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ON THE GAUSS MAP OF AN AREA-MINIMIZING HYPERSURFACE

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0. Introduction

Let S be an area-minimizing hypersurface of $\mathbf{R}^N$. By *hypersurface*, we mean a codimension one, locally integral current. By *area-minimizing*, we mean that, without introducing boundary, no compact piece of S can be replaced by a piece having less area. The main concern of this paper is the relationship between S and its spherical image under the Gauss map. For a more precise treatment of our terminology, here and below, the reader should refer to §1.

In the 1960's, area-minimizing hypersurfaces provided the focus for a great deal of research. Indeed, a major accomplishment of that period was the discovery that, upon imposing the dimensional restriction $N < 8$, one guarantees to S several very strong properties which, generally, fail to hold as soon as $N > 7$. Paramount among these properties are two: *interior regularity* and, its alter ego, the *parametric Bernstein property*. The former states that $\operatorname{spt}(S) \sim \operatorname{spt}(\partial S)$ is a codimension one, real analytic submanifold of $\mathbf{R}^N$; the latter, that if $\partial S = 0$, then each connected component of $\operatorname{spt}(S)$ is a hyperplane.

Our paper here introduces a new condition which guarantees both of these properties, independently of dimension. That is, we show that the dimensional hypothesis $N < 8$ can be *replaced* by a different condition, one which involves the topology and Gauss image of $\operatorname{spt}(S) \sim \operatorname{spt}(\partial S)$. We thereby obtain a regularity result (Theorem 3), and a parametric Bernstein result (Theorem 5), which for example, imply the following: *Suppose* $H^1(\operatorname{reg} S) = 0$ *and* $\operatorname{Gauss}(S) \subset S^{N-1}$ *omits a thickened great* S^{N-3}. *Then* $\operatorname{spt}(S) \sim \operatorname{spt}(\partial S)$ *is smooth, and if* $\partial S = 0$, *consists of affine hyperplanes.* A local version of this also holds, estimating curvature when $\partial S \neq 0$ (Theorem 4).

Like our results, our methods follow a pattern developed in the classical (dimensionally restricted) case. Recall that there, regularity is proved by induction on N. The inductive step requires that a smooth minimal hypersurface M in S^{N-1} be an equator if the cone $0 \times\!\!\!\!\times M$ is area-minimizing. That

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this fact *fails* when $N > 7$ is entirely responsible for the breakdown of regularity/Bernstein in higher dimensions. Quite analogously, Theorem 2 here, the basic tool we use to fashion our regularity result, is also proved by induction on N. Moreover, the inductive step again requires that certain smooth minimal hypersurfaces in S^{N-1} be equators. This time, however, the necessary fact, provided by Lemma 1 of §2, holds in every dimension. Consequently, so do all our results. Of course, our main contribution is to the case $N > 7$.

Lemma 1 has an additional consequence which, although incidental to our main line of reasoning, we include as Theorem 1. This is a spherical Bernstein result; stated roughly, *if* $M \subset S^{N-1}$ *is minimal*, $H^1(M) = 0$, *and* Gauss(M) *omits a great* S^{N-3}, *then* M *is an equator*. Recall that the spherical Bernstein problem [15, #99] asks whether a minimal S^{N-2} in S^{N-1} is necessarily equatorial. That the answer is in general negative follows from actual counterexamples, due to W. Y. Hsiang [9]. Thus, to obtain an affirmative result, one needs to impose extra hypotheses. For example, the assumption $N \leqslant 4$ was shown to suffice by Almgren [1]. Since $H^1(S^{N-2}) = 0$ in all higher dimensions, the Gauss map hypothesis of Theorem 1 provides an alternative sufficient condition.

This Gauss map condition, which appears throughout our paper, may evoke in the reader a sense of déjà vu. Indeed, the requirement here, that a thickened, totally geodesic, codimension two subvariety of the appropriate Grassmannian be omitted, is not new. It appears in well-known papers of Osserman [11], [12], Chern [4], and Chern & Osserman [5], in connection with minimal surfaces. But while these authors obtain very sharp parametric Bernstein results which are valid in any codimension, their methods and conclusions are strictly two-dimensional. There seems to be little relationship to our work here, beyond the obvious formal resemblance.

