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COHOMOLOGICAL STRUCTURE IN SOLITON EQUATIONS AND JACOBIAN VARIETIES

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Abstract

The structure of the orbits of the dynamical system defined by the total KP hierarchy is studied. It is shown that every orbit is locally isomorphic to a certain cohomology group associated with a commutative algebra. The KP dynamical system restricted to each orbit determines a dynamical system of linear motions on it with respect to the linear structure of the cohomology group. Remarkably, it is proved that an orbit is finite dimensional if and only if it is essentially a Jacobian variety of an algebraic curve. Using this fact, the problem of characterization of Jacobians among Abelian varieties (*Schottky Problem*) is solved. It is also shown that our cohomology group describes complete families of iso-spectral deformations of linear ordinary differential operators.

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Introduction

The purpose of this paper is to describe the geometric structure of solutions of soliton equations. It is well-known ([4], [6], [7]) that some algebraic curves solve soliton equations. For example, every hyper-elliptic curve gives a solution to the Korteweg-de Vries equation

$$u_t - \frac{1}{4}u_{xxx} - 3uu_x = 0, \tag{1}$$

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where $u = u(x, t)$, $u_t = \frac{\partial u}{\partial t}$ and $u_x = \frac{\partial u}{\partial x}$. Moreover, it is known that *any* algebraic curve gives a solution to the Kadomtsev-Petviashvili equation

$$(2) \qquad \frac{3}{4}u_{yy} - \left(u_t - \frac{1}{4}u_{xxx} - 3uu_x\right)_x = 0$$

for $u = u(x, y, t)$. A solution which comes from a non-singular curve is called a *quasi-periodic* solution. If the curve is rational with only ordinary double points, then the corresponding solution is a *soliton* solution. Furthermore, if the curve is unicursal, then it gives a *rational* solution. The *soliton number* of a soliton solution is nothing but the virtual genus of the corresponding singular curve, and hence its time invariance is obvious. Every quasi-periodic solution determines a dynamical system of a linear flow on the Jacobian variety of the corresponding curve. So the time evolution defined by (1) or (2) is very simple on these Jacobian varieties. They are just *linear* motions, even though the defining equations are *nonlinear* differential equations. Now we can ask the following questions. *Why do curves and their Jacobians appear as solutions to soliton equations? Why do nonlinear equations such as* (1) *and* (2) *determine dynamical systems of linear flows? Are there any other solutions of soliton equations which do not come from algebraic curves?*

I would like to give an answer to these questions in this paper. We will deal only with the KP equation because it is known that it generates several soliton equations such as KdV, modified KdV, Boussinesq and so on ([2], [12]). We consider, however, not only the single equation (2) but also the system of total hierarchy of the KP equations, because the geometric structure of soliton equations becomes much clearer when we deal with the total hierarchy ([2], [8]). Now let X be the space of all formal solutions to the KP hierarchy. Since the total hierarchy involves infinitely many time variables $t_1, t_2, t_3, \cdots$, the infinite dimensional vector group $T = \varinjlim_n K^n$ acts on X to make it into a dynamical system (X, T), where K denotes either the real number field $\mathbf{R}$ or the complex number field $\mathbf{C}$. The structure of the KP dynamical system (X, T) was studied first by M. Sato [12] and later by his colleagues [2]. According to Sato [12], every subdynamical system of (X, T) corresponds to a distinct soliton equation and its orbits correspond to the solutions of this soliton equation.

In this paper we study the geometric structures of orbits of the KP dynamical system. We prove that every orbit of (X, T) is locally isomorphic to a certain cohomology group $H^1(A)$ defined by a commutative algebra A which is essentially a commutative algebra consisting of mutually commuting linear ordinary differential operators. We give a classification of all the orbits of (X, T) in terms of these commutative algebras. Let us note that every orbit of

(X, T) has a linear structure induced by the natural linear structure of the cohomology group $H^1(A)$. It is remarkable that the dynamical system restricted on each orbit determines just *linear flows* with respect to this linear structure. Moreover, if the orbit is finite dimensional, then it is essentially a generalized Jacobian variety! Thus we have obtained the following situation: All the Jacobian varieties appear as orbits of the KP dynamical system. But no other general Abelian varieties appear in this stage, because *every* finite dimensional orbit should be a (generalized) Jacobian variety. This enables us to give a characterization of the Jacobian varieties among all the Abelian varieties. This problem has been long known as the *Schottky Problem* ([10], [14]). I would like to propose an answer to this problem in the following manner: *An Abelian variety is a Jacobian variety if and only if it can be an orbit of the KP dynamical system*. In connection with Novikov's conjecture (cf. [11]), we can say that an Abelian variety is a Jacobian variety if and only if its Riemann theta function solves the *total hierarchy* of the KP equation.

The algebras which will be used in the classification of orbits of the KP dynamical system are subalgebras of the field $K((\lambda^{-1}))$ of quotients of the formal power series ring $K[[\lambda^{-1}]]$ in one variable λ^{-1} having no negative order elements. It turns out that such algebras are realized as algebras consisting of mutually commuting linear ordinary differential operators. There has been a long history in studying commuting ordinary differential operators since Burchnall and Chaundy [1]. What is interesting here is that every such algebra studied by them can be embedded into the commutative field $K((\lambda^{-1}))$ as a subalgebra. The variable λ is identified with the differential operator d/dx in x. We can associate a *cohomology group* $H^1(A)$ with a subalgebra A of $K((\lambda^{-1}))$ satisfying the condition

$$A \cap K\left[[\lambda^{-1}]\right] \cdot \lambda^{-1} = 0.$$

Our cohomology group describes not only the local structure of each orbit of the KP dynamical system but also the family of all the possible *infinitesimal iso-spectral deformations* of a given linear ordinary differential operator. Let L be a linear ordinary differential operator. Then we can construct a *complete* family of infinitesimal iso-spectral deformations of L on the cohomology group $H^1(A)$, where A is the image of the embedding into $K((\lambda^{-1}))$ of the commutative algebra consisting of all linear ordinary differential operators which commute with L. The problem of finding all the possible deformations of L leads us to the KP hierarchy.

This paper is organized as follows. We describe a representation of commutative algebras of ordinary differential operators into $K((\lambda^{-1}))$ to study their algebraic structures in §1. In §2, we introduce the cohomology groups and give their algebro-geometric realization when they are finite dimensional. Actually, our cohomology group $H^1(A)$ is isomorphic to $H^1(C, \mathcal{O}_C)$ of a complete algebraic curve C defined over K which is a one-point completion of the affine curve Spec(A). The iso-spectral deformations of ordinary differential operators are studied in §3. In the process of determining all the possible deformations, we introduce the KP hierarchy as a system of deformation equations. The algebraic structure of the KP hierarchy is studied in [8] and its results are extracted in §4 without giving proofs. In §5, which constitutes the core of this paper together with the next section, we give the classification of orbits of the KP dynamical system in terms of commutative algebras studied in §1. The structure of finite dimensional orbits is studied in §6. Each of them is an open subset of the generalized Jacobian variety (or the connected component of the Picard variety)

$$\mathrm{Pic}^0(C) \cong H^1(C, \mathcal{O}_c)/H^1(C, \mathbf{Z})$$

associated with a complete algebraic curve C defined over K. A proposed solution to the Schottky Problem is given in this section.

I deal only with formal solutions in this paper. But every formal solution of the KP hierarchy corresponding to a finite dimensional orbit converges absolutely to give a globally defined meromorphic solution. The study of convergent solutions requires more lengthy machinery, and so is omitted in this paper. The convergence condition of formal solutions and the topology of the solution space will be given elsewhere.

I would like to thank Professor Mikio Sato for introducing me to the current subject. Actually, I was deeply inspired by his beautiful lectures [13], which are unfortunately unpublished. The idea of classifying orbits of the KP dynamical system by using commutative algebras is essentially due to him.

My special thanks are due to Deborah DeWitt for her careful correction of my English and for typing the manuscript.

List of notation.

$\mathbf{Z}$ = the set of all the integers.

$\mathbf{N}$ = the set of all nonnegative integers.

$K[x]$ = the ring of all the polynomials in x with coefficients in K.

$K[[x]]$ = the ring of all the formal power series in x with coefficients in K.

1. The representation of commutative algebras of ordinary differential operators

In this section we study the algebraic structure of commutative algebras consisting of linear ordinary differential operators and give their simultaneous representations (i.e. injective K-algebra homomorphisms) into the universal commutative algebra.

First of all let us determine our stage. We fix an arbitrary field K of characteristic zero throughout this paper. As a function space we take a commutative differential algebra R defined over K with the unity 1 and a derivation $\partial: R \to R$ satisfying the following conditions:

(i) R is closed under indefinite integration, i.e. for every $f \in R$ there exists $g \in R$ such that $\partial g = f$,

(ii) R is closed under exponentiation, i.e. for any $f \in R$ the expression $\sum_{n=0}^{\infty} f^n/n!$ gives an element in R.

An element $r \in R$ is called a *constant* of R with respect to the derivation ∂ if $\partial r = 0$. We denote by R_{const} the set of all the constants of R.

A typical example of such a differential algebra is the ring $R = K[[x]]$ of all the formal power series in one variable x together with the derivation $\partial = d/dx$. If K is a normed field such as the real number field $\mathbf{R}$ or the complex number field $\mathbf{C}$, then the convergent power series ring also gives an example. In both cases the set R_{const} of all the constants coincides with the basic field K.

