

PATH-CONNECTED YANG-MILLS MODULI SPACES

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Abstract

Min-max techniques in the calculus of variations are used to prove that the moduli spaces of self-dual connections on principal $SU(2)$ or $SU(3)$ bundles over S^4 are path-connected.

1. Introduction

On a principal bundle $P \rightarrow S^4$ whose structure group, G , is a compact, simple and simply connected Lie group, there are distinguished connections. These are the connections whose curvature is self-dual with respect to the Hodge dual of the metric on T^*S^4 which is induced from the identification $S^4 = \{x \in \mathbf{R}^5: |x|^2 = 1\}$. (This metric is called the standard metric.)

The moduli space of self-dual connections on P ,

$$\mathfrak{M}(P) = (P_s \times \{\text{smooth, self-dual connections on } P\}) / \text{Aut } P,$$

is a smooth manifold. Here P_s is the fibre of P at $s = \text{south pole}$, and $\text{Aut } P$ is the group of smooth automorphisms of P . The isomorphism class of P is specified by its integer degree, $k(P)$ [4]. (For $G = SU(2)$, $k(P) = -c_2(P \times_{SU(2)} \mathbf{C}^2)$.) If $k(P) < 0$, then $\mathfrak{M}(P) = \emptyset$; if $k(P) = 0$, then $\mathfrak{M}(P) = \text{point}$; and if $k(P) > 0$, then $\mathfrak{M}(P)$ is nontrivial.

Although these spaces have been the subject of much recent study, [4], [11], [14], relatively little is known of their global structure. A small advance is made in this article with the following theorem.

Theorem 1.1. *Let $P \rightarrow S^4$ be a principal $G = SU(2)$ or $SU(3)$ bundle with positive degree. Then $\mathfrak{M}(P)$ is path-connected.*

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The theorem is known to be true for a few specific cases. If $G = \text{SU}(2)$, and if $k(P) > 0$, then $\mathfrak{M}(P)$ is $8k(P)$ -dimensional. For $G = \text{SU}(2)$, $\mathfrak{M}(P(k = 1)) \simeq \mathbf{R}^5 \times \text{SO}(3)$ [4], which is path-connected, and Hartshorne has established that $\mathfrak{M}(P(k = 2))$ is path-connected [16]. If $G = \text{SU}(3)$, and if $k(P) > 0$, then $\dim \mathfrak{M}(P) = 12k(P)$. For $G = \text{SU}(3)$, it is known that $\mathfrak{M}(P(k = 1)) \simeq \mathbf{R}^5 \times \text{SU}(3)/U(1)$ [4], a path-connected space. For no other principal $\text{SU}(2)$ or $\text{SU}(3)$ bundles had $\pi_0(\mathfrak{M})$ been previously established. Theorem 1.1 is consistent with the conjectures in [5] and [30] that the inclusion of $\mathfrak{M}(P)$ in the space $\hat{\mathfrak{B}}(P)$ of $\text{Aut } P$ orbits of $(P_s \times \{\text{smooth connections on } P\})$ is effective in low-dimensional homotopy.

The proof of Theorem 1.1 is an application of the first Morse inequality on the infinite-dimensional space, $\hat{\mathfrak{B}}(P)$. (The first Morse inequality is the so-called “mountain pass lemma.”)

Consider the following example as a model application of the Morse inequalities. Let $X \subset S^2$ be a subset which is $f^{-1}(0)$ for a C^3 function $f: S^2 \rightarrow [0, 1]$ having no critical points outside of X with Morse index less than 2. Under these circumstances, the first Morse inequality implies that X is connected. In this simple case, the idea behind the proof is easy to describe. Given two points, $p_0, p_1 \in X$, consider the space, Θ , of continuous paths $\phi: [0, 1] \rightarrow S^2$ which connect p_0 to p_1 . Associated to Θ is the number

$$f_\infty = \inf_{\phi \in \Theta} \sup_{t \in [0, 1]} f(\phi(t)).$$

This number is the altitude, as defined by f , of the lowest mountain pass between p_0 and p_1 . Whatever the value of f_∞ , a mini-max argument (Ljusternik-Šnirelman theory [18]) will provide a critical point, p_* , of f where the Morse index of f is zero or one. Hence, $p_* \in X$. But the continuity of f insures that $f(p_*) = f_\infty$, and so $f_\infty = 0$. But if $f_\infty = 0$, then p_0 and p_1 are in the same connected component of X .

The proof of Theorem 1.1 requires an infinite-dimensional analog of this example. Consider the Yang-Mills functional on $\hat{\mathfrak{B}}(P)$. At a connection A on P , the Yang-Mills functional is $1/2$ times the L^2 -norm of the curvature of A , F_A :

$$(1.1) \quad \mathfrak{YM}(A) = \frac{1}{2} \int_{S^4} |F_A|^2(x) \, d\text{vol}(x).$$

In (1.1), the pointwise norm on F_A is the $\text{Aut } P$ invariant norm on the vector bundle $\text{Ad } P \otimes \wedge_2 T^*S^4$ which is induced from the Killing form on $\mathfrak{g} = \text{Lie Alg } G$, and from the standard metric on T^*S^4 .

A smooth connection A which is a critical point of \mathfrak{YM} satisfies the Yang-Mills equations,

$$(1.2) \quad D_A * F_A = 0.$$

Here, $*$ is the standard metric Hodge dual on $\wedge_2 T^*S^4$, and D_A is the covariant exterior derivative defined by A . A self-dual connection is one whose curvature satisfies

$$(1.3) \quad F_A = *F_A.$$

An anti-self-dual connection has $F_A = -*F_A$. These can be found on bundles with negative degree and they are obtainable from self-dual connections by reversing the orientation of S^4 . The Bianchi identities force a self-dual connection to satisfy the Yang-Mills equations.

As \mathcal{YM} is Aut P invariant, it descends to a functional on $\hat{\mathfrak{B}}$. A critical point is \mathcal{YM} on $\hat{\mathfrak{B}}$ is the Aut P orbit of a pair $(h, A) = (\text{point in } P_s, \text{ solution to (1.2)})$.

For those P with nonnegative degree, $k \geq 0$, the Yang-Mills functional restricts to \mathfrak{M} with the constant value k ; this is the infimum of \mathcal{YM} on $\hat{\mathfrak{B}}$. In fact, a connection A on such a principal bundle is self-dual if and only if $\mathcal{YM}(A) = k$. (This defines the normalization of the metric on $\text{Ad } P \otimes \wedge_2 T^*S^4$.)

Bourguignon, Lawson and Simons have shown that if $P \rightarrow S^4$ has structure group $SU(2)$ or $SU(3)$, then every local minimum of \mathcal{YM} on $\hat{\mathfrak{B}}(P)$ is self-dual if $k(P) \geq 0$ and anti-self-dual if $k(P) \leq 0$ [8]. Presently more information about the index of the hessian of \mathcal{YM} is available. This is summarized by

Theorem 1.2 (*C. H. Taubes [30]*). *Let $P \rightarrow S^4$ be a principal $SU(n)$ -bundle with $n = 2$ or 3 . Let $k(P) = \text{degree of } P$. If $b \in \hat{\mathfrak{B}}(P)$ is a critical point of \mathcal{YM} which is neither self-dual nor anti-self-dual, then the index of the hessian of \mathcal{YM} at b is at least $2n|k(P)| + 2$.*

If the functional \mathcal{YM} were to satisfy the Palais-Smale condition [20] on $\hat{\mathfrak{B}}(P)$, then Theorem 1.2 and a Ljusternik-Šnirelman argument would yield not only Theorem 1.1, but isomorphisms $\pi_l(\mathfrak{M}(P)) \simeq \pi_l(\hat{\mathfrak{B}}(P))$ for $l \leq 2nk(P)$. However, the Palais-Smale condition is not satisfied by \mathcal{YM} (and for $G = SU(2)$, $\pi_2(\mathfrak{M}(P(k=1))) = (0)$ while $\pi_2(\hat{\mathfrak{B}}(P(k=1))) \simeq \mathbf{Z}_2$).

The failure of the Palais-Smale condition is not always the final word. Sachs and Uhlenbeck [23] teach that it is potentially profiting to ponder the ways by which the Ljusternik-Šnirelman procedure fails. Concerning the Ljusternik-Šnirelman procedure for \mathcal{YM} on $\hat{\mathfrak{B}}$, the pondering produced the two theorems below.

Theorem 1.3. *Let $P \rightarrow S^4$ be a principal $SU(2)$ or $SU(3)$ bundle with degree $k \geq 0$. Suppose that there exists $\phi \in C^0([0, 1]; \hat{\mathfrak{B}})$ with $\phi(\{0, 1\}) \subset \mathfrak{M}$ for which*

$$(1.4) \quad \sup_{y \in [0, 1]} \mathcal{YM}(\phi(y)) < k + 2.$$

Then ϕ is homotopic rel $\{0, 1\}$ to a map into \mathfrak{M} .

Theorem 1.3 motivated the investigation into the behavior of paths in $\hat{\mathfrak{B}}$ whose endpoints lie in \mathfrak{M} . This investigation produced the following theorem.

Theorem 1.4. *Let $P \rightarrow S^4$ be a principal $G = \text{SU}(2)$ or $\text{SU}(3)$ bundle with degree $k > 0$. Suppose that for every principal G -bundle $P' \rightarrow S^4$ with degree $1 \leq k' < k$, $\mathfrak{M}(P')$ is path-connected. Then any two points in $\mathfrak{M}(P)$ are joined by a path in $\hat{\mathfrak{B}}(P)$ which obeys (1.4).*

Armed with Theorems 1.3 and 1.4 and the a priori knowledge that $\pi_0(\mathfrak{M}(P(k = 1))) \simeq (0)$, Theorem 1.1 is obtained by the obvious induction on the degree of the principal $\text{SU}(2)$ or $\text{SU}(3)$ bundle.

The threshold phenomenon that is exhibited in Theorem 1.3 appears to be common to many elliptic variational problems for which the Palais-Smale condition just fails [9]. The phenomenon is observed in the Yang-Mills-Higgs equations on \mathbf{R}^3 in [28] and [15]. It is seen in more classical problems as well, cf. [10].

This article has two distinct parts. The first part, §§2–5, describes the Ljusternik-Šnirelman procedure for $\mathfrak{Y}\mathfrak{M}$ on $\hat{\mathfrak{B}}(P)$, and the result is a proof of Theorem 1.3. This Ljusternik-Šnirelman procedure attempts to study the homotopy groups $\pi_l(\mathfrak{M})$ for $l \geq 0$ via the inclusion map $\mathfrak{M} \hookrightarrow \hat{\mathfrak{B}}$. The procedure presumes to determine from the critical point set of $\mathfrak{Y}\mathfrak{M}$, whether for $\mathfrak{N} \subset \mathfrak{M}$, and $l \geq 1$, a given path component $\Theta \subseteq C^0((D^l, S^{l-1}); (\hat{\mathfrak{B}}, \mathfrak{N}))$ has elements which map into \mathfrak{M} (Θ defines an element in $\pi_l(\hat{\mathfrak{B}}, \mathfrak{N})$). These considerations give import to the number

$$(1.5) \quad \mathfrak{A}(\Theta) = \inf_{\phi \in \Theta} \left\{ \sup_{y \in D^l} \frac{1}{2}(\mathfrak{Y}\mathfrak{M}(\phi(y)) - k(P)) \right\}.$$

The Ljusternik-Šnirelman procedure for $\mathfrak{Y}\mathfrak{M}$ is not wholly successful, but it does yield the following theorem.

Theorem 1.5. *Let $P \rightarrow S^4$ be a principal G -bundle with nonnegative degree. For $\mathfrak{N} \subseteq \mathfrak{M}$ and $l > 0$, let $\Theta \subseteq C^0((D^l, S^{l-1}); (\hat{\mathfrak{B}}, \mathfrak{N}))$ be a path-component. (1) If $\mathfrak{A}(\Theta) = 0$, then Θ has an element which maps into \mathfrak{M} . (2) If $\mathfrak{A}(\Theta) \notin \mathbf{Z}$, then there exists a principal G -bundle, $P' \rightarrow S^4$, and a smooth, critical point of $\mathfrak{Y}\mathfrak{M}$ on $\hat{\mathfrak{B}}(P')$ which is not an absolute minimum of $\mathfrak{Y}\mathfrak{M}$ on $\hat{\mathfrak{B}}(P')$. (3) At this critical point, the hessian of $\mathfrak{Y}\mathfrak{M}$ has index l or less.*

Observe that Theorems 1.2 and 1.5 immediately yield Theorem 1.3 as a corollary. The proof of Theorem 1.5 occupies §§3–5. §2 is an introduction to the notations and conventions that are used in this article. There, also, certain useful Sobolev-class Banach manifolds are introduced with their relevant properties.

The proof of Theorem 1.5 begins in earnest in §3. The theorem is proved in three steps. In §3, a gradient flow for $\mathfrak{Y}\mathfrak{M}$ is integrated to prove that $\epsilon > 0$

exists for which $\mathfrak{Y}\mathfrak{M}([k, k + \epsilon])$ is a tubular neighborhood of $\mathfrak{M} = \mathfrak{Y}\mathfrak{M}^{-1}(k)$. This is stated as Proposition 3.1; the proposition establishes Theorem 1.5 in the case when $\mathfrak{A}(\Theta) = 0$. (In [12], Donaldson examines the gradient flow for $\mathfrak{Y}\mathfrak{M}$ when restricted to 1-1 connections on a stable, holomorphic vector bundle over a Kaehler 4-manifold.)

The examination of the convergence of mini-max sequences in $\hat{\mathfrak{B}}$ constitutes Step 2 in the proof of Theorem 1.5. A mini-max sequence is a sequence $\{\bar{\phi}_i\} \subset \hat{\mathfrak{B}}$ which is obtained from a sequence $\{\phi_i, \bar{\phi}_i\} \subset X$, where

$$(1.6) \quad X \equiv \{(\phi, \bar{\phi}) \in \Theta \times \hat{\mathfrak{B}}: \bar{\phi} \in \phi(D') \text{ and for all } y \in D', \mathfrak{Y}\mathfrak{M}(\bar{\phi}) \geq \mathfrak{Y}\mathfrak{M}(\phi(y))\}.$$

The sequence $\{\bar{\phi}_i\}$ must also be minimizing in the sense that $\mathfrak{Y}\mathfrak{M}(\bar{\phi}_i) \geq \mathfrak{Y}\mathfrak{M}(\bar{\phi}_{i+1}) \searrow 2\mathfrak{A}(\Theta) + k(P)$. Step 2 occupies §4, and Proposition 4.4 establishes assertion (2) of Theorem 1.5.

To prove the third assertion of Theorem 1.5, control must be had on the hessian of $\mathfrak{Y}\mathfrak{M}$ at the limiting point of a mini-max sequence in $\hat{\mathfrak{B}}$. This control is gained in §5 where the final declaritive of Theorem 1.5 is established. The idea here is that if at this limiting point, the hessian for $\mathfrak{Y}\mathfrak{M}$ had more than l -negative directions, then a “covariant hessian” defined as in [25] would have this same property for all but a finite number of points in the mini-max sequence. Then, under such conditions, a deformation along one of these negative directions would produce a disc in Θ with $\mathfrak{Y}\mathfrak{M} < \mathfrak{A}(\Theta) + k$ and a contradiction (see Proposition 4.2 and Lemma 5.1).

The second part of this article consists of §§6–8, and it includes, in §§6 and 7, the proof of Theorem 1.4. This theorem is proved with a calculation of the “force” between two connections. For computing this force, a digression in §6 is required to define the subtraction of {a pair (P', b') of principal G -bundle $P' \rightarrow S^4$ and point $b' \in \hat{\mathfrak{B}}(P')$ } from {a like pair, (P, b) } to obtain a principal G -bundle $P - P' \rightarrow S^4$ and a point $b - b' \in \hat{\mathfrak{B}}(P - P')$. This subtraction amounts to gluing P to $\alpha^{-1}P'$ via an identification of P with P' over the equator of S^4 , where $\alpha: S^4 \rightarrow S^4$ is inversion through the fixed, equatorial S^3 . The bundle $P - P'$ has degree equal to degree $P -$ degree P' , and if b and b' have their curvatures concentrated near the north pole of S^4 , then $\mathfrak{Y}\mathfrak{M}(b - b') \sim \mathfrak{Y}\mathfrak{M}(b) + \mathfrak{Y}\mathfrak{M}(b')$. Some topological conclusions are drawn with this subtraction procedure in [5] and [26].

When $P' \rightarrow S^4$ has degree $+1$, then degree $(P - P')$ has degree $P - 1$. The question arises whether it is possible to choose self-dual $b' \in \mathfrak{M}(P')$ so that $\mathfrak{Y}\mathfrak{M}(b - b') < \mathfrak{Y}\mathfrak{M}(b) + 1$. The answer here is affirmative if b is contained in an open, dense set $\Omega \subset \hat{\mathfrak{B}}$ of orbits $[h, A]$ such that $(F_A + *F_A)(s) \neq 0$, cf. Proposition 6.2.

In general, b' as in the preceding paragraph cannot be chosen to depend continuously on $b \in \mathfrak{Q}$. There are topological obstructions to doing this; certain 2-dimensional submanifolds exist in \mathfrak{Q} along which b' cannot be made to depend continuously on b . But for a point, $\phi \in \mathfrak{Q}$, there exists a degree +1 principal G -bundle $P_1 \rightarrow S^4$ and a point $b \in \mathfrak{M}(P_1)$ for which $\mathfrak{Y}\mathfrak{M}(\phi - b) < \mathfrak{Y}\mathfrak{M}(\phi) + 1$. This is proved in Proposition 6.2 (In fact, for a continuous path, $\phi(\cdot): [0, 1] \rightarrow \mathfrak{Q}(P)$, there exists a continuous path $b(\cdot): [0, 1] \rightarrow \mathfrak{M}(P_1)$ for which $\mathfrak{Y}\mathfrak{M}(\phi(\cdot) - b(\cdot)) < \mathfrak{Y}\mathfrak{M}(\phi(\cdot)) + 1$. This fact is relevant for studying $\pi_1(\mathfrak{M}(P))$.)

The path for Theorem 1.4 is constructed in §§6 and 7 with the observations from the preceding paragraph, and the outline of this construction follows: Let $m_0, m_1 \in \mathfrak{M}(P) \cap \mathfrak{Q}(P)$ be given. There exists a principal G -bundle, $P_1 \rightarrow S^4$, of degree 1, and self-dual points $b_0, b_1 \in \mathfrak{M}(P_1)$ for which $m_0 - b_0$ and $m_1 - b_1$ have $\mathfrak{Y}\mathfrak{M} < k + 1$. From the results in §§3–5 and Theorem 1.2, one obtains via a mini-max argument (Proposition 6.4) that for $i \in (0, 1)$, there exists a path $\gamma_i \in C^0([0, 1], \mathfrak{B}(P - P_1))$ with $\gamma_i(0) = m_i - b_i$, $\gamma_i(1) \in \mathfrak{M}(P - P_1)$ and such that $\mathfrak{Y}\mathfrak{M}(\gamma_i) < k + 1$. Under the given assumption that $\mathfrak{M}(P - P_1)$ is path-connected, no generality is lost by assuming that $\gamma_0(1) = \gamma_1(1)$. Let $\gamma = \gamma_0 \cdot \gamma_1^{-1}$, where “ \cdot ” is the usual composition for paths, and $\gamma_1^{-1}(t) = \gamma_1(1 - t)$.

On the principal G -bundle $\alpha^{-1}P_1 \rightarrow S^4$ (of degree -1), there exists a continuous curve $\phi(t)$, $t \in [0, 1]$, of *anti-self-dual* points in $\mathfrak{B}(\alpha^{-1}P_1)$ with very small scale-size such that $\gamma(t) - \phi(t) \in C^0([0, 1]; \mathfrak{B}(P))$ has $\mathfrak{Y}\mathfrak{M} < k + 2$ (see Lemma 6.5). The endpoint, $\gamma(0) - \phi(0)$, can be thought of as m_0 with an anti-self-dual connection, $-b_0$, of small scale-size, grafted on near $s \in S^4$; and then a self-dual connection, $-\alpha^*b_0$, of still smaller scale-size grafted on near s again [14, §6]. The point $(m_0 - b_0) - \alpha^*b_0$ can be connected to m_0 by a continuous curve $\eta_0 \in C^0([0, 1]; \mathfrak{B}(P))$ by cancelling $-b_0$ against $-\alpha^*b_0$; this is described in Lemma 6.6. Crucial is the fact that $\mathfrak{Y}\mathfrak{M}(\eta_0(\cdot)) < k + 2$. Similarly, one obtains the curve η_1 . The required path for Theorem 1.4 is $\eta_1 \cdot (\gamma - \phi) \cdot \eta_0^{-1}$.

§8 of this article considers two extensions of the results of §§2–7. The first result concerns π_0 for moduli spaces of self-dual connections on principal G -bundles $P \rightarrow S^4$ where $\text{rank } G > 2$: If $k(P) > 0$ and $\pi_0(\mathfrak{M}(P)) \neq (1)$, then there exists a nonflat critical point of $\mathfrak{Y}\mathfrak{M}$ on $S^4 \times G$ which is not reducible as a direct sum of a self-dual and an anti-self-dual connection.

The second result in §8 concerns the moduli space of self-dual connections on degree 1, principal $SU(2)$ bundles over 4-manifolds, M , with the following properties: (1) M is compact, oriented and Riemannian. (2) M is 1-connected. (3) The intersection pairing on $H_2(M; \mathbf{Z})$ is definite (cf. [14]). For such M and

$P \rightarrow M$, define $\mathfrak{M}(P)$ as follows: Choose $x \in M$ and let

$$\mathfrak{M}(P) = (P_x \times \{\text{self-dual connections on } P\}) / \text{Aut } P.$$

The spaces $\mathfrak{M}(P)/G$ are studied in detail by Donaldson [11] and also in [14]. In §8, the proof of Theorem 1.6, below, is given.

Theorem 1.6. *Let M be a compact, oriented, 1-connected 4-manifold with definite intersection pairing on $H_2(M; \mathbf{Z})$. Let $P \rightarrow M$ be a degree 1 principal $SU(2)$ -bundle. For an open, dense set of smooth metrics on T^*M , the following is true: Either $\pi_0(\mathfrak{M}(P)) = (1)$, or there exists a connection on P or $M \times SU(2)$ which is a non-self-dual critical point of \mathfrak{M} .*

There is a short appendix to this article which contains the proof of a technical result from §4 about the convergence of mini-max sequences.

2. Background

Let M be a compact, oriented Riemannian 4-manifold, and let G be a simple, simply connected, compact Lie group. Fix a principal G -bundle, $P \rightarrow M$, and denote by $\mathfrak{C}(P)$ the space of smooth connections on P . The space \mathfrak{C} is an affine space, and its topology is defined by any affine isomorphism of \mathfrak{C} with $\Gamma(\text{Ad } P \otimes T^*)$, the space of smooth sections of $\text{Ad } P \otimes T^*$. Let $s \in M$ be a fixed point, and by fiat $s = \text{south pole}$ when $M = S^4$. The group of automorphisms of P , $\text{Aut } P$, acts continuously and freely on $\mathfrak{C} = P_s \times \mathfrak{C}$; let $\mathfrak{B} = \mathfrak{C} / \text{Aut } P$ and give \mathfrak{B} the quotient topology.

For technical reasons, it is convenient to consider \mathfrak{C} as a dense subset of the Banach spaces \mathfrak{C}_p^1 , $2 \leq p < \infty$, of connections on P . These are connections whose connection form and its first derivatives are locally in L^p . For $A \in \mathfrak{C}_p^1$, its curvature $F_A \in L^p(\text{Ad } P \otimes \wedge_2 T^*)$ (see, e.g., [14], [21] for an account).

The group $\mathfrak{G} = \mathfrak{G}_2^3$ of continuous automorphisms of P whose first and second derivatives are in L^3 acts smoothly and freely on $\mathfrak{C} = P_s \times \mathfrak{C}_1^3$. The quotient,

$$(2.1) \quad \mathfrak{B} = (P_s \times \mathfrak{C}_1^3) / \mathfrak{G},$$

is a smooth, L_1^3 -Banach manifold and the sequence $1 \rightarrow \mathfrak{G} \rightarrow \mathfrak{C} \rightarrow \mathfrak{B} \rightarrow 1$ defines a smooth, principal \mathfrak{G} -bundle [14]. It is a fact that the inclusion $\mathfrak{B} \hookrightarrow \mathfrak{B}$ is a homotopy equivalence.

Observe that both \mathfrak{B} and \mathfrak{B} admit smooth G actions; multiplication on the right induced by the action of G on P_s . These are never free G actions and they are not often free $G/\text{Center } G$ actions. Nonetheless, the quotients \mathfrak{B}/G and \mathfrak{B}/G exist as topological spaces and they appear in a few places in this article.

Two principal G -bundles, $P, P' \rightarrow M$, are isomorphic if and only if they have the same integer degree; indeed $[M; BG] \approx \mathbf{Z}$ [29]. The degree of P is denoted $k(P)$. An isomorphism from P to P' is a global section of $\text{iso}(P, P') = P' \otimes_G P$. If P and P' are isomorphic, then $\mathfrak{B}(P)$ and $\mathfrak{B}(P')$ are canonically identified. As the Yang-Mills equations and all of the constructions in this article are isomorphism equivariant, $\mathfrak{B}(P)$ and $\mathfrak{B}(P')$ will be implicitly identified for isomorphic P, P' .

The given Riemannian metric on T^* and the Killing metric on \mathfrak{g} define \mathfrak{G} -invariant metrics on $\text{Ad } P \otimes \wedge_k T^*$, $k \in (0, \dots, 4)$. With the volume form of the metric, one obtains \mathfrak{G} -invariant, L^p ($p \geq 1$) metrics on $\Gamma(\text{Ad } P \otimes \wedge_k T^*)$. These \mathfrak{G} -invariant metrics are the only ones used in this article. It is this L^2 -metric which defines the Yang-Mills functional, (1.1), but with integration over M .

The functional \mathfrak{YM} on \mathfrak{C}_1^3 descends to define a smooth, G -invariant functional on \mathfrak{B} . But all critical points of \mathfrak{YM} on \mathfrak{B} , weak solutions to (1.2) on M , lie in $\hat{\mathfrak{B}}$ [32].

Let $*$ denote the Hodge dual of the metric. A connection, A , is self-dual if and only if $F_A = *F_A$. Let

$$\mathfrak{M}(P) = (P_s \times \{\text{self-dual connections on } P\}) / \text{Aut } P.$$

It is to be topologized by the inclusion $\mathfrak{M} \hookrightarrow \mathfrak{B}$.