In fact, the two-dimensional theory makes one aspect of our work seem quite unexpected. We refer to the hypothesis "$H^1 = 0$", which is shared in some form by all the results here. Such an assumption appears nowhere in the former theory. But in §6, we give some examples to document the fact that, far from being an artifact of our methods, this hypothesis cannot be deleted from the results we prove.

Finally, we wish to make some acknowledgements. It was mentioned above that our methods follow a pattern of argument dating back to the 1960's. The papers of Fleming [8], Almgren [1], and Simons [13] are especially pertinent in this regard. Short but lucid accounts of how these, along with several other papers, successively pieced together the classical regularity/Bernstein theory, can be found in the introductions to chapter five of Federer's book [6], and to the remarkable paper of Bombieri, De Giorgi, and Giusti [2]. In a more personal vein, we want to thank the Mathematical Sciences Research Institute, Berkeley, for its hospitality and support. This paper represents work which took place in its first year of operation, when the author was fortunate to be a member there. During that time, we benefitted from conversations with many colleagues, but we especially wish to thank R. Schoen for his frequent sage advice, and M. Micallef for helping us clarify the proof of Lemma 1.

1. Preliminaries

For easy reference, we collect here several remarks concerning our terminology in this paper. For usage not covered here, the reader may consult Federer's book [6], which is a basic reference for our work.

Let $N > 1$ be an integer. We shall denote by

$$U(p, R), \quad B(p, R),$$

respectively, the open and closed balls of radius $R > 0$ and center p in $\mathbf{R}^N$.

Our basic objects of study in this paper are the *area-minimizing hypersurfaces* of $\mathbf{R}^N$. The term "*hypersurface*" will generally refer to a current,

$$S \in \mathbf{I}_{N-1}^{\mathrm{loc}}(U)$$

for some open $U \subset \mathbf{R}^N$, though we will use the phrase "*smooth hypersurface*" to signify a smoothly immersed submanifold of codimension one. While the distinction may sometimes blur, hopefully our reasoning will not.

We say a hypersurface S is *area-minimizing* in an open set $U \subset \mathbf{R}^N$ if, whenever $Z \in \mathbf{I}_{N-1}(U)$ and $\partial Z = 0$, we have

$$\|S\|B(0, R) \leqslant \|S + Z\|B(0, R),$$

for all sufficiently large $R > 0$. The most obvious area-minimizing hypersurfaces are the *hyperplanes*, i.e. those hypersurfaces S such that $\mathrm{spt}(S)$ is an affine hyperplane in the linear algebraic sense.

With regard to any hypersurface S, a point $p \in \mathrm{spt}(S) \sim \mathrm{spt}(\partial S)$ is called *regular* if it has a neighborhood in which $\mathrm{spt}(S) \sim \mathrm{spt}(\partial S)$ is an embedded smooth hypersurface. If $p \in \mathrm{spt}\, S \sim \mathrm{spt}\, \partial S$ is not regular, it is termed *singular*. We will denote the regular and singular sets of S respectively by $\mathrm{reg}\, S$, and $\mathrm{sing}\, S$. It is well known [7] that if S is area-minimizing, the Hausdorff dimension of $\mathrm{sing}\, S$ does not exceed $N - 8$.

For the purposes of this paper, the *Gauss map* on a hypersurface S is a smooth unit normal vector field defined on $\mathrm{reg}\, S$;

$$\nu_S\colon \mathrm{reg}\, S \to S^{N-1}.$$

It is characterized uniquely by the requirement that for $\|S\|$-all $x \in \operatorname{reg} S$,

$$\nu_S = \vec{\mathbf{E}}^N \llcorner \vec{S}\,.$$

The image of ν_S will be denoted by Gauss(S), and we will typically impose the condition that Gauss(S) omit a neighborhood of a totally geodesic S^{N-3} in S^{N-1}. For the sake of brevity, this will usually be expressed as "omitting a thickened great S^{N-3}."

