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MANIFOLDS WITH ALMOST EQUAL DIAMETER AND INJECTIVITY RADIUS

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1. Introduction

In this paper, we will give some constraints on the topology of compact, connected Riemannian manifolds whose injectivity radii and diameters are close to each other, in terms of their sectional curvature. For notations and definitions, we refer to §3, Besse [3], Cheeger & Ebin [6], and Gromoll, Klingenberg & Meyer [11].

The case of the spherical cut locus of a point in a compact Riemannian manifold and also the stronger case of the equality of the diameter and injectivity radius have been studied by various authors. Let (1.1) represent "M has the integral cohomology ring of one of compact, irreducible symmetric spaces of rank 1" and (1.2) represent "M^n has the same cohomology groups as that of $\mathbf{R}P^n$ and $\tilde{M}^n$ is homeomorphic to S^n".

Warner [22], has shown that if $\exists p \in M$, a compact, simply connected Riemannian manifold, for which each point of the spherical conjugate locus in TM_p is regular, then that has the same multiplicity as conjugate points which are $\geqslant 1$, and either M is homeomorphic to a sphere or (1.1) holds.

Theorem (*Nakagawa & Shiohama* [15], [16]). *Let M be a compact, connected Riemannian manifold with $K_M \leqslant 1$, such that $\exists p \in M$ with spherical cut locus, i.e., $i_p = d_p = l$. Then the following hold. $l \geqslant \frac{1}{2}\pi$. If $l = \frac{1}{2}\pi$, then M^n is isometric to $\mathbf{R}P^n$ with $K_M \equiv 1$. If $\frac{1}{2}\pi < l < \pi$, then* (1.2) *holds. If $\pi_1(M) = 1$, then $l \geqslant \pi$. If the cut locus of p is not contained in the first conjugate locus Q_p of p, then the tangential cut locus of p is disjoint from the first tangential conjugate locus of p, and hence* (1.2) *holds. Furthermore, if we also assume that $l = \pi/\sqrt{\mathrm{Max}(K_M)}$, then every geodesic segment starting from p with length $2l$ is a geodesic loop at p, and for any $q \in Q_p$, the multiplicity of p and q as a conjugate pair is constant λ, where $\lambda = 0, 1, 3, 7$ or $n - 1$. If $\pi_1(M) \neq 1$, then* (1.2) *and*

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$\lambda = 0$ *hold. If* $\pi_1(M) = 1$, *then either* (1.1) *holds for* $\lambda = 1, 3, 7$, *or* M *is isometric to a sphere of constant sectional curvature* $\mathrm{Max}(K_M)$ *for* $\lambda = n - 1$.

In Besse [3, p. 137], it is shown that a point $p \in M$, where M is C^∞, has a spherical cut locus if and only if M is a pointed Blaschke manifold at p. There is an extensive theory for Blaschke manifolds [3]; especially, the Bott-Samelson Theorem ([3, Chapter 7], [4], [17]) states that they satisfy (1.1), (1.2), or more.

Berger [1, p. 236], has shown that if $\exists$ a Blaschke Riemannian structure on S^n, then this Riemannian structure is isometric to the standard one on S^n, up to a multiplicative factor. Also the analogue is true for $\mathbf{R}P^n$.

Conjecture (*Blaschke*). Any Blaschke manifold M^n (i.e., $i_M = d_M$ by Besse [3, p. 138]) is isometric to one of the following: S^n, $\mathbf{R}P^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$, $\mathbf{Ca}P^2$ with their standard metrics, up to a constant factor (see [3]).

Recently, Gluck, Warner & Yang [9] have shown that for $\dim M^n = n \leqslant 9$, Blaschke manifolds have the correct homeomorphism types.

The theorems above show that the condition $i_p = d_p$ for some $p \in M$ is a very rigid restriction. A very natural question to consider is: If we allow some flexibility in this condition, such as "i_p is close to d_p" in some sense, then what can be said about M? This cannot be done arbitrarily (see §8, Example 2). Furthermore, the known theorems above for the equality case do not seem to generalize in this direction, because of the nature of their proofs.

The problem of finding quantitative topological restrictions on even dimensional manifolds with $1 \leqslant K_M \leqslant 4 + \varepsilon$, for some $\varepsilon > 0$, makes this situation of i_M being close to d_M interesting. Grove & Shiohama [12] have shown that if also $d_M > \frac{1}{2}\pi$ then M is homeomorphic to a sphere. Gromoll & Grove [10] extended this result: if also $d_M = \frac{1}{2}\pi$, then either M is homeomorphic to S^n or $\tilde{M}$ is isometric to a symmetric space of rank 1. By Klingenberg's Lemma ([6, pp. 96, 98], [11, p. 277]) $i_M \geqslant \pi/\sqrt{4+\varepsilon}$. The case of $\pi/\sqrt{4+\varepsilon} \leqslant i_M \leqslant d_M < \frac{1}{2}\pi$ seems to be resolved recently by Berger [2]: "$\exists \delta = \delta(n) \in \mathbf{R}$, $0 < \delta < 1/4$, such that any compact Riemannian manifold M^n, with n even, $\pi_1(M) = 1$, and $\delta \leqslant K_M \leqslant 1$, is necessarily homeomorphic to S^n or diffeomorphic to a symmetric space of rank 1."

The primary goal of this paper is to construct some universal constants such that if i_p or i_M is close to d_p or d_M in terms of these constants, then there will be some topological constraints on such compact Riemannian manifolds M. These universal constants depend only on the lower bound of the sectional curvature K_M of M, and sometimes on the dimension.

In §2, we state the main results and some theorems which are used as main tools. The basic notation and definitions are given in §3. Theorems 1–5 are proved in §§4–7. §8 contains some examples.

The results in this paper had also appeared in the dissertation of the author [7]. The author wishes to thank D. Gromoll for his guidance during the research and completion of this work; and J. Cheeger for encouraging and helpful discussions. Theorem 5 was known to J. Cheeger, independently; and the main tool in its proof is Lemma 11, and was brought to the attention of the author by J. Cheeger and D. Gromoll.

2. The main results and tools

In the rest of this paper, M^n denotes a compact, connected, smooth Riemannian manifold with no boundary, and with dimension $n \geqslant 2$. In $d_M^2 \cdot K_M \geqslant C$, *$C$ is always taken to be negative or zero* and there is no loss of generality in doing so, since if $K_M \geqslant C' > 0$, then obviously $K_M \geqslant 0$. However, it follows from the proofs of the theorems that if $i_M^2 \cdot K_M \geqslant C' > 0$, then the δ's can be made bigger for positive C'.

Theorem 1. $\forall C \in \mathbf{R}$, $\exists \delta_1(C) > 0$, *such that for any compact Riemannian manifold M^n, if $d_M^2 \cdot K_M \geqslant C$ and $\exists p \in M$ with $i_p/d_p > 1 - \delta_1(C)$, then $\pi_1(M, p) = 1$ or $\mathbf{Z}_2$.*

Theorem 2. $\forall C \in \mathbf{R}$, $\exists \delta_2(C) > 0$, *such that for any compact Riemannian manifold M^n, if $d_M^2 \cdot K_M \geqslant C$, and $\exists p \in M$ with $i_M/d_p > 1 - \delta_2(C)$ and $\pi_1(M, p) = \mathbf{Z}_2$, then*:

(i) *M^n is oriented if and only if n is odd, and*

(ii) *$\forall n \geqslant 2$, $H^*(M^n, \mathbf{Z}) \simeq H^*(\mathbf{R}P^n, \mathbf{Z})$ induced by a map of local degree ± 1, from $\mathbf{R}P^n$ onto M^n; furthermore, M^n has the homotopy type of $\mathbf{R}P^n$.*

Theorem 3. $\forall C \in \mathbf{R}$, $\exists \delta_3(C) > 0$, *such that for any compact Riemannian manifold M^n, if $d_M^2 \cdot K_M \geqslant C$, and $\exists p \in M$ with $i_M/d_p > 1 - \delta_3(C)$ and $\exp_p|\bar{B}_{d_p}(0, TM_p)$ is of maximal rank, then $\pi_1(M, p) = \mathbf{Z}_2$, and $\tilde{M}^n$ is homeomorphic to S^n.*

Theorem 4. *Let $\sigma_k = \arccos(-1/k)$ for $k \geqslant 1$.* $\forall C \in \mathbf{R}$, $\forall \alpha \in (0, \pi)$, $\exists \delta_4(\alpha, C) > 0$, *such that for any compact Riemannian manifold M^n, if $d_M^2 \cdot K_M \geqslant C$, and $\exists p \in M$ with $i_M/d_p > 1 - \delta_4(\sigma_4, C)$ and $\exp_p|\bar{B}_{d_p}(0, TM_p)$ is of maximal rank, then*:

(i) *$C_p = V_1 \cup V_2 \cup V_3$, where V_i are disjoint smooth submanifolds of codimension i, open in their dimensions*;

(ii) *if $n = 2$ or σ_4 is replaced by σ_3 in the hypothesis, then $V_3 = \varnothing$; and,*

(iii) *if σ_4 is replaced by σ_2 in the hypothesis, then $V_3 = V_2 = \varnothing$, and hence, $C_p = V_1$ is a compact, smooth $n-1$ dimensional submanifold of M^n, without* *boundary. Hence M^n is homeomorphic to a nonsimply connected pointed Blaschke manifold.*

Theorem 5. $\forall C \in \mathbf{R}$, $\forall n \geqslant 2$, $\exists \delta_5(n, C) > 0$, *such that for any compact Riemannian manifold M^n, if $d_M^2 \cdot K_M \geqslant C$ and $\exists p \in M$ with $i_M/d_p > 1 - \delta_5(n, C)$, then $d_p > \pi/2\sqrt{K}$, where $K = \mathrm{Max}(K_M)$. Obviously, if $K \leqslant 0$, then $\forall p \in M$, $i_M/d_p \leqslant 1 - \delta_5(n, C)$.*

Remark. The δ's of Theorems 1–5 are explicitly constructed, their existences are not ideal. The proof of the following theorem will appear elsewhere, since its proof is different in nature. Although it seems to generalize Theorem 2, the δ exists ideally.

Theorem. $\forall C \geqslant 0$, $\forall n \geqslant 2$, $\exists \delta(C, n) > 0$, *such that for any compact Riemannian manifold M^n, if $|d_M^2 \cdot K_M| \leqslant C$, $\pi_1(M) = \mathbf{Z}_2$, and $i_M/d_M > 1 - \delta(C, n)$, then, M^n is homeomorphic to $\mathbf{R}P^n$.*

The following results will be used in proving Theorems 1–5. Toponogov's Theorem is our main tool.