The Lie algebra D of all the ordinary differential operators with coefficients in R is defined by

$$D = R[\partial] = \left\{ \sum_{n=0}^{\text{finite}} p_n \partial^n \,\middle|\, p_n \in R \right\}.$$

To calculate fractional powers of elements in D, we need to introduce an extension of D. So we define the Lie algebra E of all the formal ordinary micro- (i.e. pseudo-) differential operators with coefficients in R by

$$\begin{aligned} E &= R((\partial^{-1})) \\ &= \left\{ \sum_{-\infty < \nu \ll +\infty} p_\nu \partial^\nu \,\middle|\, p_\nu \in R, \nu \text{ is bounded from above} \right\}. \end{aligned}$$

The extended Leibniz rule

$$\partial^\nu \cdot f = \sum_{i=0}^{\infty} \binom{\nu}{i} (\partial^i f) \partial^{\nu - i} \quad \text{for } \nu \in \mathbf{Z} \text{ and } f \in R$$

gives an associative algebra structure in E, hence it has a natural Lie algebra structure, where $\partial^\nu \cdot f$ denotes the composition of two *operators* ∂^ν and f in E and $\partial^i f$ denotes the ith derivative of f which is an element in R.

An element $P \in E$ has *order* ν if its coefficient of ∂^ν is nonzero and every coefficient of ∂^μ with $\mu > \nu$ is zero. We denote $\nu = \text{ord}(P)$. Let $E^{(\nu)}$ be the set of all the elements in E of order at most ν. Then we have a left R-module direct sum decomposition

$$E = D \oplus E^{(-1)}. \tag{1.1}$$

According to this decomposition, we write

$$P = P_+ + P_-, \tag{1.2}$$

where P is an arbitrary element of E, $P_+ \in D$ is its differential operator part and $P_- \in E^{(-1)}$ is its negative order part. This decomposition is essential in our theory.

The set $R_{\text{const}}((\partial^{-1}))$ of constant-coefficient operators gives a maximal commutative K-subalgebra of E. What we want to do is to construct a representation (K-algebra isomorphism) from every commutative K-subalgebra in D into $R_{\text{const}}((\partial^{-1}))$.

Now let $B \subset D$ be a commutative K-subalgebra with the unity 1. From now on, we always assume that B has at least one *monic* (i.e. the top order coefficient is the unity) differential operator of order greater than zero. This assumption is not a severe restriction because in usual cases, in which R is the formal or convergent power series ring, we can construct a monic differential operator from an arbitrary one by changing the independent variable x suitably.

We identify two commutative algebras B_1 and B_2 in D if there exists an invertible element $r \in R$ such that

$$B_1 = r \cdot B_2 \cdot r^{-1}.$$

Since we required R to be closed under integration and exponentiation, we can always assume the following.

Assumption 1.1. The commutative subalgebra B in D has an element of the form

$$\partial^n + b_{n-2}\partial^{n-2} + b_{n-3}\partial^{n-3} + \cdots + b_0 \tag{1.3}$$

with $n \geqslant 1$.

Lemma 1.1. *For every commutative subalgebra $B \subset D$ satisfying Assumption* 1.1, *there exists a K-subalgebra A in $R_{\text{const}}((\partial^{-1}))$ with the property*

$$A \cap R_{\text{const}}[[\partial^{-1}]] \cdot \partial^{-1} = \{0\} \tag{1.4}$$

and a K-algebra isomorphism

$$B \xrightarrow{\sim} A \subset R_{\text{const}}((\partial^{-1})). \tag{1.5}$$

Proof. Let us take a monic element $L_n \in B$ of the form (1.3) and set $L = (L_n)^{1/n} \in E$, which is of the form

$$L = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots.$$

According to Lemma 4.1 which we will prove in §4, there is an invertible integral operator

$$S = 1 + s_{-1}\partial^{-1} + s_{-2}\partial^{-2} + \cdots \in 1 + E^{(-1)}$$

such that $L = S \cdot \partial \cdot S^{-1}$. Now define $A = S^{-1} \cdot B \cdot S$. It is evident that A is isomorphic to B as a K-algebra. So let us prove that A is contained in $R_{\text{const}}((\partial^{-1}))$.

Take any $P \in B$. Since $[P, L_n] = P \cdot L_n - L_n \cdot P = 0$, we have

$$[S^{-1} \cdot P \cdot S, \partial] = S^{-1}[P, S \cdot \partial \cdot S^{-1}]S = S^{-1}[P, L]S = 0.$$

This proves that $A \subset R_{\text{const}}((\partial^{-1}))$. Since the inner automorphism of E by S does not change the order of operators in E, A has no negative order element. Hence (1.4) holds. q.e.d.

Remark. This isomorphic representation is not unique because it depends on the choice of $L_n \in B$.

By this lemma, the problem of studying the structure of commutative algebras consisting of ordinary differential operators is reduced to that of studying K-subalgebras in $R_{\text{const}}((\partial^{-1}))$ with the condition (1.4).

Let us now consider the converse. Suppose we have a subalgebra $A \subset K((\partial^{-1}))$ with the condition

$$A \cap K[[\partial^{-1}]] \cdot \partial^{-1} = \{0\}.$$

We want to find an integral operator S so that

$$S \cdot A \cdot S^{-1} \subset D. \tag{1.6}$$

Let $L = S \cdot \partial \cdot S^{-1}$. Then (1.6) is equivalent to

$$b(L) \in D \quad \text{for every } b(\partial) \in A \subset K((\partial^{-1})). \tag{1.7}$$

If we write

$$L = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots,$$

then the condition $b(L) \in D$ by an element $b(\partial) \in A$ of order n implies that every u_ν with $\nu \leqslant -n$ is determined by a differential polynomial in $u_{-1}, u_{-2}, \cdots, u_{-n+1}$.

Lemma 1.2. *Condition* (1.7) *reduces to only finitely many nonlinear ordinary differential equations on finitely many unknown functions* $u_{-1}, u_{-2}, \cdots, u_{-m+1}$ *for some* $m > 0$.

Proof. Take a monic element $a(\partial) \in A$ of the lowest order $m > 0$. By the condition $a(L) \in D$, $(m-1)$ functions $u_{-1}, u_{-2}, \cdots, u_{-m+1}$ determine all the rest of the coefficients of L completely. As we will see in the next section, there is an algebraic relation $f_b(a, b) = 0$ for every $b = b(\partial) \in A$, where $f_b(a, b)$ is a polynomial in a and b with coefficients in K. Let $b(L) = b(L)_+ + b(L)_-$ be the decomposition of $b(L) \in E$ into the differential operator part $b(L)_+ \in D$ and the negative order part $b(L)_- \in E^{(-1)}$.

Now let us impose sufficiently many but only a finite number of differential equations on $u_{-1}, u_{-2}, \cdots, u_{-m+1}$ so that $b(L)_-$ is of order $-r$. We can make this r sufficiently large such that

$$r > m, \tag{1.8}$$

$$f_b(a(L), b(L))_+ = f_b(a(L), b(L)_+). \tag{1.9}$$

Comparing the negative part of

$$0 = [a(L), b(L)] = [a(L), b(L)_+] + [a(L), b(L)_-],$$

we obtain

$$0 = [a(L), b(L)_-] \in E^{(-1)}.$$

This implies that the top order coefficient c of $b(L)_-$ is constant because $a(L)$ is a monic differential operator. Now look at the top order coefficient of

$$f_b(a(L), b(L))_- = f_b(a(L), b(L)) - f_b(a(L), b(L)_+).$$

It is zero because $f_b(a(L), b(L))$ vanishes identically. On the other hand it is a nonzero constant times c because the top order coefficients of $a(L)$ and $b(L)_+$ are both constant. Therefore $c = 0$. This means that $b(L)_-$ vanishes identically. Since A is a finite $K[a(\partial)]$-module (see Corollary 2.1), condition (1.7) can be reduced to only finitely many ordinary differential equations for $u_{-1}, u_{-2}, \cdots, u_{-m+1}$. q.e.d.

Example. If we take $A = K[\partial^2, \partial^3] \subset K((\partial^{-1}))$, then condition (1.7) reduces to only one ordinary differential equation for only one unknown function u_{-1}:

$$\partial^2 u_{-1} + 6u_{-1}^2 = 0.$$

Still it is not clear whether our system of ordinary differential equations which is equivalent to (1.7) *has* a solution or not in general. This inverse problem, however, was studied by a quite different approach. As we will show in the next section, the algebra A determines a complete algebraic curve C over K. Then one might construct a commutative algebra

$$B = S \cdot A \cdot S^{-1} \subset D$$

by using an algebro-geometric technique on the curve C. However, since it has been studied in detail by several people ([1], [4] and [9]), we will not discuss this problem here any more.

2. The first cohomology group and its geometric realization

In this section we study the algebraic structure of each unitary K-subalgebra A in $K((\partial^{-1}))$ with the property

$$A \cap K[[\partial^{-1}]] \cdot \partial^{-1} = \{0\}. \tag{2.1}$$

We define the first cohomology group $H^1(A)$ associated with A and give an algebro-geometric realization of this cohomology group. We denote by $\mathscr{A}$ the set of all the unitary K-subalgebras in $K((\partial^{-1}))$ with the property (2.1).

