differential of f at m on $h \in T_\Theta|_m$ is given by

$$df|_m(h) = 2(d(\omega_1, da_1), d(\omega_2, da_2))|_p.$$

Since a_1, a_2 are arbitrary sections and neither ω_1 nor ω_2 vanish at p, Lemma B.2 follows.

Acknowledgements. I am extremely grateful to S. K. Donaldson for the many discussions we had on this subject. I thank Professor M. F. Atiyah for inviting me to the Mathematical Institute in Oxford, where this article was written, and I thank Professor Atiyah and the Mathematical Institute for their hospitality.

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