To study \mathfrak{M} via its inclusion in \mathfrak{B} , it is convenient to use the functional $\mathfrak{A} = \frac{1}{2}(\mathfrak{YM} - k(P))$. Let $P_{\pm} = \frac{1}{2}(1 \pm *)$ denote the pointwise self-dual and anti-self-dual projections on $\wedge_2 T^*$. The normalization of the metric on $\text{Ad } P$ is set by the requirement that the value of \mathfrak{A} at $b = [h, A] \in \mathfrak{B}$ is

$$(2.2) \quad \mathfrak{A}(b) = \frac{1}{2} \int_M |P_- F_A|^2(x) \, d\text{vol}(x).$$

Thus, if $\mathfrak{M}(P) \neq \emptyset$, then $\mathfrak{M}(P) = \mathfrak{A}^{-1}(0)$. (In general, the infimum of \mathfrak{A} over $\mathfrak{B}(P)$ is zero if $k(P) \geq 0$ [29].)

The gradient of \mathfrak{A} , $\nabla \mathfrak{A}$, is defined as a smooth, linear form on the tangent bundle, $T\mathfrak{B} \rightarrow \mathfrak{B}$. This bundle is defined via the vector bundle exact sequence over \mathfrak{B} ,

$$0 \rightarrow \overline{\mathfrak{C}} \times_{\mathfrak{G}} L_2^3(\text{Ad } P) \rightarrow \overline{\mathfrak{C}} \times_{\mathfrak{G}} (\mathfrak{g} \times L_1^3(\text{Ad } P \otimes T^*)) \rightarrow T\mathfrak{B} \rightarrow 0.$$

But $T\mathfrak{B}$ itself is not as convenient to work with as is the vector bundle

$$(2.3) \quad \mathfrak{B} = \overline{\mathfrak{C}} \times_{\mathfrak{G}} (\{0\} \times L_1^3(\text{Ad } P \otimes T^*)) \rightarrow \mathfrak{B},$$

together with the vector bundle map $\Pi: \mathfrak{B} \rightarrow T\mathfrak{B}$. By pull-back via Π , the gradient of \mathfrak{A} defines a smooth, linear functional on \mathfrak{B} .

There is an alternative way to define $\nabla \mathfrak{A} \in \mathfrak{B}^*$. The affine structure of the space of connections on P provides a map $\mathfrak{f}: \mathfrak{B} \rightarrow \mathfrak{B}$ which sends $v = [h, A, \hat{v}] \in \mathfrak{B}_b$ (the fibre over $b = [h, A] \in \mathfrak{B}$) to the point $\mathfrak{f}(v) = [h, A + \hat{v}] \in \mathfrak{B}$. Then

$$\nabla \mathfrak{A}_b(v) = \left. \frac{d}{dt} \mathfrak{A}(\mathfrak{f}([h, A, t\hat{v}])) \right|_{t=0}.$$

With \mathfrak{f} , one can define a ‘‘covariant hessian’’ of \mathfrak{A} . This section, \mathfrak{H} , of $(\text{Sym}_2 \mathfrak{B})^* \rightarrow \mathfrak{B}$ is defined on $v, v_1 \in \mathfrak{B}_b$ by [25]

$$\mathfrak{H}_b(v, v_1) = \left. \frac{d^2}{ds dt} \mathfrak{A}(\mathfrak{f}([h, A, t\hat{v} + s\hat{v}_1])) \right|_{s=t=0}.$$

When b is a critical point of \mathfrak{A} , then \mathfrak{H}_b descends to $T\mathfrak{B}_b$ as the usual hessian of \mathfrak{A} .

For $v, v_1 \in \mathfrak{B}_b$ as above,

$$(2.4) \quad \nabla \mathfrak{A}_b(v) = \nabla \mathfrak{A}_A(\hat{v}) = \langle P_- D_A \hat{v}, P_- F_A \rangle_2,$$

and

$$(2.5) \quad \mathfrak{H}_b(v, v_1) = \mathfrak{H}_A(\hat{v}, \hat{v}_1) = \langle P_- D_A \hat{v}, P_- D_A \hat{v}_1 \rangle_2 + \langle P_- [\hat{v}, \hat{v}_1], P_- F_A \rangle_2.$$

Here, $D_A: L^2_1(\text{Ad } P \otimes \wedge_k T^*) \rightarrow L^2(\text{Ad } P \otimes \wedge_{k+1} T^*)$ denotes the covariant exterior derivative that is defined by A . (Unless otherwise noted, $\nabla_A: L^2_1(\text{Ad } P \otimes \wedge_k T^*) \rightarrow L^2(\text{Ad } P \otimes \wedge_k T^* \otimes T^*)$ denotes the covariant derivative that is defined by A and the standard metric’s Riemannian connection.)

A point $b \in \mathfrak{B}$ is a critical point of \mathfrak{A} if and only if $\nabla \mathfrak{A}_b = 0 \in \mathfrak{B}^*$.

It is convenient to define a norm on \mathfrak{B} . For $A \in \mathfrak{C}^2_1(P)$ and $u \in L^2_1(\text{Ad } P \otimes T^*)$, let

$$(2.6) \quad \|u\|_A^2 = \|\nabla_A u\|_2^2 + \|u\|_2^2.$$

The properties of this norm are listed below.

Proposition 2.1. *As a map from $\mathfrak{C}^2_1 \times L^2_1(\text{Ad } P \otimes T^*)$ to $[0, \infty)$, the assignment $(A, u) \rightarrow \|u\|_A^2$ is smooth. In addition, there exists $z < \infty$ which is independent of (A, u) such that $\|u\|_A < z\|u\|_A$.*

Further, for all $(A, u, a) \in \mathfrak{C}^2_1 \times_2 L^2_1(\text{Ad } P \otimes T^)$,*

$$(2.7) \quad \left| \|u\|_{A+a} - \|u\|_A \right| \leq 4\|u\|_A \|a\|_A \leq 4z^2 \|u\|_A \|a\|_A.$$

The symbol $\langle \cdot, \cdot \rangle_A$ will denote the metric on $L^2_1(\text{Ad } P \otimes T^*)$ which $\|\cdot\|_A$ induces by polarization.

The utility of $\|\cdot\|_A$ for the Yang-Mills problem is due to the following a priori estimates.

Proposition 2.2. *The Yang-Mills functional is smooth on \mathfrak{G}_1^2 and there exists a constant, $z < \infty$, which is independent of $A \in \mathfrak{G}_1^2$ and $a, u, v \in L_1^2(\text{Ad } P \otimes T^*)$ such that*

$$(2.8) \quad \begin{aligned} (1) \quad & |\mathfrak{A}(A + a) - \mathfrak{A}(A)| \leq z \|a\|_A (1 + \|a\|_A^3), \\ (2) \quad & |\mathfrak{A}(A + a) - \mathfrak{A}(A) - \nabla \mathfrak{A}_A(a)| \leq z \|a\|_A^2 (1 + \|a\|_A^2), \\ (3) \quad & |\mathfrak{A}(A + a) - \mathfrak{A}(A) - \nabla \mathfrak{A}_A(a) - \frac{1}{2} \mathfrak{S}_A(a, a)| \\ & \leq z \|a\|_A^3 (1 + \|a\|_A), \end{aligned}$$

$$(2.9) \quad \begin{aligned} (1) \quad & |\nabla \mathfrak{A}_{A+a}(u) - \nabla \mathfrak{A}_A(u)| \leq z(1 + \mathfrak{A}(A))^{1/2} \|u\|_A \|a\|_A (1 + \|a\|_A^2), \\ (2) \quad & |\mathfrak{S}_{A+a}(u, v) - \mathfrak{S}_A(u, v)| \\ & \leq z(1 + \mathfrak{A}(A))^{1/2} \|u\|_A \|v\|_A \|a\|_A (1 + \|a\|_A). \end{aligned}$$

Both Propositions 2.1 and 2.2 will be proved at the end of this section.

As the assignment of $(A, u) \in \mathfrak{G}_1^3 \times L_1^3(\text{Ad } P \otimes T^*)$ to $\|u\|_A^2$ is \mathfrak{G} -equivariant, there is a natural norm that is induced on the vector bundle $\mathfrak{B} \rightarrow \mathfrak{B}$. At $b = [h, A] \in \mathfrak{B}$, the norm induced on \mathfrak{B}_b by $\|\cdot\|_A$ will be denoted $\|\cdot\|_b$. With this norm are defined the following measures of $\nabla \mathfrak{A}$ and \mathfrak{S} on \mathfrak{B}_b :

Definition 2.3. Let $b \in \mathfrak{B}$. Denote the $\|\cdot\|_b$ dual norm of $\nabla \mathfrak{A}_b$ on \mathfrak{B}_b by $\|\nabla \mathfrak{A}_b\|_*$; that is,

$$(2.10) \quad \|\nabla \mathfrak{A}_b\|_* = \inf_{0 \neq v \in \mathfrak{B}_b} \left(\frac{\nabla \mathfrak{A}_b(v)}{\|v\|_b} \right).$$

For $l \geq 1$, define λ'_b by

$$(2.11) \quad \lambda'_b = \inf_{E \subset \mathfrak{B}_b} \left[\max_{0 \neq v \in E} \left(\mathfrak{S}_b(v, v) / \|v\|_b^2 \right) \right],$$

where the infimum above is over all l -dimensional subspaces $E \subset \mathfrak{B}_b$.

Observe that $\lambda'_b < 0$ if and only if there exists an l -dimensional subspace in \mathfrak{B}_b on which $\mathfrak{S}_b < 0$.

The Yang-Mills equations and functional are invariant under pointwise conformal changes of the metric on T^* . When $M = S^4$ with its standard metric, the group of conformal diffeomorphisms is isomorphic to $SO(5, 1)$. It is convenient to exploit the invariance of $\mathfrak{Y}\mathfrak{M}$ under this group action. Until further notice, restrict $M = S^4$ with its standard metric. Let $C = SO(4) \times \mathbf{R}^* \times \mathbf{R}^4$ denote the subgroup of the group of conformal diffeomorphisms of S^4 which fix $s = \text{south pole} \in S^4$. The group C acts on \mathfrak{B} by sending $b = [h, A]$

to $tb = [t^{-1}h, t^*A]$ for $t \in C$. That is, $[t^{-1}h, t^*A]$ is in $\mathfrak{B}(t^{-1}P) = \mathfrak{B}(P)$. This action commutes with the action of G on \mathfrak{B} and so C acts on \mathfrak{B}/G also. (In fact, the full conformal group, $SO(5, 1)$, acts on \mathfrak{B}/G .)

The action of C on \mathfrak{B} lifts to an action on \mathfrak{B} given by $t[h, A, a] = [h, t^*A, t^*a]$. The functionals \mathfrak{A} on \mathfrak{B} , and $\nabla \mathfrak{A}$, \mathfrak{S} on \mathfrak{B} are C -invariant.

It is convenient to define a C -invariant inner product on \mathfrak{B} . This is done by exploiting the conformal equivalence between $S^4 \setminus s$ and \mathbf{R}^4 which is induced by stereographic projection. Stereographic coordinates $y = \{y^\nu\}: S^4 \setminus s \rightarrow \mathbf{R}^4$ are the Cartesian coordinates on \mathbf{R}^4 . For $a \in \mathfrak{C}_1^2(P)$ and $u, v \in \Gamma(\text{Ad } P \otimes T^*)$ with compact support in $S^4 \setminus s$, define

$$|u|_A^2 = \langle \overline{\nabla}_A u, \overline{\nabla}_A u \rangle_2.$$

Here, $\overline{\nabla}_A$ is the covariant derivative on $\text{Ad } P \otimes T^*(S^4 \setminus s)$ which comes from the connection A and the flat, Euclidean connection on $T^*(S^4 \setminus s)$ which pulls back under y from $T^*\mathbf{R}^4$. The properties of $|\cdot|_A$ are listed in the propositions below; their proof are at the end of this section.

Proposition 2.4. *For fixed $A \in \mathfrak{C}_1^2(P)$, $|\cdot|_A$ extends to a continuous norm on $L_1^2(\text{Ad } P \otimes T^*)$, which is equivalent to the usual L_1^2 -norm. In addition, there is a fixed $z \in (1, \infty)$ which is independent of P and $(A, u) \in \mathfrak{C}_1^2(P) \times L_1^2(\text{Ad } P \otimes T^*)$ such that*

$$(2.12) \quad z^{-1} \|u\|_A < |u|_A < z \|u\|_A.$$

Proposition 2.5. *Let $M = S^4$ with its standard metric. Then the statements of Propositions 2.1 and 2.2 are true when $|\cdot|_A$ replaces $\|\cdot\|_A$.*

As the norm $|\cdot|_A$ is \mathbb{G} -equivariant, it too defines a norm, $|\cdot|_b$, on \mathfrak{B}_b for $b = [h, A] \in \mathfrak{B}$, which as b varies gives a continuous map from \mathfrak{B} onto $[0, \infty)$. The advantage of $|\cdot|_b$ over $\|\cdot\|_b$ is that the former is C -equivariant: If $u \in \mathfrak{B}_b$ and $t \in C$, then $|tu|_{tb} = |u|_b$. Proposition 2.5 allows this C -equivariance to be exploited through the following definition/convention.

Definition 2.6. When $M = S^4$ with its standard metric, then $\|\nabla \mathfrak{A}_b\|_*$ and λ'_b , $l > 0$, are to be defined by (2.10) and (2.11), respectively, but with $|\cdot|_b$ replacing $\|\cdot\|_b$ therein.

Now let M be unrestricted again. In the present context, the two most important properties of the maps $b \rightarrow \|\nabla \mathfrak{A}_b\|_*$ and $b \rightarrow \lambda'_b$ are stated in the following proposition.

Proposition 2.7. *The assignments of $b \in \mathfrak{B}$ to $\|\nabla \mathfrak{A}_b\|_*$ and λ'_b for $l > 0$ define continuous, G -invariant functions on \mathfrak{B} . When $M = S^4$ with the standard metric, these functions are all C -invariant.*

Proof of Proposition 2.7, given Propositions 2.1, 2.2 and 2.5. The continuity of the maps in question is a direct consequence of (2.7) and (2.9). When $M = S^4$ with its standard metric, C -invariance is by construction.

The functions $\|\nabla \mathfrak{A}(\cdot)\|_*$ and $\lambda'(\cdot)$ on \mathfrak{B} are G -equivariant and therefore they descend as continuous functions on \mathfrak{B}/G . It will be convenient in §§4 and 5 to consider these functions on \mathfrak{B}/G .

Now turn to the proofs of Proposition 2.1, 2.2, 2.4 and 2.5. The primary tool is Kato’s inequality, which states that if ∇ is a metric compatible connection on a Riemannian vector bundle E over a Riemannian manifold M , and if $v \in L^2_{1;\text{loc}}(E)$, then

$$(2.13) \quad |\nabla v|(x) \geq |d|v|(x) \quad \text{a.e.}$$

Proof of Proposition 2.1. The fact that the assignment $(A, u) \rightarrow \|u\|_A^2$ is smooth is now standard [21]. The L^2 estimate follows from (2.13) and the Sobolev embedding, $L^2_1(M) \rightarrow L^4(M)$. The last assertion follows from the identity $\nabla_{A+a}\phi = \nabla_A\phi + [a, \phi]$ for $A \in \mathfrak{C}$, $a \in \Gamma(\text{Ad } P \otimes T^*)$ and $\phi \in \Gamma(\text{Ad } P)$.

Proof of Proposition 2.2. This is standard, given Proposition 2.1; use the identities $F_{A+a} = F_A + D_A a + \frac{1}{2}[a, a]$ and $D_{A+a}u = D_A u + [a, u]$ for $a, u \in \Gamma(\text{Ad } P \otimes T^*)$.

Proof of Proposition 2.4. For smooth $A \in \mathfrak{C}$ and $u \in \Gamma(\text{Ad } P \otimes T^*)$ which is compactly supported on $S^4 \setminus s$, there exist the two inequalities

$$(2.14) \quad |u|_A \geq \frac{2}{3}\|u\|_{4;\mathbf{R}^4} = \frac{2}{3}\|u\|_{4;S^4}, \quad |u|_A \geq \left\| |y|^{-1}u \right\|_{2;\mathbf{R}^4}.$$

Here, the subscript “ ; \mathbf{R}^4 ” means that the norm is defined with Euclidean metric on \mathbf{R}^4 , while “ ; S^4 ” means that the norm is defined by the standard metric on S^4 . The first inequality above uses (2.13), a standard Sobolev embedding [6, Theorem 2.14], and the conformal invariance of the L^4 norm in 4-dimensions. The second inequality is proved as in [22, Lemma 5.4]. With (2.13) and (2.14), Proposition 2.4 is a straightforward exercise left to the reader.

Proof of Proposition 2.5. Equation (2.7) with $|\cdot|_A$ instead of $\|\cdot\|_A$ is

$$\left| |u|_{A+a} - |u|_A \right| \leq 4\|u\|_4 \|a\|_4 \leq 9|u|_A |a|_A,$$

with the right-hand inequality due to (2.14). To obtain (2.8) and (2.9) with $|\cdot|_A$ instead of $\|\cdot\|_A$, use the invariance of the left-hand sides of these equations under pull-back by a conformal diffeomorphism; specifically $y^{-1}: \mathbf{R}^4 \rightarrow S^4 \setminus s$. When pulled back to \mathbf{R}^4 via y^{-1} , (2.8) and (2.9) with $|\cdot|_A$ become self-evident using (2.14).

3. The gradient flow for $\mathfrak{Y}\mathfrak{M}$

As in §2, let M be a compact, oriented, 4-dimensional Riemannian manifold. Let $P \rightarrow M$ be a principal G -bundle with $k(P) \geq 0$, and suppose that $\mathfrak{M}(P)$ is nonempty. It is important to establish the circumstances under which there

exists $\epsilon > 0$ such that $\mathfrak{B}_\epsilon = \mathfrak{X}^{-1}([0, \epsilon])$ is a nice tubular neighborhood of \mathfrak{M} . For this purpose, define for $A \in \overline{\mathfrak{C}}$ the unbounded operators

$$(3.1) \quad \begin{aligned} \mathfrak{D}_A &= \sqrt{2} P_- D_A : L^2(\text{Ad } P \otimes T^*) \rightarrow L^2(\text{Ad } P \otimes P_- \wedge_2 T^*), \\ \mathfrak{D}_A^* &= \sqrt{2} D_A^* : L^2(\text{Ad } P \otimes P_- \wedge_2 T^*) \rightarrow L^2(\text{Ad } P \otimes T^*). \end{aligned}$$

These are formal L^2 -adjoints of each other. The formally positive, self-adjoint operator $\mathfrak{D}_A \mathfrak{D}_A^*$ on $L^2(\text{Ad } P \otimes P_- \wedge_2 T^*)$ is discussed in [31]. For $c = [A] \in \mathfrak{B}/G$, let $\mu(c) = \inf \text{spectrum}(\mathfrak{D}_A \mathfrak{D}_A^*)$.

Proposition 3.1. *Give T^*S^4 its standard metric, and let $P \rightarrow S^4$ be a principal G -bundle with $k(P) \geq 0$. Then there exists $\epsilon > 0$ and a strong deformation retract of B_ϵ onto \mathfrak{M} . More generally, let $P \rightarrow M$ be a principal G -bundle with $k(P) \geq 0$, $\mathfrak{M}(P) \neq \emptyset$ and be such that there exists $\mu, \delta > 0$ with the property that $\mu(c) \geq \mu$ for all $c \in \mathfrak{B}_\delta(P)/G$. Then there exists $\epsilon > 0$ and a strong, deformation retract of $\mathfrak{B}_\epsilon(P)$ onto $\mathfrak{M}(P)$.*

Proposition 3.1 is proved with the aid of the gradient flow for the functional \mathfrak{X} on \mathfrak{B} ; its construction and the proof of the proposition are the subjects of this section.

The gradient flow for \mathfrak{X} is obtained by integrating a smooth vector field on $\mathfrak{B}_\epsilon \equiv \mathfrak{X}^{-1}([0, \epsilon])$. Formally, this vector field is obtained by composing the vector bundle map $\Pi: \mathfrak{B} \rightarrow T\mathfrak{B}$ with the section $a: \mathfrak{B}_\epsilon \rightarrow \mathfrak{B}$ defined as follows: As a \mathfrak{G} -equivariant map from $\overline{\mathfrak{C}}_\epsilon = \mathfrak{X}^{-1}([0, \epsilon]) \cap \overline{\mathfrak{C}}$ to $L^2_1(\text{Ad } P \otimes T^*)$,

$$(3.2) \quad a((h, A)) = a(A) = -\sqrt{2} \mathfrak{D}_A^* (\mathfrak{D}_A \mathfrak{D}_A^*)^{-1} P_- F_A.$$

The retract of \mathfrak{B}_ϵ onto \mathfrak{M} is obtained from the flow $\Psi: [0, \infty) \times \mathfrak{B}_\epsilon \rightarrow \mathfrak{B}_\epsilon$ whose defining equation is

$$(3.3) \quad \frac{d\Psi}{dt} = \Pi \cdot a \quad \text{and} \quad \Psi(0, b) = b.$$

This flow satisfies

$$(3.4) \quad \mathfrak{X}(\Psi(t, b)) = \mathfrak{X}(b)e^{-2t}.$$

The retraction $\Phi: [0, 1] \times \mathfrak{B}_\epsilon \rightarrow \mathfrak{B}_\epsilon$ is formally given by

$$\Phi(s, b) = \Psi(-\ln(1 - s), b).$$

One obtains Proposition 3.1 by proving that this map Φ is continuous. Four parts comprise this task: (1) Establish the existence of $\epsilon > 0$ and the existence of the smooth section $a: \mathfrak{B}_\epsilon \rightarrow \mathfrak{B}$ of (3.2). (2) Integrate the vector field $\Pi \cdot a: \mathfrak{B}_\epsilon \rightarrow T\mathfrak{B}_\epsilon$ to obtain the flow Ψ for small time. (3) For fixed $b \in \mathfrak{B}_\epsilon$, establish the existence for all time of the curve $t \mapsto \Psi(t, b)$ and establish that as $t \rightarrow \infty$, $\Psi(t, b)$ converges strongly to $\Psi(\infty, b) \in \mathfrak{M}$. (4) Establish that the set of maps $\{\Psi(t, \cdot); t \in [0, \infty)\}$ defines a map $\Psi \in C^0([0, \infty] \times \mathfrak{B}_\epsilon; \mathfrak{B}_\epsilon)$.

The first part above is obtained with

Proposition 3.2. *Give T^*S^4 its standard metric, and let $P \rightarrow S^4$ be a principal G -bundle with $k(P) \geq 0$, or let M , and $P \rightarrow M$, be as specified by Proposition 3.1. Then there exists $\epsilon > 0$ such that (1) for all $A \in \mathbb{G}_{1;\epsilon}^3 \equiv \mathbb{G}_1^3 \cap \mathfrak{X}^{-1}([0, \epsilon])$, the map $\mathfrak{D}_A \mathfrak{D}_A^*: L_2^3(\text{Ad } P \otimes P_- \wedge_2 T^*) \rightarrow L_0^3(\text{Ad } P \otimes P_- \wedge_2 T^*)$ is an isomorphism. (2) In fact, there exists $\lambda > 0$ such that for each $(A, u) \in \mathbb{G}_{1;\epsilon}^3 \times L_1^2(\text{Ad } P \otimes P_- \wedge_2 T^*)$,*

$$(3.5) \quad \|\mathfrak{D}_A^* u\|_2^2 \geq \lambda (\|\nabla_A u\|_2^2 + \|u\|_2^2).$$

(3) *The map $(A, u) \mapsto \mathfrak{D}_A^*(\mathfrak{D}_A \mathfrak{D}_A^*)^{-1}u$ from $\mathbb{G}_{1;\epsilon}^3 \times L_0^3(\text{Ad } P \otimes P_- \wedge_2 T^*)$ to $L_1^3(\text{Ad } P \otimes T^*)$ is smooth.*

Proof of Proposition 3.2. If M is a compact, oriented, Riemannian 4-manifold and $P \rightarrow M$ is a principal G -bundle, then the map $(A, u) \mapsto \mathfrak{D}_A \mathfrak{D}_A^* u$ from $\mathbb{G}_1^3 \times L_2^3(\text{Ad } P \otimes P_- \wedge_2 T^*) \rightarrow L_0^3(\text{Ad } P \otimes P_- \wedge_2 T^*)$ is smooth; while for fixed $A \in \mathbb{G}_1^3$, the operator $\mathfrak{D}_A \mathfrak{D}_A^*$ on $L^2(\text{Ad } P \otimes P_- \wedge_2 T^*)$ is unbounded, self-adjoint with discrete spectrum, and as a bounded, linear map from L_2^3 to L_0^3 , $\mathfrak{D}_A \mathfrak{D}_A^*$ is Fredholm. See §3 of [31]. The Fredholm alternative implies that if for fixed A , $\inf \text{spectrum}(\mathfrak{D}_A \mathfrak{D}_A^*) > 0$, then $\mathfrak{D}_A \mathfrak{D}_A^*$ is an isomorphism between L_2^3 and L_0^3 . If there exists $\delta > 0$ such that for all $A \in \mathbb{G}_{1;\delta}^3$, the map $\mathfrak{D}_A \mathfrak{D}_A^*$ is an isomorphism from L_2^3 to L_0^3 , then according to §3 of [31], the map $(A, u) \mapsto \mathfrak{D}_A^*(\mathfrak{D}_A \mathfrak{D}_A^*)^{-1}u$ from $\mathbb{G}_{1;\delta}^3 \times L_2^3$ to $L_1^3(\text{Ad } P \otimes T^*)$ is smooth. Thus, statements (1) and (3) of Proposition 3.2 follow from statement (2). Statement (2) of Proposition 3.2 is proved with the Weitzenböch formula for $\mathfrak{D}_A \mathfrak{D}_A^*$ [29]: When $A \in \mathbb{G}_1^3$ and $u \in L_2^2(\text{Ad } P \otimes P_- \wedge_2 T^*)$, then

$$(3.6) \quad \mathfrak{D}_A \mathfrak{D}_A^* u = \nabla_A^* \nabla_A u + \frac{1}{6} \mathfrak{R}(u) + \{P_{-F_A}, u\}.$$

Here, ∇_A^* is the formal L^2 -adjoint of ∇_A ; $\{P_{-F_A}, u\}$ has components $\{P_{-F_A}, u\}^1 = -\sqrt{2}([P_{-F_A}^2, u^3] - [P_{-F_A}^3, u^2]) \cdots$, etc. with respect to a local, orthonormal frame for $P_- \wedge_2 T^*$; and $\mathfrak{R} \in \Gamma(\text{End } P_- \wedge_2 T^*)$ is a linear combination of the scalar and anti-self-dual Weyl curvatures of the metric on T^* . When there exists $\mu, \delta > 0$ such that $\|\mathfrak{D}_A^* u\|_2^2 \geq \mu \|u\|_2^2$ for all $(A, u) \in \mathbb{G}_{1;\delta}^3 \times L_1^2(\text{Ad } P \otimes P_- \wedge_2 T^*)$, then one obtains (3.5) by first contracting both sides of (3.6) with u , integrating over M and applying Hölder's inequality to obtain

$$\|\mathfrak{D}_A^* u\|_2^2 \geq \|\nabla_A u\|_2^2 - \frac{1}{6} \|\mathfrak{R}\|_\infty \|u\|_2^2 - 6 \|P_{-F_A}\|_2 \|u\|_4^2.$$

Now, by assumption, $\|\mathfrak{D}_A^* u\|_2^2 \geq \mu \|u\|_2^2$. And, by (2.13) and the $L_1^2(M) \rightarrow L^4(M)$ Sobolev embedding, $\|\nabla_A u\|_2^2 + \|u\|_2^2 \geq \zeta \|u\|_4^2$ with $\zeta > 0$ independent

of $A \in \mathcal{C}_{1,\varepsilon}^3$. Thus, one obtains from a rearrangement of the preceding equation that

$$\|\mathcal{D}_A^*u\|_2^2 \geq z(1 - z'\|P_{-F_A}\|_2)(\|\nabla_A u\|_2^2 + \|u\|_2^2).$$

Here, z, z' depend only on μ and the metric on T^* . This final equation implies the existence of the required $\varepsilon \in (0, \delta]$ and $\lambda > 0$.