Lemma 2.1. *Every $A \in \mathscr{A}$ has a K-linear basis with indices in*

$$N_A = \{n \in \mathbf{N} \mid \exists P \in A \text{ such that } \operatorname{ord}(P) = n\}.$$

Proof. Since A is a K-algebra, there is a monic element P_n in A of order n for every $n \in N_A$. We put $P_0 = 1$. Let us prove that $\{P_n | n \in N_A\}$ forms a K-linear basis of A.

Take an arbitrary $P \in A$ and let $\operatorname{ord}(P) = m$. If $m = 0$, then $P \in K$ by (2.1). Hence $P = P \cdot P_0$. So suppose that $m > 0$ and that every element in A of order less than m can be represented as a K-linear combination of P_n's for $n < m$. Let $c \in K$ be the top order coefficient of P. Since the order of $P - cP_m$ is less than m, it is a linear combination of P_n's for $n < m$. Therefore P can be represented as a K-linear combination of P_n's for $n \leqslant m$.

Lemma 2.2. *For every $A \in \mathscr{A}$, there exists a number $r \in \mathbf{N}$ called the* **rank** *of A and a finite subset $F \subset \mathbf{N}$ such that*

$$N_A = r(\mathbf{N} - F). \tag{2.2}$$

Proof. For every integer m and n in $\mathbf{N}$, we denote by $\operatorname{GCD}(m, n)$ their greatest common divisor. The rank r of A is defined by

$$r = \min\{\operatorname{GCD}(\operatorname{ord}(P), \operatorname{ord}(Q)) \mid P, Q \in A\}.$$

By this definition, $N_A \subset r\mathbf{N}$ holds. If $r = 0$, then $A = K$ and F is arbitrary. So we assume that A has at least one element of order greater than zero. Take $P \in A$ of order m and $Q \in A$ of order n so that $r = \operatorname{GCD}(m, n)$. Let $m = m'r$ and $n = n'r$. We want to show that $kr \in N_A$ for every $k \gg 0$ (sufficiently large $k \in \mathbf{N}$). Now choose positive integers a and b so that $r = am - bn$. Let $M = bm'n$. Then we obtain

$$\begin{aligned}
M &= m'bn = \operatorname{ord}(Q^{m'b}) \in N_A, \\
M + r &= am + (m' - 1)bn = \operatorname{ord}(P^a \cdot Q^{(m'-1)b}) \in N_A, \\
M + 2r &= 2am + (m' - 2)bn = \operatorname{ord}(P^{2a} \cdot Q^{(m'-2)b}) \in N_A, \\
&\vdots \\
M + m'r &= m'am + (m' - m')bn = \operatorname{ord}(P^{m'a}) \in N_A.
\end{aligned}$$

Since

$$M + lm = \mathrm{ord}\left(P^l \cdot Q^{m'b}\right) \in N_A$$

for every $l \geqslant 0$, we can conclude that $M + kr \in N_A$ for every $k \geqslant 0$. We define $F = \{ f \in \mathbf{N} \mid fr \notin N_A \}$. Then F is a finite set and satisfies (2.2).

Corollary 2.1. *Let P and Q be elements in $A \in \mathscr{A}$ of positive rank r such that $r = \mathrm{GCD}(\mathrm{ord}(P), \mathrm{ord}(Q))$. Then*

(i) $\dim_K(A/K[P, Q]) < +\infty$,

(ii) *A is a finite $K[P]$-module. Moreover there is a nontrivial polynomial $f(x, y) \in K[x, y]$ such that $f(P, Q) = 0$.*

Proof. Note that $K[P, Q] \in \mathscr{A}$ and $K[P, Q] \subset A$. We can construct a K-linear basis of A with indices in $(N_A - N_{K[P,Q]}) \cup N_{K[P,Q]}$. Since we have proved that $N_A - N_{K[P,Q]}$ is finite, the statement (i) holds. Applying Lemma 2.1 to $K[P, Q]$, we conclude that P and Q satisfy an algebraic relation $f(P, Q) = 0$. q.e.d.

Thus we know that every $A \in \mathscr{A}$ has transcendence degree one over K if $A \neq K$.

Definition 2.1. The cohomology group $H^1(A)$ of $A \in \mathscr{A}$ is defined as the first cohomology group of the following complex:

$$0 \to A \oplus K\left[\left[\partial^{-1}\right]\right] \cdot \partial^{-1} \to K\left(\left(\partial^{-1}\right)\right) \to 0.$$

In the rest of this section we study an algebro-geometric realization of this cohomology group. Let us first associate a complete algebraic curve defined over K with an algebra $A \in \mathscr{A}$. So define

$$A^{(n)} = \left\{P \in A \mid \mathrm{ord}(P) \leqslant n\right\}.$$

Note that $A^{(\nu)} = 0$ for all $\nu \leqslant -1$ and $A^{(0)} = K$ by (2.1). Let I denote the identity operator $1 \in A$. We introduce a new valuation called *degree* in A as follows:

(i) $\deg(I) = 1$,

(ii) every element in $K - \{0\}$ has degree zero,

(iii) $\deg(P) = \mathrm{ord}(P)$ for every $P \in A$ of order greater than zero.

Now define a graded K-algebra by

$$\mathrm{gr}(A) = \bigoplus_{n=0}^{\infty} A_n, \tag{2.3}$$

where

$$A_n = A^{(n)} \oplus A^{(n-1)} \cdot I \oplus A^{(n-2)} \cdot I^2 \oplus \cdots \oplus A^{(0)} \cdot I^n.$$

This graded algebra defines a complete algebraic variety

$$C = \mathrm{Proj}(\mathrm{gr}(A)) \tag{2.4}$$

over K. Since A is a finite $K[P]$-module for some monic element $P \in A$, $\mathrm{gr}(A)$ is finite over $K[I, P]$. Hence C is one-dimensional over K.

Following Mumford [9], our curve C is covered by two affine open sets

$$\begin{aligned} D_+(I) &= \text{affine open set defined by } I \neq 0 \\ &= \mathrm{Spec}\left(\mathrm{gr}(A)\left[\frac{1}{I}\right]_0\right) \cong \mathrm{Spec}(A), \\ D_+(P) &= \text{affine open set defined by } P \neq 0 \\ &= \mathrm{Spec}\left(\mathrm{gr}(A)\left[\frac{1}{P}\right]_0\right). \end{aligned}$$

Take another monic element $Q \in A$ so that

$$r = \text{rank of } A = \mathrm{GCD}(\mathrm{ord}(P), \mathrm{ord}(Q)).$$

Choose positive integers a and b such that $r = a\,\mathrm{ord}(P) - b\,\mathrm{ord}(Q)$ as before. Then

$$\frac{Q^b \cdot I^r}{P^a} \in \mathrm{gr}(A)\left[\frac{1}{P}\right]_0.$$

Therefore the completion of $\mathrm{gr}(A)[1/P]_0$ in the I-adic topology is $K[[Q^b/P^a]]$. Note that

$$Q^b/P^a = \partial^{-r} + (\text{lower order terms}) \in K((\partial^{-1})).$$

Now let us suppose $r = 1$. This is equivalent to the condition that requires $H^1(A)$ to be of finite dimension over K. Since

$$Q^b/P^a = \partial^{-1} + (\text{lower order terms}),$$

we obtain

$$K[[Q^b/P^a]] = K[[\partial^{-1}]].$$

This shows that Q^b/P^a gives a local coordinate of $D_+(P)$. Therefore C is a one-point completion of the affine algebraic curve $\mathrm{Spec}(A)$ by a smooth K-rational point $\{Q^b/P^a = 0\}$ of C. We denote this point by ∞.

Theorem 2.1. *Suppose A has rank one. Then our cohomology group $H^1(A)$ is isomorphic to the cohomology group $H^1(C, \mathcal{O}_C)$ defined on the complete algebraic curve C with coefficients in the structure sheaf $\mathcal{O}_C$ of C.*

Proof. Let us compute the cohomology group $H^1(C, \mathcal{O}_C)$ according to the affine covering $C = D_+(I) \cup D_+(P)$. We obtain

$$H^1(C, \mathcal{O}_C) \cong \Gamma(D_+(P) - \infty, \mathcal{O}_C)/[\Gamma(D_+(I), \mathcal{O}_C) + \Gamma(D_+(P), \mathcal{O}_C)].$$

We can compute the right-hand side in $K((\partial^{-1}))$ because

$$\Gamma(D_+(P), \mathcal{O}_C) \subset K[[Q^b/P^a]] = K[[\partial^{-1}]],$$
$$\Gamma(D_+(P) - \infty, \mathcal{O}_C) \subset K((Q^b/P^a)) = K((\partial^{-1})),$$
$$\Gamma(D_+(I), \mathcal{O}_C) = A \subset K((\partial^{-1})).$$

Therefore,

$$H^1(C, \mathcal{O}_C) \cong K((\partial^{-1}))/(A + K[[\partial^{-1}]]) = H^1(A).$$

q.e.d.