Theorem (*Sugahara* [19, *Theorem* B]). *For any compact Riemannian manifold M^n, if there exists a point p in M such that the first tangential conjugate locus of p is disjoint from the tangential cut locus of p, and the number of the minimal geodesics from p to any point on its cut locus is* 2, *then $\pi_1(M) = \mathbf{Z}_2$, and $\tilde{M}^n$ is homeomorphic to S^n.*

Theorem (*Weinstein* [3, *pp.* 137, 231]; [22]). *If M^n is of the form $M^n = \overline{D}^n \cup_a E$, where $\overline{D}^n$ is the n-dimensional closed ball, E is a C^∞ closed k-disc bundle over an $n - k$ dimensional compact C^∞ manifold, with ∂E diffeomorphic to S^{n-1}, and a: $\partial \overline{D}^n \to \partial E$ an attaching diffeomorphism, then there exists a Riemannian metric on M, such that M becomes a pointed Blaschke manifold at p, which is the center of $\overline{D}^n$.*

Theorem (*Toponogov* [20], [21], [6, pp. 42–49], [11, pp. 184 +]). (*The following form is as it appears in* [6].) *Let M^n be a complete Riemannian manifold with $K_M \geqslant C$.*

(a) *Let $(\gamma_1, \gamma_2, \gamma_3)$ determine a geodesic triangle in M; and with indices* mod 3, *α_i be $\measuredangle(-\gamma'_{i+1}(l_{i+1}), \gamma'_{i+2}(0))$. Suppose γ_1, γ_3 are minimal; and if $C > 0$, suppose $l(\gamma_2) \leqslant \pi/\sqrt{C}$. Then in M_C, there exists a geodesic triangle $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ such that $l(\gamma_i) = l(\bar{\gamma}_i)$ and $\bar{\alpha}_1 \leqslant \alpha_1$; $\bar{\alpha}_3 \leqslant \alpha_3$. Except in the case $C > 0$ and $l(\gamma_i) = \pi/\sqrt{C}$ for some i, the triangle in M_C is uniquely determined (up to congruencies of M_C).*

(b) *Let γ_1, γ_2 be geodesic segments in M such that $\gamma_1(l_1) = \gamma_2(0)$ and $\alpha := \measuredangle(-\gamma_1'(l_1), \gamma_2'(0))$. We call such a configuration a hinge L and denote it $(\gamma_1, \gamma_2; \alpha)$. Let γ_1 be minimal, and if $C > 0$, $l(\gamma_2) \leqslant \pi/\sqrt{C}$. Let $\bar{\gamma}_1, \bar{\gamma}_2 \subset M_C$ be* *such that* $\bar{\gamma}_1(l_1) = \bar{\gamma}_2(0)$, $l(\gamma_i) = l(\bar{\gamma}_i) = l_i$ *and* $\measuredangle(-\bar{\gamma}_1'(l_1), \bar{\gamma}_2'(0)) = \alpha$. *Then* $d_M(\gamma_1(0), \gamma_2(l_2)) \leqslant d_{M_C}(\bar{\gamma}_1(0), \bar{\gamma}_2(l_2))$.

3. Basic notation and definitions

For the basic notions of manifolds and Riemannian geometry, we refer to Cheeger & Ebin [6], Gromoll, Klingenberg & Meyer [11], and Kobayashi & Nomizu [13]; and for facts about Blaschke manifolds, refer to Besse [3]. Our notation and definitions are the same as in [6], and for Blaschke manifolds as in [3]. In the following, we give the most frequently used or exceptional ones.

In this text, M^n always denotes a compact, smooth, connected, n-dimensional Riemannian manifold, without boundary; and TM, UM are its tangent and unit sphere bundles, with respect to the Riemannian metric $\langle\ ,\ \rangle_p$ on TM_p, $p \in M$. $d_M(\cdot, \cdot)$ is the Riemannian distance function on M. For any metric space X and $x \in X$, $B_r(x, X) = \{y \in X | d_X(x, y) < r\}$ and $\bar{B}_r(x, X) = \{y \in X | d_X(x, y) \leqslant r\}$. K_M denotes the sectional curvature of the Riemannian connection on M.

All coordinate systems around any point are taken to be normal. Let $p, q \in M$ be fixed and γ be any geodesic from p to q. Unless otherwise specified, the following are assumed. γ is parametrized by its arclength, i.e., $\|\gamma'(t)\| = 1$ $\forall t$; and $l(\gamma)$ denotes its length. If γ is said to be a "mg(p, q)", then γ is a minimal geodesic from p to q, i.e., $l(\gamma) = d_M(p, q)$. The set of all mg(p, q) is denoted by MG(p, q) and if furthermore γ is the unique minimal geodesic from p to q, then it is denoted by "umg(p, q)". For $v_1, v_2 \in TM_p - 0$, the angle between v_1 and v_2, $\measuredangle(v_1, v_2)$ is to be $\arccos(\langle v_1, v_2\rangle_p/\|v_1\| \cdot \|v_2\|)$.

$\exp_p$: $TM_p \to M$ is the exponential map. $\forall p \in M$, $\forall v \in UM_p$, the *cut value in the direction of* v, $c_p(v)$, is to be $\operatorname{Max}\{\lambda \in \mathbf{R} | \lambda > 0, d(p, \exp_p \lambda v) = \lambda\}$ and the *fundamental region*, A_p, to be $\{v \in TM_p | d(p, \exp_p v) = \|v\|\}$. The *tangential cut locus of* p, $\tilde{C}_p$, is defined to be ∂A_p, and the *cut locus of* p, C_p, to be $\exp_p \tilde{C}_p$. $c_p(v)$ depends on p and v continuously, $0 < c_p(v) < \infty$, and ∂A_p, A_p, $\operatorname{int}(A_p)$ are homeomorphic to S^{n-1}, n-dimensional closed disc $\bar{D}^n$ and open disc D^n, respectively, since M is compact (see [6, p. 94], [11]).

The *injectivity radius at* p, i_p, is $\operatorname{Min}\{c_p(v) | v \in UM_p\}$ and the *injectivity radius of* M, i_M, is $\operatorname{Min}\{i_p | p \in M\}$. $d_p = \operatorname{Max}\{c_p(v) | v \in UM_p\}$ is the *distance to the furthest point from* p, and $d_M = \operatorname{Max}\{d_p | p \in M\}$ is the *diameter of* M.

(3.1) Let $\tilde{M}$ be the universal cover of M and ρ: $\tilde{M} \to M$ be the natural projection map. There is a natural Riemannian structure on $\tilde{M}$ by pulling back the structure on M by the local homeomorphism ρ, and with this structure on

$\tilde{M}$, ρ becomes a local isometry and $\forall \tilde{p} \in \tilde{M}$, $\forall \tilde{v} \in T\tilde{M}_{\tilde{p}}$, $\forall t \in \mathbf{R}$, $\rho(\exp_{\tilde{p}} t\tilde{v}) = \exp_{\rho(\tilde{p})} t\rho_*(\tilde{v})$.

For $p \in M$, the *first tangential conjugate locus* $\tilde{Q}_p$ *of* p is defined to be

$$\left\{ v \in TM_p \,\middle|\, \begin{array}{l} (\exp_p)_*(tv) \colon T(TM_p)_{tv} \to TM_{\exp_p tv} \text{ is of maximal rank} \\ \text{for } 0 \leqslant t < 1 \text{ and not maximal for } t = 1. \end{array} \right\}$$

The *first conjugate locus* Q_p *of* p is to be $\exp_p(\tilde{Q}_p)$.

For any $C \in \mathbf{R}$, M_C denotes the simply connected two-dimensional complete Riemannian manifold of constant sectional curvature C; i.e. a space form [6, p. 40].

For any $p \in M$, p is to have a *spherical cut locus* if and only if $i_p = d_p$. The *link* $\Lambda(p, q)$ *from* p *to* q is to be $\{v \in UM_q | \exp_q(d(p, q) \cdot v) = p\}$. A compact Riemannian manifold M is called a *pointed Blaschke manifold at* p, for some $p \in M$, if $\forall q \in C_p$, $\Lambda(p, q)$ is the intersection of UM_q with a subspace of TM_q. M is called a *Blaschke manifold* if it is a pointed Blaschke manifold at p for all p in M.

4. A description of the universal cover

Let M be nonsimply connected, $\tilde{M}$ be its Riemannian universal cover, and ρ: $\tilde{M} \to M$ be the natural Riemannian covering map (see (3.1)). For any given $p \in M$, fix $p_0 \in \tilde{M}$ with $\rho(p_0) = p$. For any $\omega_i \in \pi_1(M, p)$, let p_i be $\omega_i(p_0)$, where ω_i is also representing the corresponding deck transformation. There is a natural bijection between the set of p_i's and $\pi_1(M, p)$.

Let $U = M - C_p$. U is homeomorphic to an open n-dimensional disc and $\exp_p|\mathrm{int}(A_p)$: $\mathrm{int}(A_p) \to U$ is a diffeomorphism ([6, p. 95], [11]). So, there exists a unique open connected set $U_i \subset \tilde{M}$, for each i, such that $p_i \in U_i$ and $\rho|U_i$: $U_i \to U$ is a homeomorphism, where $\omega_i \in \pi_1(M, p)$ is any class. Clearly, if $\omega_i \neq \omega_j$, then $U_i \cap U_j = \varnothing$, $\bigcup_{\omega_i \in \pi_1(M)} \overline{U_i} = \tilde{M}$ and $\omega_i|_{U_0}$: $U_0 \to U_i$ is an isometry.

One can easily show that $\Lambda = \{\omega_i \in \pi_1(M) | \overline{U_0} \cap \overline{U_i} \neq \varnothing\}$ is a set of generators for $\pi_1(M, p)$.

Lemma 1. *∂U_0 is connected.*

Proof. ρ is a local isometry, i.e. $\forall v \in T\tilde{M}_{p_0}$, $\rho(\exp_{p_0} v) = \exp_p(\rho_*(v))$. Let $h(v) = \exp_{p_0}((\rho_*(p_0)^{-1})(v))$ $\forall v \in TM_p$. h is a homeomorphism from $\mathrm{int}(A_p)$ onto U_0, which are both open. Since M is compact, A_p is compact. So, $h(A_p)$ is closed, hence it is $\overline{U_0}$. Therefore, $\partial U_0 = \overline{U_0} - U_0 = h(\partial A_p)$ is connected, since ∂A_p is homeomorphic S^{n-1} for compact M (see [6, p. 94]).

Remark. $\partial \overline{U}_0$ is not necessarily connected.

The proofs of Lemmas 2 and 3 are elementary, and they are left to the reader.

Lemma 2. *For $C \leqslant 0$. Let two geodesic triangles in M_C be given with sides of length A_1, B_1, C_1 and A_2, B_2, C_2, respectively. Let $\alpha_i, \beta_i, \gamma_i$ be the angles between the sides of length B_i, C_i; A_i, C_i; A_i, B_i, respectively for $i = 1, 2$.*

(i) *If $A_1 = A_2$, $C_1 = C_2$ and $B_1 < B_2$ then $\beta_1 < \beta_2$.*

(ii) *If $A_1 > A_2$, $B_1 = B_2$, $C_1 = C_2$ and $\beta_1 > \frac{1}{2}\pi$ then $\beta_1 < \beta_2$ (see* [11, *p.* 195]).

Lemma 3. *Let $x_1, x_2, \cdots, x_k$ be distinct unit vectors in $\mathbf{R}^N$, with the standard inner product, such that $\measuredangle(x_i, x_j) > \arccos(-1/n)$, i.e. $\langle x_i, x_j \rangle < -1/n$, for $x_i \neq x_j$. Then $k < n + 1$. (Consider $\|\Sigma x_i\|^2 \geqslant 0$.)*

5. The fundamental group

In this section Theorem 1 will be proved, so *its hypothesis is assumed everywhere in* §5.

Proof of Theorem 1. Construction of $\delta_1(C)$: Let $C \in \mathbf{R}$ be given. *Case for* $C \leqslant 0$: Let $x \in [0, 1)$. Consider two geodesic triangles with sides of length $1 + x, 1 + x, 2$ and $1 + x, 1 + 3x, 2$ in M_C. Let $\beta_1(x)$ and $\beta_2(x)$ be the angles between the sides of length $1 + x$ in the first triangle and $1 + x$ and $1 + 3x$ in the second one, respectively. There exists unique $x_0(C)$ such that $\beta_1(x_0(C)) + 2\beta_2(x_0(C)) = 2\pi$. By Lemma 2(ii), $\beta_1(x_0(C)) > 2\pi/3$. Let q_1, q_2, q_3 be points in M_C such that $d(q_i, q_j) = 1$ for $1 \leqslant i < j \leqslant 3$, and γ_1 be the umg(q_2, q_3), with $\gamma_1(0) = q_2$, $\gamma_1(1) = q_3$. Set $q_4 = \gamma_1(1 + 2x_0(C))$. Let γ_2, γ_3 be the umg(q_4, q_1) and the umg(q_3, q_1), respectively. Set $\alpha_1 = \measuredangle(-\gamma_1'(q_4), \gamma_2'(q_4))$, then define $\delta_1'(C)$ to be $\mathrm{Min}(x_0(C), \beta_1(C)^{-1}(\pi - \alpha_1(C)))$ and $\delta_1(C) = 1 - (1 + \delta_1'(C))^{-1}$. Also, let $\alpha(C)$ be $\beta_1(\delta_1'(C)) = \mathrm{Max}(\pi - \alpha_1(C), \beta_1(x_0))$. *Case for* $C > 0$: Let $\delta_1(C) = \delta_1(0)$. A straightforward calculation shows that $0 < x_0(C) < 1/10$, for all $C \in \mathbf{R}$.