When $M = S^4$ with its standard metric, statement (2) of Proposition 3.2 is obtained as follows: The endomorphism \mathfrak{R} is here multiplication by the constant, positive scalar curvature, R . In this case, contraction of both sides of (3.6) with u yields

$$\|\mathcal{D}_A^*u\|_2^2 \geq \|\nabla_A u\|_2^2 + \frac{1}{6}R\|u\|_2^2 - 6\|P_{-F_A}\|_2\|u\|_4^2,$$

and the argument now proceeds essentially as before.

Proposition 3.2 insures that (3.2) is defining a smooth map from $\overline{\mathcal{C}}$ to $L_1^3(\text{Ad } P \otimes T^*)$, cf. §3 of [31]. This map is \mathfrak{G} -equivariant so (3.2) defines the smooth section $a: \mathfrak{B}_\varepsilon \rightarrow \mathfrak{B}$.

The small time existence of the flow Ψ is a standard construction (cf. [1, Theorem 4.1.13]). Indeed, for each $b = [h, A] \in \mathfrak{B}_\varepsilon$, there exists $t(b) > 0$ and a unique, smooth curve $\Psi(t, b): [0, t(b)) \rightarrow \mathfrak{B}_\varepsilon$ which satisfies (3.3). In addition, for $t < t(b)$, Ψ defines a smooth map from $[0, t) \times \{\text{a neighborhood of } b \text{ in } \mathfrak{B}_\varepsilon\}$ into \mathfrak{B}_ε .

The utility of the flow Ψ stems from the fact that if one lifts $\Psi(t, b)$ to $(h, A(t)) \in \overline{\mathcal{C}}_\varepsilon = \mathfrak{A}^{-1}([0, \varepsilon]) \cap \overline{\mathcal{C}}$, then for $t \in [0, t(b))$,

$$(3.7) \quad \frac{d}{dt}P_{-F_{A(t)}} = -P_{-F_{A(t)}} \quad \text{a.e.}$$

As a consequence of (3.7),

$$(3.8) \quad \|P_{-F_{A(t)}}\|_p = \|P_{-F_{A(0)}}\|_p e^{-t}$$

for $p \in (0, 3]$ and $t \in [0, t(b))$.

For step 3 in the proof of Proposition 3.1, the existence of the flow $\Psi(t, b)$ for $t \in [0, \infty)$ must be established. It is convenient to do this upstairs, \mathfrak{G} -equivariantly on $\overline{\mathcal{C}}_\varepsilon$. Let $\mathfrak{U} \subset \mathfrak{B}_\varepsilon$ be an open neighborhood, and suppose that $\Psi \in C^0([0, \tau) \times \mathfrak{U}; \mathfrak{B}_\varepsilon)$ exists for some $\tau > 0$. Let $\mathcal{A} \subset \overline{\mathcal{C}}_\varepsilon$ be the image of \mathfrak{U} by a local section of the fibration $\overline{\mathcal{C}}_\varepsilon \rightarrow \mathfrak{B}_\varepsilon$. The flow Ψ lifts to a flow $\Lambda \in C^0([0, \tau) \times \mathcal{A}; \overline{\mathcal{C}}_\varepsilon)$ and there is no loss of generality to suppose that with $c = (h, A) \in \mathcal{A}$,

$$(3.9) \quad \Lambda(t, c) = (h, A + \alpha(t, c)) \equiv (h, A(t)) \equiv c(t),$$

where $\alpha \in C^0([0, \tau) \times \mathcal{A}, L_1^3(\text{Ad } P \otimes T^*))$ satisfies

$$(3.10) \quad \frac{d\alpha}{dt} = a(c(t)), \quad \alpha(0, c) = 0.$$

Proposition 3.1 follows from the two lemmas below.

Lemma 3.3. *For fixed $c \in \mathcal{A}$, $\alpha(\cdot, c)$ converges strongly in $L^3_1(\text{Ad } P \otimes T^*)$ as $t \rightarrow \tau \in (0, \infty]$.*

Lemma 3.4. *The convergence of $\alpha(\cdot, c)$ described in Lemma 3.3 is uniform on closed L^3_1 balls in \mathcal{A} for $\tau \in (0, \infty]$.*

Proof of Proposition 3.1, given Lemmas 3.3 and 3.4. The uniform, strong convergence of $\alpha(\cdot, c)$ implies that Λ extends to $C^0([0, \tau] \times \mathcal{A}; \overline{\mathcal{C}}_\epsilon)$. If $\tau < \infty$, then an open/closed argument for the half-line $[0, \infty)$ allows one to conclude that $\Lambda \in C^0([0, \infty) \times \mathcal{A}; \overline{\mathcal{C}}_\epsilon)$. Since $\tau = \infty$ is admissible in Lemmas 3.3 and 3.4, $\Lambda \in C^0([0, \infty] \times \mathcal{A}; \overline{\mathcal{C}}_\epsilon)$. As Λ is \mathfrak{G} -equivariant by construction, the existence of $\Psi \in C^0([0, \infty) \times \mathfrak{U}; \overline{\mathcal{B}}_\epsilon)$ is obtained. Because the small time evolution for the ODE in (3.3) is unique, the flows from two open sets $\mathfrak{U}_1, \mathfrak{U}_2 \subset \mathfrak{B}_\epsilon$ agree on $\mathfrak{U}_1 \cap \mathfrak{U}_2$. Thus, the solution, Ψ , (3.3), is in $C^0([0, \infty] \times \mathfrak{B}_\epsilon; \mathfrak{B}_\epsilon)$ as required.

The proofs of Lemmas 3.3 and 3.4 require a priori estimates on the vector field $a(c) \in T_{\overline{\mathcal{C}}_\epsilon}$ of (3.2). These estimates are provided by Lemmas 3.5–3.7 below; the estimates are obtained from the identity below, (3.12).

Because the vector field $a(c)$ and the flow $\alpha(t, c)$ factor through the projection $\overline{\mathcal{C}} \rightarrow \mathcal{C}^3_1$, the P_s -dependence will be suppressed by writing $a = a(A)$ and $\alpha = \alpha(t, A)$ for $A \in \mathcal{C}^3_{1;\epsilon}$.

Define the elliptic, first-order operator

$$(3.11) \quad \delta_A = (\mathfrak{D}_A, D^*_A): L^3_1(\text{Ad } P \otimes T^*) \rightarrow L^3(\text{Ad } P \otimes (P_- \wedge_2 T^* \oplus \mathbf{R})).$$

The vector field $a(A)$ on $\mathcal{C}^3_{1;\epsilon}$ satisfies

$$\delta_A a(A) = Q(A) \equiv \sqrt{2}(-P_- F_A, 2*(P_- F_A \wedge u(A) - u(A) \wedge P_- F_A)),$$

where

$$(3.12) \quad u(A) \equiv (\mathfrak{D}_A \mathfrak{D}^*_A)^{-1} P_- F_A.$$

Let $r_0 > 0$ be the injectivity radius of M .

Lemma 3.5. *There exists a constant $z < \infty$ which is independent of $A \in \mathcal{C}^3_{1;\epsilon}$ such that*

$$\|u(A)\|_\infty \leq z \cdot \|P_- F_A\|_2^2 (1 + \|P_- F_A\|_2^4).$$

Lemma 3.6. *There exist constants $\rho \in (0, r_0]$, $z < \infty$ which are independent of $A \in \mathcal{C}^3_{1;\epsilon}$ and $r \in (0, \rho)$ such that*

$$v_r(A) \equiv \sup_{x \in M} \int_{\text{dist}(x, y) < r} \frac{|a(A)|^2(y)}{|\text{dist}(x, y)|^2} d \text{vol}(y) \leq z \|P_- F_A\|_2^2 (1 + \|P_- F_A\|_3)^5.$$

Lemma 3.7. *There exist constants $\kappa > 0$, $z < \infty$ which are independent of $A \in \mathfrak{G}_{1;\epsilon}^3$ and $r \in (0, \rho)$ such that if*

$$(3.13) \quad \int_{\text{dist}(x, y) < 2r} |F_A|^2(y) \, d\text{vol}(y) < \kappa,$$

then

$$\int_{\text{dist}(x, y) < r} |\nabla_A a(A)|^2(y) \, d\text{vol}(y) \leq z \|P_- F_A\|_2^2 (1 + \|P_- F_A\|_3)^6.$$

Proof of Lemma 3.3 and 3.4, assuming Lemmas 3.5–3.7. In the sequel, $z \in (0, \infty)$ will always denote a constant which is independent of connections in $\mathfrak{G}_{1;\epsilon}^3$. Its actual value may vary from equation to equation.

By assumption, the curve $\alpha(t, A)$ for $A \in \mathfrak{G}_{1;\epsilon}^3$ is defined for $t \in [0, \tau)$ for some $\tau \in (0, \infty]$. Along the curve $A(t)$, let $a(t)$ denote $a(A(t))$ and let $F_A(t)$ denote $F_{A(t)}$. Observe that L^2 convergence of $\alpha(t, A)$ to $\alpha(\tau, A)$ is immediate from Lemma 3.6 and (3.8) as

$$\|\alpha(t, A) - \alpha(t', A)\|_2 \leq \int_{t'}^t \|a(s)\|_2 \, ds \leq z \|P_- F_A\|_2 (1 + \|P_- F_A\|_3)^3 |e^{-t} - e^{-t'}|.$$

Observe that the convergence above is uniform in a neighborhood of A in $\mathfrak{G}_{1;\epsilon}^3$.

To obtain the L^3 convergence of $\alpha(t, A)$ one requires a bump function: For $r \in (0, r_0)$ and $x \in M$, let $0 \leq \beta_r^x \in C^\infty(M)$ be a bump function with (1) $\beta_r^x(y) = 1$ if $\text{dist}(x, y) < r$; (2) $\beta_r^x(y) = 0$ if $\text{dist}(x, y) > 3/2r$; and (3) $|d\beta_r^x(y)| \leq 4r^{-1}$.

Lemma 3.8. *There exists a constant $z < \infty$ which is independent of $A \in \mathfrak{G}_{1;\epsilon}^3$, $r \in (0, \rho)$, $x \in M$, $\tau \in [0, \infty)$ and $t, t' \in [0, \tau)$ such that*

$$\left| \|\beta_r^x P_+ F_A(t)\|_2^2 - \|\beta_r^x P_+ F_A(t')\|_2^2 \right| \leq z \|P_- F_A\|_2 (1 + \|P_- F_A\|_3)^4 |e^{-t} - e^{-t'}|.$$

Proof of Lemma 3.8. For notational convenience, write $\beta = \beta_r^x$. Using the Bianchi identity ($D_A F_A = 0$ for all $A \in \mathfrak{G}_1^3$), one obtains the identity

$$(3.14) \quad \begin{aligned} \frac{d}{dt} \|\beta P_+ F_A(t)\|_2^2 &\leq -4 \langle d\beta \wedge a(t), \beta P_+ F_A(t) \rangle_2 \\ &\quad + 4 \langle d\beta \wedge a(t), \beta P_- F_A(t) \rangle_2 \\ &\quad - 2 \langle \beta^2 P_- F_A(t), P_- F_A(t) \rangle_2. \end{aligned}$$

Now, use the fact that

$$(3.15) \quad \|d\beta \wedge a(t)\|_2^2 \leq z v_{2r}(A(t))$$

with Lemma 3.6 and (3.8) and (3.14). The result is

$$(3.16) \quad \left| \frac{d}{dt} \|\beta_r P_+ F_A(t)\|_2^2 \right| \leq z \|P_- F_A\|_2^2 (1 + \|P_- F_A\|_3)^3 e^{-2t} \\ + z \|P_- F_A\|_2 (1 + \|P_- F_A\|_3)^3 e^{-t} \|\beta_r P_+ F_A(t)\|_2.$$

The lemma follows by integration of (3.16).

The import of Lemma 3.8 is in its implication, namely

Lemma 3.9. *Let $\kappa > 0$ be as in Lemma 3.6 and let $A' \in \mathbb{C}_{1;\epsilon}^3$. There exists a neighborhood $\mathfrak{N} \subset \mathbb{C}_{1;\epsilon}^3$ of A' and a number $r(\mathfrak{N}) \in (0, \rho)$ which is independent of $[t, \tau)$ such that for all $A \in \mathfrak{N}$*

$$\sup_{x \in M} \int_{\text{dist}(x, y) < r} |F_A(t)|^2(y) \, d\text{vol}(y) < \kappa.$$

Proof of Lemma 3.9. If $\tau < \infty$, then the lemma is immediate from Lemma 3.8. If $\tau = \infty$, there exists $t' \in [0, \infty)$ for which

$$(3.17) \quad e^{-t'} z \|P_- F_{A'}\|_2 (1 + \|P_- F_{A'}\|_3)^4 < \frac{1}{4} \kappa.$$

The set of $A \in \mathbb{C}_{1;\epsilon}^3$ which satisfy (3.17) is open; this equation is satisfied on a neighborhood $\mathfrak{N}' \ni A'$ in $\mathbb{C}_{1;\epsilon}^3$. On a possibly smaller neighborhood, $\mathfrak{N} \subseteq \mathfrak{N}'$ containing A' , one can choose $r = r(\mathfrak{N})$ so that for all $A \in \mathfrak{N}$,

$$(3.18) \quad \sup_{x \in M} \|\beta_r^x P_+ F_A(t')\|_2^2 < \frac{1}{2} \kappa.$$

The lemma now follows from (3.18) and Lemma 3.8.

Observe that Lemma 3.9 gives the uniform L_1^2 convergence of $\alpha(t, A)$:

Lemma 3.10. *Let $A' \in \mathbb{C}_{1;\epsilon}^3$. There exists a neighborhood $\mathfrak{N} \subseteq \mathbb{C}_{1;\epsilon}^3$ of A' and a constant $c(\mathfrak{N}) < \infty$ which is independent of τ and $t, t' \in [0, \tau)$ such that for all $A \in \mathfrak{N}$,*

$$\|\nabla_A(\alpha(t, A) - \alpha(t', A))\|_2 \leq c(\mathfrak{N}) |e^{-t} - e^{-t'}|.$$

Proof of Lemma 3.10. Let $\mathfrak{N} \subseteq \mathbb{C}_{1;\epsilon}^3$ be the neighborhood of A from Lemma 3.9. Observe first that $\alpha(t, A)$ converges strongly in L^4 as $t \rightarrow \tau$, uniformly on \mathfrak{N} . Indeed,

$$(3.19) \quad \|\alpha(t, A) - \alpha(t', A)\|_4 \leq \int_{t'}^t \|a(A(s))\|_4 \, ds.$$

Now, due to Lemmas 3.7, 3.9 and (3.8),

$$(3.20) \quad \|\nabla_{A(s)} a(A(s))\|_2 + \|a(A(s))\|_2 \leq c_1(\mathfrak{N}) e^{-s}$$

if $A \in \mathfrak{N}$. From (3.20) and Proposition 2.1, one can conclude that if $A \in \mathfrak{N}$,

then

$$(3.21) \quad \|a(A(s))\|_4 \leq c_2(\mathfrak{N})e^{-s}.$$

From (3.19) and (3.21) it follows that

$$(3.22) \quad \|\alpha(t, A) - \alpha(t', A)\|_4 \leq c_2(\mathfrak{N})|e^{-t} - e^{-t'}|$$

if $A \in \mathfrak{N}$.

Next, consider that when $A \in \mathfrak{N}$,

$$(3.23) \quad \begin{aligned} \|\nabla_A(\alpha(t, A) - \alpha(t', A))\|_2 &\leq \int_{t'}^t ds \|\nabla_A a(A(s))\|_2, \\ &\leq \int_{t'}^t ds (\|\nabla_{A(s)} a(A(s))\|_2 + 2\|a(A(s))\|_4 \|\alpha(s, A)\|_4). \end{aligned}$$

Lemma 3.10 follows now from (3.20)–(3.23).

To complete the proofs of Lemmas 3.3 and 3.4, one must examine the L^3 -convergence of $\nabla_A \alpha(t, A)$. The result is stated in the next lemma.

Lemma 3.11. *Let $A' \in \mathfrak{G}_{1;\epsilon}^3$. There exists a neighborhood $\mathfrak{N} \subset \mathfrak{G}_{1;\epsilon}^3$ of A' and a constant $c(\mathfrak{N})$ which is independent of τ and of $t, t' \in [0, \tau]$ such that for all $A \in \mathfrak{N}$,*

$$\|\nabla_A(\alpha(t, A) - \alpha(t', A))\|_3 \leq c(\mathfrak{N})|e^{-t} - e^{-t'}|.$$

The proof of Lemma 3.11 requires

Lemma 3.12. *Let $A' \in \mathfrak{G}_{1;\epsilon}^3$. There exists a neighborhood $\mathfrak{N} \subset \mathfrak{G}_{1;\epsilon}^3$ of A' and a constant, $\xi(\mathfrak{N}) > 0$, such that for every $v \in L_1^3(\text{Ad } P \otimes T^*)$ and for all $A \in \mathfrak{N}$,*

$$\|\delta_A v\|_3 + \|v\|_3 \geq \xi(\mathfrak{N})\|\nabla_A v\|_3.$$

Proof of Lemma 3.12. The operator δ_A for $A \in \mathfrak{N}$ of (3.11) is uniformly elliptic, first order; the lemma is essentially Theorem 6.26 of [19].

Proof of Lemma 3.11. To begin, observe that for $A \in \mathfrak{N}$, if $f(t)$ denotes $\|\nabla_A \alpha(t, A)\|_3$ then

$$(3.24) \quad \begin{aligned} \left| \frac{df}{dt} \right| &\leq \|\nabla_A a(A(t))\|_3 \\ &\leq \xi(\mathfrak{N})^{-1} (\|\delta_A a(A(t))\|_3 + \|a(A(t))\|_3) \\ &\leq z\xi^{-1}(\mathfrak{N}) (\|\delta_{A(t)} a(A(t))\|_3 + \|\alpha(t, A)\|_4 \|a(A(t))\|_3 + \|a(A(t))\|_3). \end{aligned}$$

Here, the second line utilizes Lemma 3.12. The quantity $\|\delta_{A(t)} a(A(t))\|_3$ is bounded using (3.8), (3.12) and Lemma 3.5. In (3.24), $\|a(A(t))\|_3$ is bounded

using (3.20). As for the term $q(t) \equiv \|\alpha(t, A)\|_3$ in (3.24), one uses first Hölder's inequality

$$q(t) \leq \|a(A(t))\|_4 \|\alpha(t, A)\|_{12},$$

and then (2.13) and a Sobolev inequality ($L^3_1(M) \hookrightarrow L^{12}(M)$) to deduce that

$$(3.25) \quad q(t) \leq z \|a(A(t))\|_4 (\|\nabla_A \alpha(t, A)\|_3 + \|\alpha(t, A)\|_3).$$

With (3.22) and (3.25), one obtains from (3.24) the inequality

$$(3.26) \quad \left| \frac{df}{dt} \right| \leq c(\mathfrak{R}) e^{-t} (1 + f).$$

Here, $c(\mathfrak{R})$ is constant which is independent of t . Integrating the inequality in (3.26) yields

$$(3.27) \quad \left| \ln \left(\frac{1 + f(t)}{1 + f(t')} \right) \right| \leq c(\mathfrak{R}) |e^{-t} - e^{-t'}| \quad \text{for } t, t' \in [0, \tau].$$

Equation (3.27) implies that $f(t)$ converges to $f(\tau)$ as $t \rightarrow \tau$. Now, replace $f(t)$ in (3.24) by $\hat{f}(t) = \|\nabla_A(\alpha(t, A) - \alpha(t', A))\|_3$ for $t, t' \in [0, \tau]$. Since $d\hat{f}/dt = df/dt$, one obtains from (3.26) and (3.27) the bound $|d\hat{f}/dt| \leq c(\mathfrak{R}) e^{-t}$, which when integrated, gives

$$\|\nabla_A(\alpha(t, A) - \alpha(t', A))\|_3 \leq c(\mathfrak{R}) |e^{-t} - e^{-t'}|,$$

as required.

Equation 3.22 and Lemma 3.11 complete the proof of Lemmas 3.3 and 3.4.

Now, turn to the proofs of Lemmas 3.5–3.7.

Proof of Lemma 3.5. Due to (3.5), the norm of $u(A) = (\mathfrak{D}_A \mathfrak{D}_A^*)^{-1} P_{-} F_A$ satisfies

$$(3.28) \quad \|\nabla_A u\|_2^2 + \|u\|_2^2 \leq \lambda^{-1} \|P_{-} F_A\|_2^2.$$

From (3.6), one obtains for $|u|^2(x)$ the equation

$$(3.29) \quad -\Delta \frac{|u|^2}{2} + |\nabla_A u|^2 + \frac{1}{6}(u, \mathfrak{R}(u)) - (u, F_{-}(u)) = 0,$$

where $-\Delta$ is the positive Laplacian on $C^\infty(M)$. Let $x \in M$ be given, and let $\{y^\nu\}_{\nu=1}^4$ by Gaussian coordinates centered at x . Let $\beta = \beta_{r_0}^x$. Multiply both sides of (3.29) by $\beta|y|^{-2}$ and integrate over the set where $|y| \leq 2r_0$. Here, $|\cdot|$ denotes the Euclidean distance. Remember that $(4\pi^2)^{-1}|x - y|^{-2}$ is the Green's function for $-\Delta_{\mathbf{R}^4}$, and that $-\Delta = -\Delta_{\mathbf{R}^4} + y \cdot$ (lower order terms) for $|y| \sim 0$

(near the point x). One obtains from (3.29) the inequality

$$|u|^2(x) \leq z \cdot \left\{ \int_{|y| < 2r_0} |u|^2 dy + \int_{|y| < 2r_0} \frac{|P_{-F_A}|}{|y|^2} |u|^2 d^4y \right\}.$$

As $|y|^3 \in L^1(\{y \in \mathbf{R}^4: |y| < 2r_0\})$, an application of Hölder's inequality to the second term above is justified, and this yields with (3.28),

$$|u|^2(x) \leq z \cdot \left\{ \|P_{-F_A}\|_2^2 + \|P_{-F_A}\|_3 \left(\int_{|y| < 2r_0} \frac{|u|^3}{|y|^3} dy \right)^{2/3} \right\}.$$

As $|y|^{-15/4} \in L^1(\{y \in \mathbf{R}^4: |y| < 2r_0\})$, one obtains from this last equation,

$$|u|^2(x) \leq z \cdot \left\{ \|P_{-F_A}\|_2^2 + \|P_{-F_A}\|_3 \left(\int_{|y| < 2r_0} |u|^{15} \right)^{2/15} \right\};$$

and then

$$(3.30) \quad |u|^2(x) \leq z \cdot \left\{ \|P_{-F_A}\|_2^2 + \|P_{-F_A}\|_3 \|u\|_4^{8/15} \|u\|_\infty^{22/15} \right\}.$$

The choice of x was an arbitrary one, so (3.30) yields

$$(3.31) \quad \|u\|_\infty^2 \leq z \cdot \left\{ \|P_{-F_A}\|_2^2 + \|P_{-F_A}\|_3 \|u\|_4^{8/15} \|u\|_\infty^{22/15} \right\}.$$

Now, use (2.7), the Sobolev inequality and (3.28) to obtain from (3.31) the final bound

$$\|u\|_\infty^2 \leq z \cdot \|P_{-F_A}\|_2^2 (1 + \|P_{-F_A}\|_3^{15/4}).$$

The lemma follows from this last bound.

Proof of Lemma 3.6. Return to (3.29), and multiply both sides by $\beta|y|^{-2}$ where, now, $\beta = \beta_r^x$ with $r \in (0, r_0]$ and $x \in M$. Now integrate over the set where $|y| < 2r$. Use the fact that

$$\left| \int_{|y| < 2r} (-\beta|y|^{-2} \Delta |u|^2) - 4\pi^2 |u|^2(0) \right| \leq z \cdot r^{-4} \int_{|y| < 2r} |u|^2.$$

With the above inequality, (3.29) yields

$$(3.32) \quad \int_{|y| < r} \frac{|\nabla_A u|^2}{|y|^2} d^4y \leq z \cdot \left\{ \|u\|_\infty^2 + \int_{|y| < 2r} \frac{|u|^2 |P_{-F_A}|}{|y|^2} \right\} \\ \leq z \cdot \|u\|_\infty^2 (1 + \|P_{-F_A}\|_3).$$

Lemma 3.6 follows immediately from Lemma 3.5, (3.32), and the fact that $|a(A)|(x) \leq z |\nabla_A u(A)|(x)$ a.e.

Proof of Lemma 3.7. The proof requires the following result (compare Lemma 3.12).

Lemma 3.13. *There exists $z, \kappa > 0$ and $\rho \in (0, r_0)$ such that for all $A \in \mathfrak{G}_1^3$, $r \in (0, \rho)$ and $x \in M$ the following is true: Let $B = \{y \in M: \text{dist}(x, y) \leq r\}$. If $\|F_A\|_{2;B}^2 < \kappa$, then every $\omega \in L_1^2(\text{Ad } P \otimes T^*)$ with compact support in B obeys $\|\delta_A \omega\|_2 \geq z \|\nabla_A \omega\|_2$.*

Proof of Lemma 3.13. This follows easily from the Weitzenböch formula for δ_A (cf. [29], [14]): Indeed, as M has bounded Riemannian curvature, the Weitzenböch formula with Hölder's inequality gives

$$(3.33) \quad \|\delta_A \omega\|_2^2 \geq \|\nabla_A \omega\|_2^2 - z(\|P_+ F_A\|_2 + r^2) \|\omega\|_4^2.$$

Here, z is a numerical constant. Now, if ω has compact support in B , then

$$(3.34) \quad \|\omega\|_4^2 \leq z \|d|\omega|\|_2^2 \leq z_1 \|\nabla_A \omega\|_2^2.$$

The first line above is a Sobolev inequality, while line 2 uses (2.7). Here, z_1 is a constant which only depends on the metric of M . The lemma is now a consequence of (3.33) and (3.34).

