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A VANISHING THEOREM FOR SEMIPOSITIVE LINE BUNDLES OVER NON-KÄHLER MANIFOLDS

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We prove in this paper the following vanishing theorem. If M is a compact complex manifold and L is a Hermitian holomorphic line bundle whose curvature form is everywhere semipositive and is strictly positive outside a set of measure zero, then $H^q(M, LK_M)$ vanishes for $q \geqslant 1$, where K_M is the canonical line bundle of M. In view of the results of Grauert-Riemenschneider [4] this is equivalent to the statement that any compact complex manifold M which admits such a line bundle L must be Moišezon (in the sense that the transcendence degree of the meromorphic function field of M must equal the complex dimension of M). This vanishing theorem is motivated by the conjecture of Grauert-Riemenschneider [4, p. 277]. [11, Conjecture I] which is still an open problem. The difficulty with the conjecture is how to prove the following special case.

Conjecture of Grauert-Riemenschneider. Let M be a compact complex manifold which admits a Hermitian holomorphic line bundle L whose curvature form is positive definite on a dense subset G of M. Then M is Moišezon.

The conjecture of Grauert-Riemenschneider was originally introduced for the purpose of characterizing Moišezon spaces by quasipositive torsion-free sheaves. Since then a number of other characterizations of Moišezon spaces have been obtained [11], [17], [16], [2], [10] which circumvent the difficulty of proving the Grauert-Riemenschneider conjecture by stating the characterizations in such a way that a proof can be obtained by using blow-ups, Kodaira's vanishing and embedding theorems, or L^2 estimates of $\bar\partial$ for complete Kähler manifolds.

Our vanishing theorem is equivalent to the confirmation of the conjecture of Grauert-Riemenschneider for the special case where $M - G$ is of measure zero

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in M. As a consequence it gives a characterization of Moišezon spaces in the spirit of Kodaira [7] and Grauert [3].

Some special cases of our vanishing theorem were proved earlier. Riemenschneider [12] proved it under the assumption that M is Kähler and the curvature form of L is positive at some point. It was also proved when the set of points where the curvature form of L is not positive definite is contained in a complex-analytic subvariety of dimension zero [11] or one [14]. A special case with additional assumptions on the eigenvalues of the curvature form of L was proved in [15].

As a way of proving the Grauert-Riemenschneider conjecture, Peternell is trying to develop, in the case of *degenerate* Kähler metrics, a theory to represent cohomology classes with coefficients in line bundles by bundle-valued harmonic forms and has obtained some partial results [8].

We sketch below our method of proof. By the theorem of Hirzebruch-Riemann-Roch (which for the case of a general compact complex manifold is a consequence of the index theorem of Atiyah-Singer [1]),

$$\sum_{q=0}^{n} (-1)^q \dim H^q(M, L^k) \geqslant ck^n$$

for some positive constant c when k is sufficiently large. To prove that M is Moišezon, it suffices to show that $\dim \Gamma(M, L^k) \geqslant ck^n/2$ for k sufficiently large. For that purpose it suffices to show that for any given positive number ε and for $q \geqslant 1$ one has $\dim H^q(M, L^k) \leqslant \varepsilon k^n$ for k sufficiently large. This one obtains by using the ideas of Poincaré [9] and Siegel [13] in the following way. Give M a Hermitian metric and represent elements of $H^q(M, L^k)$ by harmonic forms. By using the L^2 estimates of $\bar\partial$, one obtains a linear map from the space of harmonic forms to the space of cocycles. Take a lattice of points with distances $k^{-1/2}$ apart in a small neighborhood W of $M - G$. Then one uses the Schwarz lemma to show that any cocycle coming from a harmonic form via the linear map and vanishing at all the lattice points must vanish identically, otherwise its norm is so small that the $\bar\partial$-closed form constructed from it by using a partition of unity would have a norm smaller than that of the harmonic form in its cohomology class. It follows that $\dim H^q(M, L^k)$ is dominated by the number of lattice points (which is comparable to the volume of W times k^n), otherwise there is a nonzero linear combination of cocycles coming from a basis of harmonic forms via the linear map and vanishing at all the lattice points. The reason why such a lattice of points is chosen is that the pointwise square norm of a local holomorphic section of L^k is of the form $|f|^2 e^{-k\varphi}$, where f is a holomorphic function and φ is a plurisubharmonic function corresponding to the Hermitian metric of L. The factor $e^{-k\varphi}$ is an obstacle to applying the Schwarz lemma. To overcome the obstacle, one chooses a local trivialization of L so that φ as well as $d\varphi$ vanishes at a point. Then on the ball of radius $k^{-1/2}$ centered at that point, $e^{-k\varphi}$ is bounded below from zero and from above by constants independent of k.

This method of proof gives us a way of producing holomorphic sections, over a compact complex Hermitian manifold, of a Hermitian holomorphic line bundle which is not semipositive. All that is required is the assumption that, outside a set whose measure is small compared to some constants constructed from the manifold, the curvature form of the line bundle has a positive lower bound which is large compared to its upper bound on the set of small measure and to the torsion tensor of the manifold.

One can consider a stronger version of the Grauert-Riemenschneider conjecture which assumes only that the curvature form of L is positive semidefinite everywhere and positive definite at some point. This stronger form can be proved if one can prove the following conjecture concerning eigenvalues.

The eigenvalue conjecture. Let M be a compact complex manifold and L a Hermitian holomorphic line bundle over M whose curvature form is positive semidefinite everywhere and positive definite at some point. Then $\inf_{k>0} \lambda(M, L^k) > 0$, where $\lambda(M, L^k)$ is the smallest positive eigenvalue of the Laplacian $\bar{\partial}^*\bar{\partial}$ on the Hilbert space of all global L^2 sections of L^k over M.

This conjecture is very plausible, because the smallest positive eigenvalue should increase when the line bundle is more positive. The larger k is, the more positive L^k is. In the case of a strictly positive line bundle over a compact Kähler manifold, the square of the smallest positive eigenvalue is no less than the lower bound of the quadratic form which is the sum of the curvature form of the bundle and the Ricci curvature form of the manifold. Unfortunately, no one has yet found a way to estimate from below the smallest positive eigenvalue in the case of a semipositive line bundle. The only known lower eigenvalue estimates are for the Laplace-Beltrami operator for square integrable real-valued functions on compact Riemann manifolds. Even for the space of forms on a compact Riemannian manifold such estimates are unknown, because all the methods used so far for such estimates involve some kind of maximum principle for real-valued functions. In the last part of this paper we show how one derives the stronger form of the Grauert-Riemenschneider conjecture from the eigenvalue conjecture. We hope that this relationship between the two conjectures will provide some motivation and incentive to investigate the lower bound of the first eigenvalue for square integrable sections of a holomorphic line bundle.

Table of Contents

1. Schwarz's lemma

In this section we prove a Schwarz lemma for L^2 norms of sections of the kth power of a line bundle. It is formulated in such a way as to avoid dependence on k.

(1.1) Let M be a compact complex manifold of complex dimension n and L be a Hermitian holomorphic line bundle over M. We cover M by a finite number of coordinate charts $U_1, \cdots, U_m$ with the following property. There exist positive numbers R_0, C_0 and open subsets $U_j' \Subset U_j$ $(1 \leqslant j \leqslant m)$ with $\bigcup_{j=1}^m U_j' = M$ such that

(i) for every point x of U_j' the open ball with center x and radius R_0 with respect to the coordinate patch U_j is relatively compact in U_j,

(ii) for every point x of U_j there exists a trivialization of $L|U_j$ so that the Hermitian metric of $L|U_j$, when put in the form $e^{-\varphi}$ with respect to this trivialization, satisfies the condition that $d\varphi$ vanishes at x and all the second-order derivatives of φ with respect to the coordinate chart U_j are bounded by C_0 on all of U_j.

For $x \in U_j'$ and $0 < r \leqslant R_0$ we denote by $B_j(x, r)$ the ball with center x and radius r with respect to the coordinate patch U_j. Let $C_1 = \exp(4n^2 C_0)$. For a section s of L^k over an open subset G of M we denote by $\|s\|$ the nonnegative-valued function on G which is the pointwise norm of s.