Let M^n and $p \in M^n$ be as in the hypothesis. By multiplying the metric with $1/i_p$, the hypothesis becomes; (i) $K_M \geqslant \mathrm{Min}(C, 0)$, since $i_p \leqslant d_M$; (ii) $1 = i_p \leqslant d_p < 1 + \delta_1'(C)$.

Let p_0, p_i, U_0, U_i be constructed as in §4. Suppose that order$(\pi_1(M, p)) \geqslant 3$. By the connectedness of $\tilde{M}$, we can choose U_{i_0}, U_{i_1} such that $U_0 \cap U_{i_0} = U_0 \cap U_{i_1} = \varnothing$, $U_{i_0} \cap U_{i_1} = \varnothing$, $\overline{U}_0 \cap \overline{U}_{i_0} \neq \varnothing$ and $(\overline{U}_0 \cup \overline{U}_{i_0}) \cap \overline{U}_{i_1} \neq \varnothing$. If $\overline{U}_{i_1} \cap \overline{U}_0 \neq \varnothing$, then set $U_1 = U_{i_0}$ and $U_2 = U_{i_1}$. If $\overline{U}_{i_1} \cap \overline{U}_0 = \varnothing$, then set

$U_1 = \omega_{i_0}^{-1}(U_0)$ and $U_2 = \omega_{i_0}^{-1}(U_{i_1})$, where $\omega_{i_0}: \tilde{M} \to \tilde{M}$ is the deck transformation with $\omega_{i_0}(p_0) = p_{i_0}$. So, we can choose $U_0, U_1, U_2 \subset \tilde{M}$ such that $U_i \cap U_j = \varnothing$ for $0 \leqslant i < j \leqslant 2$ and $\overline{U}_0 \cap \overline{U}_i \neq \varnothing$ for $i = 1, 2$.

(5.1) If $p_i \neq p_j$, then $d_M(p_i, p_j) \geqslant 2i_p = 2$ since the image of any $\mathrm{mg}(p_i, p_j)$ under ρ is a geodesic loop at p in M.

(5.2) Let U_i, U_j be such that $\overline{U}_i \cap \overline{U}_j \neq \varnothing$ and $U_i \cap U_j = \varnothing$. Let r be in $\overline{U}_i \cap \overline{U}_j$ and θ_i, θ_j be $\mathrm{mg}(p_i, r)$ and $\mathrm{mg}(p_j, r)$, respectively. Then $\measuredangle(\theta_i'(r), \theta_j'(r)) > \beta_1(\delta_1') > 2\pi/3 = \arccos(-\frac{1}{2})$. To prove this, let θ be any $\mathrm{mg}(p_i, p_j)$. Consider a geodesic triangle in M_C with sides of length $l(\theta_i)$, $l(\theta_j)$, and $l(\theta)$; and P be the angle between the sides of length $l(\theta_i)$ and $l(\theta_j)$. We have $l(\theta_k) \leqslant d_p < 1 + \delta_1'(C)$ for $k = i, j$, and $l(\theta) \geqslant 2$, by (5.1). Consider another geodesic triangle in M_C with sides of length $1 + \delta_1'(C)$, $1 + \delta_1'(C)$, and 2; in this triangle, the angle between the sides of length $1 + \delta_1'(C)$ is $\beta_1(\delta_1'(C))$, by the construction of $\delta_1'(C)$. To compare P and $\beta_1(\delta_1'(C))$, apply Lemma 2 three times, changing one side at a time. Hence $P > \beta_1(\delta_1'(C))$. Apply Toponogov's Theorem (§2) to the geodesic triangle in $\tilde{M}$ with the vertices p_i, p_j, r and the sides given by the minimal geodesics θ_i, θ_j, and θ and the first triangle above, and obtain the $\measuredangle(\theta_i'(r), \theta_j'(r)) \geqslant P$. Hence,

$$\measuredangle\left(\theta_i'(r), \theta_j'(r)\right) \geqslant P > \beta_1\left(\delta_1'(C)\right) \geqslant \beta_1\left(x_0(C)\right) > 2\pi/3.$$

(5.3) If U_i, U_j, U_k are distinct, then $\overline{U}_i \cap \overline{U}_j \cap \overline{U}_k = \varnothing$. The existence of any point in $\overline{U}_i \cap \overline{U}_j \cap \overline{U}_k$ would give a contradiction with (5.2) and Lemma 3.

Remark. $\partial\overline{U}_0$ is not necessarily connected. If it is connected, then (5.3) is enough to prove Theorem 1.

Let $q \in \tilde{M} - \overline{U}_i$ be any point for some fixed i, and θ be any $\mathrm{mg}(p_i, q)$, with $\theta(0) = p_i$ and $\theta(d(p_i, q)) = q$. Define $t_r = \mathrm{Max}\{t \leqslant d(p_i, q) | \theta(t) \in \overline{U}_i\}$, and also set $r = \theta(t_r)$. Obviously, $0 < t_r < d(p_i, q)$ and $p_i \neq r \neq q$.

(5.4) $t_r = c_p(\rho_*(\theta'(p_i)))$, that is

$$\{r\} = \theta\left([0, d(p_i, q)]\right) \cap \partial\overline{U}_i = \theta\left([0, d(p_i, q)]\right) \cap \partial U_i.$$

Proof of (5.4). $r \in \partial\overline{U}_i \subset \partial U_i = \exp_{p_i}(\rho_*(p_i)^{-1}(\partial A_p))$; see Lemma 1. $\exists v \in \rho_*(p_i)^{-1}(\partial A_p)$, such that $\exp_{p_i} v = r$. Let $v' = v/\|v\|$. $\exp_p(t \cdot \rho_*(v'))$ is a geodesic in M starting from p, so it is a minimal geodesic to any point on its image for $0 \leqslant t \leqslant c_p(\rho_*(v')) = \|v\|$, before its cut point. Hence, its lift $\exp_{p_i}(tv')$ to $\tilde{M}$ from p_i is a minimal geodesic from p_i to any point on its image for $0 \leqslant t \leqslant \|v\|$. Hence, $\theta(t)$ and $\exp_{p_i} tv'$ are two $\mathrm{mg}(p_i, r)$, $r = \theta(t_r) = \exp_{p_i} v$. So, $\|v\| = t_r$. Since θ is a $\mathrm{mg}(p_i, q)$, for a fixed τ with $0 < \tau < d(p_i, q)$, θ is the $\mathrm{umg}(p_i, \theta(\tau))$, especially for $\tau = t_r < d(p_i, q)$. Therefore, $\forall t$, $\theta(t) = \exp_{p_i} tv'$, and hence, $v' = \theta'(p_i)$. $t_r = \|v\| = c_p(\rho_*(v')) = c_p(\rho_*(\theta'(p_i)))$. Obviously, $\theta((t_r, d(p_i, q)]) \cap \overline{U}_i = \varnothing$. $\forall t \in \left[0, c_p\left(\rho_*(v')\right)\right)$, $\exp_p t\rho_*(v') \in U$; so, $\theta(t) = \exp_{p_i} tv' \in U_i$, and $\theta(t) \notin \partial U_i \supset \partial\overline{U}_i$. So, the rest of (5.4) follows.

Lemma 4. *Let $q \in \tilde{M} - \overline{U}_i$ be any point, and θ be any* $\mathrm{mg}(p_i, q)$. *Let r be the unique element in $\partial\overline{U}_i \cap \theta([0, d(p_i, q)])$. By* (5.3), *there exists a unique U_{j_0} with $U_{j_0} \neq U_i$, such that $r \in \overline{U}_i \cap \overline{U}_{j_0}$. Then*:

(i) $A := \{\theta(t_r + t) | 0 < t \leqslant \mathrm{Min}(2x_0(C), d_{\tilde{M}}(q, r))\} \subset \mathrm{int}(\overline{U}_{j_0})$, *and*

(ii) *if* $d_{\tilde{M}}(q, r) > 2x_0$, *then* $\{\theta(t_r + t) | 2x_0 < t \leqslant \mathrm{Min}(\frac{1}{2}, d_{\tilde{M}}(q, r))\} \subset U_{j_0}$.

Proof of Lemma 4. (i) $A \cap \overline{U}_i = \varnothing$ by (5.4). Set Σ to be $\{U_k | \overline{U}_k \cap A \neq \varnothing, \omega_k(U_0) = U_k, \omega_k \in \pi_1(M, p)\}$. $A \neq \varnothing$, so $\Sigma \neq \varnothing$. $r \in \overline{A} \subset \bigcup_{k \in \Sigma} \overline{U}_k$, hence $\exists U_{k_0} \in \Sigma$ such that $r \in \overline{U}_{k_0}$. $r \in \overline{U}_i \cap \overline{U}_{j_0} \cap \overline{U}_{k_0}$. Since $U_i \neq U_{k_0}$, by $U_i \notin \Sigma$; (5.3) implies that $U_{k_0} = U_{j_0}$, $U_{j_0} \in \Sigma$, $\overline{U}_{j_0} \cap A \neq \varnothing$. Now suppose that $A \not\subset \mathrm{int}(\overline{U}_{j_0})$. Then there exists t_0 in $(0, \mathrm{Min}(2x_0, d_{\tilde{M}}(q, r))]$, such that $\theta(t_r + t_0) \in \partial\overline{U}_{j_0}$. Hence, there exists U_{j_1} such that $U_{j_1} \neq U_{j_0}$, $U_{j_1} \neq U_i$ and $\theta(t_r + t_0) \in \overline{U}_{j_0} \cap \overline{U}_{j_1}$. Let θ_0, θ_1, θ_2 be any $\mathrm{mg}(\theta(t_r + t_0), p_{j_0})$, $\mathrm{mg}(\theta(t_r + t_0), p_{j_1})$, and $\mathrm{mg}(p_i, p_{j_0})$, respectively. Consider the geodesic triangle in $\tilde{M}$ given by the geodesics θ from p_i to $\theta(t_r + t_0)$, θ_0 from $\theta(t_r + t_0)$ to p_{j_0} and θ_2 from p_i to p_{j_0}. By a similar argument as in the proof of (5.2), using Lemma 2 three times, Toponogov's Theorem and the second triangle in the construction of $\delta_1(C)$, we conclude that $\measuredangle(-\theta'(t_r + t_0), \theta_0'(0)) > \beta_2(x_0)$, since $d(p_i, \theta(t_r + t_0)) < 1 + 3x_0$, $d(p_i, p_{j_0}) \geqslant 2$, and $d(p_{j_0}, \theta(t_r + t_0)) < 1 + x_0$. Similarly, $\measuredangle(-\theta'(t_r + t_0), \theta_1'(0)) > \beta_2(x_0)$, and by (5.2), $\measuredangle(\theta_0'(0), \theta_1'(0)) > \beta_1(\delta_1'(C)) \geqslant \beta_1(x_0(C))$. Hence

$$\measuredangle(-\theta'(t_r + t_0), \theta_0'(0)) + \measuredangle(\theta_0'(0), \theta_1'(0)) + \measuredangle(\theta_1'(0), -\theta'(t_r + t_0)) > 2\beta_2(x_0) + \beta_1(x_0) = 2\pi.$$

This gives a contradiction with the fact that $\forall v_1, v_2, v_3 \in \mathbf{R}^3 - 0$ (hence in $\mathbf{R}^n$, $\forall n \in \mathbf{N}^+$), $\sum_{1 \leqslant i < j \leqslant 3} \measuredangle(v_i, v_j) \leqslant 2\pi$. So Lemma 4(i) holds: $A \subset \mathrm{int}(\overline{U}_{j_0})$.