In the present context, use Lemma 3.13 with $\omega = \beta a$, with $\beta = \beta_r^x$ and with r such that (3.13) is obeyed. Then

$$(3.35) \quad \begin{aligned} \|\delta_A \omega\|_2 &\leq \|\beta \delta_A a\|_2 + z \|d\beta a\|_2, \\ &\leq z \cdot \{ \|P_- F_A\|_2 (1 + \|u\|_\infty) + \nu_r(A) \}. \end{aligned}$$

In (3.35), z is a constant which is independent of A . The derivation of the second line uses (3.12) and (3.15). Lemma 3.7 follows from (3.35) with Lemmas 3.5 and 3.6.

4. Ljusternik-Šnirelman theory

Let M be a compact, oriented Riemannian 4-manifold, and let $P \rightarrow M$ be a principal G -bundle for which $\mathfrak{M}(P) \neq \emptyset$. To study the topology of \mathfrak{M} via the inclusion $\mathfrak{M} \hookrightarrow \mathfrak{B}$, consider for a given $\mathfrak{N} \subseteq \mathfrak{M}$ and $l > 0$, a path component $\Theta \subseteq C^0((D^l, S^{l-1}); (\mathfrak{B}, \mathfrak{N}))$. Let X be the obvious generalization of the space defined by (1.6). Certain sequences in X are more useful than others; prior to defining these good sequences, some notation is required. For a sequence $\{\phi_i, \bar{\phi}_i\} \subset X$, denote by $\mathfrak{A}_i, \nabla \mathfrak{A}_i, \mathfrak{S}_i$ and $\|\cdot\|_{(i)}$ the functionals $\mathfrak{A}(A + (\cdot))$, $\nabla \mathfrak{A}_b(\cdot)$, $\mathfrak{S}_b(\cdot, \cdot)$ and $\|\cdot\|_b$ on $\mathfrak{B}_i \equiv \mathfrak{B}_b$ when $b = [h, A] = \bar{\phi}_i$. For such a sequence in X , or just for a sequence $\{\bar{\phi}_i\}$ in \mathfrak{B} or in \mathfrak{B}/G , denote by $\|\nabla \mathfrak{A}_i\|_*$ and λ'_i the numbers $\|\nabla \mathfrak{A}_b\|_*$ and λ'_b when $b = \bar{\phi}_i$.

Definition 4.1. A sequence of points $\{[A_i]\} \in \mathfrak{B}/G$ is a *good* sequence if $\{\mathfrak{A}(A_i)\}$ is bounded and if

$$\lim_{i \rightarrow \infty} \|\nabla \mathfrak{A}_i\|_* \rightarrow 0.$$

For given $\mathfrak{R} \subseteq \mathfrak{M}$, $l > 0$, and path component $\Theta \subseteq C^0((D^l, S^{l-1}); (\mathfrak{B}, \mathfrak{R}))$, a sequence $\{\phi_i, \bar{\phi}_i\} \subset X$ is *good* if (1) $\lim_{i \rightarrow \infty} \mathfrak{A}(\bar{\phi}_i) \rightarrow \mathfrak{A}(\Theta)$; (2) $\lim_{i \rightarrow \infty} \|\nabla \mathfrak{A}_i\|_* \rightarrow 0$; and (3) $\lim_{i \rightarrow \infty} \lambda_i^{+1} \geq 0$.

The existence of good sequences in X is provided by Proposition 4.2, below. (This proposition has antecedents in [25].) The proof of Proposition 4.2 is given in §5.

Proposition 4.2. For $\mathfrak{R} \subseteq \mathfrak{M}$ and $l > 0$, let $\Theta \subseteq C^0((D^l, S^{l-1}); (\mathfrak{B}, \mathfrak{R}))$ be a path component. There exist good sequences in X as defined by Definition 4.1.

Presented with a good sequence $\{\phi_i, \bar{\phi}_i\} \subset X$, the convergence of $\{\bar{\phi}_i\} \subset \mathfrak{B}$ is analyzed following Sedlacek [24]. The following notion of convergence is relevant.

Definition 4.3. Let $P, P' \rightarrow M$ be principal G -bundles. Let $[A] \in \mathfrak{B}(P')/G$ and let $\{[A_i]\}_{i=1}^\infty \in \mathfrak{B}(P)/G$. Let $\Omega = \{x_k\}_{k=1}^n \subset M$. The sequence $\{[A_i]\}$ is said to converge weakly (strongly) in $L^2_{1;\text{loc}}$ on $N = M \setminus \Omega$ to $[A]$ if there exists a sequence $\{g_i\} \in L^3_2(\text{iso}(P'; P)|_N)$ such that in any domain $U \subset N$ with compact closure in N , the sequence $\{g_i^* A_i - A\} \subset L^2_1(\text{Ad } P' \otimes T^*|_U)$ converges weakly (strongly) to zero in the L^2_1 -topology. A sequence $\{A_i\} \in \mathfrak{C}^3_1(P)$ or a sequence $\{b_i = [h_i, A_i]\} \in \mathfrak{B}(P)$ is said to converge to $[A] \in \mathfrak{B}(P')/G$ weakly (strongly) in $L^2_{1;\text{loc}}$ on N if $\{[A_i]\}$ converges appropriately to $[A]$.

The convergence of a good sequence is discussed in Proposition 4.4, below. Proposition 4.4 is in many respects analogous to the existence theorems for harmonic maps from S^2 that are derived by Sacks and Uhlenbeck [23], and Siu and Yau [27].

In the statement of Proposition 4.4, the conformal group, C , was defined in §2.

Proposition 4.4. Let M be a compact, oriented, Riemannian 4-manifold. Let $P \rightarrow M$ be a principal G -bundle with degree $k \geq 0$. Let $\{[A_i]\} \subset \mathfrak{B}/G$ be a good sequence for which $\lim_{i \rightarrow \infty} (A_i) \rightarrow \mathfrak{Y} \mathfrak{M}_\infty$. There exists a subsequence of $\{[A_i]\}$, also denoted $\{[A_i]\}$, and a finite set of pairs $\{(P_\alpha, A_\alpha)\}_{\alpha=0}^n$, where $P_0 \rightarrow M$ is a principal G -bundle and A_0 is a smooth connection on P_0 and a solution to the Yang-Mills equations on M , while for $\alpha > 0$, each $P_\alpha \rightarrow S^4$ is a principal G -bundle and A_α is a smooth connection on P_α which is a solution to the Yang-Mills equations on S^4 for the standard metric on T^*S^4 . These data have the following properties:

- (1) $\{[A_i]\}$ converges strongly in $L^2_{1;\text{loc}}$ of $M \setminus \{\text{finite set}\}$ to $[A_0]$.

- (2) For $\alpha > 0$, A_α is not flat.
- (3) $\sum_{\alpha=0}^n \mathfrak{Y} \mathfrak{M}(A_\alpha) = \mathfrak{Y} \mathfrak{M}_\infty$.
- (4) $\sum_{\alpha=0}^n k(P_\alpha) = k$.
- (5) Suppose that $\lambda > 0$, and $\{l_\alpha\}_{\alpha=1}^n \in (0, 1, \dots)$ exist such that for each $\alpha \in (0, \dots, n)$, $\lambda'_{[A_\alpha]} < -\lambda$. Let $l = \sum_{\alpha=0}^n l_\alpha$. Then for all i sufficiently large, $\lambda'_i < -\frac{1}{2} \cdot \lambda$.

Though not relevant in this article, it is a fact that if $M = S^4$ with its standard metric, then in addition to assertions (1)–(5), there exist sequences $\{t_i^\alpha\}_{\alpha=0}^n \subset C$ such that for each α , $\{t_i^\alpha[A_i]\}$ converges strongly to $[A_\alpha]$ in L^2_1 of $S^4 \setminus \{\text{finite set}\}$.

Proof of assertions (2), (3) of Theorem 1.5, given Propositions 4.2, 4.4. According to Proposition 4.2, a good sequence $\{\phi_i, \bar{\phi}_i\} \subset X$ can be found. Let $\{(P_\alpha, A_\alpha)\}_{\alpha=0}^n$ be the limiting pairs of principal G -bundle and Yang-Mills solution as provides by Proposition 4.4. If each A_α were either self-dual or anti-self-dual, then for each α one would have $\mathfrak{Y} \mathfrak{M}(A_\alpha) = |k_\alpha|$. In such case, statements (3) and (4) of Proposition 4.4 would require that $\mathfrak{A}(\Theta) = \frac{1}{2} \sum_{\alpha=0}^n (|k_\alpha| - k_\alpha) \in \mathbf{Z}$. This contradicts the given assumptions of Theorem 1.5 so at least one of the A_α must be nonminimal. Statement (5), above, insures that $\gamma_{A_\alpha}^{l+1} \geq 0$; for if $\gamma_{A_\alpha}^{l+1} < -2\delta < 0$, then for all i large enough, $\gamma'_i < -\delta$, and this contradicts the fact that $\{[A_i]\}$ is a good sequence.

The proof of Proposition 4.4 exploits the compactness results in Proposition 4.5, below.

Proposition 4.5. *Let $P \rightarrow M$ be a principal G -bundle of degree $k \geq 0$. Let $\{[A_i]\} \in \mathfrak{B}(P)/G$ be a good sequence. There exists (1) a bundle $P' \rightarrow M$ and a smooth critical point, A , of \mathfrak{A} on P' ; (2) a finite set $\Omega = \{x_i\}_{i=1}^n \subset M$; and (3) a subsequence of $\{[A_i]\}$ which converges strongly in $L^2_{1;\text{loc}}$ on $M \setminus \Omega$ to $[A]$. The set Ω is characterized as follows: There exists a constant $\kappa > 0$ which is universal for M such that if for an open set $U \subset M$,*

$$(4.1) \quad \lim\text{-inf}_{i \rightarrow \infty} \|F_{A_i}\|_{2;U} < \kappa,$$

then $U \cap \Omega = \emptyset$.

S. Sedlacek proves a similar result in [24] with a weaker notion of convergence that “strongly in $L^2_{1;\text{loc}}$ ”. He uses K. Uhlenbeck’s weakcompactness theorems in [32] and her removable singularity theorem in [33]. Sedlacek’s proof can be appropriated almost word for word to prove Proposition 4.5, once a technical extension of Theorem 3.6 of K. Uhlenbeck in [32] is established. The proof of Proposition 4.5 is presented in the Appendix.

Proof of Proposition 4.4. The proposition has analogies with Proposition 2 of Siu and Yau [27] concerning energy-minimizing harmonic maps from S^2 . The proof here is modelled on Siu and Yau’s proof of their Proposition 2.

It is proved in [29], an essentially proved in [13] (see also §3), that any nontrivial solution to the Yang-Mills equations with the standard metric on S^4 has $\mathfrak{Y}\mathfrak{M} > \eta > 0$ for some fixed η . Let κ be the constant in (4.1). Let l be the smallest integer for which $\mathfrak{Y}\mathfrak{M}_\infty < \frac{1}{4}l \cdot \min(\eta, \kappa^2)$. Proposition 4.4 is proved by induction on l . The case $l = 1$ is true, see Proposition 4.5. It is necessary to prove the case $l = n + 1$ under the assumption that the case $l = n$ is true.

Let $\{[A_i]\} \in \mathfrak{B}(P)/G$ be the given sequence, and let $(\Omega, P_0, A = A_0)$ be the data generated by the convergence of $\{[A_i]\}$ as described in Proposition 4.5. Here, $\Omega \subset M$ is a finite set specified by (4.1), $P_0 \rightarrow M$ is a principal G -bundle, and A is a smooth connection on P_0 which satisfies the Yang-Mills equations on M . Denote by $\{[A_i]\}$ the subsequence of the original sequence which converges strongly in $L^2_{1;loc}$ of $M \setminus \Omega$ to $[A]$. Proposition 4.5 gives Proposition 4.4 when $\Omega = \emptyset$ so assume $\Omega \neq \emptyset$.

Let r_0 be the injectivity radius of M , and let $r \in (0, r_0)$ be such that the balls $B_r(x)$ of radius r about $x \in \Omega$ are disjoint. Let

$$\text{tr}(\cdot \wedge \cdot) : \text{Sym}_2(\text{Ad } P \otimes \wedge_2 T^*) \rightarrow \mathbf{R}$$

denote the induced Killing form from the Killing form on \mathfrak{g} .

In order to investigate the behavior of $\{[A_i]\}$ near points $x \in \Omega$, define

$$\begin{aligned} \mathfrak{S}(x) &= \lim_{i \rightarrow \infty} \frac{1}{2} \int_{B_r(x)} (|F_i|^2 - |F|^2), \\ \mathfrak{Q}(x) &= \lim_{i \rightarrow \infty} c(G) \int_{B_r(x)} (\text{tr}(F_i \wedge F_i) - \text{tr}(F \wedge F)), \end{aligned} \tag{4.2}$$

where $F_i, F = F_{A_i}, F_A$, respectively, and $c(G)$ is group theoretic constant which is defined so that $k(P')$ is given via the Chern-Weil formula [4]:

$$c(G) \int_M \text{tr}(F \wedge F) = k(P'). \tag{4.3}$$

Choose a subsequence of $\{[A_i]\}$ for which the limits in (4.4) are well defined, and rename this subsequence $\{[A_i]\}$. Due to (4.1) and Proposition 4.5, $\mathfrak{S}(x) \geq \frac{1}{2}\kappa^2$ if $x \in \Omega$. In addition,

Lemma 4.6. *Let $\mathfrak{Y}\mathfrak{M}_\infty$ and k be as specified by Proposition 4.4. The following sum rules are obeyed:*

$$\begin{aligned} \mathfrak{Y}\mathfrak{M}(A) + \frac{1}{2} \sum_{x \in \Omega} \mathfrak{S}(x) &= \mathfrak{Y}\mathfrak{M}_\infty, \\ k(P_0) + \sum_{x \in \Omega} \mathfrak{Q}(x) &= k. \end{aligned}$$

Proof of Lemma 4.6. To obtain the first sum rule, use the following facts: (1) $\mathfrak{YM}(A_i) \rightarrow \mathfrak{YM}_\infty$. (2) The $L^2_{1,\text{loc}}$ strong convergence of $\{[A_i]\}$ to $[A]$ on $M \setminus \Omega$ implies that for each $\varepsilon > 0$ and $\rho \in (0, r)$ there exists $i(\varepsilon, \rho) < \infty$ such that for all $i > i(\varepsilon, \rho)$.

$$(4.4) \quad \int_{M \setminus \cup_{x \in \Omega} B_\rho(x)} \left| |F_i| - |F| \right|^2 < \varepsilon.$$

(3) Since the limit, A , is smooth, for any $\rho \in (0, r)$, and for any $x \in M$,

$$(4.5) \quad \int_{B_\rho(x)} |F|^2 \sim O(\rho^4).$$

The second sum rule is obtained similarly with (4.3).

Choose $x \in \Omega$. It is necessary to isolate that part of $\{[A_i]\}$ which is responsible for $\mathfrak{S}(x), \mathfrak{Q}(x)$. This is accomplished in part by constructing two new sequences from $\{[A_i]\}$.

The construction begins by trivializing P_0 over $B = B_r(x)$ so that with respect to the flat, product connection θ on $P_0|_B, A = \theta + a$ where $a \in \Gamma(T^*B_r \times \mathfrak{g})$ satisfies

$$(4.6) \quad a(x) = 0, v_x \lrcorner a = 0,$$

where $v_x \in \Gamma(T^*B_r)$ is tangent to the radial geodesics through x . This is the polar gauge for A [33].

Due to the strong $L^2_{1,\text{loc}}$ convergence of $\{[A_i]\}$ to $[A]$ in $B \setminus x$, there exists a sequence $\{g_i\} \subset L^2_2(\text{iso}(B \times G, P)|_{B \setminus x})$ with the following property: Given $\lambda \in (0, r)$, there exists $i(\lambda) < \infty$ such that for all $i > i(\lambda)$, the \mathfrak{g} -valued 1-forms $a_i = g_i^* A_i - A$ satisfy

$$(4.7) \quad \int_{B \setminus B_{\lambda/2}(x)} \left\{ |\nabla_\theta(a - a_i)|^2 + |a - a_i|^4 \right\} < \lambda.$$

For $n \geq 1$, let $\lambda_n = 2^{-n-1}r$. Let $\beta_n(\cdot) = \beta(\lambda_n^{-1}(|\cdot - x|)) \in C^\infty_0(B)$ be the usual bump function ($\beta(t) \geq 0$ satisfies $\beta = 1$ if $t < 1$ and $\beta = 0$ if $t > 3/2$). For simplicity, set $B_n = B_\rho(x)$ when $\rho = \lambda_n$. For each $n \geq 1$, define $i_n = i(\lambda_n)$ as above by (4.7), and let $I_n = [i_n, i_{n+1})$. It is convenient to take as $\{[A_i]\}$ the subsequence with $1 \in I_1$.

Choose Gaussian coordinates, $y: B_{r_0}(x) \rightarrow \mathbf{R}^4$, centered at x . Identify \mathbf{R}^4 with $S^4 \setminus s$ with the stereographic projection from the south pole, and identify B with $y(B) \subset S^4 \setminus s$. This identifies x with the north pole.

For each i , define principal G -bundles $P_i^1 \rightarrow S^4$ and $P_i^2 \rightarrow M$ as follows. Define $P_i^1 \simeq P$ over B and $P_i^1 \simeq (S^4 \setminus n) \times G$ over $S^4 \setminus n$, where g_i identifies $(S^4 \setminus n) \times G$ with P over $(B \setminus x)$. For each i , define $P_i^2 = P$ over $M \setminus x$ and $P_i^2 = B \times G$ where g_i identifies $B \times G$ with P over $B \setminus x$.

For each $\alpha = (1, 2)$, and for each i , define connections A_i^α on P_i^α as follows: When $i \in I_n$, set

$$(4.8) \quad A_i^1 = \begin{cases} \theta + (\beta_1(a_1 - a) + \beta_n a) & \text{over } S^4 \setminus B_{n+1}, \\ A_i & \text{over } B_n, \end{cases}$$

$$(4.9) \quad A_i^2 = \begin{cases} A_i & \text{over } M \setminus B_{n-1}, \\ \theta + (1 - \beta_n)(a_i - a) + a & \text{over } B_{n-2}. \end{cases}$$

The relevant properties of $\{(P_i^\alpha, A_i^\alpha)\}$, $\alpha = 1, 2$, are listed in the next two lemmas.

Lemma 4.7. *There exists $\hat{i} < \infty$ such that for all $i > \hat{i}$, each P_i^1 (P_i^2) is isomorphic to a fixed, principal G -bundle $P^1 \rightarrow S^4$ ($P^2 \rightarrow M$). These two fixed bundles satisfy $k(P^1) = \mathfrak{Q}(x)$ and $k(P^2) = k - \mathfrak{Q}(x)$.*

Rename as $\{[A_i]\}$ the subsequence for which $\hat{i} = 1$, and consider, for each α , $\{[A_i^\alpha]\} \subset \mathfrak{B}(P^\alpha)/G$.

Lemma 4.8. *The sequence $\{[A_i^1]\} \subset \mathfrak{B}(P^1)/G$ is a good sequence for S^4 with its standard metric and $\lim_{i \rightarrow \infty} \mathfrak{Y}\mathfrak{M}(A_i^1) = \mathfrak{S}(x)$. The sequence $\{[A_i^2]\} \subset \mathfrak{B}(P^2)/G$ is a good sequence and $\lim_{i \rightarrow \infty} \mathfrak{Y}\mathfrak{M}(A_i^2) = \mathfrak{Y}\mathfrak{M}_\infty - \mathfrak{S}(x)$.*

Proof of Lemma 4.7. The isomorphism class of P_i^α is specified by $k(P_i^\alpha)$ [29], which can be computed with A_i^α using (4.3). Using (4.5) and (4.7) with the fact that $a \in \Gamma(T^*B \times \mathfrak{g})$, one obtains for $i \in I_n$ that

$$k(P_i^1) = c(G) \int_{B_r} [\text{tr}(F_i \wedge F_i) - \text{tr}(F \wedge F)] + O(\lambda_n).$$

As $k(P_i^1)$ must be an integer, (4.4) implies that for all i sufficiently large, $k(P_i^1) = \mathfrak{Q}(x)$. The argument proving that $k(P_i^2) = k - \mathfrak{Q}(x)$ for all i sufficiently large is similar.

Proof of Lemma 4.8. As $y: B_{r_0}(x) \rightarrow \mathbf{R}^4$ are Gaussian coordinates, the pullback metric $y^*ds_{\text{Euclidean}}^2$ and the given metric on T^*M differ in C^0 near $p \in B$ by $O(|y(p)|^2)$ and they differ in C^1 near p by $O(|y(p)|)$ (cf. [31], [14]). Using this fact, (4.5), (4.7) and the fact that a is smooth, one obtains for $i \in I_n$ that

$$\frac{1}{2} \int_{S^4} |F_{A_i^1}|^2 = \frac{1}{2} \int_{B_r} (|F_i|^2 - |F|^2) + O(\lambda_n).$$

Here, the norm on the left-hand side above is the S^4 -norm and those on the right-hand side are the norms on M . This calculation is similar to the calculations in §§8, 9 or [14]. Thus, (4.2) implies that $\lim_{i \rightarrow \infty} \mathfrak{Y}\mathfrak{M}(A_i^1) = \mathfrak{S}(x)$. A similar argument proves that $\lim_{i \rightarrow \infty} \mathfrak{Y}\mathfrak{M}(A_i^2) = \mathfrak{Y}\mathfrak{M}_\infty - \mathfrak{S}(x)$.

It is necessary to establish that for each $\alpha = 1, 2$, the sequence $\{\|\nabla \mathfrak{A}_i^\alpha\|_* = \|\nabla \mathfrak{A}_c\|_*$ with $c = A_i^\alpha\}$ has limit zero. Consider $\alpha = 1$, as the other case is similar. For $i \in I_n$, let $v \in L_1^3(\text{Ad } P_i^1 \otimes T^*S^4)$ have compact support in B_n . Then $\nabla \mathfrak{A}_i^1(v) = \nabla \mathfrak{A}_i(v)$. Because of the C^1 -closeness of $y^*ds_{\text{Euclidean}}^2$ to the given metric on B_n , the following is true: Let $|\cdot|_{(1,i)} = |\cdot|_c$ with $c = A_i^1$. There is a constant, $z < \infty$, which is independent of v as above such that

$$\|v\|_{(i)} \leq (1 + z\lambda_n)|v|_{(1,i)}.$$

Therefore, for v as above, and $i \in I_n$,

$$(4.10) \quad |\nabla \mathfrak{A}_i^1(v)| \leq \|\nabla \mathfrak{A}_i\|_*(1 + z\lambda_n)|v|_{(1,i)}.$$

Consider $v \in L_1^3(\text{Ad } P_i^1 \otimes T^*S^4)$ with compact support in $S^4 \setminus y(B)$. For such v , $\nabla \mathfrak{A}_i^\alpha(v) = 0$. Finally, consider those v with compact support in the annulus $B_{2r}(x) \setminus B_{n+1}$. For these v , (4.7) and (4.8) imply that

$$(4.11) \quad |\nabla \mathfrak{A}_i^\alpha(v)| \leq \zeta \cdot |v|_{(1,i)} \lambda_n^{1/2},$$

where ζ is a constant which is independent of n and i . Again, one can ignore the difference between the metric $y^*ds_{\text{Euclidean}}^2$ and the given metric to order λ_n in B_{n-1} . Equations (4.10), (4.11) imply that $\lim_{i \rightarrow \infty} \|\nabla \mathfrak{A}_i^1\|_* \rightarrow 0$.

Lemma 4.9. *Suppose that $\lambda > 0$ and $p, q \in (0, 1, \dots)$ exist such that for all i sufficiently large, $\lambda_{[A_i^p]}^p, \lambda_{[A_i^q]}^q < \lambda$. Let $l = p + q$. Then given $\epsilon > 0$, there exists i_0 such that for all $i > i_0$, $\lambda_i^l \leq -\lambda$.*

Proof of Lemma 4.9. Fix i_0 sufficiently large so that $\lambda_A^q, \lambda_A^p < -\lambda$ with $A = A_i^\alpha$ for $\alpha = 1, 2$ and all $i > i_0$. The proof requires the following a priori estimate.

Lemma 4.10. *Given $\epsilon > 0$, there exists $M < \infty$ and $\delta = \delta(\lambda) > 0$ with the following property. For all $i \in I_n > i_0$, there exist p - and q -dimensional vector spaces $E_i^1 \subset L_1^2(\text{Ad } P_i^1 \otimes T^*S^4)$ and $E_i^2 \subset L_1^2(\text{Ad } P_i^2 \otimes T^*)$ such that:*

- (1) *Let $A = A_i^1$. Then for all $v \in E_i^1$, $\mathfrak{S}_A(v, v) \leq -\lambda(1 - \epsilon)\|v\|_A^2$ and $\|\beta_n v\|_{4+4\delta} < M\|v\|_A$.*
- (2) *Let $A = A_i^2$. Then for all $v \in E_i^2$, $\mathfrak{S}_A(v, v) \leq -\lambda(1 - \epsilon)\|v\|_A^2$ and $\|(1 - \beta_{n-1})v\|_{4+4\delta} < M\|v\|_A$.*

The proof of Lemma 4.10 is given at the end of this section; assume for now its validity.

Lemma 4.10 is used in the proof of Lemma 4.9 as follows: Fix $i \in I_n > i_0$. Let $v \in E_i^1$, and write $A = A_i^1$. Observe that

$$(4.12) \quad \|\beta_n v\|_i^2 = \|\beta_n v\|_A^2 \leq \|v\|_A^2 (1 + zM^2\lambda_n^\delta),$$

$$(4.13) \quad \mathfrak{S}_i(\beta_n v, \beta_n v) \leq (-\lambda + zM^2\lambda_n^\delta)\|v\|_A^2.$$

Here, $z < \infty$ is independent of i and n . Both estimates follow from an application of Hölder's inequality. Similar estimates are valid for $i \in I_n > i_0$ and for $(1 - \beta_{n-1})v$ when $v \in E_i^2$. These estimates imply the following: Given $\epsilon > 0$, there exists $i_1 < \infty$ such that for all $i > i_1$, (1) the span of $E_i^1 \cup E_i^2$ is $l = p + q$ dimensional ((4.12)). (2) For all $v \in \text{Span}(E_i^1 \cup E_i^2)$, $\mathfrak{S}_i(v, v) < -\lambda(1 - \epsilon)\|v\|_i^2$ (this is (4.13)). Therefore, $\lambda'_i < -\lambda(1 - \epsilon)$ when $i > i_1$ as claimed.