(1.2) Lemma (*Schwarz's lemma for L^2 sections*). *For any integer k, any numbers $0 < r < R_0$ and $0 < \lambda \leqslant 1/2$ with $1 \leqslant k \leqslant 1/r^2$, if s is a holomorphic section of L^k over the ball $B_j(0, r)$ with $0 \in U_j'$ and if s vanishes at 0 of order l, then for $P \in B_j(0, \lambda r)$*

$$\int_{B_j(0, \lambda r)} \|s\|^2 \leqslant (2\lambda)^{2l+2n} C_1 \int_{B_j(0, r)} \|s\|^2,$$

where the integration is with respect to the Euclidean volume form of the coordinates of U_j.

Proof. According to the choice of $U_1, \cdots, U_m$ we can find a trivialization of $L|U_j$ so that the Hermitian metric of $L|U_j$, when put in the form $e^{-\varphi}$ with respect to this trivialization, satisfies the conditions that $d\varphi$ vanishes at 0 and all the second-order derivatives of φ with respect to the coordinate chart U_j are bounded by C_0 on all of U_j. Let $x_1, \cdots, x_{2n}$ be the real coordinates of the coordinate chart U_j. Since $d\varphi$ vanishes at 0, by Taylor expansion we have for $P \in B_j(0, r)$

$$\varphi(P) = \varphi(0) + \frac{1}{2} \sum_{\mu, \nu=1}^{2n} \frac{\partial^2 \varphi}{\partial x_\mu \partial x_\nu}(P') x_\mu(P) x_\nu(P),$$

where P' is a point on the line-segment joining P to 0 with respect to the coordinates of U_j. Thus for $P, Q \in B_j(0, r)$

$$\begin{aligned} \varphi(P) - \varphi(Q) = & \frac{1}{2} \sum_{\mu, \nu=1}^{2n} \frac{\partial^2 \varphi}{\partial x_\mu \partial x_\nu}(P') x_\mu(P) x_\nu(P) \\ & - \frac{1}{2} \sum_{\mu, \nu=1}^{2n} \frac{\partial^2 \varphi}{\partial x_\mu \partial x_\nu}(Q') x_\mu(Q) x_\nu(Q), \end{aligned}$$

where Q' is a point on the line joining Q to 0 with respect to the coordinates of U_j. It follows that for $P, Q \in B_j(0, r)$

$$|\varphi(P) - \varphi(Q)| \leqslant (2n)^2 C_0 r^2.$$

With respect to the trivialization of $L|U_j$ the section s becomes a holomorphic function on $B_j(0, r)$ which we denote by f. By the usual Schwarz lemma applied to the holomorphic function f we have

$$|f(P)| \leqslant (2\lambda)^l \sup_{Q \in B_j(0, r/2)} |f(Q)|$$

for $P \in B_j(0, \lambda r)$. By the subharmonicity of $|f|^2$, we have for $Q \in B_j(0, r/2)$

$$|f(Q)|^2 \leqslant \frac{n!}{\left(\pi (r/2)^2\right)^n} \int_{B_j(Q, r/2)} |f|^2,$$

where the integration is with respect to the Euclidean volume form of the coordinates of U_j. Hence for $P \in B_j(0, \lambda r)$,

$$|f(P)|^2 \leqslant (2\lambda)^{2l} \frac{2^{2n} n!}{\pi^n r^{2n}} \int_{B_j(0, r)} |f|^2.$$

Since $\|s\|^2 = |f|^2 e^{-k\varphi}$, it follows that for $P \in B_j(0, \lambda r)$,

$$\begin{aligned}\|s(P)\|^2 &\leqslant (2\lambda)^{2l}\frac{2^{2n}n!}{\pi^n r^{2n}}e^{-k\varphi(P)}\int_{B_j(0,r)}|f|^2\\ &= (2\lambda)^{2l}\frac{2^{2n}n!}{\pi^n r^{2n}}\int_{B_j(0,r)}\|s\|^2 e^{k\varphi-k\varphi(P)}\\ &\leqslant (2\lambda)^{2l}\frac{2^{2n}n!}{\pi^n r^{2n}}\int_{B_j(0,r)}\|s\|^2 e^{k(2n)^2C_0r^2}\\ &\leqslant (2\lambda)^{2l}\frac{2^{2n}n!}{\pi^n r^{2n}}C_1\int_{B_j(0,r)}\|s\|^2\end{aligned}$$

because $k \leqslant 1/r$. The results follow from integrating $\|s(P)\|^2$ over $B_j(0, \lambda r)$.

2. Leray isomorphism with L^2 estimates

The Leray isomorphism establishes a correspondence between line-bundle-valued harmonic forms and cocycles with coefficients in the bundle. In this section we use the $\bar{\partial}$ estimates to keep track of the L^2 estimates in the correspondence and also study the dependence of the L^2 estimates on the size of the covering.

(2.1) Take $0 < d < R_0/3n$ and $1 \leqslant j \leqslant m$. We consider the set of all $\nu = (\nu_1, \cdots, \nu_{2n}) \in \mathbf{Z}^{2n}$ such that the ball with center $(\nu_1 d, \cdots, \nu_{2n} d)$ and radius $3nd$ with respect to the real coordinates of the coordinate patch U_j is relatively compact in U_j. For such a multi-index ν let $a_{j\nu}$ be the point whose real coordinates are $(\nu_1 d, \cdots, \nu_{2n} d)$ with respect to the coordinate patch U_j. Let $B_{j\nu} = B_j(a_{j\nu}, 2nd)$, $B'_{j\nu} = B_j(a_{j\nu}, nd)$ and $B''_{j\nu} = B_j(a_{j\nu}, 3nd/2)$.

Let $\tau(\lambda)$ be a nonnegative-valued function on $0 \leqslant \lambda \leqslant 1$ so that the support of τ is contained in $[0, 3/4)$ and $\tau \equiv 1$ on $[0, 1/2]$. Let $\sigma_{j\nu}$ be the function on $B''_{j\nu}$ with compact support defined by $\sigma_{j\nu}(x) = \tau(r_{j\nu}(x)/2nd)$, where $r_{j\nu}(x)$ is the distance from x to $a_{j\nu}$ measured respect to the coordinates of U_j. We can consider $\sigma_{j\nu}$ as a function on M. $\sigma_{j\nu} \equiv 1$ on $B'_{j\nu}$.

Let $\sigma = \sum_{j,\nu}\sigma_{j\nu}$. Since $U'_j \subset \bigcup_{j,\nu} B'_{j\nu}$ and $M = \bigcup_{j=1}^m U'_j$, it follows that $\sigma \geqslant 1$ on M. Let $\rho_{j\nu} = \sigma_{j\nu}/\sigma$. *At $x \in U'_i$ all derivatives of $\rho_{j\nu}$ of order $\leqslant l$ with respect to the coordinate patch U_i are bounded by $C_l d^{-l}$*, where C_l is a constant depending only on l and independent of j, ν and d, because clearly we have such a conclusion when $\rho_{j\nu}$ is replaced by $\sigma_{j\nu}$ and, moreover, at any given point of M no more than $(4n+1)^{2n}m$ of the functions $\sigma_{j\nu}$ can be nonzero.

We recall the following theorem of Hörmander [6, p. 107]. (The forms here and also in similar situations later are implicitly assumed to be locally square integrable.)

(2.2) Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$, δ be the diameter of Ω, ψ a plurisubharmonic function on Ω, and g a $\bar\partial$-closed $(0,q)$-form on Ω $(q>0)$. Then there exists a $(0,q-1)$-form u on Ω such that $\bar\partial u = g$ and

$$q\int_\Omega |u|^2 e^{-\psi} \leqslant e\delta^2 \int_\Omega |g|^2 e^{-\psi},$$

where the integration and the pointwise norm for forms are with respect to the Euclidean metric of $\mathbf{C}^n$. In particular, the Kohn solution u of $\bar\partial u = g$ which is perpendicular to all $\bar\partial$-closed, $(0,q-1)$-forms with respect to the weight function $e^{-\psi}$ satisfies the inequality above.