(ii) Let $t_0 \in (2x_0, \mathrm{Min}(\frac{1}{2}, d_{\tilde{M}}(q, r))]$ be fixed. Let θ_3 and θ_4 be any $\mathrm{mg}(\theta(t_r + t_0), p_{j_0})$ and $\mathrm{mg}(r, p_{j_0})$, respectively. Let $q_1, q_2, q_3, q_4, \gamma_1, \gamma_2$, and γ_3 be in M_C as in the construction of $\delta_1(C)$. Recall that $C \leqslant 0$. $d(q_1, q_4) > 1 + x_0$, by Toponogov's Theorem and the Law of Cosines. Let q_5 be the unique point on γ_1 between q_3 and q_4, with $d(q_1, q_5) = d_{\tilde{M}}(r, p_{j_0})$. q_5 exists by the continuity of the distance function, and $d(q_1, q_3) = 1 \leqslant d_{\tilde{M}}(r, p_{j_0}) < 1 + \delta_1'(C) < 1 + x_0(C) < d(q_1, q_4)$. q_5 is unique, since every metric ball in M_C is strongly convex. Let γ_4 be the $\mathrm{umg}(q_5, q_1)$. If $q_5 = \gamma_1(t_1)$, then set $q_6 = \gamma_1(t_1 - t_0)$. $\frac{1}{2} \leqslant t_1 - t_0 < 1$. By strong convexity, $d(q_1, q_6) < 1$. Suppose that $d(p_{j_0}, \theta(t_0 + t_r)) = l(\theta_3) \geqslant 1$. Consider the geodesic triangle with vertices q_1, q_5, and q_6 in M_C, and the geodesic triangle in $\tilde{M}$ given by the minimal geodesics θ_3, θ_4, and θ, with vertices $\theta(t_r + t_0)$, r, and p_{j_0}. By Toponogov's Theorem and Lemma 2, $\measuredangle(\theta'(t_r), \theta_4'(0)) > \measuredangle(-\gamma_1'(q_5), \gamma_4'(q_5))$, since $d(q_1, q_5) = d_{\tilde{M}}(r, p_{j_0})$, $d(q_5, q_6) = t_0 = d_{\tilde{M}}(r, \theta(t_r + t_0))$ and $d(q_1, q_6) < 1 \leqslant d(p_{j_0}, \theta(t_r + t_0))$. By

(5.2), $\measuredangle(\theta_4'(0), -\theta'(t_r)) > \beta_1(\delta_1') = \alpha$. Hence, $\measuredangle(\gamma_1'(q_5), \gamma_4'(q_5)) > \alpha$; and by the construction of $\delta_1'(C)$, $\measuredangle(-\gamma_1'(q_4), \gamma_2'(q_4)) = \alpha_1$, $\alpha_1 \geqslant \pi - \alpha$. This contradicts the Gauss-Bonnet Theorem for a geodesic triangle in M_C with $C \leqslant 0$. Hence, $d(p_{j_0}, \theta(t_r + t_0)) < 1 = i_p$; consequently, $\theta(t_r + t_0) \in U_{j_0}$. t_0 was fixed, but arbitrarily. q.e.d.

We had supposed that order$(\pi_1(M, p)) \geqslant 3$ and chosen $U_0, U_1, U_2 \subset \tilde{M}$ such that $U_i \cap U_j = \varnothing$ for $0 \leqslant i < j \leqslant 2$ and $\overline{U}_0 \cap \overline{U}_i \neq \varnothing$ for $i = 1, 2$. We will complete the proof of Theorem 1 after Lemma 5.

Lemma 5. *Let F: $\partial U_0 \to \mathbf{R}$, be defined by $F(q) = d_{\tilde{M}}(q, \overline{U}_1)$. Then*:

(i) *There does not exist any $q \in \partial U_0$ such that $F(q) = 3x_0(C)$, and*

(ii) *For any $q \in \overline{U}_0 \cap \overline{U}_2 \subset \partial U_0$, $F(q) \geqslant \frac{1}{2} - x_0(C)$, where U_i, $i = 0, 1, 2$, are as supposed to be as above.*

Proof of Lemma 5. (i) Suppose that $\exists q \in \partial U_0$ such that $d_{\tilde{M}}(q, \overline{U}_1) = 3x_0(C)$. Let θ be any mg(p_1, q). Let r be the unique point in $\partial\overline{U}_1 \cap \theta([0, d(p_1, q)])$, (5.4). $r \in \overline{U}_1$; so, $d(q, r) \geqslant 3x_0$. $\forall r' \in \partial\overline{U}_1$, $1 \leqslant d(r', p_1) < 1 + x_0$; hence, $d(q, p_1) < 1 + 4x_0$. So,

$$d(q, r) = d(q, p_1) - d(r, p_1) < 1 + 4x_0 - 1 = 4x_0,$$
$$3x_0 \leqslant d(q, r) < 4x_0 < \tfrac{1}{2}.$$

By Lemma 4(ii), $q \in U_{j_0}$ for some j_0. Hence, $q \in U_{j_0} \cap \partial U_0$. This gives a contradiction with the facts that each U_i is open, and $U_i = U_j$ if and only if $U_i \cap U_j \neq \varnothing$.

(ii) Let $q \in \overline{U}_2 \cap \overline{U}_0$ be any element, θ be any mg(p_1, q), and r be the unique point in $\partial\overline{U}_1 \cap \theta([0, d(p_1, q)])$; see (5.4). $r \neq q$, by (5.3). Let $r \in \partial\overline{U}_{i_0}$ for some i_0, $U_{i_0} \neq U_1$. By Lemma 4, $\theta(t_r + t) \in \mathrm{int}(\overline{U}_{i_0})$ for $0 < t \leqslant \mathrm{Min}(\frac{1}{2}, d_{\tilde{M}}(r, q))$. Suppose that $q \in \mathrm{int}(\overline{U}_{i_0})$, then $q \in \overline{U}_2 \cap \overline{U}_0 \cap \mathrm{int}(\overline{U}_{i_0}) \neq \varnothing$. It follows that $U_2 = U_{i_0} = U_0$, which is not the case. So, q is not in $\mathrm{int}(\overline{U}_{i_0})$, and consequently, $d_{\tilde{M}}(r, q) > \frac{1}{2}$. Finally, $d(q, \overline{U}_1) \geqslant \frac{1}{2} - x_0$ by the triangle inequalities. q.e.d.

Proof of Theorem 1 will be completed as follows. F is continuous by being a restriction of the distance function. By Lemma 1, $F(\partial U_0)$ is connected and $\subset \mathbf{R}$. $F(\overline{U}_0 \cap \overline{U}_1) = \{0\}$. $\varnothing \neq F(\overline{U}_0 \cap \overline{U}_2) \subset [\,\frac{1}{2} - x_0, \infty)$ and $3x_0 \notin F(\partial U_0)$, by Lemma 5. $\varnothing \neq \overline{U}_0 \cap \overline{U}_i \subset \partial U_0$, for $i = 1, 2$, and $0 < x_0 < 1/10$. This gives a contradiction with the existence of distinct U_0, U_1, and U_2 as above. Hence, order$(\pi_1(M, p)) \leqslant 2$. q.e.d.

We will use the following in the proof of Theorem 2. The proof follows from the proof of Theorem 1, since Lemma 5(i) and its preceding does not use the existence of U_2.

(5.5) **Proposition.** *If the hypothesis of Theorem* 1 *holds, and $\overline{U}_i \cap \overline{U}_j \neq \varnothing$, where U_i, U_j are as constructed as in* §4, *then $\forall q \in \partial U_i$, $d_{\tilde{M}}(q, \overline{U}_j) \leqslant 2x_0(C)$.*

6. The nonsimply connected case

This section is devoted to the proof of Theorem 2, so *its hypothesis is assumed everywhere in* §6.

Proof of Theorem 2. Construction of $\delta_2(C)$. Let $C \in \mathbf{R}$ be given.

Case for $C \leqslant 0$. Let $x \in [0, \frac{1}{4})$. Consider two geodesic triangles with sides of length 1, 1, $1 - 4x$; and 1, 1, $2 - 4x$ in M_C. Let $\beta_3(x)$ and $\beta_4(x)$ be the angles between the sides of length 1 in the first and second triangles, respectively. There exists a unique $x_1 \in (0, \frac{1}{4})$ such that $\beta_3(x_1(C)) + \beta_4(x_1(C)) = \pi$, by Lemma 2, and β_3, β_4 being strictly decreasing continuous functions of x. Let $x_2(C) = \mathrm{Min}(x_0(C), x_1(C))$, where $x_0(C)$ is as in Theorem 1. Let q_1, q_2, q_3, γ_1, β_1, and $\delta_1(C)$ be as in Theorem 1. Set $q_7 = \gamma_1(1 + 2x_2(C))$, and let γ_5 be the umg(q_7, q_1), and $\alpha_2(C) = \measuredangle(-\gamma_1'(q_7), \gamma_5'(q_7))$. Define $\delta_2'(C) = \mathrm{Min}(x_2(C), \beta_1^{-1}(\pi - \alpha_2))$, and $\delta_2(C) = 1 - (1 + \delta_2'(C))^{-1}$.

Case for $C > 0$. Set $\delta_2(C) = \delta_2(0)$.

Let M^n and $p \in M^n$ be as in the hypothesis. By multiplying the metric with $1/i_M$, the hypothesis becomes: (i) $K_M \geqslant \mathrm{Min}(C, 0)$, (ii) $1 = i_M \leqslant i_p \leqslant d_p < 1 + \delta_2'(C)$, (iii) $\pi_1(M) = \mathbf{Z}_2$.

Let $U = M - C_p$, and construct U_0 and U_1 in $\tilde{M}$, as in §4. We have $p_i \in U_i$, $\rho(p_i) = p$ for $i = 0, 1$, $U_0 \cap U_1 = \varnothing$ and $\overline{U_0} \cup \overline{U_1} = \tilde{M}$. We need Lemmas 6, 6′, and 7 for proving Theorem 2.

Lemma 6. $\forall w \in UM_p$, $d_M(\exp_p w, \exp_p -w) < 1 = i_M$.