Now, to complete the proof of Proposition 4.4, it is necessary to observe that if Ω contains 2 or more elements, or if $\mathfrak{Y}\mathfrak{M}(A_0) > \frac{1}{2}\mathfrak{h}$, then the induction hypothesis applied to the good sequences $\{[A_i^\alpha]\} \subset \mathfrak{B}(P^\alpha)/G$ for $\alpha = 1, 2$, together with Lemmas 4.7–4.9 establishes the proposition.

If $\mathfrak{Y}\mathfrak{M}(A_0) \leq \frac{1}{2}\mathfrak{h}$ and $\Omega = \{x\}$, then one must analyze the good sequence $\{[A_i^1]\}$ over S^4 with its standard metric. Here, the conformal invariance of $\mathfrak{Y}\mathfrak{M}$ plays an explicit role; it is used to center the distribution on the sphere of the curvature of each $[A_i^1]$.

For this purpose, fix stereographic coordinates $y: S^4 \setminus s \rightarrow \mathbf{R}^4$. Let d^4y denote the pull-back by y of the Lebesgue measure on \mathbf{R}^4 . The convention in what follows is that all norms which appear in integrals with d^4y are those that are induced from $y^*ds_{\text{Euclidean}}^2$ on $T^*(S^4 \setminus s)$.

A subgroup $T \subset C$ which is isomorphic to $\mathbf{R}^* \times \mathbf{R}^4$ is defined by its action on the coordinate functions y as follows: For $t = (\rho, x) \in \mathbf{R}^* \times \mathbf{R}^4$,

$$(4.14) \quad t^*y = \rho^{-1}(y - x).$$

For a bundle $P \rightarrow S^4$, define a map $z: (\mathfrak{B}(P)/G) \setminus \mathfrak{Y}\mathfrak{M}^{-1}(0) \rightarrow B^5$ (the unit 5-ball) by sending $[A]$ to

$$(4.15) \quad z(A) = \left(\int d^4y |F_A|^2 \right)^{-1} \left[\int d^4y \left(\frac{2y}{|y|^2 + 1}, \frac{|y|^2 - 1}{|y|^2 + 1} \right) |F_A|^2(y) \right].$$

This map is evidently continuous. The map z is used here with the following lemma.

Lemma 4.11. *For each $[A] \in \mathfrak{B}(P)$ there exists $t \in T$ such that $z(t^*[A]) = O \in B^5$.*

The proof of this lemma will be given momentarily, assume for now its validity.

It is notationally convenient to let $[A_i] = [A_i^1]$. Due to Proposition 2.7 and Lemma 4.11, there is no loss of generality in assuming that $z(A_i) = 0$ for each i . The convergence of the good sequence $\{[A_i]\}$ is analyzed with Proposition 4.5: Let (Ω, P', A) be the data generated. Here, $\Omega \subset S^4$ is a finite set specified by (4.1), $P' \rightarrow S^4$ is a principal G -bundle, and A is a connection on P' which satisfies the Yang-Mills equations on S^4 . Denote by $\{[A_i]\}$ the subsequence of the original sequence which converges strongly in $L^2_{1,\text{loc}}$ of $S^4 \setminus \Omega$ to $[A]$.

Proposition 4.4 is obtained from the induction step, Lemmas 4.7–4.9 and

Lemma 4.12. *Give T^*S^4 its standard metric. Let $\{[A_i]\}$ be a good sequence with $\lim_{i \rightarrow \infty} \mathfrak{Y}\mathfrak{M}(A_i) > 0$, which converges strongly in $L^2_{1,\text{loc}}$ of $S^4 \setminus \Omega$ to $[A']$ as described by Proposition 4.5. Assume that $z(A_i) = 0$. Then $\mathfrak{Y}\mathfrak{M}(A') \geq \eta > 0$ or Ω contains more than one point.*

Proof of Lemma 4.7. Suppose the converse were true. By Proposition 4.5, Ω can not be empty, so it must have one point, x . For $1 > \rho > 0$, let B_ρ denote the geodesic ball of radius ρ about x . According to Proposition 4.5, there exists $i(\rho)$ such that for all $i > i(\rho)$, $\int_{B_\rho} |F_{A_i}|^2 \geq \kappa^2$, while $\int_{S^4 \setminus B_\rho} |F_{A_i}|^2 < \rho$. Under these conditions, it is evident that for $i > i(\rho)$, $z(A_i) = x + O(\rho)$; here, S^4 is identified with ∂B^5 . Hence, a contradiction is obtained.

Proof of Lemma 4.11. Observe that for $t \in T$ defined by (4.14),

$$(4.16) \quad z(t^*A) = \left(\int d^4y |F_A|^2 \right)^{-1} \left[\int d^4y \left(\frac{2\rho y + x}{|\rho y + x|^2 + 1}, \frac{|\rho y + x|^2 - 1}{|\rho y + x|^2 + 1} \right) |F_A|^2(y) \right].$$

Fix $[A]$, and consider (4.16) as defining a map $z(\rho, x) : \mathbf{R}^* \times \mathbf{R}^4 \rightarrow B^5$. Because stereographic coordinates identify $S^4 \setminus s$ with \mathbf{R}^4 , z defines a map from $\mathbf{R}^* \times (S^4 \setminus s) \rightarrow B^5$. Three claims are made: First, z extends to a continuous map from $\mathbf{R}^* \times S^4 \rightarrow B^5$ with $z(\cdot, s) = s \in \partial B^5$. Second, z extends to a continuous map from $(0, \infty] \times S^4 \rightarrow B^5$ with $z(\infty, \cdot) = s \in \partial B^5$. Third, z extends to a continuous map from $[0, \infty] \times S^4 \rightarrow B^5$ with $z(0, \cdot) = \text{id}_{S^4}$. Together, these three claims imply that $z^{-1}(0) \cap \mathbf{R}^* \times \mathbf{R}^4 \neq \emptyset$ because if it were true that $|z| > \delta > 0$, then $z/|z| : [0, \infty] \times S^4 \rightarrow S^4$ would provide a continuous homotopy of a degree 1 map, id_{S^4} , with the constant map. No such homotopy exists.

The three claims are proved using the fact that $|F_A|(y) \in L^2(\mathbf{R}^4)$. Specifically, this implies that given $\delta > 0$, there exist $0 < r(\delta) < R(\delta) < \infty$ such that for any $x \in \mathbf{R}^4$,

$$(4.17) \quad \begin{aligned} (1) \quad & \int_{|y-x| < r} d^4y |F_A|^2(y) < \delta, \\ (2) \quad & \int_{|y| > R} d^4y |F_A|^2(y) < \delta. \end{aligned}$$

To establish the first claim, pick $\epsilon > 0$. If $|x| > \epsilon^{-1} + \rho R(\epsilon)$, then $|\rho y + x| < \epsilon^{-1}$ only if $y > R(\epsilon)$. By splitting the integrand in (4.16) into domains $|y| < \cdot, > R(\epsilon)$ and using (4.17.2), one observes that for $|x| > \epsilon^{-1} + \rho R(\epsilon)$,

$$(4.18) \quad z(\rho, x) = (0, 0, 0, 0, 1) + O(\epsilon) \in B^5.$$

Hence, z extends continuously to $\mathbf{R}^* \times S^4$.

To establish the second claim, pick $\epsilon > 0$ and observe that if ρ is sufficiently large so that $\rho^{-1}\epsilon^{-1} < r(\epsilon)$, then $|\rho y + x| < \epsilon^{-1}$ only if $|y + \rho^{-1}x| < r(\epsilon)$. By splitting the integrand in (4.16) into domains $|y| < r(\epsilon)$ and using (4.17.1), one observes that for $\rho > \epsilon^{-1}r(\epsilon)^{-1}$, $z(\rho, x)$ is again given by (4.18) where the $O(\epsilon)$ is independent of x . Hence, z extends as required to $(0, \infty] \times S^4$.

To establish the third claim, pick $\epsilon > 0$ and observe that if $\rho < \epsilon R^{-1}(\epsilon)$, then $|\rho y + x - x| > \epsilon$ only if $|y| > R(\epsilon)$. By splitting the integrand in (4.16) again into domains $|y| < R(\epsilon)$ one observes that if $\rho < \epsilon R^{-1}(\epsilon)$, then uniformly on S^4 , $z(\rho, \cdot) = \text{id}_{S^4} + O(\epsilon)$, and z extends as required to $[0, \infty] \times S^4$.

The proof of Proposition 4.4 is finally completed with the promised proof of Lemma 4.10:

Proof of Lemma 4.10. Consider $A = A_i^2$ for $i \in I_n > i_0$; the proof for A_i^1 is similar. The quadratic form $\mathfrak{S}_A(\cdot, \cdot)$ on L_1^2 is bounded, but not compact. Nonetheless, because there exists a q -dimensional vector space in L_1^2 on which $\mathfrak{S}_A(\cdot, \cdot) < -\lambda < 0$, a direct minimization argument obtains eigenvectors $\{v_\alpha\}_{\alpha=1}^q$ of $\mathfrak{S}_A(\cdot, \cdot)$ with respect to the metric $\langle \cdot, \cdot \rangle_A$; and each v_α has eigenvalue $-\lambda_\alpha < -\lambda$. The eigenvalue equation for $v = v_\alpha$ is

$$(4.19) \quad \lambda_\alpha \langle v, u \rangle_A + \langle P_+ D_A v, P_+ D_A u \rangle_2 + \langle P_+ F_A, [v, u] \rangle_2 = 0$$

for all $u \in L_1^2(\text{Ad } P \otimes T^*)$. As $\lambda_\alpha > \lambda$, this equation is uniformly elliptic for v and standard estimates [19, Chapter 6] imply that $v \in L_2^3(\text{Ad } P \otimes T^*)$. For $\rho \leq r/2$, let β_ρ denote the usual cut-off function. With $\delta \geq 0$, consider $u = \beta_\rho^2(|v|^2 + 1)^\delta v$. If $\delta < \min(\frac{1}{2}, 2\lambda)$, then one obtains, with $f = (1 + |v|^2)^{(1+\delta)/2}$,

$$(4.20) \quad \frac{\lambda}{(1 + \delta)^2} \langle \beta_\rho^2, |df|^2 \rangle_2 \leq z\rho^{-2} \|v\|_A^2 + \langle \beta_\rho^2 |P_+ F_A|, f^2 \rangle_2.$$

Here, $z < \infty$ is independent of A ; Proposition 2.1 is used in this derivation. As $\|f\|_2 \leq z\|v\|_A$, one obtains from (4.20) an inequality for $f_\rho = \beta_\rho f$:

$$(4.21) \quad \|f_\rho\|_{1,2}^2 \leq z(\rho^{-2} + 1) \|v\|_A^2 + \frac{(1 + \delta)^2}{\lambda} \|\beta_{2\rho} P_+ F_A\|_2 \|f_\rho\|_{1,2}^2.$$

Here, $z < \infty$ is still independent of A . As the sequence $\{A_i^2\}$ converges strongly in L_1^2 of B_r to A_0 , there exists $i_1 < \infty$ and $\rho > 0$ such that for all $i > i_1$, and $A = A_i^1$,

$$\|\beta_{2\rho} P_+ F_A\|_2 < \frac{1}{2}\lambda(1 + \delta)^{-2}.$$

For such i, f_ρ is uniformly bounded in L_1^2 and hence L^4 , and so the $L^{4+4\delta}$ norm of each $v \in E_i^1$ is uniformly bounded in β_ρ as required.

5. The hessian

It is reasonable to conjecture that mini-max over curves should produce critical points with hessian index less than 2; an experiment with a length of string on a hilly surface will illustrate this principle. One extrapolates that mini-max over an l -disc produces critical points with indices less than $l + 1$. This is Proposition 4.2's assertion, and Proposition 4.2 is proved here.

This proof has four steps. Step 1 is represented by Lemma 5.1 where, given $\epsilon > 0$, a pseudo-gradient vector field for \mathfrak{A} is used to construct a disc $(\phi, \bar{\phi}) \in X$ with $\mathfrak{A}(\bar{\phi}) < \mathfrak{A}_\infty + \epsilon$ and $\|\nabla \mathfrak{A}_{\bar{\phi}}\|_* < \epsilon$. Step 2 is represented by Lemma 5.2. Here, the pseudo-gradient vector field is used to deform ϕ to a new disc, $(\psi, \bar{\psi}) \in X$ which has $\mathfrak{A}(\bar{\psi}) < \mathfrak{A}_\infty + \epsilon$, and $\|\nabla \mathfrak{A}_{\psi(y)}\|_* < \epsilon$ on the set of y , where $\mathfrak{A}(\psi(y)) > \mathfrak{A}_\infty$. In step 3, the disc ψ is deformed along a negative direction of \mathfrak{G}_ψ to obtain finally $(\eta(\epsilon), \bar{\eta}) \in X$ which has $\mathfrak{A}_{\bar{\eta}} < \mathfrak{A}_\infty + \epsilon$, $\|\nabla \mathfrak{A}_{\bar{\eta}}\|_* < \epsilon$, and $\lambda_{\bar{\eta}}^{l+1} > -\epsilon$. Step 4 is to construct for each n , $\eta(1/n) \in \Theta$. The sequence $\{\eta(1/n), \bar{\eta}(1/n)\}$ is a good sequence.

A deformation of a disc, $\phi \in \Theta$, to a new disc, $\psi \in \Theta$, can be constructed given a section $v \in C^0(D^l; \mathfrak{B})$ which vanishes on $S^{l-1} = \partial D^l$. The procedure follows: Because D^l is contractible, the bundle $\phi^*(P_s \times \mathbb{G}_1^3) \rightarrow D^l$ is isomorphic to the product bundle $D^l \times \mathbb{G}_2^3$. Each such isomorphism defines and is defined by a lift of ϕ to a map from D^l to $P_s \times \mathbb{G}_1^3$. Choose a lift of ϕ ; a disc $(h(\cdot), A(\cdot)) \in C^0(D^l; P_s \times \mathbb{G}_1^3)$ which projects to ϕ . With respect to the induced trivialization of $\phi^*(P_s \times \mathbb{G}_1^3)$, a section $v \in \phi^*\mathfrak{B} \rightarrow D^l$ defines a map, $v(\cdot) \in C^0(D^l; L_1^3(\text{Ad } P \otimes T^*))$. Define

$$(5.1) \quad \psi(\cdot) = [h(\cdot), A(\cdot) + v(\cdot)].$$

If $v \in \Gamma_\phi \equiv \{\text{the continuous sections of } \phi^*\mathfrak{B} \rightarrow D^l \text{ which vanish on } S^{l-1}\}$, then $\psi \in \Theta$. Note that ψ is defined independently of the choice of the lift of ϕ .

Proof of Proposition 4.2. It is convenient to first introduce the following notation: If $\phi \in \Theta$, then $\mathfrak{A}_y = \mathfrak{A}(\phi(y))$; $\nabla \mathfrak{A}_y = \nabla \mathfrak{A}_{\phi(y)}$; $\|\cdot\|_{(y)} = \|\cdot\|_{\phi(y)}$, \dots , etc.

By definition, given $\epsilon > 0$, there exist $(\phi, \bar{\phi}) \in X$ with $\mathfrak{A}_{\bar{\phi}} < \mathfrak{A}_\infty + \epsilon$. This observation and the lemma below give step 1 of the proof of Proposition 4.2.

Lemma 5.1. *Let $\phi \in \Theta$. There exists a disc $(\psi, \bar{\psi}) \in X$ satisfying for all $y \in D^l$, $\mathfrak{A}(\psi(y)) \leq \mathfrak{A}(\phi(y))$ and in addition satisfying*

$$\|\nabla \mathfrak{A}_{\bar{\psi}}\|_* \leq z \left\{ (\mathfrak{A}_{\bar{\psi}} - \mathfrak{A}_\infty) + (\mathfrak{A}_{\bar{\psi}} - \mathfrak{A}_\infty)^2 \right\},$$

where $z < \infty$ is a constant which is independent of ϕ and Θ .

Proof of Lemma 5.1. With a locally, uniformly finite partition of unity of D^l , one constructs a pseudo-gradient vector field [20], $a \in \Gamma_\phi$, satisfying

$$(5.2) \quad \|a(y)\|_{(y)} = \|\nabla \mathfrak{A}_y\|_*, \quad \nabla \mathfrak{A}_y(a(y)) \leq -\frac{1}{2} \|\nabla \mathfrak{A}_y\|_*^2.$$

Let z be the constant in Proposition 2.2. Define the open sets

$$(5.3) \quad \begin{aligned} \Omega_0 &= \{y \in D^l : \mathfrak{A}_y > \mathfrak{A}_\infty\}, \\ \Omega_1 &= \left\{y \in \Omega_0 : \|\nabla \mathfrak{A}_y\|_*^2 (1 + \|\nabla \mathfrak{A}_y\|_*)^{-1} > 4(1 + 8z)(\mathfrak{A}_y - \mathfrak{A}_\infty)\right\}. \end{aligned}$$

Let $\theta(x)$ denote the usual step function; $\theta = 1$ when $x > 0$, and $\theta = 0$ when $x < 0$. Define

$$(5.4) \quad \mathfrak{A}_* = \sup_{y \in \partial\Omega_1} \mathfrak{A}_y.$$

A section, $v \in \Gamma_\phi$, is defined as follows: In $D^l \setminus \Omega_1$, set $v \equiv 0$ and for $y \in \Omega_1$, set

$$(5.5) \quad v(y) = 4 \|\nabla \mathfrak{A}_y\|_*^{-2} (\mathfrak{A}_y - \mathfrak{A}_*) \theta(\mathfrak{A}_y - \mathfrak{A}_*) \cdot a(y).$$

Observe that v is continuous, $\|v(y)\|_{(y)} \leq 1$, and $v(y) \equiv 0$ on S^{l-1} as required for an element in Γ_ϕ .

Define $\psi \in \Theta$ by (5.1) using v as defined above. When $y \notin \Omega_1$, then $\psi(y) = \phi(y)$. When $y \in \Omega_1$, one computes with (2.8.2) the following inequality:

$$(5.6) \quad \mathfrak{A}(\psi(y)) \leq \mathfrak{A}_y - (\mathfrak{A}_y - \mathfrak{A}_*) \theta(\mathfrak{A}_y - \mathfrak{A}_*) \leq \mathfrak{A}_*.$$

One concludes from (5.6) that $\bar{\psi}$ can be chosen to lie in $\overline{\phi(D^l \setminus \Omega_1)}$. Since $\psi(y) = \phi(y)$ for $y \in \overline{D^l \setminus \Omega_1}$, the lemma follows.

The next lemma summarizes step 2 of the proof of Proposition 4.2.

Lemma 5.2. *Given $\varepsilon > 0$, there exist $(\phi, \bar{\phi}) \in X$ with (1) $\mathfrak{A}_{\bar{\phi}} - \mathfrak{A}_\infty < \varepsilon$, and (2) for all $y \in D^l$ where $\mathfrak{A}_y > \mathfrak{A}_\infty$, $\|\nabla \mathfrak{A}_y\|_* < \varepsilon$.*

Proof of Lemma 5.2. Fix $1 > \delta > 0$ and choose $(\phi, \bar{\phi}) \in X$ with $\mathfrak{A}_{\bar{\phi}} < \mathfrak{A}_\infty + \delta$ and with $\|\nabla \mathfrak{A}_{\bar{\phi}}\|_* < \delta$. Let $\bar{y} \in \Phi^{-1}(\bar{\phi})$. There exists a neighborhood $U \subseteq D^l$ of \bar{y} such that for all $y \in U$,

$$(5.7) \quad \|\nabla \mathfrak{A}_y\|_* < \delta.$$

Let $0 \leq \beta \leq 1$ be a function in $C_0^\infty(U)$ with $\beta(\bar{y}) = 1$. Let $a \in \Gamma_\phi$ be the pseudo-gradient vector field of (5.2), and define Ω_0, Ω_1 by (5.6). Define $v \in \Gamma_\phi$ as follows: When $y \notin \Omega_0$, set $v \equiv 0$. When $y \in \Omega_1$, set

$$(5.8) \quad v(y) = 4(1 - \beta(y)) \|\nabla \mathfrak{A}_y\|_*^{-2} (\mathfrak{A}_y - \mathfrak{A}_\infty) \theta(\mathfrak{A}_y - \mathfrak{A}_\infty) \cdot a(y).$$

When $y \in \Omega_0 \setminus \Omega_1$, set

$$(5.9) \quad v(y) = (1 - \beta(y))(1 + 8z)^{-1} (1 + \|\nabla \mathfrak{A}_y\|_*)^{-1} \cdot a(y).$$

The continuity of v follows from the definition of Ω_1 , as does the fact that $\|v(y)\|_{(y)} < 1$. The section v vanishes on S^{l-1} because $a(\cdot)$ vanishes there.

Define $\psi \in \Theta$ by using v as defined above in (5.1). Using (2.8.2), one can make the following observations: First, $\mathfrak{A}(\psi(y)) \leq \mathfrak{A}(\phi(y))$, but $\psi(\bar{y}) = \phi(\bar{y}) = \bar{\phi}$ so one can choose $\bar{\psi} = \bar{\phi}$. For $y \notin \Omega_0$, $\psi(y) = \phi(y)$. For $y \in \text{Int}(\Omega_1 \setminus U)$,

$$(5.10) \quad \mathfrak{A}(\psi(y)) < \mathfrak{A}_y - (\mathfrak{A}_y - \mathfrak{A}_\infty)\theta(\mathfrak{A}_y - \mathfrak{A}_\infty) < \mathfrak{A}_\infty.$$

The main consequence of (5.10) is that only for $y \in \overline{(\Omega_0 \setminus \Omega_1) \cup U} \equiv B$ is it possible that $\mathfrak{A}(\psi(y)) \geq \mathfrak{A}_\infty$.

To prove the lemma, an estimate for $\|\nabla \mathfrak{A}_{\psi(y)}\|_*$ when $y \in B$ is required. For this purpose, write $\phi = [h(\cdot), A(\cdot)]$ and $\psi = [h(\cdot), A(\cdot) + v(\cdot)]$. Let $u \in L^3_1(\text{Ad } P \otimes T^*)$ be fixed. Using (2.9.1), one find that for $y \in B$,

$$(5.11) \quad \begin{aligned} & \left| \nabla \mathfrak{A}_{(A+v)(y)}(u) \right| \\ & \leq \|u\|_{(y)} \left\{ \|\nabla \mathfrak{A}_y\|_* + 2z(1 + \mathfrak{A}_\infty)^{1/2} \|v(y)\|_{(y)} (1 + \|v(y)\|_{(y)}) \right\}. \end{aligned}$$

Use (2.7), (5.3), (5.7) and (5.9) to evaluate $\|\nabla \mathfrak{A}_y\|_*$ and $\|v(y)\|_{(y)}$ when $y \in B$. The result is the bound

$$(5.12) \quad \left| \nabla \mathfrak{A}_{(A+v)(y)}(u) \right| \leq \|u\|_{(A+v)(y)} z(1 + \mathfrak{A}_\infty)^{1/2} \delta \quad \text{for } y \in B.$$

Here, $z < \infty$ is independent of ϕ , $\delta < 1$ and Θ . Lemma 5.2 is a direct consequence of (5.10) and (5.12).

Step 3 of the proof of Proposition 4.2 is summarized by

Lemma 5.3. *Given $\epsilon > 0$, there exists $(\phi, \bar{\phi}) \in X$ with (1) $\mathfrak{A}_{\bar{\phi}} < \mathfrak{A}_\infty + \epsilon$ and (2) for all $y \in D^l$ where $\mathfrak{A}_y > \mathfrak{A}_\infty$, $\|\nabla \mathfrak{A}_y\|_* < \epsilon$ and $\lambda_y^{l+1} > -\epsilon$.*

For the proof of Lemma 5.3, define with $\phi \in \Theta$ and $1 > \delta > 0$, the set

$$(5.13) \quad \Omega(\delta) = \left\{ y \in D^l : \lambda_y^{l+1} < -\delta \right\}.$$

Lemma 5.3 is a consequence of

Lemma 5.4. *Given $\phi \in \Theta$ and $\delta, \delta' \in (0, 1)$, there exists $u \in \Gamma_\phi$ which satisfies at $y \in \Omega(\delta)$, $\|u(y)\|_{(y)} = 1$; $\nabla \mathfrak{A}_y(u(y)) < \delta'$; and $\mathfrak{S}_y(u(y), u(y)) < -\frac{1}{2}\lambda_y^{l+1}$.*

Proof of Lemma 5.3, assuming Lemma 5.4. Let $(\phi, \bar{\phi}) \in X$ satisfy the statements of Lemma 5.2 with $\epsilon \in (0, 1)$. For $\delta, \delta' \in (0, 1)$ to be determined shortly, consider $v = fu \in \Gamma_\phi$, where u is given by Lemma 5.4 and $f \in C^0(D^l)$ takes values in $[0, 1]$. Let ψ be defined by (5.1) using v . From (2.8.3), one obtains the inequality

$$(5.14) \quad \mathfrak{A}(\psi(y)) < \mathfrak{A}(\phi(y)) + \delta'f - \frac{1}{4}|\lambda_y^{l+1}|f^2 + zf^3.$$

Define f more explicitly as follows: For $y \in \Omega(2\delta)$, set

$$(5.15) \quad f(y) = (1 + 16z)^{-1} \min\{1, |\lambda_y^{+1}|\}.$$

For $y \notin \Omega(2\delta)$, set

$$(5.16) \quad f(y) = 2(1 + 16z)^{-1} (|\lambda_y^{+1}| - \delta) \theta(-\lambda_y^{+1} - \delta).$$

With no loss of generality, suppose that $\epsilon 16(1 + 16z)^2 < 1$ and choose

$$(5.17) \quad \delta = (\epsilon 16(1 + 16z)^2)^{1/3}.$$

Restrict $\delta' < \epsilon$. For $y \in \Omega(2\delta)$, (5.14), (5.15) and (5.17) imply that

$$(5.18) \quad \mathfrak{A}(\psi(y)) < \mathfrak{A}(\phi(y)) + \epsilon - \frac{1}{8} \frac{1}{(1 + 16z)^2} \delta^3 \leq \mathfrak{A}(\phi(y)) - \epsilon \leq \mathfrak{A}_\infty.$$

For $y \in \Omega(\delta) \setminus \Omega(2\delta)$, (5.14), (5.16) and (5.17) imply that

$$(5.19) \quad \mathfrak{A}(\psi(y)) < \mathfrak{A}(\phi(y)) + \delta'.$$

For $y \notin \Omega(\delta)$, $\psi(y) = \phi(y)$.