Of primary concern will be the geometric relationship between ν_S and an arbitrary two-dimensional subspace of $\mathbf{R}^N$. In line with a common practice, we often will fail to distinguish explicitly between a two-plane π and the corresponding standard orthogonal projection onto π. Thus, notation such as

$$\pi \colon \mathbf{R}^N \to \pi$$

is allowed. Moreover, we will frequently identify π with $\mathbf{R}^2$ by choosing an oriented basis.

Concerning $\mathbf{R}^2$, there is a one-form commonly referred to as "$d\theta$", defined on $\mathbf{R}^2 \sim (0,0)$. This form is closed, but it is not exact, and hence to avoid confusion we will call it ω. That is, in coordinates,

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}\,.$$

Finally, let us agree that the first cohomology groups appearing in this paper and denoted by H^1, will represent de Rham cohomology.

2. Spherical Bernstein results

This section is primarily devoted to proving Lemma 1, which plays for us a role analogous to that played by Simons' calculation [13] in classical regularity theory. As discussed in the Introduction, however, an additional feature of Lemma 1 is the fact that it directly implies the following theorem of spherical Bernstein type.

Theorem 1. *Let $M \subset S^{N-1}$ be a smooth, compact minimal hypersurface. If $H^1(M) = 0$ and* Gauss(M) $\subset S^{N-1}$ *omits a great S^{N-3}, then M is an equator.*

Remark. M need not be orientable, for if not, we merely interpret Gauss(M) as referring to the Gauss image of an oriented double cover. In the oriented case, Gauss(M) refers to the image of the map

$$\nu_M \colon M \to S^{N-1},$$

defined to coincide on M with the Gauss map of the minimal cone $0 \times\!\!\!\!\times M$. Note that M is not assumed to be embedded.

Proof of Theorem 1. Using the fact that any great $S^{N-3} \subset S^{N-1}$ is the kernel of an orthogonal projection

$$S^{N-1} \subset \mathbf{R}^N \to \mathbf{R}^2,$$

the reader will easily deduce Theorem 1 from Lemma 1 below, passing to an oriented double cover if necessary. q.e.d.

Let $\pi \subset \mathbf{R}^N$ be any fixed two-dimensional subspace.

Lemma 1. *Suppose $M \subset S^{N-1}$ is a compact, oreinted, smooth minimal hypersurface. If $|\pi\nu_M| > 0$ and $(\pi\nu_M)^{\#}\omega$ is exact on M, then M is an equator.*

Proof. Assuming $|\pi\nu_M| > 0$ and $(\pi\nu_M)^{\#}\omega$ is exact, we will show that the second fundamental form of M in S^{N-1}, denoted by A, vanishes identically. Hence M is totally geodesic. Our proof twice employs the well-known fact that since M is minimal, the identity

$$(\dagger) \qquad (\Delta_M + |A|^2)(\nu_M \cdot a) = 0$$

holds for every $a \in \mathbf{R}^N$. (Here $\Delta_M = * \, d * d$ is the intrinsic Laplacian on M.) Indeed, we will exhibit $a \in \mathbf{R}^N$ such that $\nu_M \cdot a > 0$ throughout M. By integrating ($\dagger$) over M, we thus obtain a contradiction unless $A \equiv 0$ on M as desired.

For this purpose we simply take, with $p \in M$ chosen arbitrarily,

$$a = \frac{\pi\nu_M(p)}{|\pi\nu_M(p)|},$$

and define a unit vector b so that (a, b) forms an oriented orthonormal basis for the two-plane π. Letting

$$\nu_a = \nu_M \cdot a, \quad \nu_b = \nu_M \cdot b,$$

we thus have

$$(\pi\nu_M)^{\#}\omega = \frac{\nu_a \, d\nu_b - \nu_b \, d\nu_a}{\nu_a^2 + \nu_b^2} = \frac{\nu_a \, d\nu_b - \nu_b \, d\nu_a}{|\pi\nu_M|^2} = du$$

for some $u \colon M \to \mathbf{R}$, because $(\pi\nu_M)^{\#}\omega$ is exact. Using ($\dagger$), a short calculation similar to one in [3, Theorem 8] then shows

$$* \, d * |\pi\nu_M|^2 \, du = 0.$$

Multiplying by u and integrating over M by parts, we now find $|du|^2 \equiv 0$, hence $d(\nu_b/\nu_a) \equiv 0$. But this implies

$$\frac{\nu_b}{\nu_a} = \text{const} = \frac{\nu_b(p)}{\nu_a(p)} = 0.$$

Consequently $\nu_b \equiv 0$, so that $\nu_a = |\pi\nu_M| > 0$, and we have the desired positive solution of ($\dagger$).