Proof of Lemma 6. Given any $v \in U\tilde{M}_{p_0}$, let $q(v) = \exp_{p_0} v$ and

$$r(v) = \exp_{p_0}\big(v \cdot c_p(\rho_*(v))\big).$$

$$d_{\tilde{M}}(q(v), r(v)) \leqslant c_p(\rho_*(v)) - 1 \leqslant d_p - i_p < \delta_2'(C) \leqslant x_2(C).$$

Since $x_2(C) \leqslant x_0(C)$ and α_2 is constructed in a similar way to α_1, with the hypothesis of Theorem 2, x_0 can be replaced by x_2 in the proofs of Lemmas 4(ii) and 5(i), and therefore, in Proposition (5.5). So $d_{\tilde{M}}(r(v), \overline{U_1}) \leqslant 2x_2(C)$. $r(v) \notin U_1$ and $\overline{U_1}$ is compact, so $\exists s(v) \in \partial\overline{U_1}$ such that $d_{\tilde{M}}(s(v), r(v)) = d_{\tilde{M}}(r(v), \overline{U_1})$. Since $s(v)$ is in $\partial\overline{U_1} \subset \exp_{p_1}((\rho_*(p_1))^{-1}(\partial A_p))$, $\exists v' \in U\tilde{M}_{p_1}$ with $s(v) = \exp_{p_1}(v' \cdot c_p(\rho_*(v')))$. Obviously, v' depends on v and the choice of $s(v)$.

$$\begin{aligned} &d_{\tilde{M}}\big(\exp_{p_0} v, \exp_{p_1} v'\big) \\ &\qquad \leqslant d_{\tilde{M}}\big(\exp_{p_0} v, r(v)\big) + d_{\tilde{M}}(r(v), s(v)) + d_{\tilde{M}}\big(s(v), \exp_{p_1} v'\big) \\ &\qquad < \delta_2'(C) + 2x_2(C) + \delta_2'(C) \leqslant 4x_2(C). \end{aligned}$$

Let T be the nontrivial deck transformation on $\tilde{M}$, i.e. $\rho(T(m)) = \rho(m)$, $T(m) \neq m$, $T^2(m) = m$ $\forall m \in \tilde{M}$, and T is an isometry. $d_{\tilde{M}}(m, T(m)) \geqslant 2i_M$

$= 2\ \forall m \in \tilde{M}$, since, for any $\psi \in \mathrm{MG}(m, T(m))$, $\rho(\psi)$ is a geodesic loop at $\rho(m)$. Therefore,

$$d_{\tilde{M}}\big(q(v), T(\exp_{p_1} v')\big) \geqslant d_{\tilde{M}}\big(\exp_{p_1} v', T(\exp_{p_1} v')\big) - d_{\tilde{M}}\big(q(v), \exp_{p_1} v'\big)$$
$$> 2 - 4x_2(C) \geqslant 2 - 4x_1(C).$$

Let $\sigma(t) = \exp_{p_1} tv'$. Consider the geodesic triangle in $\tilde{M}$ with vertices p_0, $q(v)$ and $T(\exp_{p_1} v')$, and sides given by the minimal geodesics $\exp_{p_0} tv$, $0 \leqslant t \leqslant 1$, $T(\sigma(t))$, $0 \leqslant t \leqslant 1$, and any $\mathrm{mg}(q(v), T(\sigma(1)))$. We have $d_{\tilde{M}}(q(v), p_0) = 1$, $d(q(v), T(\sigma(1))) > 2 - 4x_1(C)$ and $d_{\tilde{M}}(p_0, T(\sigma(1))) = d_{\tilde{M}}(T(p_0), \sigma(1)) = d_{\tilde{M}}(p_1, \sigma(1)) = 1$. Consider any geodesic triangle in M_C with side lengths 1, 1, and $d_{\tilde{M}}(q(v), T(\sigma(1)))$, and let P be the angle between the sides of length 1. By Toponogov's Theorem, $\sphericalangle(v, T_*(v')) \geqslant P$, since $T(\sigma(t)) = T(\exp_{p_1} tv') = \exp_{p_0} t \cdot T_*(v')$. On the other hand, by Lemma 2, $P > \beta_4(x_1(C))$, since $d_{\tilde{M}}(q(v), T(\sigma(1))) > 2 - 4x_1(C)$. Therefore, $\sphericalangle(v, T_*(v')) > \beta_4(x_1(C))$, and hence, $\sphericalangle(-v, T_*(v')) < \pi - \beta_4(x_1) = \beta_3(x_1)$. Consider the geodesic hinge in $\tilde{M}$ with vertex p_0, the minimal geodesics $\exp_{p_0} - tv$ and $T(\exp_{p_1} tv') = T(\sigma(t))$, from p_0 to $\exp_{p_0} - v$ and $T(\sigma(1)) = T(\exp_{p_1} v')$, respectively. Also, consider a geodesic triangle with side lengths $1, 1, 1 - 4x_1(C)$ in M_C. Apply Toponogov's Theorem and Lemma 2 in a similar fashion as above to obtain that $d_{\tilde{M}}(\exp_{p_0} - v, T(\exp_{p_1} v')) < 1 - 4x_1(C)$, by taking a hinge in M_C of two minimal geodesics of length 1, starting from the same point with an angle of $\sphericalangle(-v, T_*(v'))$ between them. Let $w \in UM_p$ be any element. There exists a unique $v \in U\tilde{M}_{p_0}$ such that $\rho_*(v) = w$. Choose v' depending on v as above. Since ρ is a local isometry, $\forall m, m_1, m_2 \in \tilde{M}$, $\rho(T(m)) = \rho(m)$, $\exp_{\rho(m)}(\rho_*(\cdot)) = \rho(\exp_m(\cdot))$ on $T\tilde{M}_m$, and $d_{\tilde{M}}(m_1, m_2) \geqslant d_M(\rho(m_1), \rho(m_2))$, we have

$$\begin{aligned}
&d_M(\exp_p w, \exp_p - w)\\
&\leqslant d_M\big(\exp_p - w, \rho\big(T(\exp_{p_1} v')\big)\big) + d_M\big(\rho(\exp_{p_1} v'), \exp_p w\big)\\
&\leqslant d_{\tilde{M}}\big(\exp_{p_0}\big(\rho_*(p_0)^{-1}(-w)\big), T(\exp_{p_1} v')\big) + d_{\tilde{M}}\big(\exp_{p_1} v', \exp_{p_0}\big(\rho_*(p_0)^{-1} w\big)\big)\\
&= d_{\tilde{M}}\big(\exp_{p_0} - v, T(\exp_{p_1} v')\big) + d_{\tilde{M}}\big(\exp_{p_1} v', \exp_{p_0} v\big)\\
&< 1 - 4x_1(C) + 4x_2(C) \leqslant 1 = i_M.
\end{aligned}$$

Therefore, $d_M(\exp_p w, \exp_p - w) < 1$ and this does not depend on the choice of v'. w was arbitrary, so it is true for all w in UM_p.

Lemma 6′. $\forall v \in U\tilde{M}_{p_0}$, $d_{\tilde{M}}(T(\exp_{p_0} - v), \exp_{p_0} v) < 1 = i_M \leqslant i_{\tilde{M}}$.

The proof of this follows from above.

Lemma 7. *There exists a continuous function f: $\mathbf{R}P^n \to M^n$ such that $f|f^{-1}(B_r(p, M))$ is a diffeomorphism onto $B_r(p, M)$ for some $r > 0$, and $f(B_{r'}(a, \mathbf{R}P^n)) = B_{r'}(p, M)$, where $\{a\} = f^{-1}(p)$ and $\forall r' \leqslant r$.*

Proof of Lemma 7. Given any $w \in UM_p$, $\exists$umg$(\exp_p w, \exp_p - w)$, θ_w, since, by Lemma 6, $d_M(\exp_p w, \exp_p - w) < i_M$. $l(\theta_w) = l(\theta_{-w})$. By symmetry, $\theta_w(t) = \theta_{-w}(l(\theta_w) - t)$ and hence, $\theta_w(\frac{1}{2}l(\theta_w)) = \theta_{-w}(\frac{1}{2}l(\theta_w))$. If $w_1, w_2 \in T\mathbf{R}P^n_a$ with $\|w_i\| = \frac{1}{2}\pi$, $i = 1, 2$, then, $\exp_a w_1 = \exp_a w_2$ if and only if $w_1 = \pm w_2$, where $a \in \mathbf{R}P^n$ is any fixed point. Let ψ be an isometry of $T\mathbf{R}P^n_a$ onto TM^n_p.

$$\begin{array}{ccc}
\bar{B}_{\pi/2}(0, T\mathbf{R}P^n_a) & \xrightarrow{\quad\psi\quad} & \bar{B}_{\pi/2}(0, TM_p) \\
\Big\downarrow{\scriptstyle \exp_a} & & \Big\downarrow{\scriptstyle h} \\
\mathbf{R}P^n & \xrightarrow{\qquad f \qquad} & M^n
\end{array}$$

$$h(y) = \begin{cases} \exp_p y & \text{if } 0 \leqslant \|y\| \leqslant 1, \\ \theta_{y/\|y\|}\big((\|y\| - 1) \cdot l(\theta_{y/\|y\|})/(\pi - 2)\big) & \text{if } 1 \leqslant \|y\| \leqslant \pi/2. \end{cases}$$

Let $w \in T\mathbf{R}P^n_a$ such that $\|w\| = \pi/2$. $h(\psi(w)) = \theta_{\psi(w)/\|\psi(w)\|}(\frac{1}{2}l(\theta_{\psi(w)/\|\psi(w)\|}))$ $= h(\psi(-w))$. Since, $\exp_a$ is one-to-one on the interior of $\bar{B}_{\pi/2}(0, T\mathbf{R}P^n_a)$, and by above, there exists a unique well-defined function f: $\mathbf{R}P^n \to M^n$ which makes the above diagram commutative.

(6.1) f is continuous. The continuity of f on $\exp_a(B_1(0, T\mathbf{R}P^n_a))$ is obvious. Let $w_n \in UM_p$, $n \in \mathbf{N}$, and $w_n \to w_0$ as $n \to \infty$. Let $q_n = \exp_p w_n$ and $q'_n = \exp_p - w_n$ $\forall n \in \mathbf{N}$. Since θ_{w_n} is the umg(q_n, q'_n), $q_n \to q_0$ and $q'_n \to q'_0$; $\{\theta_{w_n} | n \in \mathbf{N}^+\}$ has a convergent subsequence converging to a mg(q_0, q'_0). There exists only one such minimal geodesic, namely θ_{w_0}, and all θ_{w_n} lie in a compact set; therefore, we conclude that $\theta_{w_n} \to \theta_{w_0}$ as geodesics, i.e. if $t_n \in [0,1]$ $\forall n \in \mathbf{N}$, with $t_n \to t_0$ as $n \to \infty$, and if $r_n = \theta_{w_n}(t_n \cdot w_n \cdot l(\theta_{w_n}))$ $\forall n \in \mathbf{N}$, then $\lim_{n \to \infty} r_n = r_0$. Otherwise, if there existed two distinct limit points r_0, r'_0 of $\{r_n | 1 \leqslant n < \infty\}$, then by the continuity of the distance function and $l(\theta_w) = d(\exp_p w, \exp_p - w)$, we have that $d(r_0, q_0) = d(r'_0, q_0)$, $d(r_0, q'_0) = d(r'_0, q'_0)$ and $d(r_0, q_0) + d(r_0, q'_0) = d(q_0, q'_0)$ which will lead to two distinct mg(q_0, q'_0), one passing through r_0, the other one through r'_0; this would give a contradiction with θ_{w_0} being the umg(q_0, q'_0). The continuity of f follows this argument easily. Also see [7, pp. 44, 45].

Although f is continuous, it may not be smooth. $l(\theta_w) < 1$, and $d_M(p, \exp_p w) = d_M(p, \exp_p - w) = 1$, so θ_w never passes through p. Let $r \in \mathbf{R}$ be $\frac{1}{2}\mathrm{Min}\{d_M(p, \theta_w(t)) | w \in UM_p,\ 0 \leqslant t \leqslant l(\theta_w)\}$. Clearly, $1 \geqslant 2r > 0$. Therefore, $f^{-1}(B_r(p, M)) = B_r(a, \mathbf{R}P^n)$ and on this set f is defined by nonsingular one-to-one exponential maps; so, it is a diffeomorphism onto $B_r(p, M)$. The rest follows from the construction of f. q.e.d.