Explicit in the choice of ϕ is the requirement that $\|\nabla \mathfrak{A}_y\|_* < \epsilon$ when $\mathfrak{A}_y > \mathfrak{A}_\infty$. Choose $\delta' \in (0, \epsilon)$ so that when $\mathfrak{A}_y > \mathfrak{A}_\infty - \delta'$, then $\|\nabla \mathfrak{A}_y\|_* < 2\epsilon$. Consider the consequences: First of all, from (5.19), $\mathfrak{A}(\bar{\psi}) < \mathfrak{A}_\infty + 2\epsilon$. Second, if one defines $\Omega'_0 = \{y \in D': \mathfrak{A}(\psi(y)) \geq \mathfrak{A}_\infty\}$, then $\Omega'_0 \subset D' \setminus \Omega(2\delta)$ and

$$(5.20) \quad \|\nabla \mathfrak{A}_{\phi(y)}\|_* < 2\epsilon \quad \text{when } y \in \Omega'_0.$$

Now use (2.9.1) to estimate $\|\nabla \mathfrak{A}_{\psi(y)}\|_*$ when $y \in \Omega'_0$. Use the fact that for such y ,

$$(5.21) \quad \|v(y)\|_{(y)} \leq 2(\epsilon 16(1 + 16z)^2)^{1/3}.$$

The conclusion is that for $y \in \Omega'_0$,

$$(5.22) \quad \|\nabla \mathfrak{A}_{\psi(y)}\|_* \leq z(1 + \mathfrak{A}_\infty)^{1/2} (\epsilon + \epsilon^{1/3}).$$

Here, $z < \infty$ is independent of ϕ , ϵ and Θ . Use (2.9.2) and (5.13), (5.17) and (5.21) to estimate $\lambda_{\psi(y)}^{+1}$ for $y \in \Omega'(0)$. The result is

$$(5.23) \quad \begin{aligned} \lambda_{\psi(y)}^{+1} &\geq -2(\epsilon 16(1 + 16z)^2)^{1/3} - z'(1 + \mathfrak{A}_\infty k)^{1/2} \epsilon^{1/3} \\ &\geq -z\epsilon^{1/3}(1 + \mathfrak{A}_\infty)^{1/2}. \end{aligned}$$

In the last line, above, $z < \infty$ is independent of ϕ , ϵ and Θ .

Observe that Lemma 5.3 follows directly from (5.20), (5.22) and (5.23) by redefining ϵ .

Proof of Lemma 5.4. One reduces the problem to one of finding an appropriate section of a finite-dimensional vector bundle, $L \subset \phi^*\mathfrak{B} \rightarrow D^l$. A standard general position argument gives the required section, u . This argument fails if one were to replace λ_y^{l+1} with λ_y^k , $k \leq l$, in the statement of Lemma 5.4.

Lemma 5.5. *Let $\phi \in \Theta$. Given $\delta > 0$, there exists a vector subbundle $L \subset \phi^*\mathfrak{B} \rightarrow D^l$ with $l + 1 < \dim L < \infty$ and such that at each $y \in D^l$, the restriction of \mathfrak{S}_y to L_y has at least $l + 1$ eigenvectors with respect to $\langle \cdot, \cdot \rangle_{(y)}$ with eigenvalue less than $\lambda_y^{l+1} + \frac{1}{4}\delta$.*

Proof of Lemma 5.5. Write $\phi = [h(\cdot), A(\cdot)]$. At each $y \in D^l$, there exists an $l + 1$ -dimensional vector subspace, $E(y) \subset L_1^3(\text{Ad } P \otimes T^*)$, such that

$$(5.24) \quad \sup_{0 \neq v \in E(y)} \left\{ \mathfrak{S}_{A(y)}(v, v) - \left(\lambda_y^{l+1} + \frac{1}{4}\delta \right) \|v\|_{A(y)}^2 \right\} < 0.$$

Continuity implies the existence of a neighborhood $U(y) \subseteq D^l$ such that for all $x \in U(y)$,

$$(5.25) \quad \sup_{0 \neq v \in E(y)} \left\{ \mathfrak{S}_{A(x)}(v, v) - \left(\lambda_x^{l+1} + \frac{1}{4}\delta \right) \|v\|_{A(x)}^2 \right\} < 0.$$

As D^l is compact, there exists a finite set $\{y_j\}_{j=1}^N$ with the property that $\{U(y_j)\}_{j=1}^N$ is a cover of D^l . Define $\hat{L} = \text{Span} \bigcup_{j=1}^N E(y_j) \subset L_1^3(\text{Ad } P \otimes T^*)$, and set $L = [h(\cdot), A(\cdot), \hat{L}] \subset \phi^*\mathfrak{B}$. The reader can readily verify that L has the required properties.

For $\lambda \in [\lambda_y^{l+1} + \frac{1}{4}\delta, \lambda_y^{l+1} + \frac{1}{2}\delta]$, let $\pi(y, \lambda)$ denote the $\langle \cdot, \cdot \rangle_{(y)}$ -orthogonal projection onto the eigenspaces of the restriction of \mathfrak{S}_y to L_y with eigenvalue less than λ . When λ is not an eigenvalue of this restricted form, there exists a ball, $B(y, \lambda) \subseteq D^l$, centered at y , of positive radius such that $K(y) \equiv \{\pi(x, \lambda)L \subseteq L, x \in B(y, \lambda)\}$ defines a continuous subbundle of L over $B(y, \lambda)$. As D^l is compact, there exists a finite set $\{y_j\}_{j=1}^M \subset D^l$ such that $\{B(j) = B(y_j, \lambda_j)\}_{j=1}^M$ covers D^l .

Lemma 5.6. *Fix $\delta > 0$. Let ω^* denote the restriction of $\nabla \mathfrak{A}(\cdot)$ to L^* . Given $\delta' > 0$, there exists a continuous section, ω of L^* , which has the following properties: (1) At each $y \in D^l$, $|\omega - \omega^*|_{L^*} < \delta'$. (2) For each $j \in (1, \dots, M)$, the restriction of ω to $K(y_j)^*$ is nonvanishing.*

Proof of Lemma 5.6. This is a straightforward, general position argument of the kind discussed in Chapter 3 of [17]; see specifically Theorems 2.2 and 2.6 there. These results are applicable only because $\dim K(y_j) \geq l + 1$.

To construct the section u of Lemma 5.4, let $\{\beta_j\}$ be a partition of unity for the cover $\{B(j)\}$ of D^l . In $B(j)$, let $v_j \in C^0(B(j); K(y_j)) \subset C^0(B(j); L)$ be

dual in $K(y_j)$ to ω . Define

$$(5.26) \quad v = -\sum \beta_j v_j.$$

Observe that $v \in C^0(D^l; L)$, and at each $y \in D^l$,

$$(5.27) \quad \omega_y(v(y)) = -\sum \beta_j \omega_y(v_j(y)) < 0.$$

Hence, v is nonvanishing. Because $v_j \in K(y_j)$,

$$(5.28) \quad \mathfrak{S}_y(v(y), v(y)) \leq (\lambda_y^{+1} + \frac{1}{2}\delta) \|v(y)\|_{(y)}^2$$

at each $y \in D^k$. Finally, at each $y \in D^k$,

$$(5.29) \quad |\nabla \mathfrak{A}_y(v(y)) - \omega_y(v(y))| < \delta' \|v(y)\|_{(y)}.$$

To obtain Lemma 5.4, set $u(y) = \|v(y)\|_{(y)}^{-1} v(y)$ when $y \in \Omega(\delta)$. As $S^{l-1} \subset D^l \setminus \Omega(\delta)$, there exists a continuous extension of $u(y)$ to D^l which vanishes on S^{l-1} . The requirements of Lemma 5.5 follow from (5.28) and (5.29).

6. Constructing paths

The proof of Theorem 1.4 was outline in the Introduction; the details of it are provided here and in §7. The exposition of the proof is presented here, but the propositions in this section have their proofs in §7.

Let M be a compact, oriented Riemannian 4-manifold. A preliminary digression is required for the purpose of explaining how to subtract a pair (P', b') of principal G -bundle $P' \rightarrow S^4$ and point $b' \in \mathfrak{B}(P')$ from a pair (P, b) of principal G -bundle $P \rightarrow M$ and point $b \in \mathfrak{B}(P)$ to obtain a principal G -bundle, " $P - P'$ " $\rightarrow M$ and a point " $b - b'$ " $\in \mathfrak{B}(P - P')$.

This subtraction procedure requires the orientation reversing map $\alpha: S^4 \rightarrow S^4$, the inversion of S^4 through its equatorial S^3 , which is fixed. The procedure also requires the choice of a point $s \in M$ and a Gaussian coordinate system, $z: B \rightarrow \mathbf{R}^4$, centered at s and defined on a ball B about s . It is convenient to identify $S^4 \setminus n$ with \mathbf{R}^4 via stereographic projection from n , and then to identify B with $z(B) \subset S^4 \setminus n$, and $s \in M$ with $s = \text{south pole} \in S^4$ with $0 \in \mathbf{R}^4$. It is no less of generality to assume that B is the set $\{z : |z| < 1\} \subset \mathbf{R}^4 \simeq S^4 \setminus n$. An identification of P with $\alpha^{-1}P'$ over $U = B \setminus s$ is canonically defined, given b, b' . Indeed, each $(h, A) \in P_s \times \mathfrak{C}(P)$ defines a section $\phi(h, A) \in \Gamma(P|_B)$ by the parallel transport of h by A along the short, radial geodesics through s . This is an $\text{Aut } P$ equivariant map from $P_s \times \mathfrak{C}$ to $\Gamma(P|_B)$; hence it is a continuous section of the fibre bundle

$$(P_s \times \mathfrak{C}) \times_{\text{Aut } P} \Gamma(P|_B) \rightarrow \mathfrak{B}(P).$$

With this trivialization, each $b \in \hat{\mathfrak{B}}(P)$ defines a trivialization of $P \rightarrow B$. Similarly, $b' = [h', A']$ defines a trivialization of P' over $S^- = S^4 \setminus n$ and hence, one of $\alpha^{-1}P'$ over $S^+ = S^4 \setminus s$. By identifying the two trivializations over $B \setminus s$, one obtains a principal G -bundle which is, by definition, $P - P'$. Note that $\text{degree}(P - P') = \text{degree } P - \text{degree } P'$. Here, the notation is cryptic, because although the isomorphism class of $P - P'$ is independent of the choice $(b, b') \in \hat{\mathfrak{B}}(P) \times \hat{\mathfrak{B}}(P')$, the actual bundle is not independent of this choice.

Next, a family of points, $\{(b - b')_\rho \in \hat{\mathfrak{B}}(P - P') : \rho \in (0, \frac{1}{4})\}$, will be defined. For this purpose, introduce the family of smooth, radial bump functions, $\{\eta_\rho(z) = \eta(|z|/\sqrt{\rho}) ; \rho \in (0, \infty) \text{ and } z \in \mathbf{R}^4\}$. Require that $\eta = 0$ if $|z| < \frac{1}{2}$ and that $\eta = 1$ if $|z| > 1$. Introduce the subgroup $T \subset C$ of (4.2) and define the family of pure dilations, $\{t_\rho = (\rho, 0) \in \mathbf{R}^* \times \mathbf{R}^4 = T : \rho \in (0, \frac{1}{16})\}$. Thus, $t_\rho^*z = \rho z$.

It is important to note that $P - P'$ as constructed from (b, b') has a canonical product structure over $U = B \setminus s$. Let θ denote the induced flat, product connection on $P - P'|_U$. Suppose that $b = [h, A]$ and that $b' = [h', A']$. A family of connections $\{(A - A')_\rho : \rho \in (0, \frac{1}{16})\}$ on $P - P'|_U$ is given by setting

$$(6.1) \quad (A - A')_\rho = \theta + t_{\rho^{-1}} \left[\eta_\rho t_\rho^* \phi^*(h, A)A + \alpha^*(\eta_\rho t_\rho^* \phi^*(h', A')A') \right].$$

On the set $M^+ = \{p \in M : \text{dist}(p, s) > \frac{1}{2}\}$, $P - P'$ is canonically identified with P , and a family of connections, $\{(A - A')_\rho : \rho \in (0, \frac{1}{16})\}$ on $P - P'|_{M^+}$ is given by setting

$$(6.2) \quad (A - A')_\rho = A.$$

Similarly, on $B^- = \{z \in \mathbf{R}^4 : |z| < \frac{1}{2}\rho^{3/2}\}$, $P - P'$ is canonically identified with $\alpha^{-1}P'$, and a family of connections, $\{(A - A')_\rho : \rho \in (0, \frac{1}{16})\}$ on $P - P'|_{B^-}$ is given by setting

$$(6.3) \quad (A - A')_\rho = \alpha^* t_{\rho^2}^* A'.$$

These three families agree where the domains of definition overlap, and so define a family of connections, $\{(A - A')_\rho : \rho \in (0, \frac{1}{16})\}$ on $P - P'$.

To complete the definition of $\{(b - b')_\rho : \rho \in (0, \frac{1}{16})\}$, it is necessary to fix once and for all a half great circle on S^4 running between s and n . Denote it by l . Let $l(h', A')$ denote the point in P'_n obtained by the parallel transport of $h' \in P'_s$ along l by the connection A' .

Define $\{(b - b')_\rho : \rho \in (0, \frac{1}{16})\}$ by setting

$$(6.4) \quad (b - b')_\rho = [\alpha^{-1}l(h', A'), (A - A')_\rho].$$

It is an exercise that is left to the reader to show that the map from $(0, \frac{1}{16}) \times \mathfrak{B}(P) \times \mathfrak{B}(P')$ to $\mathfrak{B}(P - P')$ which sends (ρ, b, b') to $(b - b')_\rho$ is continuous. (Recall that if P_1, P_2 are isomorphic principal G -bundles, then $\mathfrak{B}(P_1)$ and $\mathfrak{B}(P_2)$ are canonically identified.)

An estimate for $\mathfrak{Y}\mathfrak{M}((b - b')_\rho)$, $\rho \in (0, \frac{1}{16})$, is provided in Proposition 6.1 as an expansion in powers of ρ . For the present purposes, a precise estimate is necessary only for the case where $P' \rightarrow S^4$ has $k(P') > 0$, T^*S^4 has its standard metric, and $b' \in \mathfrak{M}(P')$. To state this proposition, it is necessary to introduce (\cdot, \cdot) to denote the metric on $(\text{Ad } P \otimes P_+ \wedge_2 T^*M)_s$, and to remark that the Gaussian coordinate chart identifies $T^*M|_s$ isometrically with $T^*S^4|_s$ and isometrically with $T^*\mathbf{R}^4|_0$.

Proposition 6.1. *Let S^4 have its standard metric. Let M be a compact, oriented Riemannian 4-manifold. Let $P \rightarrow M$ and $P' \rightarrow S^4$ be principal G -bundles. Let $b = [h, A] \in \mathfrak{B}(P)$, let $b' = [h', A'] \in \mathfrak{B}(P')$, and let $\rho \in (0, \frac{1}{4})$. Then*

$$\mathfrak{Y}\mathfrak{M}((b - b')_\rho) = \mathfrak{Y}\mathfrak{M}(b) + \mathfrak{Y}\mathfrak{M}(b') + I_\rho(b, b'),$$

where $|I_\rho(b, b')| = O(\rho^4 \ln|\rho|)$. However, if $k(P') > 0$ and $b' \in \mathfrak{M}(P')$, then

$$I_\rho(b, b') = -\pi^2 \rho^4 (h'^{-1}P_+F_{A'}(s)h', h^{-1}P_+F_A(s)h) + \rho^4 O(\rho^{1/4}).$$

An important corollary to Proposition 6.1 is stated below in the next proposition. To state the result requires

$$(6.5) \quad \mathfrak{Q}^c(P) = \{b = [h, A] \in \mathfrak{B}(P) : P_+F_A(s) = 0\} \quad \text{and} \quad \mathfrak{Q} = \mathfrak{B} \setminus \mathfrak{Q}^c.$$

Proposition 6.2. *Let M and S^4 be as in Proposition 6.1. Let $P \rightarrow M$ be a principal G -bundle and let $b \in \mathfrak{Q}(P)$. Let $P_1 \rightarrow S^4$ be a principal G -bundle of degree 1, and let $[A] \in \mathfrak{M}(P_1)/G$. There exists $b_1 = [h_1, A] \in \mathfrak{M}(P_1)$ and $\rho_0 \in (0, \frac{1}{16})$ such that for all $\rho \in (0, \rho_0)$,*

$$(6.6) \quad \mathfrak{Y}\mathfrak{M}((b - b_1)_\rho) < \mathfrak{Y}\mathfrak{M}(b) + 1.$$

For the remainder of this section, $M = S^4$ with the standard metric. To begin the proof of Theorem 1.4, let $P \rightarrow S^4$ be the given principal G -bundle and let $m_0, m_1 \in \mathfrak{M}(P)$. To use Propositions 6.1, 6.2, one requires

Lemma 6.3. *Let $P \rightarrow S^4$ be a principal G -bundle of positive degree. Then $\mathfrak{Q}(P) \cap \mathfrak{M}(P)$ is dense in $\mathfrak{M}(P)$.*

It is therefore no loss of generality to assume that $m_0, m_1 \in \mathfrak{Q} \cap \mathfrak{M}$. By invoking Proposition 6.2, one obtains $[A] \in \mathfrak{M}(P_1)/G$ and points $\{b_i = [h_i, A] \in \mathfrak{M}(P_1)\}_{i=0}^1$ and $\rho_0 \in (0, \frac{1}{16})$ such that for all $\rho \in (0, \rho_0)$ and for each $i \in \{0, 1\}$,

$$(6.7) \quad \mathfrak{Y}\mathfrak{M}((m_i - b_i)_\rho) < k + 1 = (k - 1) + 2.$$

Each $(m_i - b_i)_\rho$ is in $\hat{\mathfrak{B}}(P - P_1)$. Further progress requires

Proposition 6.4. *Let $P \rightarrow S^4$ be a principal $G = \text{SU}(2)$ or $\text{SU}(3)$ bundle of nonnegative degree, k . Every point $b \in \mathfrak{B}(P)$ such that $\mathfrak{Y}\mathfrak{M}(b) < k + 2$ is connected to $\mathfrak{M}(P)$ by a path γ in \mathfrak{B} for which $\mathfrak{Y}\mathfrak{M}(\gamma(\cdot))$ is monotone decreasing.*

Apply Proposition 6.4 to $(m_i - b_i)_\rho$ for $i = 0$, then 1. The proposition supplies two paths, $\gamma_i = \gamma_i[\rho](\cdot)$, $i \in \{0, 1\}$, with one endpoint $(m_i - b_i)_\rho$ and with the other endpoint in $\mathfrak{M}(P - P_1)$. As $\text{degree } P - P_1 = k - 1$, $\pi_0(\mathfrak{M}(P - P_1)) = (1)$, by assumption, so there is no loss of generality in assuming that $\gamma_0(1) = \gamma_1(1) \in \mathfrak{M}(P - P_1)$. By density, one can take $\gamma_i \subset \hat{\mathfrak{B}}$.

Define $\gamma = \gamma[\rho](\cdot) \in C^0([0, 1]; \hat{\mathfrak{B}}(P - P_1))$ by $\gamma = \gamma_1^{-1} \cdot \gamma_0$. Thus, $\gamma(i) = (m_i - b_i)_\rho$ for $i \in \{0, 1\}$, and due to Proposition 6.4, $\mathfrak{Y}\mathfrak{M}(\gamma(\cdot)) \leq \sup\{\mathfrak{Y}\mathfrak{M}((m_i - b_i)_\rho)\}_{i=0}^1 < k + 1$.

The next step is to use the subtraction procedure to translate γ back to $\hat{\mathfrak{B}}(P)$. This notion requires the definition of $\alpha b \in \hat{\mathfrak{B}}(\alpha^{-1}P)$, for each $b \in \hat{\mathfrak{B}}(P)$, where $\alpha : S^4 \rightarrow S^4$ is the inversion: For each $b = [h, A] \in \hat{\mathfrak{B}}(P)$, define $\alpha b \in \hat{\mathfrak{B}}(\alpha^{-1}P)$ by $\alpha b = [\alpha^{-1}l(h, A), \alpha^*A]$. Observe that $\alpha : \hat{\mathfrak{B}}(P) \rightarrow \hat{\mathfrak{B}}(\alpha^{-1}P)$ is continuous, $\alpha^2 = 1$, and α maps orbits of self-dual connections on P one-to-one onto orbits of anti-self-dual connections on $\alpha^{-1}P$.

Let $\phi \in C^0([0, 1]; \mathfrak{M}(P_1))$ be a given path. According to Proposition 6.1, there exists $r(\cdot) : (0, \frac{1}{16}) \rightarrow (0, 1)$ such that for all $\rho \in (0, \rho_0)$ and $r \in (0, r(\rho))$, the path $\psi[\rho, r](\cdot) = (\gamma[\rho](\cdot) - \alpha\phi(\cdot))_r \in C^0([0, 1]; \hat{\mathfrak{B}}(P))$ satisfies $\mathfrak{Y}\mathfrak{M}(\psi(\cdot)) < k + 2$. As G is path-connected, one can choose such a path ϕ to satisfy $\phi(i) = t_\rho b_i$ for $i \in \{0, 1\}$. Thus, one obtains

Lemma 6.5. *Let $P \rightarrow S^4$ be a principal $G = \text{SU}(2)$ or $\text{SU}(3)$ bundle of degree > 0 . Let $m_0, m_1 \in \mathfrak{M}(P) \cap \mathfrak{Q}(P)$. There exists $\rho_0 \in (0, \frac{1}{16})$ and $r(\cdot) : (0, \rho_0) \rightarrow (0, 1)$ and $b_0, b_1 \in \mathfrak{M}(P)$ with the following property: For all $\rho \in (0, \rho_0)$ and $r \in (0, r(\rho))$, there exists $\psi[\rho, r](\cdot) \in C^0([0, 1]; \hat{\mathfrak{B}}(P))$ satisfying for $i \in \{0, 1\}$, $\psi(i) = ((m_i - b_i)_\rho - \alpha t_\rho b_i)_r$, and satisfying for all $t \in [0, 1]$, $\mathfrak{Y}\mathfrak{M}(\psi(t)) < k + 2$.*

To finish the proof of Theorem 1.4, the endpoint $\psi(i)$ of ψ must be joined to its respective m_i by a path on which $\mathfrak{Y}\mathfrak{M} < k + 2$. One's ability to do this is asserted in the final lemma.

Lemma 6.6. *Let $P \rightarrow S^4$ be a principal $\text{SU}(2)$ bundle of degree > 0 . Let $m \in \mathfrak{Q}(P)$, and let $b \in \mathfrak{M}(P_1)$ and $\rho_0 \in (0, \frac{1}{16})$ be such that for all $\rho \in (0, \rho_0)$, $\mathfrak{Y}\mathfrak{M}((m - b)_\rho) < \mathfrak{Y}\mathfrak{M}(m) + 1$. Then there exists $r(\cdot) : (0, \rho_0) \rightarrow (0, 1)$ with the following property: For all $\rho \in (0, \rho_0)$ and $r \in (0, r(\rho))$ there exists $\eta[\rho, r](\cdot) \in C^0([0, 1]; \hat{\mathfrak{B}}(P))$ satisfying $\eta(0) = ((m - b)_\rho - \alpha t_\rho b)_r$, $\eta(1) = m$ and for all $t \in [0, 1]$, $\mathfrak{Y}\mathfrak{M}(\eta(t)) < \mathfrak{Y}\mathfrak{M}(m) + 2$.*

A path connecting m_0 to m_1 in $\mathfrak{Y}\mathfrak{M}^{-1}([k, k + 2]) \cap \hat{\mathfrak{B}}(P)$ is provided by Lemmas 6.5 and 6.6. The proof of Propositions 6.1, 6.2 and 6.4 and of Lemmas 6.3 and 6.6 are provided in §7.

7. Properties of subtraction

The proof of Proposition 6.4 requires analysis which is different from the proofs of the other assertions in the last section. The proof requires the mini-max theory for free homotopy classes of spheres in $\hat{\mathfrak{B}}$ which is summarized by Proposition 7.1, below. Proposition 6.4 follows as a corollary.

Let M be a compact, oriented Riemannian 4-manifold, and let $P \rightarrow M$ be a principal G -bundle with $k(P) \geq 0$.

For $l \geq 0$, let $\phi \in C^0(S^l; \mathfrak{B}(P))$ be a fixed sphere. To construct the Ljusternik-Šnirelman procedure for spheres homotopic to ϕ , define

$$\Lambda(\phi) \equiv \{ \lambda \in C^0([0, 1] \times S^l; \mathfrak{B}(P)) : \lambda(0, \cdot) = \phi \}.$$

For each $\lambda \in \Lambda$, define $t(\lambda) \in [0, 1]$ by

$$t(\lambda) \equiv \text{g.l.b.} \left\{ t \in (0, 1] : \sup_{y \in S^l} \lambda(t, y) > \sup_{y \in S^l} \lambda(0, y) \right\}$$

if the bound exists or $t(\lambda) = 1$ if it does not. To each $\lambda \in \Lambda$, associate the number

$$\mathfrak{A}_\lambda \equiv \inf_{t \in [0, t(\lambda)]} \left[\sup_{y \in S^l} \mathfrak{A}(\lambda(t, y)) \right],$$

and to ϕ , associate

$$\mathfrak{A}_\infty(\phi) \equiv \inf_{\lambda \in \Lambda(\phi)} \mathfrak{A}(\lambda).$$

A mini-max sequence for $\Lambda(\phi)$ is by definition a sequence $\{(\lambda_i, \bar{\lambda}_i)\}$ in the space $\bar{Y} = \{(\lambda, \bar{\lambda}) \in \Lambda \times \mathfrak{B} : \exists(\bar{t}, \bar{y}) \in [0, t(\lambda)] \times S^l \text{ with } \bar{\lambda} = \lambda(\bar{t}, \bar{y}) \text{ and } \mathfrak{A}(\bar{\lambda}) = \sup_{y \in S^l} \mathfrak{A}(\lambda(\bar{t}, y)) = \mathfrak{A}_\lambda\}$ for which

$$(7.1) \quad \mathfrak{A}(\bar{\lambda}_i) \searrow \mathfrak{A}_\infty(\phi).$$

By mimicking the proof of Theorem 1.5, one obtains

Proposition 7.1. *Let $P \rightarrow M$ be as described above, and for $l \geq 0$, let $\phi \in C^0(S^l; \mathfrak{B}(P))$.*

(1) *If $\mathfrak{A}_\infty(\phi) \notin \mathbf{Z}$, then over M or over S^4 with its standard metric there exists a principal G -bundle P' and a smooth critical point of $\mathfrak{Y}\mathfrak{M}$ on $\mathfrak{B}(P')$ which is not an absolute minimum of $\mathfrak{Y}\mathfrak{M}$ on $\mathfrak{B}(P')$.*

(2) *At this critical point, the hessian of $\mathfrak{Y}\mathfrak{M}$ has index l or less.*

(3) If M satisfies Proposition 3.1's conditions and if $\mathfrak{A}_\infty(\phi) = 0$, then $\lambda \in \Lambda(\phi)$ exists with $\lambda(1, \cdot) \in C^0(S^1; \mathfrak{M}(P))$ and for all $x \in [0, 1] \times S^1$, $\mathfrak{A}(\lambda(x)) \leq \sup_{y \in S^1} \mathfrak{A}(\phi(y))$.