3. A closure lemma

A crucial link between the spherical differential geometry of §2, and our regularity/Bernstein results in §5, is forged by Lemma 2 here. This lemma enables us to take limits within certain spaces of hypersurfaces which are precisely tailored to our needs. We refer to the spaces

$$\mathcal{Q}(N, \pi, R) \subset \mathbf{I}_{N-1}^{\mathrm{loc}}(\mathbf{R}^N)$$

which are defined as follows.

Fix $\eta > 0$. Let $0 < R < \infty$, and abbreviate $U(0, R) = U_R$. Recalling from §1 the definitions of ω and ν_S, we select any two-dimensional subspace $\pi \subset \mathbf{R}^N$, and proceed to define $\mathcal{Q}(N, \pi, R)$ as the collection of hypersurfaces $S \in \mathbf{I}_{N-1}(U_R)$ which satisfy the following four conditions:

(1) $S = \partial(\mathbf{E}^N \llcorner M) \llcorner U_R$ for some measurable $M \subset \mathbf{R}^N$,

(2) S is area-minimizing in U_R,

(3) $|\pi\nu_M| \geqslant \eta$ on reg S,

(4) $(\pi\nu_S)^{\#}\omega$ is exact on reg S.

Let us also define $\mathcal{Q}(N, \pi, \infty) \subset \mathbf{I}_{N-1}^{\mathrm{loc}}(\mathbf{R}^N)$, by assigning it all hypersurfaces S such that $S \llcorner U_R \in \mathcal{Q}(N, \pi, R)$ for every $0 < R < \infty$.

Although our notation suppresses the fact, these spaces do of course depend on the choice of $\eta > 0$. Our intention is simply that the reader assume a fixed value for η until indicated otherwise.

We may now state our closure lemma.

Lemma 2. *Suppose $\pi \subset \mathbf{R}^N$ is a two-dimensional subspace, and let* $0 < R \leqslant \infty$. *Then $\mathcal{Q}(N, \pi, R)$ is closed in $\mathbf{I}_{N-1}^{\mathrm{loc}}$.*

Proof. Let us assume $R < \infty$, as the case $R = \infty$ then follows without difficulty. We leave the reader to verify that the first two defining conditions for $\mathcal{Q}(N, \pi, R)$ are themselves closed. (In this regard [6, 5.4.2] may be helpful.) It then remains to establish the closedness of conditions (3) and (4) on the set defined by (1) and (2). The main content of Lemma 2 resides in condition (4).

Accordingly, let $S \in \mathbf{I}_{N-1}$ be a boundary point of $\mathcal{Q}(N, \pi, R)$. Then S is an area-minimizing hypersurface satisfying condition (1). By a standard argument based on the classical regularity theory [6, 5.3.14], we consequently obtain the following fact:

> For every $p \in \operatorname{reg} S$ and $\epsilon > 0$, there exists $\delta > 0$ and a neighborhood $\mathcal{O}$ of S in $\mathbf{I}_{N-1}$, with the following properties. For each $T \in \mathcal{O} \cap \mathcal{Q}(N, \pi, R)$, $\operatorname{spt}(T) \cap B(p, \delta)$ is the graph of a smooth function over the hyperplane tangent to reg S at p. Moreover,
>
> $$|\nu_S - \nu_T| < \epsilon \quad \text{in } B(p, \delta).$$

Though we mainly seek to apply this in reference to condition (4), we may note in passing that S is now easily seen to satisfy condition (3).