Now let us go back to commutative algebras consisting of ordinary differential operators. Let A be an element of $\mathscr{A}$ of rank one and let $S \in 1 + E^{(-1)}$ be an invertible integral operator satisfying

$$B = S \cdot A \cdot S^{-1} \subset D. \tag{2.5}$$

We can associate a graded K-algebra $\mathrm{gr}(B)$ with B as in (2.3). Similarly, D has a graded K-algebra structure $\mathrm{gr}(D)$. Then, $\mathrm{gr}(D)$ has a right $\mathrm{gr}(A)$-module structure via (2.5). Since $\mathrm{gr}(D)$ is naturally a left R-module, it has a module structure over $R \otimes_K \mathrm{gr}(A)$ whose rank is precisely the rank r of A, hence in our case, one. In short, every $S \in 1 + E^{(-1)}$ satisfying (2.5) gives $\mathrm{gr}(D)$ a locally free $(R \otimes_K \mathrm{gr}(A))$-module structure. Let $\mathscr{L}$ denote the corresponding rank one sheaf over $\mathrm{Spec}\, R \times_K C$. Note that the right B-module structure of D and the $(\tilde{R} \otimes_K \mathcal{O}_C)$-module structure of $\mathscr{L}$ on $\mathrm{Spec}\, R \times_K C$ are in one-to-one correspondence. Therefore we have an injective map

$$\{S \in G/G_{\mathrm{const}} \text{ satisfying } (2.5)\} \to H^1(\mathrm{Spec}\, R \times_k C, \mathcal{O}^*), \tag{2.6}$$

where $G = 1 + E^{(-1)}$ and $G_{\mathrm{const}} = \{S \in G \| [S, \partial] = 0\}$.

3. Iso-spectral deformation of ordinary differential operators

Let us consider the following problem: *Find a complete family of infinitesimal iso-spectral deformations of a given linear ordinary differential operator.* Here by *complete family* we mean a family involving all the possible infinitesimal iso-spectral deformations. As for geometric structures defined on a manifold, their deformations are usually described by certain first cohomology groups. In our case, even though differential operators are different from geometric structures on manifolds, the deformations are again related to a cohomological structure.

In this section we show that the cohomology group defined in the previous section describes an *effective* (i.e. smallest possible) complete family of infinitesimal iso-spectral deformations of a given ordinary differential operator. In the rest of this paper, we always assume that $R_{\mathrm{const}} = K$.

Let us start with a monic nth order ordinary differential operator $L_n \in D$ of the form (1.3). According to P. Lax [5], every one-parameter family $L_n(y)$ of infinitesimal iso-spectral deformations of L_n should satisfy the *Lax type equation*

$$\frac{\partial L_n(y)}{\partial y} = [Z(y), L_n(y)] \tag{3.1}$$

for some differential operator $Z(y)$ depending on $y \in K$. Therefore finding all the possible iso-spectral deformations of L_n is equivalent to determining all the permissible Lax type equations for L_n and to solve them.

Before talking about Lax type equations, we have to determine the deformation parameter dependence of elements in R, D and E. So we introduce the following notations

$$\mathscr{R} = R[[t_1, t_2, t_3, \cdots]], \qquad \mathscr{D} = \mathscr{R}[\partial], \qquad \mathscr{E} = \mathscr{R}((\partial^{-1})).$$

We introduce the weighted degree in $\mathscr{R}$ by defining $\deg t_n = n$. Note here that $\mathscr{R}$ satisfies the same conditions (i) and (ii) which we imposed on R in §1. The space T of all deformation parameters $t_1, t_2, \cdots$ is defined by $T = \varinjlim_n K^n$ together with the inductive limit topology.

Let $\mathscr{E}^{(\nu)}$ be the set of all operators in $\mathscr{E}$ of order at most ν. Then corresponding to (1.1) we have a left $\mathscr{R}$-module direct sum decomposition.

$$\mathscr{E} = \mathscr{D} \oplus \mathscr{E}^{(-1)}. \tag{3.2}$$

We use the notation P_+ for the differential operator part of $P \in \mathscr{E}$ and P_- for its negative order part.

What we want to do now is to determine all permissible Lax type equations for L_n. So let us look at (3.1). It is an equation in $\mathscr{D}$. We can also write it in $\mathscr{E}$ level as follows:

$$\frac{\partial (L_n(y))^{1/n}}{\partial y} = \left[Z(y), (L_n(y))^{1/n}\right]. \tag{3.3}$$

Since the $1/n$ power of L_n is of the form

$$(L_n(y))^{1/n} = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots,$$

the left-hand side of (3.3) is an element of $\mathscr{E}^{(-1)}$.

Definition 3.1. We call a first order monic operator in $\mathscr{E}$ of the form

$$L = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots$$

a *Lax operator*. For a differential operator $P \in \mathscr{D}$ and a Lax operator L, the pair (P, L) is called a *Lax pair* if their commutator $[P, L] = P \cdot L - L \cdot P$ belongs to $\mathscr{E}^{(-1)}$.

Let L be a Lax operator. Then for every $n \in \mathbf{N}$ the pair $((L^n)_+, L)$ is a Lax pair since

$$[(L^n)_+, L] = -[(L^n)_-, L] \in [\mathscr{E}^{(-1)}, \mathscr{E}^{(1)}] \subset \mathscr{E}^{(-1)}.$$

Lemma 3.1. *Let L be a Lax operator and (P, L) be a Lax pair. Then $P \in \mathscr{D}$ is a linear combination of $(L^n)_+$'s with coefficients in $\mathscr{R}_{\mathrm{const}}$ = constants in $\mathscr{R}$ with respect to ∂.*

Proof. Suppose the lemma is true for all $P \in \mathscr{D}$ of order less than n. Take a Lax pair (P, L) with $\operatorname{ord}(P) = n$. Since L is monic and $[P, L] \in \mathscr{E}^{(-1)}$, the top order coefficient p_n of P is a constant. Hence $(P - p_n(L^n)_+, L)$ gives another Lax pair with $\operatorname{ord}(P - p_n(L^n)_+) < n$. This proves that P is a linear combination of $1, (L)_+, (L^2)_+, \cdots, (L^n)_+$ with coefficients in $\mathscr{R}_{\mathrm{const}}$. q.e.d.

Thus we have determined all the *candidates* for the right-hand side of the permissible Lax type equations (3.3) for a Lax operator L. Since we want only *independent* infinitesimal deformations of L, we need not take any t-dependent combinations but just K-linear combinations of $(L^n)_+$'s. The following system of infinitely many Lax type equations is called the *Kadomtsev-Petviashvili hierarchy* and known as a *generic* system of several kinds of *soliton equations*:

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \qquad n = 1, 2, 3, \cdots. \tag{3.4}$$

Let us examine how this system (KP hierarchy) generates soliton equations. First we note that the original Kadomtsev-Petviashvili (or 2-dimensional KdV) equation

$$\frac{3}{4}\frac{\partial^2 u_{-1}}{\partial t_2^2} - \partial\left(\frac{\partial u_{-1}}{\partial t_3} - \frac{1}{4}\partial^3 u_{-1} - 3u_{-1}\partial u_{-1}\right) = 0$$

follows from the system (3.4). Secondly, if we impose an auxiliary condition $L^2 \in \mathscr{D}$, then we immediately obtain from (3.4) the system

$$\frac{\partial L^2}{\partial t_{2n-1}} = [(L^{2n-1})_+, L^2], \qquad n = 1, 2, 3, \cdots, \tag{3.5}$$

which describes the complete family of infinitesimal iso-spectral deformations of the one-dimensional Schrödinger operator $L^2 = \partial^2 + 2u_{-1}$. This system is nothing but the Korteweg-de Vries hierarchy which appeared in [5]. The original KdV equation

$$\frac{\partial u_{-1}}{\partial t_3} = \frac{1}{4}\partial^3 u_{-1} + 3u_{-1}\partial u_{-1}$$

follows from (3.5) by taking $n = 2$.

Similarly the auxiliary condition $L^3 \in \mathscr{D}$ gives the total Boussinesq hierarchy, and so on.

To get a deformation family, we have to solve (3.4). The initial value problem of system (3.4) was solved in my previous paper [8] which is briefly reviewed in the next section. For the given operator $L_n \in D$, we can solve (3.4) with the initial value

$$L(0) = (L_n)^{1/n} \in E^{(1)},$$

according to Theorem 4.1 of §4. We denote by $L(t)$ the unique solution defined on T. Since the nth power $L_n(t) = (L(t))^n$ is automatically a differential operator by Lemma 5.1 of §5, we obtain a complete family $L_n(t)$ of infinitesimal iso-spectral deformations of $L_n = L_n(0)$ defined on T.

The remaining problem is to eliminate the irrelevant deformation parameters. In the KdV case, for example, all the even number parameters are irrelevant because

$$\frac{\partial L^2}{\partial t_{2n}} = \left[(L^{2n})_+, L^2\right] = [L^{2n}, L^2] = 0.$$

Thus, some of the deformation parameters $t_1, t_2, t_3, \cdots$ might be irrelevant for the given L_n. It is our cohomology group that picks up all the relevant nontrivial deformation parameters.

Lemma 3.2. *The set of all the differential operators in D which commute with a given operator $L_n \in D$ is itself a commutative subalgebra of D.*

Proof. Let $B = \{P \in D \,|\, [P, L_n] = 0\}$ and take an integral operator $S \in 1 + E^{(-1)}$ such that

$$(L_n)^{1/n} = S \cdot \partial \cdot S^{-1}.$$

Then the argument in the proof of Lemma 1.1 shows that

$$S^{-1} \cdot B \cdot S \hookrightarrow K((\partial^{-1})).$$

Hence B is itself commutative.