(2.3) We now assume that the curvature form of the Hermitian metric of L is semipositive everywhere on M. Let $\mathscr{B}$ denote the Stein cover $\{B_{j\nu}\}$ of M. Take a $\bar\partial$-closed L^k-valued $(0,q)$-form ω on M. We construct an element f of $Z^q(\mathscr{B}, L^k)$ corresponding to ω in the following way.

Let $\mathscr{A}^l(L^k)$ be the sheaf of germs of L^k-valued $(0,l)$-forms on M. For notational simplicity we use the single index μ to replace the double index (j,ν). For $0 \leqslant l \leqslant q-1$ construct $\eta^l = \{\eta^l_{\mu_0\cdots\mu_l}\} \in (\mathscr{B}, \mathscr{A}^{q-l}(L^k))$ with $\eta^l_{\mu_0\cdots\mu_l} \in \Gamma(B_{\mu_0} \cap \cdots \cap B_{\mu_l}, \mathscr{A}^{q-l}(L^k))$ such that

(i) $\qquad \omega = \bar\partial \eta^0_{\mu_0} \quad \text{on } B_{\mu_0},$

(ii) $\qquad (\delta\eta^l)_{\mu_0\cdots\mu_{l+1}} = \bar\partial\eta^{l+1}_{\mu_0\cdots\mu_{l+1}} \quad \text{on } B_{\mu_0} \cap \cdots \cap B_{\mu_{l+1}}\ (0 \leqslant l \leqslant q-2),$

where

(a) δ means the coboundary operator and $(\delta\eta^l)_{\mu_0\cdots\mu_{l+1}}$ is the value of $\delta\eta^l$ at $(B_{\mu_0}, \cdots, B_{\mu_{l+1}})$,

(b) in solving the $\bar\partial$ equations in (i) and (ii) the Kohn solution with estimates given by Hörmander's theorem (2.2) is used and B_{μ_0} or $B_{\mu_0} \cap \cdots \cap B_{\mu_{l+1}}$ is regarded as a subdomain of the coordinate patch U_j with the smallest j which contains it.

Let $f = \delta\eta^{q-1} \in Z^q(\mathscr{B}, L^k)$. Then the map$\omega \mapsto f$ is $\mathbf{C}$-linear. Moreover, we have the following estimate

$$(2.3.1) \qquad \int_{B_{\mu_0}\cap\cdots\cap B_{\mu_q}} \left\| f_{\mu_0\cdots\mu_q} \right\|^2 \leqslant C^* d^{2q} \sup_{0\leqslant i\leqslant q} \int_{B_{\mu_i}} \|\omega\|^2,$$

where C^* is independent of d, k, and $\mu_0, \cdots, \mu_q$. Here $\|\cdot\|$ means the pointwise norm with respect to the Hermitian metric of L^k and, in the case of ω, also with respect to the Euclidean metric of U_j with the smallest j which contains B_{μ_0}. The integration is with respect to the Euclidean volume form of U_j with the smallest j which contains $B_{\mu_0} \cap \cdots \cap B_{\mu_q}$ and, respectively, B_{μ_i}. The factor d^{2q} on the right-hand side comes from applying (2.2) to solve the $\bar{\partial}$ equation q times.

(2.4) Now we want to reverse the process. Given $f = \{f_{\mu_0 \cdots \mu_q}\} \in Z^q(\mathscr{B}, L^k)$, we want to produce a $\bar{\partial}$-closed L^k-valued $(0, q)$-form ω on M. In general, these two processes are not the inverses of each other, but, of course, at the cohomology level they give the two directions of the Leray isomorphism between the Dolbeault and the Čech cohomology groups. For $0 \leqslant l \leqslant q-1$ construct $\xi^l = \{\xi^l_{\mu_0 \cdots \mu_{q-l-1}}\} \in C^{q-l-1}(\mathscr{B}, \mathscr{A}^l(L^k))$ with $\xi^l_{\mu_0 \cdots \mu_{q-l-1}} \in \Gamma(B_{\mu_0} \cap \cdots \cap B_{\mu_{q-l-1}}, \mathscr{A}^l(L^k))$ such that

$$\text{(i)} \qquad f_{\mu_0 \cdots \mu_q} = \left(\delta \xi^0\right)_{\mu_0 \cdots \mu_q} \quad \text{on } B_{\mu_0} \cap \cdots \cap B_{\mu_q},$$

$$\text{(ii)} \qquad \bar{\partial} \xi^l_{\mu_0 \cdots \mu_{q-l-1}} = \left(\delta \xi^{l+1}\right)_{\mu_0 \cdots \mu_{q-l-1}}$$
$$\text{on } B_{\mu_0} \cap \cdots \cap B_{\mu_{q-l-1}} \ (0 \leqslant l \leqslant q-1),$$

where the δ-equations are solved by using the partition of unity $\{\rho_\mu\}$ constructed in (2.1). More precisely

$$\xi^0_{\mu_0 \cdots \mu_{q-1}} = \sum_\lambda \rho_\lambda f_{\lambda \mu_0 \cdots \mu_{q-1}},$$
$$\xi^{l+1}_{\mu_0 \cdots \mu_{q-l-2}} = \sum_\lambda \rho_\lambda \bar{\partial} \xi^l_{\lambda \mu_0 \cdots \mu_{q-l-2}} \qquad (0 \leqslant l \leqslant q-1).$$

Finally, we set $\omega = \bar{\partial} \xi^{q-1}_{\mu_0}$ on B_{μ_0}. The map $f \mapsto \omega$ is $\mathbf{C}$-linear and the following estimate holds:

$$\text{(2.4.1)} \qquad \int_{B_{\mu_0}} \|\omega\|^2 \leqslant C^{\#} d^{-2q} \sup_{\mu_0, \cdots, \mu_q} \int_{B''_{\mu_0} \cap \cdots \cap B''_{\mu_q}} \|f\|^2,$$

where $C^{\#}$ is independent of d, k, and μ_0. The norm $\|\cdot\|$ and the integration carry the same meaning as in the estimate of (2.2). On the right-hand side we have integration over $B''_{\mu_0} \cap \cdots \cap B''_{\mu_q}$ instead of $B_{\mu_0} \cap \cdots \cap B_{\mu_q}$ because the support of ρ_μ is contained in B''_μ. The factor d^{-2q} on the right-hand side comes from applying the $\bar{\partial}$ operator q times and from the fact that the factor d^{-l} occurs in the estimate of the derivatives of ρ_μ of order $\leqslant l$.

3. Bochner-Kodaira formula for non-Kähler manifolds

We now use the Bochner-Kodaira formula for non-Kähler manifolds to show that for large k a harmonic form of unit L^2 norm with values in the kth power of a semipositive line bundle has small L^2 norm on the set where the curvature form of the line bundle is strictly positive.

(3.1) We give the compact complex manifold M a Hermitian metric $g_{\alpha\bar{\beta}}$. The torsion tensor $T^{\alpha}_{\beta\gamma}$ is given by

$$T^{\alpha}_{\beta\gamma} = \sum_{\lambda} g^{\alpha\bar{\lambda}}\left(\partial_{\beta} g_{\gamma\bar{\lambda}} - \partial_{\gamma} g_{\beta\bar{\lambda}}\right).$$

For any smooth L^k-valued $(0, q)$-form

$$\phi = \frac{1}{q!}\sum \phi_{\bar{\alpha}_1\cdots\bar{\alpha}_q} dz^{\bar{\alpha}_1} \wedge \cdots \wedge dz^{\bar{\alpha}_q}$$

on M we have the following formula of Bochner-Kodaira type

$$\begin{aligned}(\bar{\partial}\phi, \bar{\partial}\phi)_M + (\bar{\partial}^*\phi, \bar{\partial}^*\phi)_M = (\overline{\nabla}\phi, \overline{\nabla}\phi)_M + k(\theta\phi, \phi)_M + (\operatorname{Ric}\phi, \phi)_M \\ + 2\operatorname{Re}([\bar{\partial}, T^*]\phi, \phi)_M - ([T, T^*]\phi, \phi)_M,\end{aligned}$$

where

(i) $(\cdot,\cdot)_M$ means the inner product for L^k-valued tensors corresponding to the global L^2 norm over M,