Let $K \subset \operatorname{reg} S$ be any compact subset. The above fact then conveniently provides us with the existence of neighborhoods

$$U \text{ of } K \text{ in } \mathbf{R}^N, \quad \mathcal{V} \text{ of } S \text{ in } \mathbf{I}_{N-1},$$

such that, first, $(\operatorname{spt}(S) \cap U) \subset \operatorname{reg} S$. Secondly, whenever $T \in \mathcal{V} \cap \mathcal{Q}(N, \pi, R)$, there is a smooth function

$$f_T\colon (\operatorname{spt}(S) \cap U) \to \mathbf{R}$$

so that $\operatorname{spt}(T) \cap U$ is the graph of f_T with respect to ν_S. That is,

$$\operatorname{spt}(T) \cap U = \mathbf{R}^N \cap \{x + f_T(x)\nu_S(x)\colon x \in \operatorname{spt}(S) \cap U\}.$$

Using this, we then fix a particular $T \in \mathcal{V} \cap \mathcal{Q}(N, \pi, R)$ with the additional property that for all $x \in \operatorname{spt}(S) \cap U$,

$$|\nu_T(x + f_T(x)\nu_S(x)) - \nu_S(x)| < \eta.$$

But now there is an obvious homotopy,

$$H\colon (\operatorname{spt}(S) \cap U) \times [0,1] \to \mathbf{R}^2,$$

defined via the formula

$$H(x,t) = (1-t)\pi\nu_S(x) + t\pi\nu_T(x + f_T(x)\nu_S(x)),$$

and the above inequality implies

$$H((\operatorname{spt}(S) \cap U) \times [0,1]) \subset \mathbf{R}^2 \sim (0,0).$$

$H^{\#}\omega$ is therefore well defined. Since $H(\cdot, 1)^{\#}\omega = (\pi\nu_T)^{\#}\omega$ is by hypothesis exact, the homotopy formula for differential forms [6, 4.1.9] allows us to conclude that on $\operatorname{spt}(S) \cap U$ (and hence on K) so is $H(\cdot, 0)^{\#}\omega = (\pi\nu_S)^{\#}\omega$.

It now follows immediately that condition (4) holds for S, as can be seen by considering the case in which K "captures" any given closed loop in $\operatorname{reg} S$. The proof of Lemma 2 is thus complete.

Corollary. *$\mathcal{Q}(N, \pi, \infty)$ is a closed cone.*

Proof. In calling $\mathcal{Q}(N, \pi, \infty)$ a cone, we mean that when $S \in \mathcal{Q}(N, \pi, \infty)$, the homothetic image $\mu_{r\#}S$ also belongs to $\mathcal{Q}(N, \pi, \infty)$ for every $r \geq 0$. Since $\mu_{r\#}\mathcal{Q}(N, \pi, R)$ clearly equals $\mathcal{Q}(N, \pi, rR)$, the corollary is immediate.

4. The basic tool

Theorem 2 of this section is the basic tool we shall use in the following section to obtain our main results.

As before, we fix an arbitrary $\eta > 0$, and let $\pi \subset \mathbf{R}^N$ be a two-dimensional subspace.

Theorem 2. *All cones in $\mathcal{Q}(N, \pi, \infty)$ are hyperplanes.*

Proof. Let $C \in \mathcal{Q}(N, \pi, \infty)$ be a cone. It will suffice to show that $\operatorname{spt}(C) \sim 0$ (0 being the origin of $\mathbf{R}^N$) is in fact a *smooth* hypersurface. For in this case, $\operatorname{spt}(C) \cap S^{N-1}$ is smooth in S^{N-1}, and satisfies the hypotheses of Lemma 1 in §2.

We will establish the desired smoothness inductively, assuming $N > 2$ and that Theorem 2 (which is elementary when $N = 2$) holds in dimension $N - 1$. Moreover, with the help of [6, 5.4.6], we can reduce our problem to that of showing, whenever $p \in \operatorname{spt}(C) \sim 0$, that every oriented tangent cone to C at p is a hyperplane.

Selecting such a tangent cone, at some $p \in \operatorname{spt}(C) \sim 0$, we label it P. To simplify matters, let us adopt the convention that $\mathbf{R}^{N-1}$ is the subspace of $\mathbf{R}^N$ on which the Nth coordinate function vanishes. Without loss of generality, we then assume p is a unit vector orthogonal to $\mathbf{R}^{N-1}$. According to well-known theory [6, 4.3.16, 5.4.8], P is a cylinder (with direction p) on an area-minimizing cone $Q \in \mathbf{I}_{N-1}^{\mathrm{loc}}(\mathbf{R}^{N-1})$. The corollary to Lemma 2 now implies $P \in \mathcal{Q}(N, \pi, \infty)$. Subject to verification of the following claim, our proof may therefore be completed by induction on N.