Definition 3.2. Let $L_n(y_1, y_2, y_3, \cdots)$ be a family of iso-spectral deformations of $L_n(0)$ depending on deformation parameters $y = (y_1, y_2, y_3, \cdots) \in K^m$, where m is a finite number or ∞. Then the parameters $y_1, y_2, \cdots$ are called *effective* if the K linear map from the tangent space $T_0(K^m)$ to $E^{(-1)}$ defined by

$$T_0(K^m) \ni \frac{\partial}{\partial y_i} \mapsto \left.\frac{\partial (L_n(y))^{1/n}}{\partial y_i}\right|_{y=0} \in E^{(-1)}$$

is injective.

Let $A \in \mathscr{A}$. We now want to construct a natural linear map from $H^1(A)$ to T. Since

$$H^1(A) = K((\partial^{-1}))/A \oplus K[[\partial^{-1}]] \cdot \partial^{-1}$$
$$\cong K[\partial] \cdot \partial/(A/K[[\partial^{-1}]]),$$

we can take finitely or infinitely many elements

$$h_i = \sum_{j=1}^{n_i} h_{ij}\partial^j \in K[\partial] \cdot \partial, \qquad i = 1, 2, 3, \cdots, \tag{3.6}$$

as a K-linear basis of $H^1(A)$. Let $y = (y_1, y_2, y_3, \cdots)$ be the coordinate system of $H^1(A)$ with respect to the basis $h_1, h_2, \cdots$. Then the *canonical linear map*

$$f\colon H^1(A) \ni y \mapsto f(y) = t \in T \tag{3.7}$$

is defined by

$$t_j = \sum_{i \geqslant 1} h_{ij} y_i. \tag{3.8}$$

This definition does not depend on the choice of a basis of $H^1(A)$. Note that the canonical linear map has maximal rank.

Theorem 3.1. *Let B be the commutative algebra consisting of all the differential operators which commute with the given differential operator $L_n \in D$ of order n and $A \in \mathscr{A}$ be its representation*

$$A = S^{-1} \cdot B \cdot S \subset K((\partial^{-1})),$$

where $S \in 1 + E^{(-1)}$ is an integral operator satisfying $(L_n)^{1/n} = S \cdot \partial \cdot S^{-1}$. Let $L(t)$ be the solution of (3.4) *starting at $L(0) = (L_n)^{1/n}$ and let $L_n(t) = (L(t))^n \in \mathscr{D}$. Then the family $L_n(f(y))$ defined on $H^1(A)$ gives an effective complete family of infinitesimal iso-spectral deformations of L_n, where $f\colon H^1(A) \ni y \mapsto f(y) = t \in T$ is the canonical linear map.*

Proof. This theorem follows directly from the results of §5. We have only to prove that the orbit M of the KP dynamical system corresponding to $L(t)$ is an A-maximal orbit (in the notations given in §5). So it is sufficient to prove that our A coincides with A_M of (5.5). Let $S(t)$ be a gauge operator of $L(t)$ satisfying the gauge equation (4.9) (see §4) and put $B(t) = S(t) \cdot A \cdot S(t)^{-1}$. This algebra coincides with B at $t = 0$. Obviously, the commutative algebra B_M of (5.4) is contained in $B(t)$ because every element in B_M commutes with $L(t)$. So let us show the converse. Now take an element $P(t) = S(t) \cdot a(\partial) \cdot S(t)^{-1} \in B(t)$, where $a(\partial) \in A \subset K((\partial^{-1}))$. Since $B(t) \subset \mathscr{D}$, we have $P(t) = a(L(t)) = a(L(t))_+$. Hence we can write

$$a(L(t))_+ = \sum_{n=0}^{\text{finite}} a_n (L(t)^n)_+,$$

where the a_n's are constants. By the KP hierarchy (3.4),

$$\sum_{n\geqslant 1} a_n \frac{\partial L(t)}{\partial t_n} = \sum_{n\geqslant 0} a_n\left[L(t)_+^n, L(t)\right] = \left[P(t), L(t)\right] = 0.$$

This means $P(t) \in B_M$. Therefore we can conclude that M is an A-maximal orbit.

4. The algebraic structure of the Kadomtsev-Petviashvili equation

In [8] we gave a method for solving the initial value problem for the KP hierarchy (3.4). Here we review briefly the results therein, without giving proofs.

The key idea of the arguments in [8] is the *linearization* of the nonlinear evolution equation (3.4). In that paper we gave a system of linear partial differential equations with constant coefficients which is equivalent to (3.4). Since all the solutions to this linear system are easily determined, we can solve the original system (3.4). The basic tool we need there is the formal Lie groups consisting of infinite order ordinary (micro-) differential operators. We use the following notations:

$$\hat{\mathscr{D}} = \left\{ \sum_{\nu=0}^{\infty} p_\nu \partial^\nu \,\middle|\, p_\nu \in \mathscr{R}, \exists N \in \mathbf{N}, \operatorname{ord}_t(p_\nu) > \nu - N \,\forall \nu \gg 0 \right\},$$

$$\hat{\mathscr{E}} = \left\{ \sum_{\nu\in\mathbf{Z}} p_\nu \partial^\nu \,\middle|\, p_\nu \in \mathscr{R}, \exists N \in \mathbf{Z}, \operatorname{ord}_t(p_\nu) > \nu - N \,\forall \nu \gg 0 \right\},$$

where $\operatorname{ord}_t(p_\nu)$ denotes the order of p_ν as a formal power series in $t \in T$. These algebras are the formal completions of $\mathscr{D}$ and $\mathscr{E}$ respectively. Their *formal Lie groups* are defined by

$$\hat{\mathscr{D}}^\times = \left\{ P \in \hat{\mathscr{D}} \mid P|_{t=0} = 1, \exists P^{-1} \in \hat{\mathscr{D}} \right\},$$

$$\hat{\mathscr{E}}^\times = \left\{ P \in \hat{\mathscr{E}} \mid P|_{t=0} \in 1 + E^{(-1)}, \exists P^{-1} \in \hat{\mathscr{E}} \right\},$$

where $P|_{t=0}$ is the restriction of P at $t = 0$. The set $G = 1 + E^{(-1)}$ has a group structure and we regard this as the formal Lie group of the Lie algebra $E^{(-1)}$. Similarly $\mathscr{G} = 1 + \mathscr{E}^{(-1)}$ is the formal Lie group of the Lie algebra $\mathscr{E}^{(-1)}$. Corresponding to the direct sum decomposition of the Lie algebra (3.2), we have "a kind of" direct product decomposition in group level as follows;

(4.1) $$\hat{\mathscr{E}}^\times = \mathscr{G} \cdot \hat{\mathscr{D}}^\times.$$

This means that for every $U \in \hat{\mathscr{E}}^\times$, there exist unique elements S in $\mathscr{G}$ and Y in $\hat{\mathscr{D}}^\times$ such that

(4.2) $$U = S^{-1} \cdot Y.$$

Lemma 4.1 [16]. *For every Lax operator $L \in \mathcal{E}^{(-1)}$, there is an integral operator $S \in \mathcal{G}$ called a* ***gauge operator*** *of L such that*

$$L = S \cdot \partial \cdot S^{-1}. \tag{4.3}$$

Proof. Put $L = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots$ and solve (4.3) in $S = 1 + s_{-1}\partial^{-1} + s_{-2}\partial^{-2} + \cdots$. The equation is equivalent to the following system:

$$\partial s_\nu = -\sum_{\mu \leqslant -1} \sum_{i \geqslant 0} \binom{\mu}{i} u_\mu \left(\partial^i s_{\nu-\mu+i}\right), \qquad \nu = -1, -2, -3, \cdots. \tag{4.4}$$

Note that the right-hand side involves only $s_{-1}, s_{-2}, \cdots, s_{\nu+1}$. So we can solve this system successively by taking indefinite integrals only once at each step to determine all the coefficients s_ν of S.

Remark. (i) The proof is valid for $L \in E^{(1)}$ to determine $S \in G$ satisfying (4.3). We have used this fact in §1.

(ii) In each step of the integration of (4.4) we have one constant c_ν of integration with respect to the derivation ∂. So define

$$S_{\text{const}} = 1 + c_{-1}\partial^{-1} + c_{-2}\partial^{-2} + \cdots.$$

If S is a gauge operator of L, then so is $S \cdot S_{\text{const}}$ because

$$(S \cdot S_{\text{const}}) \cdot \partial \cdot (S \cdot S_{\text{const}})^{-1} = S \cdot \partial \cdot S^{-1} = L.$$

The gauge operator $S \cdot S_{\text{const}}$ is a solution of (4.4) taking all the constants c_ν of integration into account. Conversely, if S_1 and S_2 are gauge operators of the same L, then there exists $S_{\text{const}} \in \mathcal{G}$ with $[S_{\text{const}}, \partial] = 0$ such that $S_2 = S_1 \cdot S_{\text{const}}$. Let $\mathcal{G}_{\text{const}} = \{S \in \mathcal{G} \| [S, \partial] = 0\}$. Then the Lax operator $L \in \mathcal{E}^{(1)}$ determines a unique element in $\mathcal{G}/\mathcal{G}_{\text{const}}$ and vice versa. Therefore we identify $\mathcal{G}/\mathcal{G}_{\text{const}}$ with the set of all Lax operators. Similarly we identify G/G_{const} with the set of all monic first order operators of the form $\partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots \in E^{(1)}$.