(ii) $\overline{\nabla}$ denotes covariant differential in the $(0, 1)$-direction,

(iii) $\theta\phi = (1/q!)\sum \theta^{\bar{\lambda}}_{\bar{\alpha}_1}\phi_{\bar{\lambda}\bar{\alpha}_2\cdots\bar{\alpha}_q} dz^{\bar{\alpha}_1} \wedge \cdots \wedge dz^{\bar{\alpha}_q}$ with $\theta^{\bar{\lambda}}_{\bar{\alpha}_1}$ = the curvature tensor of L with the first index raised,

(iv) $(\operatorname{Ric}\phi, \phi)_M$ is defined analogous to $(\theta\phi, \phi)$ with $\theta^{\bar{\lambda}}_{\bar{\alpha}_1}$ replaced by the Ricci tensor $R^{\bar{\lambda}}_{\bar{\alpha}_1}$ with the first index raised,

(v) $T\phi = (1/(q-1)!)\sum \overline{T^{\lambda}_{\alpha_0\alpha_1}}\phi_{\bar{\lambda}\bar{\alpha}_2\cdots\bar{\alpha}_q} dz^{\bar{\alpha}_0} \wedge dz^{\bar{\alpha}_1} \wedge \cdots \wedge dz^{\bar{\alpha}_q}$,

(vi) $\bar{\partial}^*$ and T^* are respectively the adjoint operators of $\bar{\partial}$ and T,

(vii) for two operators A and B, $[A, B]$ means $AB + BA$.

This formula was given by Griffiths [5, p. 429, (7.14)]. In particular, when ϕ is harmonic, we have

$$0 = (\overline{\nabla}\phi, \overline{\nabla}\phi)_M + k(\theta\phi, \phi)_M + (\operatorname{Ric}\phi, \phi)_M - ([T, T^*]\phi, \phi)_M,$$

because

$$([\bar{\partial}, T^*]\phi, \phi)_M = (\bar{\partial}T^*\phi, \phi)_M + (T^*\bar{\partial}\phi, \phi)_M = (T^*\phi, \bar{\partial}^*\phi)_M + (T^*\bar{\partial}\phi, \phi)_M.$$

Hence for any harmonic L^k-valued $(0, q)$-form ϕ on M we have

$$k(\theta\phi, \phi)_M \leqslant -(\operatorname{Ric}\phi, \phi)_M + ([T, T^*]\phi, \phi)_M \leqslant C_2(\phi, \phi)_M,$$

where C_2 is a constant depending only on the Hermitian metric of M and is independent of k and ϕ.

As a consequence we have the following lemma.

(3.2) Lemma. *Let M be a Hermitian compact complex manifold and L a Hermitian holomorphic line bundle over M such that the curvature form of L is semipositive everywhere on M and is positive definite on an open subset G of M. Let K be a compact subset of G. Then there exists a positive constant C_K such that for every positive integer k and for every harmonic L^k-valued $(0, q)$-form ϕ on M one has $\int_K \|\phi\|^2 \leqslant (1/k) C_K \int_M \|\phi\|^2$, where $\|\cdot\|$ is the pointwise norm of ϕ with respect to the Hermitian metrics of M and L, and the integration is with respect to the volume form of the Hermitian metric of M.*

4. Estimates for dimensions of cohomology groups

For $q \geqslant 1$ we estimate in terms of k the dimension of the qth cohomology group with coefficients in the kth power of a semipositive line bundle whose curvature form is strictly positive outside a set of measure zero.

(4.1) We now assume that M and L are as in (3.2) and use the notations of §§1–3. Fix an open neighborhood W of $M - G$ and fix $\eta > 0$. Let $K = M - W$ and k_0 be the smallest integer $\geqslant C_K \eta^{-1}$. Fix $k \geqslant k_0$ and $q \geqslant 1$. Take an L^k-valued harmonic $(0, q)$-form ω on M with unit global L^2 norm.

Take $0 < d < R_0/3n$ (and other restrictions will be put on d later). From ω we can construct by (2.3) an element f of $Z^q(\mathscr{B}, L^k)$. Let γ be a number greater than all the ratios (and their reciprocals) of the volume forms of the Hermitian metric of M and the Euclidean metrics of the coordinates of U_j $(1 \leqslant j \leqslant m)$. Let $C' = (8n)^{2nq}(4n+1)^{2n} m \gamma^2 C^*$. Since $\int_M \|\omega\|^2 = 1$, it follows from (3.2) that $\int_K \|\omega\|^2 \leqslant \eta$. Since no point of M can belong to more than $(4n+1)^{2n} m$ of the sets B_μ and since no B_λ can intersect more than $(8n)^{2n}$ of the sets B_μ, from the estimate (2.3.1) we have

$$(4.1.1) \qquad \sum{}' \int_{B_{\mu_0} \cap \cdots \cap B_{\mu_q}} \left\| f_{\mu_0 \cdots \mu_q} \right\|^2 \leqslant C' d^{2q} \eta,$$

the summation Σ' being over $B_{\mu_0}, \cdots, B_{\mu_q}$ disjoint from W, and

$$(4.1.2) \qquad \sum{}'' \int_{B_{\mu_0} \cap \cdots \cap B_{\mu_q}} \left\| f_{\mu_0 \cdots \mu_q} \right\|^2 \leqslant C' d^{2q},$$

the summation Σ'' being over $B_{\mu_0}, \cdots, B_{\mu_q}$ not all disjoint from W.

(4.2) Let γ^* be a positive number such that for any $1 \leqslant j, k \leqslant m$ the Euclidean metric of the coordinates of U_j is $\leqslant \gamma^*$ times the Euclidean metric of the coordinates of U_k at every point of $U_j' \cap U_k'$.

Choose $0 < \delta < d/8\gamma^*$. Consider $B = B_{\mu_0} \cap \cdots \cap B_{\mu_q}$ such that $B_{\mu_0} \cap \cdots \cap B_{\mu_q}$ is not disjoint from W. Let j be the smallest integer such that B is contained in U_j'. For $\sigma = (\sigma_1, \cdots, \sigma_{2n}) \in \mathbf{Z}^{2n}$ let b_σ be the point in B whose coordinates with respect to U_j are $(\sigma_1\delta, \cdots, \sigma_{2n}\delta)$ if b_σ is of distance $> 6n\delta$ from $U_j - B$ with respect to the Euclidean metric of the coordinates of U_j. For each b_σ let $D_\sigma = B_j(b_\sigma, 6n\delta)$ and $D_\sigma' = B_j(b_\sigma, 2n\delta)$. Each D_σ is contained in B and the union of all D_σ' contains the subset of all points of B whose distance with respect to the Euclidean metric of the coordinates of U_j is $> 4n\delta$ from $U_j - B$. Since $\delta < d/8\gamma^*$, it follows that the union of all D_0' contains $B_{\mu_0}'' \cap \cdots \cap B_{\mu_q}''$.

Let Ω_δ be the set of all b_σ as B ranges over all possible choices. Let N_δ be the number of all elements in Ω_δ. Let W' be the union of W and all such B's. Let $\mathrm{Vol}(\cdot)$ be the volume function with respect to the Hermitian metric of M. Let $\gamma' = (8n)^{2nq}(4n+1)^{2n}m\gamma$. Since no point of M can belong to more than $(4n+1)^{2n}m$ of the sets B_μ and since no B_λ can intersect more than $(8n)^{2n}$ of the sets B_μ, it follows that no point of M can belong to more than $(8n)^{2nq}(4n+1)^{2n}m$ such B's. Hence $\lim_{\delta\to 0} N_\delta \delta^{2n} \leqslant \gamma' \mathrm{Vol}(W)$. Since W' is contained in the set of all points of M whose distance from W with respect to the Hermitian metric of M is $< 2nd\gamma^*$, it follows that, after W is chosen, for d sufficiently small one has $\mathrm{Vol}(W') \leqslant 2\,\mathrm{Vol}(W)$. After W and d are so successively chosen, for δ sufficiently small one has $N_\delta \leqslant 4\gamma' \mathrm{Vol}(W)\delta^{-2n}$.