Proof of Proposition 7.1. One need only establish the existence of good sequences in \bar{Y} . These are mini-max sequences $\{(\lambda_i, \bar{\lambda}_i)\} \in \Lambda(\phi)$ for which

$$\lim_{i \rightarrow \infty} \|\nabla \mathfrak{A}_i\|_* \searrow 0, \quad \lim_{i \rightarrow \infty} \lambda_i^{i+1} \geq 0,$$

in addition to (7.1). By altering a few definitions in §5, one obtains the existence of good sequences. The details are left to the reader.

Proposition 6.4 is part of the following corollary:

Corollary 7.2. *On S^4 with its standard metric, let P be a principal $SU(2)$ or $SU(3)$ bundle with degree $k \geq 0$. Then a point $b \in \mathfrak{B}$ for which $\mathfrak{Y}\mathfrak{M}(b) < k + 2$ is connected to $\mathfrak{M}(P)$ by a continuous path on which $\mathfrak{Y}\mathfrak{M}$ is nonincreasing. A loop $\phi \in \mathfrak{B}(P)$ on which $\mathfrak{Y}\mathfrak{M} < k + 2$ is free-homotopic to a loop in \mathfrak{M} by a homotopy on which $\mathfrak{Y}\mathfrak{M} < k + 2$.*

Proof of Corollary 7.2. According to Proposition 7.1, the corollary must hold unless there is a nonminimal critical point of $\mathfrak{Y}\mathfrak{M}$ at which the hessian has index < 2 . Theorem 1.2 rules out the latter case.

The remainder of this section is occupied with the proof of Propositions 6.1, 6.2 and Lemmas 6.3, 6.6.

Proof of Proposition 6.1. Consider the self-dual curvature of $\Phi = (A - A')_\rho$ over M , and more generally (with Lemma 6.6 in mind) the self-dual curvature of $\Phi = \Phi(\rho, r)$ which is defined for $\rho \in (0, \frac{1}{16})$ and $r \in (\rho^{1/2}, 1)$ by (6.1)–(6.4) with the replacement of η_ρ by $\eta_{\rho r}$ but all else unchanged.

On B , let $a = \phi^*(h, A)A$ and on $S^4 \setminus s$, let $a' = \alpha^* \iota_\rho^* \phi^*(h', A')A'$. Let $\eta_1 = \eta_{\rho^3 r}$ and $\eta_2 = \alpha^* \eta_{\rho^{-1} r}$. Then on $B \setminus s$,

$$(7.2) \quad \begin{aligned} P_+ F_\Phi &= \eta_1 P_+ F_{\theta+a} + P_+ d\eta_1 \wedge a + \frac{1}{2} \eta_1 (\eta_1 - 1) P_+ [a, a] + \eta_2 P_+ F_{\theta+a'} \\ &+ P_+ d\eta_2 \wedge a' + \frac{1}{2} \eta_2 (\eta_2 - 1) P_+ [a', a'] = \eta_1 \eta_2 [a, a']. \end{aligned}$$

Note that P_+ is the self-dual projection on $\wedge_2 T^*$ as defined by the given metric on M .

Let $z : B \rightarrow \mathbf{R}^4$ be the Gaussian coordinates as in §6 which map $s \in B$ to $\{0\} \in \mathbf{R}^4$ and as in §6, identify B with $\{z \in \mathbf{R}^4 : |z| < 1\}$ with the southern hemisphere of S^4 . Let $|\cdot|_e$ denote norms with respect to the Euclidean metric, $ds_{\text{Euclidean}}^2$ on B . Finally, let $\hat{P}_+ : \wedge_2 T^* \rightarrow \wedge_2 T^*$ denote the self-dual projection from $ds_{\text{Euclidean}}^2$.

For $b = [h, A] \in \mathfrak{B}(P)$, define $\zeta(b)$ to be the $\sup\{|F_A|_e(z) : z \in B\}$. For $b' = [h', A'] \in \mathfrak{B}(P')$, define $\zeta(b') = \sup\{(1 + |z|^2)^2 |F_{A'}|_e(z) : z \in \mathbf{R}^4\}$. Alternatively, $\zeta(b')$ is the L^∞ -norm of $F_{A'} \in \Gamma(\text{Ad } P' \otimes T^*S^4)$.

Lemma 7.3. *Let b, b' be as stated in Proposition 6.1. For $\rho \in (0, \frac{1}{16})$ and $r \in [\rho^{1/2}, 1]$, let P_+F_Φ be given by (7.2). Let $\zeta(b) + \zeta(b') = N$. Then*

$$\left| \|P_+F_\Phi\|_{2;M}^2 - \|P_+F_A\|_{2;M}^2 - \|P_-F_{A'}\|_{2;S^4}^2 \right| \leq \zeta \rho^4 \ln \rho N^2 (1 + N^2),$$

where $\zeta < \infty$ is independent of b, b' and ρ .

Proof of Lemma 7.3. To obtain the lemma, one must use the following facts:

- (1) $|d\eta_1| < c \cdot (\rho^3 r)^{-1/2}$ and both $d\eta_1$ and $\eta_1(\eta_1 - 1)$ have their support in $V = \{z \in \mathbf{R}^4 : (\rho^3 r)^{-1/2} |z| \in (\frac{1}{2}, 1)\}$. Here, c is a fixed constant.
- (2) Similarly, $|d\eta_2| \leq c \cdot (r/\rho)^{1/2}$ and both $d\eta_2$ and $\eta_2(\eta_2 - 1)$ have their support in $\bar{V} = \{z \in \mathbf{R}^4 : (\rho/r)^{-1/2} |z| \in (\frac{1}{2}, 1)\}$.
- (3) For $z \in B$,

$$(7.3) \quad |P_+F_{\theta+a}|_e(z) \leq \zeta(b).$$

Since a is the connection form for A on $P|_B$ in Uhlenbeck's polar gauge [33], it is given in B by

$$(7.4) \quad a(z) = \int_0^1 D\tau \tau |z| \frac{\partial}{\partial |z|} \lrcorner F_{\theta+a}(\tau z).$$

Hence, for $z \in B$,

$$(7.5) \quad |a(z)|_e \leq \frac{1}{2} |z| \zeta(b).$$

(4) A similar argument, but using the fact that $\Phi^*(h', A')A'$ is the polar gauge for A' on $S^4 \setminus n$, yields for $|z| \in (\frac{1}{2}(\rho^3 r)^{1/2}, (\rho r)^{1/2})$ the bounds

$$(7.6) \quad \begin{aligned} |\hat{P}_+F_{\theta+a'}|_e &\leq \rho^4 (\rho^4 + |z|^2)^{-2} \zeta(b'), \\ |a'|_e &\leq \frac{1}{2} \rho^4 |z|^{-1} (\rho^4 + |z|^2)^{-1} \zeta(b'), \\ \left| |\hat{P}_+F_{\theta+a'}|_e - |P_+F_{\theta+a'}| \right| &\leq c \cdot \rho^4 |z|^2 (\rho^4 + |z|^2)^{-2} \zeta(b), \end{aligned}$$

with $c < \infty$ a metric dependent constant.

(5) Finally, observe that the data $(S^+ \times G, \theta + a')$ and $(P'|_{S^+}, \alpha^* t_\rho^* A')$ are isomorphic as bundles with connection. Thus, $\|P_+F_{\theta+a'}\|_{2;S^4}^2 = \|P_-F_{A'}\|_{2;S^4}^2$. With (7.3)–(7.6) and the preceding discussion, one readily obtains Lemma 7.3.

The first assertion of Proposition 6.1 follows from Lemma 7.3 by setting $r = 1$ and using the facts: (1) $\mathfrak{Y}\mathfrak{M}(\Phi) = \|P_+F_A\|_{2;M}^2 - (k - k')$, (2) $\mathfrak{Y}\mathfrak{M}(A) = \|P_+F_A\|_{2;M}^2 - k$, and (3) $\mathfrak{Y}\mathfrak{M}(A') = \|P_-F_{A'}\|_{2;S^4}^2 + k'$.

Now consider the situation when $b' = [h', A'] \in \mathfrak{M}(P')$. In this case, as $|P_+ - \hat{P}_+|_e \leq c|z|^2$ in B ,

$$|P_+F_{\theta+a'}|_e(z) < c \cdot \rho^4 |z|^2 (\rho^4 + |z|^2)^2 \zeta(b'),$$

and (7.2), (7.3) and (7.5), (7.6) yield

$$(7.7) \quad \|P_+ F_\Phi\|_{2;M}^2 = \|P_+ F_A\|_{2;M}^2 + 2 \int d^4 z \eta_1(\hat{P}_+ F_{\theta+a}, d\eta_2 \wedge a')_e + O(r^{-1}\rho^5).$$

To proceed requires the leading order terms of the small $|z|$ expansion for $P_+ F_{\theta+a}$, and those of the $|z|^{-1}$ expansion for a' . To obtain the expansion, it is convenient to identify \mathbf{R}^4 , and hence $S^4 \setminus n$, with the quaternion algebra, \mathbf{H} . This is accomplished in practice by choosing an orthonormal basis, $\{\tau^i\}_{i=1}^3$ for $\text{Im } \mathbf{H}$ and sending $z = \{z^\nu\}_{\nu=1}^4$ to $z^0 + \sum_{i=1}^3 z^i \tau^i$. Once this is done, an isomorphism of $S^- \times \text{Im } \mathbf{H}$ with $P_+ \wedge_2 T^* S^-$ is obtained by sending $\{\tau^i\}$ to

$$\{\omega^i = (2\sqrt{2})^{-1}(\tau^i, d\bar{z} \wedge dz)\} \subset P_+ \wedge_2 T^*|_{S^-}.$$

On $S^+ \cap S^-$ let $y = z^{-1}$. The function y extends smoothly to S^+ and gives the stereographic coordinates $y: S^+ \rightarrow \mathbf{R}^4$. With y , one obtains an isomorphism between $S^+ \times \mathbf{H}$ and $P_+ \wedge_2 T^*|_{S^+}$; the one obtained by sending $\{\tau^i\}$ to $\{\bar{\omega}^i = (2\sqrt{2})^{-1}(\tau^i, d\bar{y} \wedge dy)\} \subset P_+ \wedge_2 T^*|_{S^+}$.

Lemma 7.4. *If $|z| < 1$, then*

$$P_+ F_{\theta+a} = (h^{-1} P_+ F_A(s) h, \omega^i) \omega^i + w_1(z),$$

where $|w_1| < \xi(A) \cdot |z|$.

Proof of Lemma 7.4. As $\theta + a$ is smooth on S^- , $P_+ F_{\theta+a} = \phi^*(h, A) P_+ F_A$ has a Taylor's expansion in z about $s = \{z = 0\}$. It is a straightforward exercise for the reader to establish the leading order term above.

As for a' , one obtains

Lemma 7.5. *Let $a' = \alpha^* t_{\rho^0}^* \phi^*(h', A') A'$ with $b' = [h', A'] \in \mathfrak{M}(P')$. If $|z| > (\rho^3 r)^{1/2}$, then*

$$\hat{P}_+(d\eta_2 \wedge a') = \frac{1}{4} |z|^{-3} \left(\frac{\partial}{\partial |z|} \eta_2 \right) \rho^4 (h'^{-1} P_+ F_A(s) h', \omega^i) \omega^i + w_2(z),$$

where $|w_2| < \xi(A') \rho^6 |z|^{-4} |\partial \eta_2 / \partial |z||_e$.

Proof of Lemma 7.5. Let $\bar{a} = \phi^*(h', A') A' \in \Gamma(S^- \times \mathfrak{g})$. The 1-form \bar{a} is given by (7.4), but with \bar{a} replacing a there. A 1-term Taylor's expansion yields

$$(7.8) \quad \left| \bar{a} - \frac{1}{2} |z| \frac{\partial}{\partial |z|} \lrcorner P_+ F_{\theta+\bar{a}}(s) \right|_e \leq \xi(A') |z|^2 |dz|_e.$$

Now observe that because

$$|z| \frac{\partial}{\partial |z|} \lrcorner \omega^i = \frac{1}{\sqrt{2}} (\tau^i, \bar{z} dz),$$

and $\alpha^*z = z^{-1}$, one has

$$(7.9) \quad a' = \frac{1}{2\sqrt{2}} |z|^{-4} \rho^4 (P_+ F_A(s), \omega^i)(\tau^i, \bar{z} dz) + O(\rho^6 |z|^{-5}).$$

Now, let $\zeta \in C^\infty(S^-)$ be a function of $|z|$ only. Then $d|z|$ is real, and

$$\begin{aligned} d\zeta \wedge (\tau^i, \bar{z} dz) &= \frac{1}{2} |z|^{-1} \frac{\partial \zeta}{\partial |z|} (\tau^i, d|z| \wedge \bar{z} dz - \bar{z} dz \wedge d|z|) \\ &= \frac{1}{4} |z| \frac{\partial \zeta}{\partial |z|} (\tau^i, d\bar{z} \wedge dz - \bar{z} dz \wedge d\bar{z} z). \end{aligned}$$

Hence,

$$(7.10) \quad \hat{P}_+(d\zeta \wedge (\tau^i \bar{z} dz)) = \frac{1}{\sqrt{2}} |z| \frac{\partial \zeta}{\partial |z|} \omega^i.$$

Lemma 7.5 follows from this last equality and (7.9).

Together, (7.7) and Lemmas 7.4, 7.5 yield the following refinement of Lemma 7.3.

Lemma 7.6. *Let b, b', ρ, r and $P_+ F_\Phi$ be given as described in Lemma 7.3. Then*

$$\|P_+ F_\Phi\|_2^2 = \|P_+ F_A\|_{2;M}^2 - \pi^2 \rho^4 (h'^{-1} P_+ F_{A'}(s) h', h^{-1} P_+ F_A(s) h) + w_3,$$

where $|w_3| < z(A, A') \rho^4 (\rho/r)^{1/2}$.

Proof of Lemma 7.6. Use Lemmas 7.4, 7.5 in (7.7) with the integration by parts to identify

$$\int_0^\infty d|z| \eta_1 \frac{\partial}{\partial |z|} \eta_2 = - \int_0^\infty d|y| \frac{\partial}{\partial |y|} (1 - \eta_2) = -1.$$

The second assertion of Proposition 6.1 follows from Lemma 7.6 by setting $r = 1$ there.

Proof of Proposition 6.2. The moduli space $\mathfrak{M}(P_1)$ is known explicitly, as every point in it is the orbit under $\text{Aut } P_1$ of a connection reducible to $\text{SU}(2)$ [4]. More explicitly, let $\hat{P} \rightarrow S^4$ be a principal $\text{SU}(2)$ bundle of degree 1. Let $\tau: \text{SU}(2) \rightarrow G$ be a group homomorphism which generates $\pi_3(G)$. Then P_1 is isomorphic to $\hat{P} \times_\tau G$, and $\mathfrak{M}(P_1) = \mathfrak{M}(\hat{P}) \times_\tau G/H$, where $H \subseteq G$ is the maximal subgroup of G which commutes with $\tau(\text{SU}(2))$. If G is simple, any two such homomorphisms are conjugate in G and so $\mathfrak{M}(P_1)$ does not depend on the choice of τ .

Choose any $b_1 = [h_1, A] \in \mathfrak{M}(P_1)$, and let $b = [h, A']$. For any $\rho \in (0, \frac{1}{16})$, write the leading order (in ρ) term from Proposition 6.1 of $\mathfrak{Y}\mathfrak{M}((b - b_1)_\rho) - \mathfrak{Y}\mathfrak{M}(b) - 1$ as $-\pi^2 \rho^4 J(b, b_1)$, where

$$(7.11) \quad J(b, b_1) = (h_1^{-1} P_+ F_A(s) h_1, h^{-1} P_+ F_{A'}(s) h).$$

Let $b_1 g = [h_1 g, A]$ and let $\mu(g)$ denote Harr measure on G .

As every $\sigma \in \mathfrak{g}$ is conjugate by $\text{Ad } G$ to $-\sigma$, if there exists $g \in G$ such that $J(b, b_1g) \neq 0$, then there exists $g \in G$ for which $J(b, b_1g) > 0$. For such g , the requirements of Proposition 6.2 are satisfied with b_1g and with all ρ small enough so that $-\pi^2\rho^4J(b, b_1g)$ gives the dominant contribution to $\mathfrak{Y}\mathfrak{M}((b - b_1g)_\rho) - \mathfrak{Y}\mathfrak{M}(b) - 1$. The function $J(b, b_1(\cdot)): G \in \mathbf{R}$ does not vanish identically for the following reason: For $[h_1, A] \in \mathfrak{M}(P_1)$, $P_+F_A(s) \neq 0$ as the explicit formulae attest [7]. In fact, if $\{\omega^i\}_{i=1}^3$ is an orthonormal basis for $P_+ \wedge_2 T^*|_s$, then $\{\tau^i = (\omega^i, h_1^{-1}P_+F_A(s)h_1)\} \subset \mathfrak{g}$ are orthogonal and they determine a principle embedding of $\mathfrak{su}(2)$ in \mathfrak{g} . Let $\{\eta^i = (\omega^i, h^{-1}P_+F_A(s)h)\} \subset \mathfrak{g}$ and with no loss of generality, suppose that $\eta^1 \neq 0$. As $\text{Span}\{g\tau^1g^{-1} : g \in G\}$ is an $\text{Ad } g$ invariant subspace in \mathfrak{g} and \mathfrak{g} is a simple Lie-algebra, one has $\mathfrak{g} = \text{Span}\{g\tau^1g^{-1} : g \in G\}$. Hence, there exists $g \in G$ such that $(g\tau^1g^{-1}, \eta^1) > 0$. Let $\{\sigma^i\}_{i=1}^3 = \{g\tau^i g^{-1}\}_{i=1}^3$, and let $h_\theta = \exp \theta\sigma^1 \in G$ for $\theta \in [0, \pi)$. When $\theta = \pi/2$,

$$\sum_{i=2}^3 (h_\theta\sigma^i h_\theta^{-1}, \eta^i) = - \sum_{i=2}^3 (\sigma^i, \eta^i)$$

so there exists $\theta \in [0, \pi)$ such that

$$\sum_{i=1}^3 (h_\theta g\tau^i g^{-1} h_\theta^{-1}, \eta^i) = (g\tau^1g^{-1}, \eta^1) > 0$$

as required.¹

Proof of Lemma 6.3. Let $[A] \in \mathfrak{M}(P)/G$. As degree $P > 0$, A is not flat and it is a fact that P_+F_A is nonvanishing on an open, dense set in S^4 [3]. The second fact is that the group of rotations of S^4 , $\text{SO}(5)$, acts on $\mathfrak{M}(P)/G$ via $R[A] = [R^*A]$, where $R: S^4 \rightarrow S^4$ is a rotation. These two facts imply that for any $[A] \in \mathfrak{M}(P)/G$, there exist $R \in \text{SO}(5)$, arbitrarily close to 1, such that $P_+F_{R^*A}(s) \neq 0$.

Proof of Lemma 6.6. It is convenient to first state the following associativity property of the subtraction procedure:

Lemma 7.7. *Let $\{P_i \rightarrow S^4\}_{i=1}^3$ be principal G -bundles, and let $\{b_i = \mathfrak{S}(P_i)\}_{i=1}^3$. Then for all $\rho \in (0, \frac{1}{16})$ and $r \in (0, \rho^2)$,*

$$((b_1 - b_2)_\rho - b_3)_{\rho r} = (b_1 - t_{r^2}(b_3 - \alpha t_{\rho^2} b_2)_{\rho r})_\rho.$$

Proof of Lemma 7.7. This is a tedious untangling of definitions which is left to the reader. But the following facts may help: (1) $\alpha t_\lambda = t_{\lambda^{-1}}\alpha$; (2) $t_\lambda^* \eta_\rho = \eta_{\rho\lambda^{-2}}$; hence (3) if $r < \frac{1}{2}\rho$ then $\eta_{\rho^3}\eta_{\rho^3 r^3} = \eta_{\rho^3}$ while $(\alpha\eta_{\rho^{-1}})(\alpha\eta_{\rho^{-1}r^{-1}}) = \alpha\eta_{\rho^{-1}r^{-1}}$.

An immediate result of Lemma 7.7 is the corollary that

$$(7.12) \quad ((m - b)_\rho - \alpha b)_{\rho r} = (m - t_{r^2}(\alpha b - \alpha t_{\rho^2} b)_{\rho r})_\rho.$$

¹ This version of the proof of Proposition 6.2 is due to T. Parker.

Lemma 7.8. *Let m, b be as in Lemma 6.6, and let $b = [h, A]$. Then $P_+F_A(n) \neq 0$, and there exists $\rho_0 \in (0, \frac{1}{16})$ and for each $\rho \in (0, \rho_0)$ there exists $r(\rho) \in (0, \rho^2]$ with the following properties: For each $\rho \in (0, \rho_0)$ and $r \in (0, r(\rho))$, and for all $\lambda \in (0, \rho]$,*

- (1) $\mathfrak{Y}\mathfrak{M}(t_{r^2}(ab - \alpha t_{\rho^2}b)_{\rho r}) < 2 - \frac{1}{2}\pi^2 r^4 |P_+F_A(n)|^2,$
- (2) $\mathfrak{Y}\mathfrak{M}((m - t_{r^2}(ab - \alpha t_{\rho^2}b)_{\rho r})_{\lambda}) < \mathfrak{Y}\mathfrak{M}(m) + 2 - \frac{1}{2}\pi^2 r^4 |P_+F_A(n)|^2.$

Proof of Lemma 6.6, assuming Lemma 7.8. Due to assertion (1) of Lemma 7.8, and Proposition 6.4, for each $\rho \in (0, \rho_0)$ and $r \in (0, r(\rho))$, there exists $\phi[\rho, r](\cdot) \in C^0([0, 1]; \hat{\mathfrak{B}}(S^4 \times G))$ satisfying $\phi(0) = t_{r^2}(ab - \alpha t_{\rho^2}b)_{\rho r}$, $\phi(1) = * \mathfrak{M}(S^4 \times G)$, and for all $t \in [0, 1]$

$$\mathfrak{Y}\mathfrak{M}(\phi(t)) < 2 - \frac{1}{2}\pi^2 r^4 |P_+F_A(n)|^2.$$

Due to Lemma 7.3 and the fact that $\phi([0, 1])$ is a compact set in $\hat{\mathfrak{B}}(S^4 \times G)$, there exists $\zeta < \infty$ which is independent of $\lambda \in (0, \rho]$ with the following property: For all $t \in [0, 1]$,

$$(7.13) \quad \mathfrak{Y}\mathfrak{M}((m - \phi(t))_{\lambda}) < \mathfrak{Y}\mathfrak{M}(m) + 2 + \zeta \lambda^4 \ln|\lambda| - \frac{1}{2}\pi^2 r^4 |P_+F(n)|^2.$$

For each $\lambda \in (0, \rho]$, define a path $\eta[\rho, r, \lambda](\cdot) \in C^0([0, \frac{2}{3}]; \hat{\mathfrak{B}}(P))$ as follows: Let $\lambda(t) = (1 - 3t)\rho + 3t\lambda$. When $t \in [0, \frac{1}{3}]$, set

$$(7.14) \quad \eta(t) = (m - t_{r^2}(ab - \alpha t_{\rho^2}b))_{\lambda(t)}.$$

When $t \in [\frac{1}{3}, \frac{2}{3}]$, set

$$(7.15) \quad \eta(t) = (m - \phi[\rho, r](3t - 1))_{\lambda}.$$

Observe that due to assertion (2) of Lemma 7.8 and (7.13), $\mathfrak{Y}\mathfrak{M}(\eta(t)) < \mathfrak{Y}\mathfrak{M}(m) + 2$ for all $t \in [0, \frac{2}{3}]$. Notice that $\eta(\frac{2}{3}) = (m - *)_{\lambda}$. However, as $\lambda \rightarrow 0$, $(m - *)_{\lambda}$ converges strongly in L^2_2 on S^4 to $m \in \hat{\mathfrak{B}}(P)$. Hence, for $\lambda > 0$, but small, there exists a continuous extension of η to $C^0([0, 1]; \hat{\mathfrak{B}}(P))$ which satisfies all of the requirements of Lemma 6.6.

Proof of Lemma 7.8. As previously remarked, there is an explicit formula for $b = [h, A] \in \mathfrak{M}(P_1)$ which reveals that $P_+F(n) \neq 0$ [7].

Let $\Phi(r) = (ab - \alpha b)_r$ and let $\Phi(r, \rho) = (ab - \alpha t_{\rho^2}b)_{\rho r}$. If $\rho \in (0, \frac{1}{16})$ and $r \in (0, \frac{1}{16}\rho^2)$, then also $r \in (0, \frac{1}{16})$ and $\rho \in (r^{1/2}, 1)$. In this case, $\mathfrak{Y}\mathfrak{M}(\Phi(r, \rho))$ is estimated by Lemma 7.6 (with the orientation reversed). Assertion (1) of Lemma 7.8 follows from Lemma 7.6.

To obtain assertion (2) of Lemma 7.8, write $c[r, \rho] = t_{r^2}\Phi(r, \rho) = [h_c, A_c]$. The claim is that if $\lambda \in (0, \rho)$ and $z \in U = \{z \in \mathbf{R}^4 : |z| > \frac{1}{2}\lambda^{3/2}\}$, then

$$(7.16) \quad \alpha^* t_{\lambda^2} \phi^*(h, A) A = \alpha^* t_{\lambda^2} \phi^*(h_c, A_c) A_c.$$

Indeed, by construction, the data $(S^4 \times G, (h_c, A_c))$ and $(P_1, (h, A))$ are isomorphic on $\{z \in \mathbf{R}^4 : t_r^* z < \frac{1}{2}(\rho^3 r^3)^{1/2}\} = \{z \in \mathbf{R}^4 : z < \frac{1}{2}(\rho^3/r)^{1/2}\}$: see (6.3). Further, as $r < \frac{1}{16}\rho^2$, this set contains $V = \{z \in \mathbf{R}^4 : |z| < 2\sqrt{\rho}\}$. Thus, $\alpha^* t_{\lambda^2}^* \phi^*(h, A)A = \alpha^* t_{\lambda^2}^* \phi^*(h_c, A_c)A_c$ on $\alpha^{-1} t_{\lambda^2}^{-1} V = \{z \in \mathbf{R}^4 : |z| > \frac{1}{2}\lambda^2/\sqrt{\rho}\}$, and this contains U .