Zakharov and Shabat [18] wrote equation (3.4) in terms of *zero curvature condition*. Let us introduce the Zakharov-Shabat connections of L:

$$Z_L^{\pm} = \pm \sum_{n=1}^{\infty} (L^n)_{\pm}\, dt_n. \tag{4.5}$$

Then the Lax equation (3.4) for the KP hierarchy can be written as

$$dL = \left[Z_L^+, L\right], \tag{4.6}$$

where d denotes the exterior differentiation on T. Since $[Z_L^+, L] = [Z_L^-, L]$, we have an equivalent equation

$$dL = \left[Z_L^-, L\right]. \tag{4.7}$$

Under the definition (4.5), (4.6) or (4.7) is equivalent to the Zakharov-Shabat equation

$$dZ_L^{\pm} - \tfrac{1}{2}[Z_L^{\pm}, Z_L^{\pm}] = 0. \tag{4.8}$$

M. Sato [13] and G. Wilson [17] have introduced independently an equation on the gauge operator $S \in \mathscr{G}$ equivalent to (4.6):

Lemma 4.2. *A Lax operator $L \in \mathscr{E}^{(1)}$ satisfies* (4.6) *if and only if there exists a gauge operator $S \in \mathscr{G}$ of L satisfying the* ***gauge equation***

$$dS = Z_L^- \cdot S. \tag{4.9}$$

Now we can recall the main theorem of [8].

Theorem 4.1 [8]. *We can solve the initial value problem for the KP hierarchy* (4.6) *by the following procedures*:

Start with a given initial data $L(0) \in G/G_{\text{const}}$. By calculating infinitely many successive indefinite integrals, we obtain an integral operator $S(0) \in G$ satisfying $L(0) = S(0) \cdot \partial \cdot S(0)^{-1}$. Now define

$$U(t) = \exp(t_1\partial + t_2\partial^2 + t_3\partial^3 + \cdots) \cdot S(0)^{-1} \in \hat{\mathscr{E}}^{\times}. \tag{4.10}$$

Let $S(t)$ be the $\mathscr{G}$-part of $U(t)$ according to the decomposition (4.2). *Then the Lax operator $L(t) = S(t) \cdot \partial \cdot S(t)^{-1}$ gives the desired solution of* (4.6) *with the initial value $L(t)|_{t=0} = L(0)$.*

By this theorem we see that the solution space of (4.6) is nothing but the space G/G_{const} of initial data. The KP hierarchy (4.6) determines an infinitesimal dynamical system on it with the infinite dimensional additive time group T acting on it, which we call the *KP dynamical system* and denote by $(G/G_{\text{const}}, T)$.

5. The classification of solutions of soliton equations

From the point of view of M. Sato [12], the KP hierarchy (3.4) or (4.6) is a defining equation of the KP dynamical system on the solution space G/G_{const}, and all its subdynamical systems correspond to distinct *soliton equations* and every orbit of them gives a solution of these soliton equations.

In this section we give a classification of all the orbits of the KP dynamical system in terms of the commutative algebras and their cohomology groups studied in §§1 and 2.

Lemma 5.1. *For every algebra A in $\mathscr{A}$, the condition $S \cdot A \cdot S^{-1} \subset D$ imposed on $S \in G = 1 + E^{(-1)}$ is time invariant. Namely the unique solution*

$S(t) \in \mathcal{G}$ of the gauge equation (4.9) *which starts at $S(0) = S$ satisfies the same condition $S(t) \cdot A \cdot S(t)^{-1} \subset \mathcal{D}$.*

Proof. Let $U(t) = S(t)^{-1} \cdot Y(t)$ be the decomposition of the operator

$$U(t) = \exp\left(t_1\partial + t_2\partial^2 + t_3\partial^3 + \cdots\right) \cdot S(0)^{-1}$$

of (4.10) according to (4.2). Since $Y(t)$ belongs to $\hat{\mathcal{D}}^{\times}$ and the operator $\exp(t_1\partial + t_2\partial^2 + \cdots)$ commutes with A, we have

$$\begin{aligned} S(t) \cdot A \cdot S(t)^{-1} &= Y(t) \cdot U(t)^{-1} \cdot A \cdot U(t) \cdot Y(t)^{-1} \\ &= Y(t) \cdot S(0) \cdot A \cdot S(0)^{-1} \cdot Y(t)^{-1} \\ &\subset Y(t) \cdot D \cdot Y(t)^{-1} \subset \hat{\mathcal{D}}. \end{aligned}$$

Every element in $S(t) \cdot A \cdot S(t)^{-1}$ is of finite order, hence this set is contained in $\mathcal{D}$. q.e.d.

Thus the set

$$X_A = \left\{S \in G \middle| S \cdot A \cdot S^{-1} \subset D\right\}$$

is a time invariant subspace of G. Let

$$\tilde{X}_A = \left\{L = S \cdot \partial \cdot S^{-1} \middle| S \in X_A\right\} \subset G/G_{\text{const}}. \tag{5.1}$$

We call $(\tilde{X}_A, T)$ the *subdynamical system* of the KP dynamical system defined by $A \in \mathcal{A}$. By definition, if $A \subset A'$ for A and A' in $\mathcal{A}$, then $\tilde{X}_A \supset \tilde{X}_{A'}$. An orbit of $(\tilde{X}_A, T)$ is called an *A-maximal orbit* if it is not contained in any smaller subdynamical system $(\tilde{X}_{A'}, T)$ of $A' \supsetneq A$. What we want to show first is that every A-maximal orbit of the KP dynamical system is locally isomorphic to the cohomology group $H^1(A)$ of $A \in \mathcal{A}$.

Lemma 5.2. *Let $L(t)$ be a solution to the KP hierarchy* (4.6) *corresponding to an A-maximal orbit in $(G/G_{\text{const}}, T)$ and let $S(t)$ be a gauge operator of $L(t)$ satisfying the gauge equation* (4.9). *Then the algebra A can be recovered by*

$$A = \left\{S(0)^{-1} \cdot a(L(0))_+ \cdot S(0) \middle| a(\partial) \in K((\partial^{-1})), \left[a(L(0))_+, L(0)\right] = 0\right\}.$$

Proof. Take an arbitrary element $a(\partial) \in A \subset K((\partial^{-1}))$. Since $S(0) \cdot A \cdot S(0)^{-1} \subset D$, $a(L(0)) = S(0) \cdot a(\partial) \cdot S(0)^{-1}$ coincides with $a(L(0))_+$. Hence we have $a(L(0)) = S(0)^{-1} \cdot a(L(0))_+ \cdot S(0)$ and $[a(L(0))_+, L(0)] = 0$.

Conversely, suppose an element $a(\partial) \in K((\partial^{-1}))$ satisfies $[a(L(0))_+, L(0)] = 0$. Since it means $[S(0)^{-1} \cdot a(L(0))_+ \cdot S(0), \partial] = 0$, $S(0)^{-1} \cdot a(L(0))_+ \cdot S(0)$ is an element in $K((\partial^{-1}))$. Now let $A' = A[S(0)^{-1} \cdot a(L(0))_+ \cdot S(0)] =$ the symmetric algebra generated by $S(0)^{-1} \cdot a(L(0))_+ \cdot S(0)$ over A. This is also an element in $\mathscr{A}$ and satisfies $S(0) \cdot A' \cdot S(0)^{-1} \subset D$. Therefore the orbit determined by $L(t)$ is contained in the subdynamical system $(\tilde{X}_{A'}, T)$ of $A' \supset A$. But we assumed that this orbit was an A-maximal orbit of $(\tilde{X}_A, T)$. Hence A' must coincide with A. This means $S(0)^{-1} \cdot a(L(0))_+ \cdot S(0) \in A$. q.e.d.

Now let M_A denote an A-maximal orbit of $(\tilde{X}_A, T)$ defined by $A \in \mathscr{A}$ and let $L = L(t)$ be a corresponding solution of (4.6) starting at $L(0) \in M_A$. First we note that every K-basis ∂^n of $K[\partial] \cdot \partial$ corresponds to a distinct time evolution of L via the Lax type equation

$$\frac{\partial L}{\partial t_n} = \left[(S \cdot \partial^n \cdot S^{-1})_+, L \right],$$

where $S = S(t)$ is the corresponding solution of the gauge equation (4.9). We also know by Lemma 5.2 that every element $a \in A$ corresponds to a trivial (or stationary) time evolution because $[(S \cdot a \cdot S^{-1})_+, L] = 0$. Therefore our cohomology group $H^1(A)$ represents the *effective* time evolutions.

Definition 5.1. A subspace $Y \subset T$ containing $0 \in T$ is called an *L-effective parameter subspace* of T if the linear map from $T_0(Y)$ to $E^{(-1)}$ defined by

$$T_0(Y) \ni \frac{\partial}{\partial y} \mapsto \left. \frac{\partial L}{\partial y} \right|_{t=0} \in E^{(-1)}$$

is injective.