(4.3) Let $h = h(q, k)$ be the dimension of $H^q(M, L^k)$ over $\mathbf{C}$. Let $\omega_1, \cdots, \omega_h$ be a basis over $\mathbf{C}$ of the space of all L^k-valued harmonic $(0, q)$-forms on M. From ω_i $(1 \leqslant i \leqslant h)$ we can construct by (2.3) $f^{(i)} = \{f^{(i)}_{\mu_0\cdots\mu_q}\} \in Z^q(\mathscr{B}, L^k)$, $1 \leqslant i \leqslant h$. Choose a positive integer l. Some additional assumption will be imposed on l later. We now take a nonzero linear combination $f = \{f_{\mu_0\cdots\mu_q}\} \in Z^q(\mathscr{B}, L^k)$ of $f^{(i)}$ $(1 \leqslant i \leqslant h)$ so that $f_{\mu_0\cdots\mu_q}$ vanishes to order l at every point b_σ of Ω_δ coming from $B_{\mu_0} \cap \cdots \cap B_{\mu_q}$. Since the number of terms in a polynomial of degree $< l$ in n variables is $\binom{n+l}{l}$ and since there are no more than $4\gamma' \mathrm{Vol}(W)\delta^{-2n}$ elements in Ω_δ, it follows that we have to assume $h > 4\gamma' \mathrm{Vol}(W)\delta^{-2n}\binom{n+l}{l}$ to conclude that there is such a nonzero linear combination. Let us make such an assumption. Our purpose is to show that with a suitable choice of η, l, W, d, δ, and k this assumption will lead to a contradiction. Let ω be the linear combination of ω_i $(1 \leqslant i \leqslant h)$ corresponding to the nonzero linear combination f of f_i $(1 \leqslant i \leqslant h)$. By multiplying ω by a positive constant, we can assume without loss of generality that $(\omega, \omega)_M = 1$.

(4.4) We impose the additional assumption that $\delta \leqslant 1/6n\sqrt{k}$. Now apply Lemma (1.2) to the section $f_{\mu_0\cdots\mu_q}$ restricted to D_σ (constructed in (4.2)) with $\lambda = 1/3$. Since the union of all D_σ' coming from $B_{\mu_0} \cap \cdots \cap B_{\mu_q}$ covers

$B''_{\mu_0} \cap \cdots \cap B''_{\mu_q}$ and since no point of $B_{\mu_0} \cap \cdots \cap B_{\mu_q}$ can belong to more than $(4n)^{2n}$ such balls D_σ, it follows that

$$\int_{B''_{\mu_0} \cap \cdots \cap B''_{\mu_q}} \left\| f_{\mu_0 \cdots \mu_q} \right\|^2 \leqslant \left(\frac{2}{3}\right)^{2l+2n} C_1 (4n)^{2n} \gamma^2 \int_{B_{\mu_0} \cap \cdots \cap B_{\mu_q}} \left\| f_{\mu_0 \cdots \mu_q} \right\|^2,$$

where integration is with respect to the Hermitian metric of M.

Combining it with the inequality (4.1.2), we obtain

$$\text{(4.4.1)} \qquad \sum{}'' \int_{B''_{\mu_0} \cap \cdots \cap B''_{\mu_q}} \left\| f_{\mu_0 \cdots \mu_q} \right\|^2 \leqslant \left(\frac{2}{3}\right)^{2l+2n} C_1 (4n)^{2n} \gamma^2 C' d^{2q}.$$

We now use the procedure in (2.4) to construct from f an L^k-valued $\bar\partial$-closed $(0, q)$-form ω' on M. From (2.4.1), (4.1.1), and (4.4.1) it follows that

$$\int_M \|\omega'\|^2 \leqslant C^{\#} \max\left(C'\eta, \left(\frac{2}{3}\right)^{2l+2n} C_1 (4n)^{2n} \gamma^2 C' \right).$$

Since ω' is in the same cohomology class as ω and since a harmonic form minimizes the L^2 norm in its cohomology class, we conclude that

$$\text{(4.4.2)} \qquad 1 = \int_M \|\omega\|^2 \leqslant C^{\#} \max\left(C'\eta, \left(\frac{2}{3}\right)^{2l+2n} C_1 (4n)^{2n} \gamma^2 C' \right).$$

(4.5) We are going to derive a contradiction by successive appropriate choices of η, l, W, d, δ, and k. Fix any $\varepsilon > 0$.

(i) Choose η such that $C^{\#} C' \eta < 1$.

(ii) Choose l such that $C^{\#}(2/3)^{2l+2n} C_1 (4n)^{2n} \gamma^2 C' < 1$.

(iii) Choose W such that $4\gamma' \operatorname{Vol}(W) \binom{n+l}{l} < \varepsilon (6n)^{-2n}$. After η and W are chosen, k_0 is determined.

(iv) Choose d sufficiently small so that $\operatorname{Vol}(W') \leqslant 2 \operatorname{Vol}(W)$ is satisfied. Let $\delta_0 = d/8\gamma^*$.

Now we are free to choose w and k as long as the inequalities $\delta < \delta_0$, $k \geqslant k_0$ and $\delta \leqslant 1/6n\sqrt{k}$ are satisfied. Let k_1 be an integer greater than both k_0 and $(6n\delta_0)^{-2}$. Set $\delta = 1/6n\sqrt{k}$. Then for any choice of $k \geqslant k_1$ the inequality (4.4.2) is contradicted. Thus for such choice of η, l, W, d, δ, and k we must have

$$h \leqslant 4\gamma' \operatorname{Vol}(W) \delta^{-2n} \binom{n+l}{l} < \varepsilon (6n)^{-2n} \delta^{-2n} = \varepsilon k^n.$$

We have thus obtained a proof of the following proposition.

(4.6) Proposition. *Let M be a compact complex manifold of complex dimension n and L a Hermitian holomorphic line bundle over M such that the curvature form of L is semipositive everywhere on M and is positive definite outside a subset* *of M of measure zero. Then for every positive integer q and every positive number ε there exists a positive integer k_1 depending on ε such that* $\dim H^q(M, L^k) \leqslant \varepsilon k^n$ *for $k \geqslant k_1$.*

5. Characterization of Moišezon manifolds

Now we are ready to prove the following main result, which has as an immediate consequence the characterization of Moišezon spaces by the existence of torsion-free coherent sheaves of rank 1 which are semipositive everywhere and strictly positive outside a set of measure zero.

(5.1) Theorem. *Let M be a compact complex manifold and L a Hermitian holomorphic line bundle over M whose curvature form is everywhere semipositive and is strictly positive outside a set of measure zero. Then M is a Moišezon manifold.*

Proof. Let n be the complex dimension of M. Since the curvature form of L is everywhere semipositive and is strictly positive at some point of M, it follows that the nth power $c_1(L)^n$ of the first Chern class $c_1(L)$ of L is positive. By the theorem of Hirzebruch-Riemann-Roch (which for the case of a general compact complex manifold is a consequence of the index theorem of Atiyah-Singer [1]),

$$\sum_{q=0}^{n} (-1)^q \dim H^q(M, L^k)$$

equals the value of

$$\exp(kc_1(L)) \prod_{\nu=1}^{n} \frac{\gamma_\nu}{1-\exp(-\gamma_\nu)}$$

at the fundamental class of M, where $\gamma_1, \cdots, \gamma_n$ are the Chern roots of the tangent bundle of M. Hence for k sufficiently large,

$$\sum_{q=0}^{n} (-1)^q \dim H^q(M, L^k) \geqslant \frac{1}{2} \frac{c_1(L)^n}{n!} k^n.$$

By Proposition (4.6) for k sufficiently large

$$\dim H^q(M, L^k) \leqslant \frac{1}{4n} \frac{c_1(L)^n}{n!} k^n$$

for $q \geqslant 1$. It follows that

$$\dim \Gamma(M, L^k) \geqslant \frac{1}{4} \frac{c_1(L)^n}{n!} k^n \tag{5.1.1}$$

for k sufficiently large.