By Lemma 6$'$, $\forall v \in U\tilde{M}_{p_0}$, $d_{\tilde{M}}(\exp_{p_0} v, T(\exp_{p_0} - v)) < 1 = i_M \leqslant i_{\tilde{M}}$. Let $\tilde{\theta}_v$ be the umg$(\exp_{p_0} v, T(\exp_{p_0} - v))$. $\rho(\tilde{\theta}_v)$ is a geodesic from $\rho(\exp_{p_0} v) =$

$\exp_p(\rho_*(v))$ to $\rho(T(\exp_{p_0} - v)) = \exp_p(-\rho_*(v))$, whose length is $< 1 = i_M$. Therefore $\rho(\tilde{\theta}_v) = \theta_{\rho_*(v)}$. Define

$$\gamma_v(t) = \begin{cases} \exp_{p_0} tv & \text{if } 0 \leqslant t \leqslant 1, \\ \tilde{\theta}_v\big((t-1)\cdot\big(l(\tilde{\theta}_v)\big)/(\pi - 2)\big) & \text{if } 1 < t \leqslant \pi - 1, \\ T\big(\exp_{p_0}(-v(\pi - t))\big) & \text{if } \pi - 1 < t \leqslant \pi. \end{cases}$$

Clearly $\gamma_v(t)$ is a continuous curve from p_0 to p_1. Hence $\rho(\gamma_v(t))$ represents the nontrivial element of $\pi_1(M, p)$. Obviously, $f(\exp_a(t(\psi^{-1}(\rho_*(v))))) = \rho(\gamma_v(t))$. Hence $f_*\colon \pi_1(\mathbf{R}P^n) \to \pi_1(M)$ is bijective. By Lemma 7, $f_*\colon H_n(\mathbf{R}P^n, \mathbf{R}P^n - a) \to H_n(M, M - p)$ is an isomorphism, i.e. f has local degree ± 1 with $\mathbf{Z}$-coefficients.

The rest of the proof follows as in Samelson [17], and Berger [1, pp. 135–141]. Although the results of Samelson are obtained under different hypothesis, only the existence of a continuous function from $\mathbf{R}P^n$ to M of local degree ± 1 is used, and the rest of the arguments do not use any other assumption. These proofs are purely algebraic topological.

M^n or $\mathbf{R}P^n$ may not be orientable, so if we use $\mathbf{Z}_2$-coefficients, then f^* is an isomorphism from $H^n(M, \mathbf{Z}_2)$ onto $H^n(\mathbf{R}P^n, \mathbf{Z}_2)$ by Poincaré duality and having field coefficients.

(6.2) $f^*\colon H^*(M, \mathbf{Z}_2) \to H^*(\mathbf{R}P^n, \mathbf{Z}_2)$ is an isomorphism. f^* is injective, since for any $0 \neq e \in H^*(M, \mathbf{Z}_2)$, $\exists e' \in H^*(M, \mathbf{Z}_2)$ with $e \cup e' = [M]$ and $f^*(e) \cup f^*(e') = f^*(e \cup e') = f^*([M]) = [\mathbf{R}P^n] \neq 0$, so $f^*(e) \neq 0$. Since $f_*(\pi_1(\mathbf{R}P^n, a)) = \pi_1(M, p)$, it follows that $f^*(H^1(M, \mathbf{Z}_2)) = H^1(\mathbf{R}P^n, \mathbf{Z}_2) = \mathbf{Z}_2$. $H^*(\mathbf{R}P^n, \mathbf{Z}_2)$ is a truncated polynomial ring with one generator, namely the nontrivial element of $H^1(\mathbf{R}P^n, \mathbf{Z}_2)$. Hence, (6.2) holds.

By Proposition C of Samelson [17], M^n is oriented if and only if n is odd. Whenever n is odd, both M^n and $\mathbf{R}P^n$ are $\mathbf{Z}$-orientable; and f_* has local and global degree ± 1 with $\mathbf{Z}$-coefficients. Hence, $f^*\colon H^*(M, \mathbf{Z}) \to H^*(\mathbf{R}P^n, \mathbf{Z})$ is still injective, (see Browder [5, p. 8, Theorem I.2.5]). Also by similar proofs to Theorems D and E of Samelson [17]; for n is either odd or even, $f^*\colon H^*(M, \mathbf{Z}) \to H^*(\mathbf{R}P^n, \mathbf{Z})$ is an isomorphism.

Again using similar arguments to Samelson's proofs, a stronger conclusion can be obtained as follows. There exists a unique function $\tilde{f}\colon S^n \to \tilde{M}^n$ which makes the following diagram commutative:

$$\begin{tikzcd} S^n \arrow[r, "\tilde{f}"] \arrow[d, "\rho'"'] & \tilde{M}^n \arrow[d, "\rho"] \\ \mathbf{R}P^n \arrow[r, "f"] & M^n \end{tikzcd}$$

Since f induces an isomorphism on π_1 level, it follows that $\tilde{f}$ has local degree ± 1. By Browder [5, p. 8, Theorem I.2.5], f^*: $H^*(\tilde{M}, \mathbf{Z}) \to H^*(S^n, \mathbf{Z})$ is injective, and hence by Whitehead's Theorem (see Spanier [18]), $\tilde{M}$ is a homotopy sphere. By Lopez de Medrano [14, p. 43], M^n has the homotopy type of $\mathbf{R}P^n$ since the $\mathbf{Z}_2$ action on $\tilde{M}$, which yields M as a quotient, is a smooth action. q.e.d.

An elementary calculation shows that $\delta_2(0) = (13 - 4\sqrt{7})/57 \simeq 0.04$ and $\delta_1(0) \simeq 0.087$.

7. A special case: Tangential cut locus away from tangential conjugate locus

In this section, we prove Theorems 3, 4, and 5. They investigate the case in which the first tangential conjugate locus is bounded away from the cut locus in the tangent space of a fixed point.

Lemma 8. (*Gromoll, Klingenberg & Meyer* [11, *pp*. 198–199]). *Let M be a complete Riemannian manifold, $p \in M$, and $\exp_p$: $B_R(0, TM_p) \to M$ be of maximal rank. Given v and w in $B_R(0, TM_p)$ such that $v \neq w$, and $\exp_p v = \exp_p w =: r \in M$. For $t_0 \in [0,1]$ fixed, let $q = \exp_p t_0 v$, c_0: $[0,1] \to M$ be the geodesic given by $c_0(t) = \exp_p t t_0 v$ from p to q, and c_1: $[0,1] \to M$ be the broken geodesic given by*

$$c_1(t) = \begin{cases} \exp_p(2tw) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \exp_p((1-(2t-1)(1-t_0))v) & \text{if } \frac{1}{2} < t \leqslant 1. \end{cases}$$

For any homotopy H: $[0,1] \times [0,1] \to M$ between c_0 and c_1, fixing the end points, i.e. $H(i,t) = c_i(t)$ $\forall t \in [0,1]$, for $i = 0, 1$, and $H(s,0) = p$, $H(s,1) = q$ $\forall s \in [0,1]$, then there exists $s_0 \in [0,1]$ so that $l(c_0) + l(H(s_0, t)) \geqslant 2R$.

Lemma 9. $\forall C \in \mathbf{R}$, $\forall \alpha \in (0, \pi)$, $\exists \delta = \delta(\alpha, C) > 0$ *such that for any compact Riemannian manifold M^n with $K_M \cdot d_M^2 \geqslant C$, and if $\exists p \in M$ with* (i) $i_M/d_p > 1 - \delta(\alpha, C)$, *and* (ii) $\exp_p$: $\bar{B}_{d_p}(0, TM_p) \to M$ *is of maximal rank, then, for any $q \in C_p$ and for any two distinct* $\mathrm{mg}(p,q)$ γ_1, γ_2, *we have* $\measuredangle(\gamma_1'(q), \gamma_2'(q)) > \alpha$.

Proof of Lemma 9. Construction of $\delta(\alpha, C)$: Given $C \in \mathbf{R}$, and $\alpha \in (0, \pi)$.

Case for $C \leqslant 0$. Let $x \in [0, \infty)$ and consider a geodesic triangle in M_C with sides of length $x + \frac{1}{2}$, $x + \frac{1}{2}$, and 1, let $\beta_5(x)$ be the angle between the sides of length $x + \frac{1}{2}$. $\beta_5(x)$ is a strictly decreasing continuous function of x, by Lemma 2. $\lim_{x \to \infty} \beta_5(x) = 0$, and $\beta_5(0) = \pi$. Define $\delta'(\alpha, C) = \beta_5^{-1}(\alpha)$, and $\delta(\alpha, C) = 1 - (1 + \delta'(\alpha, C))^{-1}$.

Case for $C > 0$. Define $\delta(\alpha, C) = \delta(\alpha, 0)$.

As in the proofs of Theorems 1 and 2, by normalizing the metric by $i_M = 1$, the hypothesis becomes $K_M \geqslant \mathrm{Min}(C, 0)$, $1 = i_M \leqslant d_p < 1 + \delta'(\alpha, C)$ and the other conditions remain unchanged. Let γ_1, γ_2 be as in the hypothesis, and $l = d_M(p, q)$. Define f: $[0, l] \to \mathbf{R}$ by $f(s) = d_M(\gamma_1(s), \gamma_2(s))$. f is continuous, $f(0) = f(l) = 0$ and $f(s) > 0$, for $0 < s < l$.

(7.1) There exists $t_0 \in (0, l)$ such that $f(t_0) = 1$.

Proof of (7.1). Suppose that $f(s) < 1 = i_M$, $\forall s \in (0, l)$. For any $s \in [0, l]$, let $\theta_s(t)$ be the umg$(\gamma_1(s), \gamma_2(s))$. $\theta_s(t)$ depends on s continuously, i.e. $\lim_{s \to s_0} \theta_s(t) = \theta_{s_0}(t)$ by the uniqueness of $\theta_s(t)$ for each s. The proof of this is the same as (6.1) of Lemma 7. By definition $d_p \geqslant l$. Let $v = l \cdot \gamma_1'(0)$, $w = l \cdot \gamma_2'(0)$ and $t_0 = 0$, for applying Lemma 8. $c_0(t) = p\ \forall t$,

$$c_1(t) = \begin{cases} \exp_p 2tw & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \exp_p (2 - 2t)v & \text{if } \frac{1}{2} < t \leqslant 1. \end{cases}$$

Obviously, $\exp_p w = \exp_p v = q$. Set $I = [0, 1]$ and define a homotopy H: $I \times I \to M$ as

$$H(s, t) = \begin{cases} p & \text{if } s = 0, \\ \gamma_2(2lt) & \text{if } s > 0, \text{ and } 0 \leqslant t \leqslant \frac{1}{2}s, \\ \theta_{sl}\left(f(sl) \dfrac{t - 1 + \frac{1}{2}s}{s - 1} \right) & \text{if } 1 > s > 0, \text{ and } \frac{1}{2}s \leqslant t \leqslant 1 - \frac{1}{2}s, \\ q & \text{if } s = 1, \text{ and } t = \frac{1}{2}, \\ \gamma_1(l(2 - 2t)) & \text{if } s > 0, \text{ and } 1 - \frac{1}{2}s \leqslant t \leqslant 1. \end{cases}$$

Continuity of f follows from the continuity of γ_1 and γ_2 and the continuous dependence of $\theta_s(t)$ on s. Clearly, $\gamma_2(sl) = \theta_{sl}(f(sl))$, and $\gamma_1(sl) = \theta_{sl}(0)$. It is straightforward to show that H is continuous. $H(0, t) = p = c_0(t)$, $H(1, t) = c_1(t)\ \forall t \in I$, and $H(s, 0) = H(s, 1) = p\ \forall s \in I$. $\exp_p$: $\overline{B}_{d_p}(0, TM_p) \to M$ is of maximal rank, hence, $\exists \tau > 0$ such that $\exp_p$: $B_{d_p + \tau}(0, TM_p) \to M$ is of maximal rank. Therefore, Lemma 8 is applicable and $\exists s_0 \in I$ such that

$$(7.2) \qquad l(H(s_0, t)) + l(c_0) = l(H(s_0, t)) \geqslant 2(d_p + \tau) > 2d_p.$$

$H(s_0, t)$ is a union of broken geodesic segments with parametrizations other than arclength: from p to $\gamma_2(s_0 l)$ along γ_2; from $\gamma_2(s_0 l)$ to $\gamma_1(s_0 l)$ along $\theta_{s_0 l}$ with opposite orientation and from $\gamma_1(s_0 l)$ to p along γ_1, with the opposite orientation. Since γ_1, γ_2, and θ_s are minimal geodesics between those points,

$$\begin{aligned} &l(H(s_0,t)) \\ &\quad = d(p,\gamma_2(s_0l)) + d(\gamma_2(s_0l),\gamma_1(s_0l)) + d(\gamma_1(s_0l),p) \\ &\quad \leqslant d(p,\gamma_2(s_0l)) + d(\gamma_2(s_0l),q) + d(q,\gamma_1(s_0l)) + d(\gamma_1(s_0l),p) \\ &\quad = 2d(p,q) \leqslant 2d_p \quad \text{since } \gamma_1 \text{ and } \gamma_2 \text{ are } \mathrm{mg}(p,q). \end{aligned} \tag{7.3}$$

(7.2) and (7.3) are contradictory; therefore, such an H should not exist, and finally, (7.1) must hold: there exists $t_0 \in (0, l)$ such that $f(t_0) = 1 = i_M$.