Lemma 5.3. *The image $Y = f(H^1(A)) \subset T$ of the canonical linear map f: $H^1(A) \to T$ of (3.7) gives an L-effective parameter subspace of T of the maximal dimension.*

Proof. Let $h_i = \sum_{j=1}^{n_i} h_{ij} \partial^j \in K[\partial]$ $(i = 1, 2, 3, \cdots)$ be a basis of $H^1(A)$ as in (3.6) and let

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{n_i} h_{ij} \frac{\partial}{\partial t_j} \in T_0(Y), \tag{5.2}$$

where $y_1, y_2, y_3, \cdots$ are the coordinates of $H^1(A)$ with respect to the basis $h_1, h_2, h_3, \cdots$. Then the KP hierarchy in terms of $y_1, y_2, y_3, \cdots$ is given by

$$\frac{\partial L}{\partial y_i} = \left[(S \cdot h_i \cdot S^{-1})_+, L \right]. \tag{5.3}$$

Suppose there is a K-linear relation

$$\sum_{i=1}^{\text{finite}} c_i \left. \frac{\partial L}{\partial y_i} \right|_{t=0} = 0$$

with some constants $c_i \in K$. Then by the KP hierarchy, we have

$$\left[\left(S \cdot \sum_i c_i h_i \cdot S^{-1}\right)_+, L\right]\Bigg|_{t=0} = 0.$$

Then Lemma 5.2 says that $\sum_i c_i h_i$ + (suitable nonpositive order terms) belongs to the algebra A. Hence $\sum_i c_i h_i = 0$ as an element in $H^1(A)$. Therefore every coefficient c_i vanishes. This proves that $Y = f(H^1(A))$ is an L-effective parameter subspace of T.

Now take any

$$\frac{\partial}{\partial y} = \sum_{j \geqslant 1} k_j \frac{\partial}{\partial t_j} \in T_0(T) - T_0(Y).$$

By definition, $\sum_j k_j \partial^j$ corresponds to the zero-element of $H^1(A)$, hence it belongs to $A/(K[[\partial^{-1}]] \cap A)$. Then again by Lemma 5.2,

$$\frac{\partial L}{\partial y}\bigg|_{t=0} = \left[\left(S \cdot \sum_i k_j \partial^j \cdot S^{-1}\right)_+, L\right]\Bigg|_{t=0} = 0.$$

This shows that the L-effective parameter subspace $Y = f(H^1(A))$ has maximal dimension in T. q.e.d.

Now we consider $\partial L/\partial t_n|_{t=0}$ an element of the tangent space $T_{L(0)}(M_A)$ of the A-maximal orbit M_A corresponding to $L(t)$ at $L(0) \in M_A$. Then we have an isomorphism from $H^1(A)$ to $T_{L(0)}(M_A)$ defined by

$$H^1(A) \tilde{\to} T_0\big(f(H^1(A))\big) \ni \frac{\partial}{\partial y} \mapsto \frac{\partial L}{\partial y}\bigg|_{t=0} \in T_{L(0)}(M_A).$$

Note that this isomorphism does not depend on the specific point $L(0) \in M_A$. Thus we conclude that M_A is locally isomorphic to $H^1(A)$. We can take the coordinates $y_1, y_2, y_3, \cdots$ of $H^1(A)$ as a local coordinate system of M_A near $L(0)$. Therefore the time evolution of $L(t) \in M_A$ described in terms of the local coordinates $y_1, y_2, y_3, \cdots$ is *linear* in $t \in T$! Thus we have obtained the following.

Theorem 5.1. *Every A-maximal orbit M_A in the subdynamical system $(\tilde{X}_A, T)$ of the KP dynamical system defined by $A \in \mathscr{A}$ is locally isomorphic to the cohomology group $H^1(A)$ of A and the dynamical system restricted on M_A defines a dynamical system of linear motions with respect to the linear structure of $H^1(A)$.*

We now consider the converse direction. Let L be a solution of (4.6) which corresponds to an orbit M of the KP dynamical system and let S be a gauge operator of L satisfying the gauge equation (4.9). Then L defines an onto linear map $h\colon T_0(T) \to T_{L(0)}(M)$ by

$$h\colon T_0(T) \ni \frac{\partial}{\partial t_n} \mapsto \left.\frac{\partial L}{\partial t_n}\right|_{t=0} \in T_{L(0)}(M).$$

We define

$$B_M = K\text{-subalgebra of } \mathscr{D} \text{ generated by}$$

(5.4) $$\left\{ \sum_{n=1}^{\text{finite}} c_n (L^n)_+ \,\middle|\, c_n \in K,\ \sum_{n=1}^{\text{finite}} c_n \frac{\partial}{\partial t_n} \in \operatorname{Ker}(h) \right\}.$$

Lemma 5.4. *This algebra B_M is a commutative subalgebra in $\mathscr{D}$. Moreover, if we define A_M by*

(5.5) $$A_M = S^{-1} \cdot B_M \cdot S,$$

then this is an element of $\mathscr{A}$.

Proof. Let

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{\text{finite}} c_{ij} \frac{\partial}{\partial t_j} \qquad (i = 1, 2, 3, \cdots)$$

be a basis of Ker(h). We associate an element $Z_i = \sum_j c_{ij} (L^j)_+ \in \mathscr{D}$ with each of them. Since $\partial L/\partial y_i|_{t=0} = 0$, we have $\partial L^n/\partial y_i|_{t=0} = 0$ for every $n \in \mathbf{N}$. Taking its differential operator part, we obtain

$$\left.\frac{\partial Z_i}{\partial y_j}\right|_{t=0} = 0 \quad \text{for every } i \text{ and } j.$$

Then by the Zakharov-Shabat equation (4.8),

$$[Z_i, Z_j]|_{t=0} = \left.\frac{\partial Z_j}{\partial y_i}\right|_{t=0} - \left.\frac{\partial Z_i}{\partial y_j}\right|_{t=0} = 0.$$

Since $B_M = K[Z_1, Z_2, Z_3, \cdots]$, we know that it gives a commutative subalgebra $B_M|_{t=0} \subset D$ at $t = 0$.

Now let us turn to the algebra

$$A_M = S^{-1} \cdot B_M \cdot S = K[S^{-1} \cdot Z_1 \cdot S, S^{-1} \cdot Z_2 \cdot S, \cdots].$$

By the gauge equation $\partial S/\partial t_j = -(L^j)_- \cdot S$, we obtain

$$\begin{aligned} S^{-1} \cdot Z_i \cdot S &= S^{-1} \cdot \sum_j c_{ij} \left(L^j - (L^j)_-\right) \cdot S \\ &= \sum_j c_{ij} S^{-1} \cdot L^j \cdot S + S^{-1} \cdot \sum_j c_{ij} \frac{\partial S}{\partial t_j} \\ &= \sum_j c_{ij} \partial^j + S^{-1} \cdot \frac{\partial S}{\partial y_i}. \end{aligned}$$

Since we want to show that $S^{-1}\cdot Z_i\cdot S\in K((\partial^{-1}))$, let us first claim that $S^{-1}\cdot \partial S/\partial y_i$ does not depend on $t\in T$. Indeed,

$$\begin{aligned}\frac{\partial}{\partial t_n}\left(S^{-1}\cdot\frac{\partial S}{\partial y_i}\right) &= -S^{-1}\cdot\frac{\partial S}{\partial t_n}\cdot S^{-1}\cdot\frac{\partial S}{\partial y_i}+S^{-1}\cdot\frac{\partial}{\partial t_n}\left(\frac{\partial S}{\partial y_i}\right)\\ &= S^{-1}\cdot(L^n)_-\cdot\frac{\partial S}{\partial y_i}+S^{-1}\cdot\frac{\partial}{\partial y_i}\left(\frac{\partial S}{\partial t_n}\right)\\ &= S^{-1}\cdot(L^n)_-\cdot\frac{\partial S}{\partial y_i}-S^{-1}\cdot\frac{\partial}{\partial y_i}((L^n)_-\cdot S)\\ &= S^{-1}\cdot(L^n)_-\cdot\frac{\partial S}{\partial y_i}-S^{-1}\cdot(L^n)_-\cdot\frac{\partial S}{\partial y_i}\\ &= 0,\end{aligned}$$

and similarly all higher derivatives in t vanish at $t=0$. Now $[S^{-1}\cdot\partial S/\partial y_i, \partial]$ vanishes at $t=0$ because

$$\begin{aligned}\left[S^{-1}\cdot\frac{\partial S}{\partial y_i},\partial\right] &= S^{-1}\cdot\left[\frac{\partial S}{\partial y_i}\cdot S^{-1}, S\cdot\partial\cdot S^{-1}\right]\cdot S\\ &= S^{-1}\cdot\frac{\partial(S\cdot\partial\cdot S^{-1})}{\partial y_i}\cdot S\\ &= S^{-1}\cdot\frac{\partial L}{\partial y_i}\cdot S,\end{aligned}$$

and $\partial L/\partial y_i|_{t=0}=0$. But since $S^{-1}\cdot\partial S/\partial y_i$ does not depend on t, we conclude that $[S^{-1}\cdot\partial S/\partial y_i, \partial]$ vanishes identically. Hence

$$S^{-1}\cdot Z_i\cdot S=\sum_i c_{ij}\partial^j+S^{-1}\cdot\frac{\partial S}{\partial y_i}$$

is an element of $K((\partial^{-1}))$ for every i. Therefore $B_M=K[Z_1, Z_2, \cdots]$ is a commutative subalgebra in $\mathscr{D}$ and $A_M=S^{-1}\cdot B_M\cdot S$ is an element of $\mathscr{A}$. q.e.d.