Let Z_k be the set of points in M where every global holomorphic section of L^k over M vanishes. Let $Z = \bigcap_{k=1}^{\infty} Z_k$. Then there exists k_0 such that $Z = Z_{k_0}$. Let $F = L^{k_0}$. By a level set of $\bigcup_{k=1}^{\infty} \Gamma(M, F^k)$ at a point of $M - Z$ we mean the intersection for $1 \leqslant k < \infty$ of the level sets at that point of the maps Φ_k: $M - Z \to \mathbf{P}_{N_k}$ defined by $\Gamma(M, F^k)$. Let d be the minimum of the complex dimensions of the branches of all the level sets of $\bigcup_{k=1}^{\infty} \Gamma(M, F^k)$.

Assume that M is not Moišezon and we want to derive a contradiction. Then $d > 0$. We cover M by a finite number of open unit balls B_j $(1 \leqslant j \leqslant m)$, each in a coordinate patch, so that

(i) F is trivial on some open neighborhood of the topological closure of each B_j, and

(ii) the center a_j of each B_j is outside Z and is a regular point of a level set E_j of $\bigcup_{k=1}^{\infty} \Gamma(M, F^k)$ whose complex dimension at a_j is d.

By replacing F by a suitable power of F, we can assume without loss of generality that each E_j is the level set of $\Gamma(M, F)$. For every positive integer k the rank of Φ_k is maximum at each a_j.

Choose a positive number $r < 1$ such that the balls B_j' with center a_j and radius r $(1 \leqslant j \leqslant m)$ still cover M. Let $\|\cdot\|$ denote the pointwise norm of a section of F^k (computed from the Hermitian metric of L). By using the usual Schwarz lemma, we conclude that there exists a positive number $C > 1$ independent of k such that for every holomorphic section s of F^k over B_j which vanishes at a_j to order l,

$$\sup_{P \in B_j'} \|s\|(P) \leqslant C^k r^l \sup_{Q \in B_j} \|s\|(Q). \tag{5.1.2}$$

(The constant C is obtained from the Hermitian metric of $F|B_j$.)

Let h_k be the dimension of $\Gamma(M, F^k)$. We claim that

$$h_k \leqslant m\left(n - d + \frac{k \log C}{\log(1/r)} + 1\right)^{n-d}. \tag{5.1.3}$$

Otherwise, if we set l equal to the greatest integer not exceeding $k \log C/\log(1/r) + 1$, then

$$h_k > m\binom{n-d+l}{n-d} \tag{5.1.4}$$

and

$$C^k r^l < 1. \tag{5.1.5}$$

Since the rank of Φ_k at each a_j is equal to its maximum rank $n - d$ over $M - Z$, from (5.1.4) it follows that there is a nonzero element s of $\Gamma(M, F^k)$

which vanishes to order l at each a_j $(1 \leqslant j \leqslant m)$. From (5.1.2) and (5.1.5) we conclude that the supremum of $\|s\|$ over M must be strictly less than itself, which is a contradiction. Hence (5.1.3) holds. On the other hand (5.1.3) contradicts (5.1.1) when k is sufficiently large. It follows that M must be Moišezon.

(5.2) Remark. The method of proof given for Theorem (5.1) gives us also a way of producing global holomorphic sections for a Hermitian line bundle L over a compact complex manifold M of complex dimension n without the assumption that the curvature form θ of L is semipositive. We use the notations of the preceding sections unless the contrary is explicitly stated. We now use W to denote an open subset of M such that the curvature form θ as a quadratic form is bounded fromn below by $a > 0$ at every point of $M - W$ and by $-b$ at every point of W, where $b > 0$. We set $k = 1$. Then instead of Lemma (3.2) the arguments of (3.1) yield

$$a\int_{M-W}\|\phi\|^2 \leqslant C_2\int_M\|\phi\|^2 + b\int_W\|\phi\|^2 \leqslant (C_2 + b)\int_M\|\phi\|^2$$

for every harmonic L-valued $(0, q)$-form ϕ on M. We use $\eta = (C_2 + b)/a$. To make the arguments of §4 work, we have to assume

$$C^{\#}C'\frac{C_2 + b}{a} < 1,$$

and choose l such that

$$C^{\#}\left(\frac{2}{3}\right)^{2l+2n}C_1(4n)^{2n}\gamma^2C' < 1.$$

We then end up with

$$\dim H^q(M, L) \leqslant 4\gamma'\operatorname{Vol}(W)\left(\min\left(\tfrac{d}{8\gamma^*}, \tfrac{1}{6n}\right)\right)^{-2n}\binom{n+l}{l}$$

for $q \geqslant 1$. From the theorem of Hirzebruch-Riemann-Roch we obtain

$$\dim\Gamma(M, F) \geqslant \chi(M, L) - 2(n + 1)\gamma'\operatorname{Vol}(W)\left(\min\left(\tfrac{d}{8\gamma^*}, \tfrac{1}{6n}\right)\right)^{-2n}\binom{n+l}{l},$$

where $\chi(M, L)$ equals the value of

$$\exp(c_1(L))\prod_{\nu=1}^{n}\frac{\gamma_\nu}{1 - \exp(-\gamma_\nu)}$$

at the fundamental class of M. Thus to conclude the existence of global holomorphic sections of L over M we must assume that a is large relative to b and to other constants coming from M and its Hermitian metric and also assume that $\operatorname{Vol}(W)$ is small relative to $\chi(M, L)$ and the other constants coming from M and its Hermitian metric. Instead of working with $k = 1$ we can consider L^k for sufficiently large k and make the contribution from the torsion tensor and the Chern roots of M negligible. Then $\chi(M, L)$ can essentially be replaced by $c_1(L)^n/n!$ if $c_1(L)^n$ is positive. Of course in that case we can only draw conclusions about the existence of global holomorphic sections of L^k over M for sufficiently large k.

However, as it is, this result on the existence of global holomorphic sections for line bundles without the semipositivity assumption is highly unsatisfactory, because the estimates for the dimensions of the cohomology groups are too crude. The obstacle to getting sharper results is that, in the present method, to apply the Schwarz lemma one has to use Leray's isomorphism to convert harmonic forms to cocycles first and then back to $\bar\partial$-closed forms. In this process a lot of undesirable constants come into the picture. If there is a direct way of applying the Schwarz lemma to harmonic forms without the intermediate step of conversion to cocycles, then one can get a condition for the existence of global holomorphic sections expressed in an invariant form in terms of certain integral expressions of the curvature form, its sup norm, and the volume of the set where it fails to be positive. The Schwarz lemma is a consequence of the log subharmonicity of the absolute value of a holomorphic function. A harmonic form does not have this kind of log subharmonicity property. However, for harmonic forms the fact that one has a Schwarz lemma after using the intermediate step of conversion to cocycles indicates the possibility of formulating some Schwarz lemma type result directly in terms of harmonic forms.

A good criterion for the existence of global holomorphic sections which does not involve pointwise positivity assumptions clearly would have important and far-reaching consequences in the theory of complex manifolds.

6. Relation between the Grauert-Riemenschneider and the eigenvalue conjecture

We are going to show how the eigenvalue conjecture implies the stronger form of the Grauert-Riemenschneider conjecture.

(6.1) Let M be a compact complex manifold and L a Hermitian holomorphic line bundle over M whose curvature form is semipositive everywhere and is strictly positive at some point P_0 of M. We assume that the eigenvalue conjecture is true. Then there exists a positive number λ_0 such that for every nonnegative integer k the smallest positive eigenvalue $\lambda(M, L^k)$ of the Laplacian $\bar\partial^*\bar\partial$ on the Hilbert space of all global L^2 sections of L^k over M is greater than or equal to λ_0.