$\frac{1}{2} \leqslant t_0 \leqslant l - \frac{1}{2}$, by the triangle inequalities. Consider the geodesic triangle in M determined by the vertices $\gamma_1(t_0)$, $\gamma_2(t_0)$ and q, and minimal geodesics γ_1, γ_2 and θ_{t_0} between the appropriate points. θ_{t_0} may not be unique anymore, but any will work. $l - t_0 \leqslant l - \frac{1}{2} \leqslant d_p - \frac{1}{2} < \frac{1}{2} + \delta'(\alpha, C)$. Consider any geodesic triangle in M_C with side lengths 1, $l - t_0$ and $l - t_0$. By Toponogov's Theorem, Lemma 2 and the construction of $\beta_5(x)$, we obtain that $\measuredangle(\gamma_1'(q), \gamma_2'(q)) \geqslant \beta_5(l - t_0 - \frac{1}{2}) > \beta_5(\delta'(\alpha, C)) = \alpha$.

Proof of Theorem 3. Take $\delta_3(C) = \delta(2\pi/3, C)$. $d_M^2 \cdot K_M \geqslant C$ implies that $i_M^2 \cdot K_M \geqslant \mathrm{Min}(C, 0)$. By Lemma 9, for any q in C_p and any two distinct $\mathrm{mg}(p,q)$, γ_1, γ_2, $\measuredangle(\gamma_1'(q), \gamma_2'(q)) > 2\pi/3 = \arccos(-\frac{1}{2})$. There are at most two distinct $\mathrm{mg}(p,q)$ by Lemma 3. Since q is not conjugate to p along any $\mathrm{mg}(p,q)$, there are at least two such geodesics (for example, see [6, p. 93]). So the hypothesis of Sugahara's Theorem B [19, §2] is satisfied, and therefore, $\tilde{M}^n$ is homeomorphic to S^n and $\pi_1(M) = \mathbf{Z}_2$.

Lemma 10. *Let $w_i \in \mathbf{R}^N$, $i = 1, \cdots, k \leqslant 4$, such that $\|w_i\| = 1$ and $\langle w_i, w_j \rangle < 0$, if $i \neq j$. Then $w_1 - w_k, \cdots, w_{k-1} - w_k$ are linearly independent.*

Proof of this lemma is elementary and left to the reader.

Proof of Theorem 4. We define $N_p\colon C_p \to \mathbf{N}^+$ by $\forall q \in C_p$, $N_p(q)$ is the number of distinct $\mathrm{mg}(p,q)$'s. Since $\exp_p|\bar{B}_{d_p}(0, TM_p)$ is of maximal rank, $\exp_p$ is still nonsingular on a sufficiently small open neighborhood of $\bar{B}_{d_p}$. So, q is not conjugate to p along any minimal geodesic; we have $2 \leqslant N_p(q) < \infty$ by [6, p. 93], [19]. Set $V_i = N_p^{-1}(i+1)$. $C_p = \bigcup_{i=1}^{\infty} V_i$, $V_i \cap V_j = \varnothing$ if $i \neq j$. Take $\delta_4(\alpha, C) = \delta(\alpha, C)$ of Lemma 9.

Let $q \in C_p$ be any fixed point, and $\gamma_1, \cdots, \gamma_k$ be all of the distinct $\mathrm{mg}(p,q)$, i.e., $N_p(q) = k$ and $q \in V_{k-1}$. $\measuredangle(\gamma_i'(q), \gamma_j'(q)) > \sigma_4$ if $i \neq j$, by Lemma 9, and $k \leqslant 4$, by Lemma 3. Clearly, $V_i = \varnothing$ if $i \geqslant 4$. If σ_4 is replaced by σ_3 or σ_2, then furthermore $V_3 = \varnothing$ or $V_3 = V_2 = \varnothing$, respectively.

Set $l = d(p,q)$ and let $\tau > 0$ be such that $\exp_p|B_{d_p+\tau}$ is a local diffeomorphism. There exist an open ball $U \subset TM_p$ and an open set $U_q \subset M$ such that $0 \in U, q \in U_q, p \notin U_q, \forall i = 1, \cdots, k, U_i \supseteq l \cdot \gamma_i'(0) + U, \forall i \neq j, U_i \cap U_j = \varnothing$, $\forall i$, $U_i \subset B_{d_p+\tau}$ and $\exp_p|U_i\colon U_i \to U_q$ is a diffeomorphism. Let $f_i := (\exp_p|U_i)^{-1}\colon U_q \to U_i$ and $F_i := \|f_i\|\colon U_q \to \mathbf{R}\ \forall i$. Define $F_{ij} := F_i - F_j$ and $H_{ij}(q) := \{x \in U_q | F_{ij}(x) = 0\}$ only when $1 \leqslant i < j \leqslant k$. F_i are smooth functions on U_q, since

f_i are smooth and $0 \notin U_i$. (Grad F_i)$(q) = \gamma_i'(q)$ by Gauss' Lemma ([6], [11]) $\forall i$. $\forall i \neq j$, (Grad F_{ij})$(q) = \gamma_i'(q) - \gamma_j'(q) \neq 0$, i.e. F_{ij} is regular at q. Therefore, there exists an open neighborhood U_q' of q such that $U_q' \subset U_q$, and $\forall i \neq j$, F_{ij} is regular on U_q'. $\forall i$, $F_i(q) = l$. $H_{ij}(q) \cap U_q' = \{x \in U_q' | F_{ij}(x) = F_{ij}(q) = 0\}$ is locally a smooth submanifold of M of codimension 1, it contains q, and is open in its dimension by the Implicit Function Theorem. Furthermore, $\gamma_i'(q) - \gamma_j'(q)$ is orthogonal to $T(H_{ij}(q))_q$ which is a hyperplane in TM_q for $i \neq j$. If we set $w_i = \gamma_i'(q)$, then by Lemma 10, $\{\gamma_i'(q) - \gamma_k'(q) | i = 1, \cdots, k-1\}$ forms a linearly independent set. Hence, the set of $H_{ik}(q)$ is transversal at q, and consequently, there exists an open neighborhood U_q'' of q such that $U_q'' \subset U_q'$ and $H(q) = U_q'' \cap \bigcap_{i=1}^{k-1} H_{ik}(q)$ is an $n - k + 1$ dimensional submanifold of M locally, open in its dimension, containing q. Obviously, if $n = 2$, then $k \leqslant 3$.

(7.4) There exists an open neighborhood U_q''' of q such that $U_q''' \subset U_q''$ and $U_q''' \cap H(q) = U_q''' \cap V_{k-1} \subset C_p$. This follows from (7.5) and (7.6) below.

(7.5) There exists an open neighborhood U_q''' of q such that $U_q''' \subset U_q''$ and $U_q''' \cap H(q) \subset U_q''' \cap V_{k-1}$.

Proof of (7.5). Suppose that $\forall U_q'''$ open, $U_q''' \cap H(q) \not\subset U_q''' \cap V_{k-1}$, i.e., $\exists q_n \in (H(q) - V_{k-1}) \cap U_q''$, $\forall n \in \mathbf{N}$, such that $q_n \to q$ which is in $H(q) \cap V_{k-1} \cap U_q''$. $q_n \in H(q)$, so $\forall i = 1, \cdots, k-1$, $F_{ik}(q_n) = 0$. $\forall i = 1, \cdots, k$, define $\theta_{n,i}(t) := \exp_p(t \cdot f_i(q_n)/F_i(q_n))$ for large n (since for sufficiently large n, $q_n \neq p$, and $F_i(q_n) \neq 0$), for $0 \leqslant t \leqslant F_i(q_n)$. $\theta_{n,i}$ is a geodesic from p to q_n. For a fixed n, $\theta_{n,i}$ have the same length $F_i(q_n) = F_k(q_n)$, all are distinct for large n. Note that it is not necessary that $\theta_{n,i}$ are minimal. If $\theta_{n,i}$, $i = 1, \cdots, k$, are all of the distinct $\mathrm{mg}(p, q_n)$, then $q_n \in V_{k-1}$, which is not the case we supposed. So, there exists a minimal geodesic ψ_n distinct from all $\theta_{n,i}$, from p to q_n. Since $q_n \to q$, ψ_n has a convergent subsequence ψ_{n_m} converging to a $\mathrm{mg}(p, q)$, namely γ_{i_0}, for some i_0, $1 \leqslant i_0 \leqslant k$. Let q_m also represent the corresponding subsequence q_{n_m}. In this case, θ_{m,i_0} and ψ_m are distinct geodesics from p to q_m, and both sequences converge to γ_{i_0} as geodesics. $\exp_p | B_{d_p + \tau}$ is a local diffeomorphism, so, we conclude that $f_{i_0}(q_m) \to f_{i_0}(q)$, $\psi_m'(0) \cdot d(p, q_m) \to f_{i_0}(q)$ in TM_p and $f_{i_0}(q_m) \neq \psi_m'(0) \cdot d(p, q_m)$, since ψ_m and θ_{m,i_0} are distinct geodesics from p to q_m, and $\exp_p \psi_m'(0) \cdot d(p, q_m) = \psi_m(d(p, q_m)) = q_m = \exp_p f_{i_0}(q_m)$, for all m large. This contradicts the fact that $\exp_p | U_{i_0}$ is a diffeomorphism. So, such $\psi_n(t)$ should not exist, and for large n, q_n is in V_{k-1}; consequently, (7.5) holds.

(7.6) There exists an open neighborhood U_q''' of q such that $U_q''' \subset U_q''$ and $U_q''' \cap V_{k-1} \subset U_q''' \cap H(q)$.