Thus every orbit M of the KP dynamical system determines an algebra $A_M\in\mathscr{A}$. Note that A_M does not depend on the choice of a specific point $L(0)\in M$. It is easily verified that M is nothing but an A_M-maximal orbit of the subdynamical system $(\tilde{X}_{A_M}, T)$ defined by A_M. Now we have obtained the classification theorem of all the orbits of the KP dynamical system.

Theorem 5.2. *For every orbit M of the KP dynamical system there exists a unique commutative algebra $A\in\mathscr{A}$ such that the subdynamical system $(\tilde{X}_A, T)$ defined by A contains M as an A-maximal orbit. Thus all the orbits of the KP dynamical system and elements of $\mathscr{A}$ are in one-to-one correspondence.*

6. Finite dimensional orbits of the KP dynamical system and a characterization of the Jacobian varieties

We have seen so far that every orbit of the KP dynamical system is in one-to-one correspondence with an algebra in $\mathscr{A}$ and that the cohomology group of this algebra determines the local structure of the orbit. In the present section we describe the slightly more detailed structure of every finite dimensional orbit in terms of algebraic geometry. As a by-product, we give a characterization of all the Jacobian varieties among all the Abelian varieties defined over an algebraically closed field of characteristic zero. The problem of the characterization of Jacobian varieties among Abelian varieties is called the *Schottky Problem* ([10], [14]) and has long been unsolved. Our solution to this problem using the KP hierarchy is somewhat similar to the statement of Novikov's conjecture (cf. [11]), however, what is quite different is that we use the total KP hierarchy instead of the original single KP equation. Dubrovin [3] has also obtained a partial solution to this problem, but our method and result are completely different.

In this section we restrict ourselves to the case of $R = K[[x]]$. As we have noted in §2, the cohomology group $H^1(A)$ of $A \in \mathscr{A}$ is of finite dimension if and only if the rank of A is one. Therefore every finite dimensional orbit of the KP dynamical system corresponds bijectively to a rank one algebra in $\mathscr{A}$. What we want to show here is that *these finite dimensional orbits are essentially Jacobian varieties of complete algebraic curves.*

So let us start with a finite dimensional orbit M of the KP dynamical system. Let $L = L(t)$ be its corresponding solution of the KP hierarchy (4.6) and let $S = S(t)$ be a gauge operator of L satisfying the gauge equation (4.9) as before. As is shown in §5, M determines an algebra $A \in \mathscr{A}$ which is of rank one. Since

$$B_S = S \cdot A \cdot S^{-1}$$

is a commutative subalgebra in $\mathscr{D}$, $\mathscr{D}$ has a left $\mathscr{R}$-module and a right A-module structure via $B_S \subset \mathscr{D}$. The rank of $\mathscr{D}$ over $\mathscr{R} \otimes_K A$ is one. The argument in §2 proves that our solution L defines a line bundle $\mathscr{L} = \mathscr{L}(t)$ over $\operatorname{Spec} \mathscr{R} \times_K C$, where C denotes the complete algebraic curve defined in §2. Let $\mathscr{M}$ be the maximal ideal of $R = K[[x]]$ generated by x. Then we can restrict the line bundle $\mathscr{L}(t)$ on $(\mathscr{M} \times T) \times C \cong T \times C$ which we denote by $\mathscr{L}_0(t)$. This is a deformation family of a line bundle on C with parameters in T. Thus we have a formal map

$$T \ni t \mapsto \mathscr{L}_0(t) \in H^1(C, \mathscr{O}_C^*). \tag{6.1}$$

Let $\dim_K H^1(A) = \dim_K H^1(C, \mathcal{O}_C) = g$. According to §3 we can take a coordinate system $y_1, y_2, \cdots, y_g$ of $H^1(A)$ so that the canonical linear map f of maximal rank from $H^1(A)$ to T is defined by

$$t_j = \sum_{i=1}^{g} h_{ij} y_i, \qquad j = 1, 2, 3, \cdots,$$

as in (3.8). As we have seen in §5, these $y_1, y_2, \cdots, y_g$ are effective deformation (or time evolution) parameters of the solution $L(t)$ via f. Composing these two maps, we obtain a local isomorphism

$$H^1(A) \xrightarrow{f} T \xrightarrow{(6.1)} H^1(C, \mathcal{O}_C^*).$$

Since every point $L(t) \in M$ determines injectively a line bundle $\mathcal{L}_0(t) \in H^1(C, \mathcal{O}_C^*)$, we can conclude here that the map

$$M \ni L(t) \mapsto \mathcal{L}_0(t) \in H^1(C, \mathcal{O}_C^*)$$

sends the orbit M into $H^1(C, \mathcal{O}_C^*)$ as an open subset. More precisely, every finite dimensional orbit M is contained in $\mathrm{Pic}^0(C) = H^1(C, \mathcal{O}_C)/H^1(C, \mathbf{Z}) \subset H^1(C, \mathcal{O}_C^*)$ as an open set. Since $\mathrm{Pic}^0(C)$ has a natural linear structure which comes from $H^1(C, \mathcal{O}_C)$, the flows on M defined by the KP dynamical system is still linear with respect to the linear structure of $\mathrm{Pic}^0(C)$. Thus we have obtained the following.

Theorem 6.1. *Every finite dimensional orbit of the KP dynamical system is an open set of the Picard variety (or the generalized Jacobian variety)* $\mathrm{Pic}^0(C)$ *of a certain complete algebraic curve C over K defined by an algebra in $\mathcal{A}$ corresponding to the orbit and the KP dynamical system restricted on this orbit determines linear flows on it with respect to the linear structure of* $\mathrm{Pic}^0(C)$.

Remark. Every complete algebraic curve determines an algebra $A \in \mathcal{A}$ if it has a smooth K-rational point. Indeed, let C be a curve with a smooth K-rational point ∞ on it and let z be its local coordinate near ∞ such that $\{z = 0\}$ defines the point ∞. Then Laurent series expansion in z gives us a map $\Gamma(C - \infty, \mathcal{O}) \hookrightarrow K((z))$. Let A be its image. Then

$$A \cap K[[z]] = \Gamma(C, \mathcal{O}_C) = K.$$

If we write ∂^{-1} instead of z, the commutative subalgebra $A \subset K((\partial^{-1}))$ becomes itself an element of $\mathcal{A}$.

What is important here is that if K is algebraically closed, then every generalized Jacobian variety appears as a finite dimensional orbit of the KP dynamical system, but no other Abelian variety does:

Corollary. *An Abelian variety defined over an algebraically closed field of characteristic zero is a Jacobian variety of a certain algebraic curve if and only if it can be an orbit of the KP dynamical system.*

M. Sato [12] and his colleagues [2] have introduced the notion of the *τ-functions* for the KP hierarchy. To define τ-functions in our context, we need some new notations. First, we denote

$$\partial_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \cdots \right).$$

Secondly, we introduce polynomials $p_\nu(t)$ in $K[t_1, t_2, t_3, \cdots]$ by

$$\sum_{\nu=0}^{\infty} p_\nu(t)\lambda^\nu = \exp\left(\sum_{n=1}^{\infty} t_n \lambda^n \right).$$

Definition 6.1. Let $\tau(t)$ be an invertible element of $K[[t_1, t_2, t_3, \cdots]]$. We define a function $\tau(x, t)$ in $\mathscr{R} = (K[[x]])[[t_1, t_2, \cdots]]$ by $\tau(x, t) = \tau(x + t_1, t_2, t_3, \cdots)$. Then $\tau(t)$ is said to be a τ-function of the KP hierarchy (4.6) if and only if

$$S(x, t) = \sum_{\nu=0}^{\infty} \frac{p_\nu(-\partial_t)\tau(x, t)}{\tau(x, t)} \partial^{-\nu} \in \mathscr{G}$$

satisfies the gauge equation (4.9).

Remark. We can rewrite the nonlinear differential equations (4.9) for S in terms of τ-functions. Essentially, they are nothing other than the so-called *Hirota's bilinear equations* for τ-functions (cf. [2]).

It is well known that the τ-function corresponding to a quasi-periodic solution is essentially given by a Riemann theta function;

$$\tau(t) = \exp(\text{quadratic term in } t) \cdot \theta(\phi(t)),$$

where $\theta(z) = \theta(z_1, \cdots, z_g)$ is a theta function defined on the covering space K^g of a Jacobian variety J of dimension g and ϕ: $T \to K^g$ is a suitable onto linear map. If we combine this result with our Corollary, we obtain the following

Theorem 6.2. *An Abelian variety of dimension g defined over an algebraically closed field of characteristic zero is a Jacobian variety if and only if there exists a suitable quadratic form $q(t)$ in $t \in T$ and an onto linear map ϕ: $T \to K^g$ such that*

$$\tau(t) \underset{\text{def}}{=} \exp(q(t)) \cdot \theta(\phi(t))$$

gives a τ-function of the KP hierarchy. Here $\theta(z) = \theta(z_1, z_2, \cdots, z_g)$ denotes the Riemann theta function associated with the given Abelian variety.

It is obvious from the argument of §2 that we need only finitely many differential equations to characterize Jacobian varieties of dimension g for any given $g \in \mathbf{N}$.

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