(6.2) Lemma. *If $k \geqslant 0$ and ω is a $\bar\partial$-exact L^k-valued* $(0,1)$*-form on M, then there exists a section f of L^k over M such that $\omega = \bar\partial f$ and $(f, f)_M \leqslant (\omega, \omega)_M/\lambda_0$, where $(\cdot,\cdot)_M$ is the inner product corresponding to the global L^2 norm over M.*

Proof. Let f be the unique section of L^k over M which is perpendicular to all global holomorphic sections of L^k over M and whose image under $\bar\partial$ is ω. Since f is perpendicular to the kernel of $\bar\partial^*\bar\partial$ (which is precisely the set of all global holomorphic sections of L^k over M), it follows from the definition of $\lambda(M, L^k)$ that $(\bar\partial^*\bar\partial f, f)_M \geqslant \lambda(M, L^k)(f, f)_M$. Hence

$$(\omega, \omega)_M = (\bar\partial f, \bar\partial f)_M = (\bar\partial^*\bar\partial f, f)_M \geqslant \lambda(M, L^k)(f, f)_M \geqslant \lambda_0(f, f)_M.$$

(6.3) Let $\pi\colon L^* \to M$ be the dual bundle of L with Hermitian metric induced from L. For $v \in L^*$ let $\|v\|$ be its length with respect to the Hermitian metric.

Since the curvature form of L is strictly positive at P_0, we can choose a Stein open neighborhood G of P_0 in M, a trivialization of $L^*|G$ with fiber coordinate w, and a holomorphic coordinate system $z = (z_1, \cdots, z_n)$ of G such that, when the Hermitian metric of L^* with respect to the fiber coordinate w is written in the form e^φ, the function φ is a strictly convex function of the coordinate z at every point of G.

Let ρ be a smooth nonnegative function on G with compact support such that $\rho(P_0) > 0$ and $\varphi + \rho$ is still a strictly convex function of z on G. Let $\psi = \varphi + \rho$. Now change the Hermitian metric on $\pi^{-1}(G)$ to get a new Hermitian metric for L^* so that, with respect to the trivialization of $L|G$ with fiber coordinate w, the new Hermitian metric is e^ψ. For $v \in L^*$ let $\|v\|'$ denote its length with respect to the new Hermitian metric. For $r > 0$ let Ω_r (respectively Ω'_r) denote the set of all vectors v of L^* with $\|v\| < r$ (respectively $\|v\|' < r$).

Let $\sigma(z)$ be the complex linear function of z such that $2 \operatorname{Re} \sigma(z)$ is the linear part of the Taylor series expansion of ψ at P_0 with respect to the coordinate z. Since ψ is a strictly convex function of z on G, one has $\psi > \psi(P_0) + 2 \operatorname{Re} \sigma$ on $G - \{P_0\}$. Let $s(z) = \exp(-\frac{1}{2}\psi(P_0) - \sigma(z))$. Then $e^\psi|s|^2 > 1$ on $G - \{P_0\}$ and $e^\psi|s|^2 = 1$ at P_0. Since $\rho(P_0) > 0$, we have $e^\varphi|s|^2 < 1$ at P_0. Let V be the complex submanifold of $\pi^{-1}(G)$ defined by $w = s(z)$. Then V is disjoint from Ω'_1 but $V \cap \pi^{-1}(P_0)$ is contained in Ω_1. Since $\Omega_1 - \Omega'_1 \subset \pi^{-1}(G)$, $V \cap \Omega_1$ is a complex submanifold of Ω_1.

(6.4) Lemma. *The cohomology group $H^1(\Omega_1, \mathcal{O}_{L^*})$, with the natural topology as a quotient of a Fréchet space, is Hausdorff, where $\mathcal{O}_{L^*}$ is the sheaf of germs of holomorphic functions on L^*.*

Proof. Let $\{U_j\}_{1 \leqslant j \leqslant m}$ be a finite Stein cover of M and let $\{\rho_j\}_{1 \leqslant j \leqslant m}$ be a partition of unity subordinate to $\{U_j\}_{1 \leqslant j \leqslant m}$. Since the curvature form of L is everywhere semipositive, it follows that the boundary of Ω_1 is everywhere weakly pseudoconvex. Hence $\pi^{-1}(U_j) \cap \Omega_1$ is Stein for $1 \leqslant j \leqslant m$.

A $(0,1)$-form on an open subset of L^* is said to be holomorphic in the fiber coordinates if with respect to the local coordinates $\zeta_1, \cdots, \zeta_n$ of M and the local fiber coordinate τ of L^* it can be expressed as

$$\sum_{\nu=1}^{n} f_\nu(\zeta_1, \cdots, \zeta_n, \tau)\, d\bar{\zeta}_\nu$$

with f_ν holomorphic in τ. The cohomology group $H^1(\Omega_1, \mathcal{O}_{L^*})$ equals the quotient of the group of all smooth $\bar\partial$-closed $(0,1)$-forms on Ω_1 holomorphic in the fiber coordinates by the image under $\bar\partial$ of all smooth functions on Ω_1 holomorphic in the fiber coordinates. To show this, it suffices to verify the following two statements.

(i) For any $\bar\partial$-closed smooth $(0,1)$-form ω on Ω_1 we can find a smooth $\bar\partial$-closed $(0,1)$-form ω' on Ω_1 holomorphic in the fiber coordinates such that $\omega - \omega'$ is $\bar\partial$-exact on Ω_1.

(ii) If ω' is a smooth $(0,1)$-form on Ω_1 and f is a smooth function on Ω_1 such that $\omega' = \bar\partial f$, then f is holomorphic in the fiber coordinates.

Statement (ii) is obvious, because $d\bar\tau$ does not occur in ω'. To prove (i), we observe that due to the Steinness of $\pi^{-1}(U_j) \cap \Omega_1$ we can find a smooth function f_j on $\pi^{-1}(U_j) \cap \Omega_1$ such that $\omega = \bar\partial f_j$ on $\pi^{-1}(U_j) \cap \Omega_1$. Let $\omega' = -\Sigma_j \pi^*(\bar\partial \rho_j) f_j$. Then $\omega - \omega' = \bar\partial(\Sigma_j(\rho_j \circ \pi) f_j)$ is $\bar\partial$-exact on Ω_1. Since ω' is $\bar\partial$-closed and since $d\bar\tau$ clearly does not occur in ω', it follows that ω' must be holomorphic in the fiber coordinates.

To prove the Hausdorff property of $H^1(\Omega_1, \mathcal{O}_{L^*})$, we take a sequence of smooth $(0,1)$-forms ω_ν on Ω_1 holomorphic in the fiber coordinates such that:

(1) each ω_ν is the image under $\bar\partial$ of some smooth function f_ν on Ω_1 holomorphic in the fiber coordinates,

(2) the sequence ω_ν approaches in the Fréchet topology a smooth $\bar\partial$-closed $(0,1)$-form ω on Ω_1 holomorphic in the fiber coordinates.

We have to show that ω is $\bar\partial$-exact on Ω_1. Take $0 < r < 1$. First we show that $\omega | \Omega_r$ is $\bar\partial$-exact on Ω_r.

Consider the following power series expansions in the fiber coordinate τ of every local trivialization of L^*.

$$\omega = \sum_{k=0}^{\infty} \omega^{(k)} \tau^k, \quad \omega_\nu = \sum_{k=0}^{\infty} \omega_\nu^{(k)} \tau^k, \quad f_\nu = \sum_{k=0}^{\infty} f_\nu^{(k)} \tau^k.$$

Then $\omega^{(k)}$, $\omega_\nu^{(k)}$ are L^k-valued $(0,1)$-forms on M and $f_\nu^{(k)}$ is a section of L^k over M. Moreover, $\bar\partial f_\nu^{(k)} = \omega_\nu^{(k)}$ on M and $\lim_{\nu\to\infty}\omega_\nu^{(k)} = \omega^{(k)}$. We also have the estimates of power series coefficients

$$\left(\omega^{(k)}, \omega^{(k)}\right)_M \leqslant A_r r^{-k}$$

for some constant A_r depending on r. Being the limit of the $\bar\partial$-exact L^k-valued $(0,1)$-forms $\omega_\nu^{(k)}$ on M, the L^k-valued $(0,1)$-form $\omega^{(k)}$ must also be $\bar\partial$-exact on M. By Lemma (6.2) for every $k \geqslant 0$ there exists a section $f^{(k)}$ of L^k over M such that $\omega^{(k)} = \bar\partial f^{(k)}$ on M and

$$\left(f^{(k)}, f^{(k)}\right)_M \leqslant \frac{1}{\lambda_0}A_r r^{-k}.$$

Let $f = \sum_{k=0}^{\infty} f^{(k)}\tau^{(k)}$. Then f is a smooth function on Ω_r holomorphic in the fiber coordinates and $\bar\partial f = \omega$ on Ω_r.