Proof of (7.6). Suppose that $\forall U_q'''$ open, $U_q''' \cap V_{k-1} \not\subset U_q''' \cap H(q)$, i.e., $\exists q_n \in (V_{k-1} - H(q)) \cap U_q''$, $\forall n \in \mathbf{N}$, such that $q_n \to q$ which is in $H(q) \cap V_{k-1} \cap U_q''$. $q_n \in V_{k-1}$, so, there exists k distinct $\mathrm{mg}(p, q_n)$, say $\theta_{n,i}$, $i = 1, \cdots, k$. By Lemma 9, $\langle \theta_{n,i}'(0), \theta_{n,j}'(0) \rangle < -\frac{1}{4}$ for $i \neq j$. Therefore, the limit set of these geodesics contains at least k distinct $\mathrm{mg}(p, q)$. They have to be $\gamma_1, \cdots, \gamma_k$. For sufficiently large n, by rearranging i-indices for a fixed n, and by taking convergent subsequences, we may assume that $\theta_{n,i} \to \gamma_i$ as $n \to \infty$, as geodesics. $\theta_{n,i}'(0) \to \gamma_i'(0)$; $\theta_{n,i}'(0) \cdot d(p, q_n) \to \gamma_i'(0) \cdot d(p, q) = f_i(q)$ and obviously $f_i(q_n) \to f_i(q)$. For sufficiently large n, $\theta_{n,i}'(0) \cdot d(p, q_n) \in U_i$. $\exp_p(\theta_{n,i}'(0) \cdot d(p, q_n)) = q_n = \exp_p f_i(q_n)$. For sufficiently large n, $\theta_{n,i}'(0) \cdot d(p, q_n) = f_i(q_n)$; otherwise, this would contradict the fact that $\exp_p$ is a local diffeomorphism around $f_i(q)$. So, for sufficiently large n, and for $i = 1, \cdots, k$, $F_i(q_n) = \|\theta_{n,i}'(0) \cdot d(p, q_n)\| = d(p, q_n)$ and hence $F_{ij}(q_n) = 0$ and $q_n \in H(q)$. This gives the desired contradiction and hence it proves (7.6).

Finally, (7.4) follows from (7.5) and (7.6).

For the argument above, q was fixed but arbitrarily. For any $q \in V_{k-1}$, there exists U_q''' as in (7.4): $H(q) \cap U_q''' = V_{k-1} \cap U_q'''$, which is an open piece of an $n - k + 1$ dimensional smooth submanifold of M. This shows that V_{k-1} is an $n - k + 1$ dimensional submanifold of M, which is open in its dimension. If $q \in \overline{V}_{k-1}$, i.e. $\exists q_n \in V_{k-1}$, $\forall n \in \mathbf{N}$, $q_n \to q$ as $n \to \infty$; then, there are k distinct $\mathrm{mg}(p, q_n)$, and the limit set of them contains at least k distinct geodesics as in the proof of (7.6) or simply by $\exp_p$ being of maximal rank on $B_{d_p+\tau}$. However, there may be other $\mathrm{mg}(p, q)$; so, $q \in V_{k+m}$, $m \geqslant -1$. Hence, $\overline{V}_i - V_i \subset \bigcup_{j>i} V_j$. By Sugahara [19], V_1 is an open and dense subset of C_p. $\partial V_1 = \overline{V}_1 - V_1 = V_2 \cup V_3$. We only have $\partial V_2 \subset V_3$, since V_2 is not necessarily dense in $V_2 \cup V_3$ which may not be connected.

If σ_4 is replaced by σ_2 in the hypothesis, then $C_p = V_1$ by Lemmas 3 and 9. In this case, C_p is an $n - 1$ dimensional compact smooth submanifold of M. For any arbitrary but fixed $q \in C_p$, V_1 is locally given by $H(q) \cap U_q''' = \{x \in U_q''' \mid F_{12}(x) = 0\}$, a level set of a smooth regular function around q. $F_1(x)$ is a smooth function on U_q'''. Therefore, for $x \in C_p$, $F_1(x) = d(p, x)$ is a smooth function on C_p, and hence, $c_p(\cdot)$: $UM_p \to \mathbf{R}$ is smooth. For any μ, $0 < \mu < i_M$, $V_\mu =: \{\exp_p tv \mid v \in UM_p,\ 0 \leqslant t < c_p(v) - \mu\}$ is diffeomorphic to the open n-dimensional disc D^n and ∂V_μ is diffeomorphic to $\partial D^n = S^{n-1}$. Since $\exp_p$ is of maximal rank of $B_{d_p+\tau}$ and C_p is a smooth submanifold, locally around any $q \in C_p$ for $r \in U_q''' \cap C_p = U_q''' \cap H(q)$, for $i = 1, 2$, $(\mathrm{Grad}\ F_i)(r)$ depends on r smoothly. Hence $(\mathrm{Grad}\ F_{12})(r)$ and $\langle (\mathrm{Grad}\ F_1)(r), (\mathrm{Grad}\ F_2)(r) \rangle$ depend on r smoothly. However, $(\mathrm{Grad}\ F_1)(r) + (\mathrm{Grad}\ F_2)(r)$ is not necessarily 0 in TM_r. Hence, $M - V_\mu$ is homeomorphic (possibly diffeomorphic) to a smooth 1-disc bundle E over $V_1 = C_p$. In fact, this homeomorphism can be taken to be smooth everywhere on $M - V_\mu$ but except on C_p. So, M is homeomorphic to $\bar{D}^n \cup_a E$, where $a\colon S^{n-1} \to \partial E$ is an attaching diffeomorphism. Finally, Weinstein's Theorem, §2, is applicable, and M is homeomorphic to a nonsimply connected pointed Blaschke manifold, by Theorem 3 and $\delta_4(\sigma_2, C) = \delta_3(C)$.

Lemma 11 (*Cheeger & Gromoll*). *For any compact Riemannian manifold M^n, if $d_p < \pi/2\sqrt{K}$ for some $p \in M$, where $K = \mathrm{Max}(K_M)$, and $d_p = d(p, q)$ for some $q \in C_p$, then, there are at least $n + 1$ distinct* mg(p, q). *For $K \leqslant 0$, we mean ∞ instead of $K^{-1/2}$.*

Proof of Lemma 11. Let $\gamma_1, \cdots, \gamma_k$ be all of the distinct mg(p, q). Suppose that $k \leqslant n$. $\exists v \in TM_q$, such that $\|v\| = 1$ and $\forall i = 1, \cdots, k-1$, $\langle v, \gamma_i'(q)\rangle = 0$. We may choose w among $\pm v$ such that $\langle w, \gamma_k'(q)\rangle \geqslant 0$. Hence $\forall i = 1, \cdots, k$, $\langle w, \gamma_i'(q)\rangle \geqslant 0$. Let $\theta(t) =: \exp_q tw$, for $t \in (-1, 1)$. $\forall i$, construct f_i around q as in the proof of Theorem 4.

(7.7) $\forall i = 1, \cdots, k$, and for $t \in [0, 1]$, $F_i(\theta(t)) = \|f_i(\theta(t))\|$ is strictly increasing at $t = 0$. If $\langle w, \gamma_i'(q)\rangle > 0$, then (7.7) is obvious. If $\langle w, \gamma_i'(q)\rangle = 0$, then consider the pull-back metric from M on $B := B_{\pi/\sqrt{K}}(0, TM_p)$ by $\exp_p|B$ which is nonsingular and hence is a local diffeomorphism by [6, p. 30]. With this new metric on B, the metric ball of radius d_p $(< \pi/2\sqrt{K})$ around 0 in TM_p is strictly convex by Whitehead's Lemma [6, p. 103], [23]; and hence, (7.7) still holds. For all large $n \in \mathbf{N}$, let $q_n = \theta(1/n)$, and θ_n be any mg(p, q_n). $q_n \to q$ as $n \to \infty$; therefore, θ_n has a convergent subsequence θ_{n_m} converging to a mg(p, q), namely γ_j, for some j, $1 \leqslant j \leqslant k$. Let $r_m = q_{n_m}$ and $\psi_m = \theta_{n_m}$. For sufficiently large m, $\nu_m(t) := \exp_p tf_j(r_m)$ is not a mg(p, r_m), since for sufficiently large m,

$$l(\nu_m) = \|f_j(r_m)\| = F_j(\theta(1/n_m)) > F_j(\theta(0)) = d(p, q) = d_p \geqslant d(p, r_m).$$

So, we have $f_j(r_m) \to f_j(q)$, $\psi_m'(0) \cdot l(\psi_m) \to f_j(q)$, $f_j(r_m) \neq \psi_m(0) \cdot l(\psi_m)$ since ν_m is not a mg(p, r_m), and $\exp_p f_j(r_m) = \exp_p(\psi_m'(0) \cdot l(\psi_m)) = r_m \to q$ as $m \to \infty$. This gives a contradiction with the fact that $\exp_p$ is a local diffeomorphism around $f_j(q)$. Consequently, $k \geqslant n + 1$. q.e.d.

Proof of Theorem 5. Set $\delta_5(n, C) = \delta(\sigma_n, C)$ of Lemma 9, where $\sigma_n = \arccos(-1/n)$. Suppose that $d_p < \pi/2\sqrt{K}$. Let $q \in C_p$ be with $d(p, q) = d_p$. By Lemma 11, there should exist at least $n + 1$ distinct mg(p, q). Lemma 9 is applicable since $\exp_p$ is of maximal rank on $B_{\pi/2\sqrt{K}}(0, TM_p)$ [6, p. 30]; then by Lemma 3, there should exist at most n mg(p, q). This contradiction leads to $d_p \geqslant \pi/2\sqrt{K}$. Case for $K \leqslant 0$ follows.

8. Examples

Example 1. Let M be one of the following with their standard metrics: S^n, $\mathbf{R}P^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$, and $\mathbf{Ca}P^2$. $i_{\mathbf{R}P^n} = d_{\mathbf{R}P^n} = \frac{1}{2}\pi$, and if $M \neq \mathbf{R}P^n$, then $i_M = d_M = \pi$. Let $g(t)$ be a C^2 1-parameter family of metrics on a fixed M such that $g(0)$ is the standard one. Since, the diameter and injectivity radius depend on the metric continuously [8] and $g(0)$ has positive curvature, there exists a $\delta > 0$ such that for all $t \in (-\delta, \delta)$ and for all p in M, $i_p(g(t))/d_p(g(t)) > 1 - \delta_1(0)$.

Example 2. For any compact Riemannian manifold M^n, and any $\delta_0 > 0$, there exists a Riemannian metric g_1 on M such that $i_p(g_1)/d_p(g_1) > 1 - \delta_0$ for some $p \in M$. The construction of g_1: Let g_0 be any Riemannian metric on M, and choose $r \in \mathbf{R}$ with $0 < r < i_p(g_0)$ for any fixed p in M. There exists a smooth function ψ: $M \to [0, 1]$, with $\mathrm{Supp}(\psi) \subset B_r(p, M;\ g_0)$ and $\psi(B_{r(1-\frac{1}{2}\delta_0)}(p, M;\ g_0)) \equiv 1$. Let $d = d(M, g_0)$. Define $g_1 = (1 + (2d\psi/\delta_0 r)) \cdot g_0$. Then, $i_p(g_1) \geqslant (1 - \frac{1}{2}\delta_0) \cdot r \cdot (2d/\delta_0 r)$ and $d_p(g_1) \leqslant (2dr/\delta_0 r) + d$. Hence, $i_p(g_1)/d_p(g_1) \geqslant (2 - \delta_0)/(2 + \delta_0) > 1 - \delta_0$.

Remark. Example 2 shows that the curvature conditions of Theorems 1–4 cannot be removed. However, they might be replaced by weaker conditions. $\lim_{C \to -\infty} \delta_1(C) = 0$; since, $\delta_1(C)$ is decreasing as $C \to -\infty$, $\delta_1(C) > 0$, and the limit can not be positive by above.

Example 3. Consider the lattice $L := \mathbf{Z}e_1 + \mathbf{Z}e_2$ in $\mathbf{R}^2$, where $e_1 = (1, 0)$ and $e_2 = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$. $T^2 := \mathbf{R}^2/L$ is a flat hexagonal torus. One can show that $i_{T^2} = \frac{1}{2}$ and $d_{T^2} = 3^{-1/2}$. So, $\delta(0)$ of Theorems 1–3 cannot be made larger than $1 - \frac{1}{2}\sqrt{3}$.

Remark. Since for all p in M, any compact Riemannian manifold, $i_M \leqslant i_p \leqslant d_p \leqslant d_M$; all of the Theorems 1–5 are still valid if all of i_p and d_p are replaced by i_M and d_M, respectively.

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