Equation (7.16) implies assertion (2) as follows: Write $m = [h_m, A_m]$. Let $\lambda \in (0, \rho]$. Then on U ,

$$(7.17) \quad \theta + \eta_{\lambda^3} \phi^*(h_m, A_m)A_m + \alpha^*(\eta_{\lambda^{-1}} t_{\lambda^2}^* \phi^*(h, A)A) \\ = \theta + \eta_{\lambda^3} \phi^*(h_m, A_m)A_m + \alpha^*(\eta_{\lambda^{-1}} t_{\lambda^2}^* \phi^*(h_c, A_c)A_c),$$

and hence the two numbers

$$I_1 = \mathfrak{Y}\mathfrak{M}((m - b)_\lambda) - \mathfrak{Y}\mathfrak{M}(m) - 1 \quad \text{and} \\ I_2 = \mathfrak{Y}\mathfrak{M}((m - c[r, \rho])_\lambda) - \mathfrak{Y}\mathfrak{M}(m) - \mathfrak{Y}\mathfrak{M}(c[r, \rho])$$

are equal. This is because I_1 involves the expression on the left-hand side of (7.17), integrated over U , and in a similar way, I_2 involves the right-hand side of (7.17). By assumption, $I_1 < 0$ for all $\lambda \in (0, \rho_0]$ and so the same is true for I_2 for all $\lambda \in (0, \rho_0]$. With (7.15), this establishes assertion (2) of Lemma 7.8, given (7.16).

8. Extensions

Let M be a compact, oriented Riemannian 4-manifold which is 1-connected and which has nonnegative intersection pairing on $H_2(M; \mathbf{Z})$, and let $P \rightarrow M$ be a principal $SU(2)$ bundle of degree 1. Or, let S^4 have its standard metric, and let $P \rightarrow S^4$ be a principal G -bundle of positive degree, where $\text{rank } G > 2$. In both cases, the statement “ $\pi_0(\mathfrak{M}(P)) = (1)$ ” cannot be proved by the methods in §§2–7. This is because the analog of Theorem 1.2 is not available.

For the first case above, one obtains using §§2–7 the result stated in Theorem 1.6. For the second case, one can return to [30] to analyze why Theorem 1.2 fails. The result is Theorem 8.1 below. This section contains the proofs of Theorem 8.1 and 1.6.

Theorem 8.1. *Let S^4 have its standard metric, and let $P \rightarrow S^4$ be a principal G -bundle of positive degree, where $\text{rank } G > 2$. If $\pi_0(\mathfrak{M}(P)) \neq (1)$, there exists a connection A on a principal G -bundle $P' \rightarrow S^4$ which satisfies the following conditions:*

- (1) A solves the Yang-Mills equations on S^4 .
- (2) $0 < \mathfrak{Y}\mathfrak{M}(A) < 2$.
- (3) $\mathfrak{S}_A(\cdot, \cdot)$ has ≤ 1 negative direction.

(4) Let $x \in S^4$, and let $\{P_{\pm}F^i(x)\}_{i=1}^3$ be the components of $P_{\pm}F_A(x)$ with respect to an orthonormal frame for $P_{\pm} \wedge_2 T^*|_x$. At each $x \in S^4$, and for all i, j , $[P_+F_A^i(x), P_-F_A^j(x)] = 0$. But, at almost every $x \in S^4$ and for all $i \neq j$, $[P_+F_A^i, P_+F_A^j(x)] \neq 0$ and $[P_-F_A^i(x), P_-F_A^j(x)] \neq 0$.

It is remarked that the solution, A , above would not be a direct sum of self-dual and anti-self-dual connections, because that situation would imply that $\mathfrak{M}(A) \geq 2$.

Proof of Theorem 8.1. Let $P \rightarrow S^4$ be a principal G -bundle of smallest, positive degree such that $\pi_0(\mathfrak{M}(P)) \neq (1)$. Assume this degree is finite. Let $m_0, m_1 \in \mathfrak{M}(P)$ be in distinct path components. Let $\Theta = \{\phi \in C^0([0, 1]; \mathfrak{B}(P)) : \phi(0) = m_0 \text{ and } \phi(1) = m_1\}$. If $\mathfrak{A}(\Theta) < 2$, then assertions (1)–(3) follow from Proposition 4.4. If $\mathfrak{A}(\Theta) \geq 2$, then the proof of Theorem 1.4 has broken down. For this to happen, there must exist a point $b \in \mathfrak{B}(P - P_1)$ with $\mathfrak{A}(b) < 2$ which is not connected to $\mathfrak{M}(P - P_1)$ by a path $\phi \in C^0([0, 1]; \mathfrak{B}(P - P_1))$ with $\mathfrak{A}(\phi(\cdot)) < 2$. In this situation, Proposition 7.1 implies assertions (1)–(3).

At a connection A satisfying (1)–(3) above, neither P_-F_A nor P_+F_A can vanish identically. Let $y = \{y^{\nu}\} : S^4 \setminus s \rightarrow \mathbf{R}^4$ be stereographic coordinates. As in §7, identify $S^4 \setminus s$ with \mathbf{R}^4 using y . The sections $|y|\partial/\partial|y| \lrcorner P_+F_A$ and $(\partial/\partial y^{\nu} \lrcorner P_+F_A)_{\nu=1}^4$ of $T^*\mathbf{R}^4 \times \mathfrak{g}$ are (1) square integrable and (2) in the kernel of $\delta_A = (P_-D_A, \nabla_A^*) : L^2(T^*\mathbf{R}^4 \times \mathfrak{g}) \rightarrow L^2(P_- \wedge_2 T^* \oplus \mathbf{R} \times \mathfrak{g})$ [30]. The analysis of [30] implies that assertion (3) is true only if each of these sections is also in $\ker \mathfrak{S}_A$. Assertion (4) follows from the three conditions $P_+F_A \neq 0, P_-F_A \neq 0$ and $\{\partial/\partial y^{\nu} \lrcorner P_+F_A\} \subset \ker \delta_A \cap \ker \mathfrak{S}_A \cap L^2$; see §3 of [30].

Proof of Theorem 1.6. It is useful to list some a priori facts concerning M and principal $SU(2)$ bundles $P \rightarrow M$ of degree 1.

- (1) Because M is 1-connected, $\mathfrak{M}(M \times SU(2)) = \text{point}$ [11], [14].
- (2) For a dense set, \mathfrak{F} , of smooth metrics on T^*M , $\mathfrak{M}(P)$ is a smooth, 5-dimensional manifold [14, §3].

(3) If the metric on T^* is in \mathfrak{F} , then for all $[A] \in \mathfrak{M}(P)/G$,

(8.1)

$$\text{inf spectrum}(\mathfrak{D}_A \mathfrak{D}_A^* : L^2(\text{Ad } P \otimes P_- \wedge_2 T^*) \rightarrow L^2(\text{Ad } P \otimes P_- \wedge_2 T^*)) > 0.$$

For (3), see also [14, §3].

Lemma 8.2. *Let M be as in Theorem 1.6 and let $P \rightarrow M$ be a principal $SU(2)$ bundle of degree +1. Suppose that the metric on T^* is in \mathfrak{F} . then there exists $\epsilon > 0$ and a strong deformation retract of $\mathfrak{B}_{\epsilon}(P)$ onto $\mathfrak{M}(P)$, and of $\mathfrak{B}_{\epsilon}(M \times G)$ onto $\mathfrak{M}(M \times G)$.*

Proof of Theorem 1.6, assuming Lemma 8.2. Suppose that $\mathfrak{M}(P)$ is the disjoint union of nonempty sets $\mathfrak{M}_0, \mathfrak{M}_1$. Let Θ be the space of continuous

maps of $[0, 1]$ into $\mathfrak{B}(P)$ sending $\{0\} \rightarrow \mathfrak{M}_0$ and $\{1\} \rightarrow \mathfrak{M}_1$. If $\mathfrak{A}(\Theta) < 2$, then Proposition 4.4, Lemma 8.2 and Theorem 1.2 imply that there exists a critical point, b , of $\mathfrak{Y}\mathfrak{M}$ in either $\mathfrak{B}(P)$ or $\mathfrak{B}(M \times G)$. The critical point b cannot be self-dual, but the hessian there has at most 1 negative direction.

If $\mathfrak{A}(\Theta) \geq 2$, then one does not have Theorem 1.4 for $\mathfrak{M}(P)$, and if this is the case, it is because one does not have Proposition 6.4 for $\mathfrak{B}(M \times G)$. Indeed, given Proposition 6.4, the construction of a path $\gamma \in \Theta$ with $\mathfrak{A}(\gamma(\cdot)) < 2$ proceeds as in the case $M = S^4$ which is detailed in §§6 and 7. If Proposition 6.4 fails for $\mathfrak{B}(M \times G)$, then Proposition 7.1 and Lemma 8.1 imply that there exists a critical point b of $\mathfrak{Y}\mathfrak{M}$ in $\mathfrak{B}(M \times G)$ which is not flat, but it is a local minimum of $\mathfrak{Y}\mathfrak{M}$.

Proof of Lemma 8.2. Due to Proposition 3.1, it is sufficient to establish the existence of $\mu, \delta > 0$ such that for all $[A] \in \mathfrak{B}_\delta(P)$ or $\mathfrak{B}_\delta(M \times G)$, $\inf \text{spectrum}(\mathfrak{D}_A \mathfrak{D}_A^*) > \mu > 0$. Consider the case for $P \rightarrow M$ and assume the contrary. Then a sequence $\{[A_i]\} \subset \mathfrak{B}(P)/G$ exists with (1) $\mathfrak{A}([A_i]) \searrow 0$ and (2) $\mu_i \equiv \inf \text{spectrum}(\mathfrak{D}_{A_i} \mathfrak{D}_{A_i}^*) \searrow 0$. As \mathfrak{A} has uniform second derivatives (Proposition 2.2) then as in §5 one can readily show that $\|\nabla \mathfrak{A}_i\|_* \searrow 0$ also. Hence, $\{[A_i]\}$ is a good sequence. Let $\{A_\alpha\}_{\alpha=0}^n$ be the limiting connections as given by Proposition 4.4. Either $n = 0$ and $[A_0] \in \mathfrak{M}(P)/G$ or $n = 1$ and A_0 is flat, while A_1 is self-dual on a degree 1, principal $SU(2)$ bundle over S^4 . Consider the sequence $\{\mu_i\}$. In the case $n = 0$ above, $\{[A_i]\}$ converges strongly in L^2_1 on M and so $\{\mu_i\}$ converges to $\mu(A_0)$ which is positive by assumption.

In the case $n = 1$, above, the eigenvalue estimates in [29] and [31] can be applied here to show that $\lim \mu_i > 0$ also. But the limit of $\{\mu_i\}$ was 0 by assumption. This contradiction establishes the lemma for P . The proof for $M \times G$ is similar and left as an exercise.

Appendix: Uhlenbeck’s compactness theorem

The purpose here is to prove Proposition 4.5. First, consider a sequence $\{[A_i]\} \in \mathfrak{B}(P)/G$ for which $\mathfrak{Y}\mathfrak{M}(A_i) < K$ for all i . According to Sedlacek [24], there exists a constant $\kappa > 0$ which is a function only of the metric on M ; a finite set $\Omega \in M$; a cover of $M \setminus \Omega$ by open balls, $\{U_\alpha\}$; and a subsequence of $\{[A_i]\}$, now renamed $\{[A_i]\}$ such that on each U_α ,

$$(A.1) \quad \int_{U_\alpha} |F_{A_i}|^2 < \kappa^2.$$

Using Theorem 2.1 of [32], Sedlacek deduces the existence, for each α , of sequences $\{g_{\alpha i}\} \in L^2_3(\text{iso}(M \times G, P)|_{U_\alpha})$ such that if one defines θ to be the

product connection on $M \times G$, and if, for each α, i , one defines $a_{\alpha i} = g_{\alpha i}^* A_i - \theta$, then

$$(A.2) \quad \begin{aligned} (1) \quad & \int_{U_\alpha} \{ |\nabla_\theta a_{\alpha i}|^2 + |a_{\alpha i}|^2 \} \leq \zeta \int_{U_\alpha} |F_{\alpha i}|^2, \\ (2) \quad & D_\theta * a_{\alpha i} = 0 \quad \text{in } U_\alpha, \\ (3) \quad & i^*(* a_{\alpha i}) = 0 \quad \text{in } U_\alpha \text{ where } i: \partial U_\alpha \rightarrow U_\alpha \text{ is the inclusion.} \end{aligned}$$

Here, $\zeta < \infty$ depends only on the metric on M .

In each nonempty $U_\alpha \cap U_\beta$, define sequences $\{g_{\alpha\beta i} = g_{\alpha i}^{-1} g_{\beta i}\} \in L^2_2(U_\alpha \cap U_\beta; G)$. In each nonempty $U_\alpha \cap U_\beta$,

$$(A.3) \quad a_{\alpha i} = g_{\alpha\beta i} a_{\beta i} g_{\alpha\beta i}^{-1} + g_{\alpha\beta i} d g_{\alpha\beta i}^{-1}.$$

By a diagonalization argument, Sedlacek obtains with (A.2), (A.3) a subsequence of $\{[A_i]\}$, now renamed $\{[A_i]\}$ such that in each U_α , $\{a_{\alpha i}\}$ converges weakly in $L^2_1(\mathfrak{g} \otimes T^*|_{U_\alpha})$ to a_α , and in each nonempty $U_\alpha \cap U_\beta$, $\{g_{\alpha\beta i}\}$ converges weakly in $L^2_2(U_\alpha \cap U_\beta; G)$ to $g_{\alpha\beta}$. Sedlacek deduced that in each U_α , a_α satisfies (A2); while in each nonempty $U_\alpha \cap U_\beta$,

$$(A.4) \quad a_\alpha = g_{\alpha\beta} a_\beta g_{\alpha\beta}^{-1} + g_{\alpha\beta} d g_{\alpha\beta}^{-1};$$

and in each nonempty $U_\alpha \cap U_\beta \cap U_\gamma$,

$$(A.5) \quad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1_G \quad \text{a.e.}$$

For fixed $\psi \in L^2_1(\mathfrak{g} \otimes T^*|_{U_\alpha})$ with compact support, the map $a \in L^2_1(\mathfrak{g} \otimes T^*|_{U_\alpha}) \mapsto \nabla \mathfrak{A}_{\theta+a}(\psi) \in \mathbf{R}$ is weakly continuous. Hence, if one now assumes additionally that $\{\|\nabla \mathfrak{A}_i\|_*\}$ has limit zero, then $\theta + a_\alpha$ is a weak solution to the Yang-Mills equations in U_α .

Equation (A.2.2) for a_α and the equation $\nabla \mathfrak{A}_{\theta+a_\alpha}(\cdot) = 0$ form a uniformly elliptic system for a_α on U_α , so the standard arguments imply that each a_α is C^∞ [32]. Equation (A.5) implies that where $U_\alpha \cap U_\beta \neq \emptyset$, the map $g_{\alpha\beta}$ is also C^∞ . Hence, the data $\{U_\alpha, g_{\alpha\beta}, a_\alpha\}$ define a C^∞ principal G -bundle, $P' \rightarrow M \setminus \Omega$, and a smooth connection A on P' which satisfies the Yang-Mills equations. K. Uhlenbeck's removable singularity theorem states that (P', A) extends as a C^∞ bundle with connection over M , where A satisfies the Yang-Mills equations [33].

Although not discussed by Sedlacek, it is nonetheless a fairly standard boot-strap argument to deduce from the condition $\|\nabla \mathfrak{A}_i\|_* \searrow 0$ and from (A.2) that $\{a_{\alpha i}\}$ converges strongly to a in $L^2_{1,loc}(\mathfrak{g} \otimes T^*|_{U_\alpha})$. Via (A.3), this implies that where $U_\alpha \cap U_\beta \neq \emptyset$, the sequence $\{g_{\alpha\beta i}\}$ converges strongly to $g_{\alpha\beta}$ in $L^2_{2,loc}(U_\alpha \cap U_\beta; G)$.

Thus, Sedlacek’s argument yields here a form of local convergence of the sequence $\{[A_i]\} \in \mathfrak{B}(P)/G$ to $[A] \in \mathfrak{B}(P')/G$. But this convergence is somewhat weaker than the L^2_1 -strong convergence as given in Definition 4.3 because it is not obtained by pulling A_i back from $P|_{M \setminus \Omega}$ to $P'|_{M \setminus \Omega}$ by $g_i \in L^3_2(\text{iso}(P', P)|_{M \setminus \Omega})$. To complete the proof of Proposition 4.5, it is necessary to construct these g_i . These problem amounts to finding a sequence of splittings, $\{\eta_{\alpha i}\} \subset L^2_2(U_\alpha; G)$ such that (1) where $U_\alpha \cap U_\beta \neq \emptyset$,

$$(A.6) \quad \eta_{\alpha i} g_{\alpha \beta i} \eta_{\beta i}^{-1} = g_{\alpha \beta};$$

and (2) $\{\eta_{\alpha i}\}$ converges strongly to 1_G in $L^2_{2;\text{loc}}(U_\alpha; G)$.

If the convergence of $\{g_{\alpha \beta i}\}$ to $g_{\alpha \beta}$ were in

$$C^0_{\text{loc}}(U_\alpha \cap U_\beta; G) \cap L^2_{2;\text{loc}}(U_\alpha \cap U_\beta; G),$$

then the existence of this splitting is established in §3 of [32], cf. Proposition 3.2 there. The fact that in $U_\alpha \cap U_\beta$, $\{g_{\alpha \beta i}\}$ converges strongly to $g_{\alpha \beta}$ would make irrelevant the restriction in said proposition that the cover $\{U_\alpha\}$ is finite. Indeed, if $\Omega \neq \emptyset$, then for each i , one can restrict attention to a compact subset $M_i \subset M \setminus \Omega$ on which Proposition 3.2 of [32] is applicable. Due to the strong convergence of $\{g_{\alpha \beta i}\}$ to $g_{\alpha \beta}$ in $C^0 \cap L^2_2$, Proposition 3.2 of [32] allows for $M_{i-1} \subseteq M_i$ and $U_i M_i = M \setminus \Omega$. One can assume with no loss of generality that for all i sufficiently large, $(M \setminus \Omega) \setminus M_i = \cup_{\Omega} S^3 \times (0, 1)$. Thus, for all i sufficiently large, there is no obstruction to extending the isomorphism over M_i given by Proposition 3.2 of [32] to an isomorphism over $M \setminus \Omega$ which agrees with the constructed one over M_i .

A propos the discussion above, Proposition 4.4 follows from an argument which establishes the C^0_{loc} convergence in $U_\alpha \cap U_\beta$ of $\{g_{\alpha \beta i}\}$ to $g_{\alpha \beta}$. In particular,

Lemma A.1. *Let U be an open ball in a Riemannian 4-manifold. Let $\mathfrak{F} = \{(g, a, b) \in \times L^2_2(U; G) \times_2 L^2_1(\mathfrak{g} \otimes T^*); a = gb^{-1} + gdg^{-1}$ and both a and b satisfy (A.2.2) in $U\}$. The projection $\mathfrak{F} \rightarrow L^2_2(U; G)$ sending (g, a, b) to g factors continuously through $C^0_{\text{loc}}(U)$.*

Proof of Lemma A.1. Consider the case where U is the unit ball in \mathbf{R}^4 ; a nonflat metric adds no essential complication and this generalization is left to the reader. It is convenient to represent G as a subgroup of $SU(n)$ for some $n < \infty$. Then G and \mathfrak{g} are realized as submanifolds of \mathbf{M}_n , the space of $n \times n$ complex matrices. In this regard, $\zeta \eta$ for $\zeta, \eta \in G$ or \mathfrak{g} is to be interpreted as matrix multiplication in \mathbf{M}_n .

Observe that because of (A.2.2), g satisfies

$$(A.7) \quad d^* dg = gb \cdot b - 2a \cdot gb + a \cdot ag,$$

where $\cdot : \oplus_2 \mathbf{M}_n \times T^*U \rightarrow \mathbf{M}_n$ is matrix multiplication and contraction by the metric on T^*U .

For $\epsilon > 0$, let $U_\epsilon \hookrightarrow U$ be the open ball of radius $1 - \epsilon$, and let $j_\epsilon : L^1(U_\epsilon) \hookrightarrow C^\infty(U)$ be a standard mollifier. For $x \in U$, let β^x be a cut-off function which at $y \in U$ is 1 if $|x - y| < \frac{1}{4} \text{dist}(x, \partial U)$ and 0 if $|x - y| > \frac{1}{2} \text{dist}(x, \partial U)$. Suppose also that $|d\beta^x(y)| \leq 8 \text{dist}(x, \partial U)^{-1}$.

To prove that g is continuous, it is helpful to mollify both sides of (A.7) with j_ϵ . This obtains

$$(A.8) \quad d^*d(j_\epsilon * g)(y) = (j_\epsilon * L)(y) \quad \text{at } y \in U_\epsilon,$$

where $L \equiv$ the right-hand side of (A.7). Let $x \in U_{2\epsilon}$, and multiply both sides of (A.8) by $|x - y|^{-2}\beta^x(y)$ and then integrate the resulting equation over U . The result after an integration by parts is

$$(A.9) \quad \begin{aligned} (j_\epsilon * g)(x) &= 2\pi^2 \int_U d^4y (j_\epsilon * L)(y) |x - y|^{-2} \beta^x(y) \\ &+ 2\pi^2 \int_U d^4y (j_\epsilon * g)(y) \left(2d\beta^x \cdot \frac{d|x - y|}{|x - y|^3} - \frac{d^*d\beta^x}{|x - y|^2} \right). \end{aligned}$$

Because $j_\epsilon * g$ converges strongly to g in $L^p_{\text{loc}}(U)$ for any $1 \leq p \leq \infty$, the continuity of g follows from a proof that the $\epsilon = 0$ of the right-hand side of (A.9) defines a continuous function from U to \mathbf{M}_n . Let $g_1^\epsilon(y), g_2^\epsilon(y)$ denote the first and second terms, there. Because $\text{dist}(x, \text{supp } d\beta^x) \geq \frac{1}{4} \text{dist}(x, \partial U)$, the $\epsilon = 0$ limit of g_2^ϵ exists. Indeed, the map $x \mapsto 2|x - y|^{-3}d\beta^x(y) \cdot d|x - y| - |x - y|^{-2}d^*d\beta^x(y)$ is a smooth map from U into $C^\infty(U)$. The implications are summarized with the following lemma.

Lemma A.2. *The map $h_2 : U \times L^1(U) \rightarrow \mathbf{R}$ defined to be*

$$h_2(x, v) = 2\pi^2 \int_U d^4yv(y) \left(2 \frac{d\beta^x \cdot d|x - y|}{|x - y|^3} - \frac{d^*d\beta^x}{|x - y|^2} \right)$$

is jointly continuous.

The analysis for $g_1^\epsilon(x)$ is more complicated. The first observation to make is that in dimension 4, the map $l : U \times L^2_1(U) \rightarrow L^2(U)$ which sends $(x, f(y))$ to $|x - y|^{-1}f(y)$ is, for fixed $x \in U$, continuous (cf. (2.15), and §6 of [30]). Therefore, the $\epsilon = 0$ limit of g_1^ϵ exists as a map $g_1 : U \rightarrow \mathbf{M}_n$. In fact, this observation establishes that the map $h_1 : U \times (L^1 \cap L^\infty) \times_2 L^2_1(U) \rightarrow \mathbf{R}$, defined as

$$h_1(x, u, v, w) = \int_U d^4y \frac{uwx}{|x - y|^2},$$

is well defined. Its properties are summarized by

Lemma A.3. *The map h_1 , above, is continuous on its domain, $K \equiv U \times (L^1 \cap L^\infty) \times_2 L^2_1(U)$.*

Proof of Lemma A.3. Let $\hat{K} = (L^1 \cap L^\infty) \times_2 L^2_1(U)$. For fixed $x \in U$, let $\{k_i = (u_i, v_i, w_i)\} \subset \hat{K}$ converge strongly to $k \in \hat{K}$. Let $m = \limsup \|u_i\|_\infty$. Observe that

$$|h_1(x, k) - h_1(x, k_i)| \leq \int_U d^4y \left[\frac{|u - u_i|vw}{|x - y|^2} + \frac{u_i|vw - v_iw_i|}{|x - y|^2} \right].$$

Let $\{q_\alpha\} \in C^\infty(U)$ converge strongly to $|x - y|^2vw$ in $L^1(U)$. For any α ,

$$|h_1(x, k) - h_1(x, k_i)| \leq c(m) \left[\|q_\alpha\|_\infty \|u - u_i\|_1 + \left\| q_\alpha - \frac{vw}{|x - y|^2} \right\|_1 + \|v - v_i\|_{1,2} \|w - w_i\|_{1,2} \right].$$

Here, $c(m)$ is a constant which depends on m and $\text{dist}(x, \partial U)$, and all integrations are integrations over U . Given $\epsilon > 0$, choose α so that $\|q_\alpha - |x - y|^{-2}vw\|_1 < c^{-1}\epsilon/3$, and choose i so that $\|u - u_i\|_1 < c^{-1}\|q_\alpha\|^{-1}\epsilon/3$ and $\|v - v_i\|_{1,2} \|w - w_i\|_{1,2} < c^{-1}\epsilon/3$. Then for all $j > i$, $|h_1(x, k) - h_1(x, k_j)| < \epsilon$.

For fixed $k \in K$, consider the continuity of $h(\cdot, k) : U \rightarrow \mathbf{R}$: Let $x \in U$ and $\eta \in U_{1-|x|}$. Then

$$(A.10) \quad |h_1(x, k) - h_1(x - \eta, K)| \leq \int_U d^4y \left(\frac{\beta^x(y)}{|y - x|^2} - \frac{\beta^x(y + \eta)}{|y - x + \eta|^2} \right) uww + \int_U d^4y |\beta^x(y + \eta) - \beta^{x-\eta}(y)| \frac{uww}{|y - x + \eta|^2}.$$

For $|\eta|$ sufficiently small, the difference $\beta^x(y + \eta) - \beta^{x-\eta}(y)$ vanishes where $|y - x| < \frac{1}{8} \text{dist}(x, \partial U)$, and specifically where $|y - x + \eta|^2$ is singular. Also, as $|\eta| \rightarrow 0$, the measure of the set where $\beta^x(y + \eta) - \beta^{x-\eta}(y)$ differs from zero vanishes. Therefore, the second term on the right-hand side of (A.10) vanishes as $\eta \rightarrow 0$.

As for the first term on the right-hand side of (A.10), define the translation map $T : U_\epsilon \times L^p_k(U) \rightarrow L^p_k(U_{1-\epsilon})$ by $T(x, v)(y) = v(y - x)$. For $|\eta| \ll \text{dist}(x, \partial U)$, that first term may be rewritten as

$$\int_U d^4y \frac{\beta^x(y)}{|x - y|^2} (uww - T(\eta, u)T(\eta, v)T(\eta, w)).$$

As T is jointly continuous (cf. [19, Lemma 3.4.2]), this first term must vanish as $\eta \rightarrow 0$ also since $h_1(x, \cdot) : \hat{K} \rightarrow \mathbf{R}$ has been established as continuous.

The continuity of $g_1(x)$ follows from Lemma A.3. This establishes that the projection from $\mathfrak{F} \rightarrow L^2_2(U; G)$ sending (g, a, b) to g factors through

$C_{\text{loc}}^0(U; G)$. Lemmas A.2 and A.3 imply as well that the map factors continuously through $C_{\text{loc}}^0(U, G)$. This completes the proof of Lemma A.1 and Proposition 4.5.

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