Take an increasing sequence of real numbers r_ν $(0 \leqslant \nu < \infty)$ with 1 as limit. The preceding argument shows that there exists a smooth function g_ν on Ω_{r_ν} holomorphic in the fiber coordinates such that $\bar\partial g_\nu = \omega$ on Ω_{r_ν}. It follows that $g_\nu - g_{\nu-1}$ is holomorphic on $\Omega_{r_{\nu-1}}$. Expand $g_\nu - g_{\nu-1}$ as a power series in the fiber coordinates and let $h_{\nu-1}$ be a partial sum of the power series with sufficient terms so that the supremum norm of $g_\nu - g_{\nu-1} - h_{\nu-1}$ on $\Omega_{r_{\nu-2}}$ is less than $2^{-\nu}$. Let g be the function on Ω_1 which is the limit of $g_\nu - \sum_{\mu=1}^{\nu-1} h_\mu$ as $\nu \to \infty$. Such a limit exists because

$$\left(g_\nu - \sum_{\mu=1}^{\nu-1} h_\mu\right) - \left(g_{\nu-1} - \sum_{\mu=1}^{\nu-2} h\right) = g_\nu - g_{\nu-1} - h_{\nu-1}$$

is bounded by $2^{-\nu}$ on $\Omega_{\nu-2}$. Clearly, we have $\bar\partial g = \omega$ on Ω_1.

(6.5) Let $W = \Omega_1 \cap \pi^{-1}(G)$. Then $\Omega_1 = \Omega_1' \cup W$ and the topological closures of $\Omega_1' - W$ and $W - \Omega_1'$ are disjoint. As part of the Mayer-Vietoris sequence we have the exact sequence

$$\begin{aligned}(6.5.1)\qquad \Gamma(\Omega_1', \mathcal{O}_{L^*}) \oplus \Gamma(W, \mathcal{O}_{L^*}) &\to \Gamma(W \cap \Omega_1', \mathcal{O}_{L^*}) \to H^1(\Omega_1, \mathcal{O}_{L^*})\\ &\to H^1(\Omega_1', \mathcal{O}_{L^*}) \oplus H^1(W, \mathcal{O}_{L^*}).\end{aligned}$$

Since W is Stein, $H^1(W, \mathcal{O}_{L^*})$ vanishes. The map $\Gamma(W, \mathcal{O}_{L^*}) \to \Gamma(W \cap \Omega_1', \mathcal{O}_{L^*})$ has dense image, because the density of the image of $\Gamma(\pi^{-1}(G), \mathcal{O}_{L^*}) \to \Gamma(\pi^{-1}(G) \cap \Omega_1', \mathcal{O}_{L^*})$ is clear by using Taylor series expansion in fiber coordinates. Since by Lemma (6.4) the cohomology group $H^1(\Omega_1, \mathcal{O}_{L^*})$ is Hausdorff, it follows from the exact sequence in (6.5.1) that the restriction map α: $H^1(\Omega_1, \mathcal{O}_{L^*}) \to H^1(\Omega_1', \mathcal{O}_{L^*})$ is injective.

Let $\mathcal{F}$ be the sheaf of germs of meromorphic functions on Ω_1 whose poles are at most simple ones contained in V. Let $\mathcal{Q}$ be the quotient sheaf $\mathcal{F}/\mathcal{O}_{L^*}$.

From the short exact sequence $0 \to \mathcal{O}_{L^*} \to \mathcal{F} \to \mathcal{Q} \to 0$ we obtain the following commutative diagram whose first row is exact:

$$\begin{tikzcd}
\Gamma(\Omega_1, \mathcal{F}) \arrow[r, "\eta"] & \Gamma(\Omega_1, \mathcal{Q}) \arrow[r, "\beta"] & H^1(\Omega_1, \mathcal{O}_{L^*}) \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & H^1(\Omega_1, \mathcal{F}) \arrow[d] \\
 & & H^1(\Omega_1', \mathcal{O}_{L^*}) \arrow[r, "\xi"] & H^1(\Omega_1', \mathcal{F})
\end{tikzcd}$$

(6.6) Lemma. *The map* η: $\Gamma(\Omega_1, \mathcal{F}) \to \Gamma(\Omega_1, \mathcal{Q})$ *is surjective.*

Proof. Take $x \in \Gamma(\Omega_1, \mathcal{F})$. It suffices to show that $\beta(x) = 0$. Clearly, $\gamma\beta(x) = 0$. It follows that $\xi\alpha\beta(x) = 0$. Since V is disjoint from Ω_1', clearly ξ is an isomorphism. From the injectivity of α we conclude that $\beta(x) = 0$.

(6.7) We are now ready to show that there are sufficiently many global holomorphic sections of $\Gamma(M, L^k)$ to make M Moišezon. Without loss of generality we can assume that the local coordinates z of M and the fiber coordinate w of $L|\pi^{-1}(G)$ are so chosen that $z(P_0) = 0$ and $s(P_0) = 1$. Let z_0 denote the function on G which is identically 1. Since η is surjective by Lemma (6.6), for $0 \leqslant j \leqslant n$ there exists a meromorphic function F_j on Ω_1 which on $\Omega_1 \cap \pi^{-1}(G)$, when expressed in terms of $z_1, \cdots, z_n, w$, is of the form

$$F_j(z, w) = \frac{z_j}{w - s(z)} + h_j(z, w), \tag{6.7.1}$$

where $h_j(z, w)$ is holomorphic on Ω_1. Let

$$h_j(z, w) = \sum_{k=0}^{\infty} h_{jk}(z) w^k \tag{6.7.2}$$

be the power series expansion of $h_j(z, w)$ in w. By the definition of Ω_1, for fixed z the radius of convergence of the power series (6.7.2) is at least $e^{-\varphi(z)/2}$. Since

$$\varphi(P_0) = \psi(P_0) - \rho(P_0) = -\rho(P_0) < 0,$$

it follows that there exists an open neighborhood D of P_0 in G and a number $R > 1$ such that the radius of convergence of the power series (6.7.2) is at least R for $z \in D$. By replacing D by a smaller neighborhood and R by a smaller number, we can assume without loss of generality that there exists a positive number A such that

$$|h_{jk}| \leqslant AR^{-k} \quad \text{on } D \text{ for } k \geqslant 0. \tag{6.7.3}$$

Since $s(P_0) = 1$, we can also assume without loss of generality (after replacing D by a smaller neighborhood) that

$$(6.7.4) \qquad |s| < R \quad \text{on } D.$$

We now expand the right-hand side of (6.7.1) in power series in w and obtain

$$F_j(z, w) = \sum_{k=0}^{\infty} \left(h_{jk}(z) - \frac{z_j}{s(z)^{k+1}} \right) w^k.$$

Fix k and consider the map from D to $\mathbf{P}_n$ defined by the homogeneous coordinates

$$\left[h_{0k}(z) - \frac{z_0}{s(z)^{k+1}}, h_{1k}(z) - \frac{z_1}{s(z)^{k+1}}, \cdots, h_{nk}(z) - \frac{z_n}{s(z)^{k+1}} \right],$$

which is the same as the map defined by the homogeneous coordinates

$$\left[z_0 - s(z)^{k+1} h_{0k}(z), z_1 - s(z)^{k+1} h_{1k}(z), \cdots, z_n - s(z)^{k+1} h_{nk}(z) \right].$$

Because of (6.7.3) and (6.7.4), as $k \to \infty$ this map approaches the one defined by the homogeneous coordinates $[1, z_1, \cdots, z_n]$. Thus for k sufficiently large the $n + 1$ elements of $\Gamma(M, L^k)$ which are the kth coefficients of F_j $(0 \leqslant j \leqslant n)$ in the power series expansion in the fiber coordinates of L^* can be used as homogeneous coordinates of some open neighborhood of D. Hence M is Moišezon.

Added in proof. The author has succeeded in refining the method used in the proof of Theorem (5.1) to obtain a proof of the stronger version of the Grauert-Riemenschneider conjecture. Details will appear in a later paper.

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