Claim. For a suitably chosen two-dimensional subspace $\tilde{\pi} \subset \mathbf{R}^{N-1}$, the area-minimizing cone Q belongs to $\mathcal{Q}(N - 1, \tilde{\pi}, \infty)$.

To verify this, we first observe that since $\nu_Q = \nu_P$ on $\operatorname{reg} Q$, and $P \in \mathcal{Q}(N, \pi, \infty)$, $(\pi\nu_Q)^{\#}\omega$ is necessarily exact on $\operatorname{reg} Q$.

Next, noting that $\dim(\pi \cap \mathbf{R}^{N-1}) > 0$, we fix a unit vector $a \in \pi \cap \mathbf{R}^{N-1}$. It is then not difficult to find an angle $\phi \in [-\pi/2, \pi/2]$, and a curve $b_t \in S^{N-1}$, satisfying, for every $t \in [0, 1]$,

$$b_t \cdot a = 0, \quad b_0 \in \mathbf{R}^{N-1},$$
$$b_t = p\sin(t\phi) + b_0\cos(t\phi),$$
$$(a, b_1) \text{ is an oriented orthonormal basis for } \pi.$$

We may consequently set

$$\pi_t = \operatorname{span}\{a, b_t\},$$

and compute

$$(\nu_Q \cdot b_t)^2 = (\nu_Q \cdot b_0)^2 \cos^2 t\phi \geqslant (\nu_Q \cdot b_0)^2 \cos^2 \phi = (\nu_Q \cdot b_1)^2.$$

Since $\pi_1 = \pi$, this yields

$$|\pi_t \nu_Q| = (\nu_Q \cdot a)^2 + (\nu_Q \cdot b_t)^2 \geqslant |\pi\nu_Q|^2.$$

In particular,

$$|\pi_0 \nu_Q|^2 \geqslant \eta \quad \text{on reg}\, Q,$$

and $H^{\#}\omega$ is well defined for the homotopy

$$H\colon \text{reg}\, Q \times [0,1] \to \mathbf{R}^2 \quad \text{via}\; H(x,t) = \pi_t \nu_Q(x).$$

But then, by the homotopy formula for differential forms [6, 4.1.9], the exactness of $(\pi_1 \nu_Q)^{\#}\omega = (\pi \nu_Q)^{\#}\omega$ on reg Q implies that of $(\pi_0 \nu_Q)^{\#}\omega$. Clearly the defining conditions for $\mathcal{Q}(N-1, \tilde{\pi}, \infty)$ are now satisfied for Q with $\tilde{\pi} = \pi_0$. This establishes our claim and completes the proof of Theorem 2.

5. Main results

We are now ready to state and prove three theorems which, together, comprise our principal reason for undertaking this paper. The first two, establishing regularity (Theorem 3), and estimating curvature (Theorem 4), are of a local nature. The third, a parametric Bernstein result (Theorem 5), is global.

Theorem 3. *Suppose $U \subset \mathbf{R}^N$ is open and $S \in \mathbf{I}^{\text{loc}}_{N-1}(U)$ is area-minimizing. If $H^1(\text{reg}\, S) = 0$ and* Gauss(S) *omits a thickened great S^{N-3} in S^{N-1}, then* sing $S = \varnothing$.

Proof. Arbitrarily choosing $0 \in \text{spt}(S) \sim \text{spt}(\partial S)$, we will show $0 \in \text{reg}\, S$. For this purpose we may assume, after suitably translating and restricting, that 0 is the origin of $\mathbf{R}^N$, that $U = U(0, R)$ for some $R > 0$, and that $\text{spt}(\partial S) \subset \partial U$. We then select a two-dimensional subspace $\pi \subset \mathbf{R}^N$ such that Gauss(S) omits a neighborhood of the great S^{N-3} defined by $\ker \pi \cap S^{N-1}$.

In this situation, there clearly exists $\eta > 0$ such that $|\pi \nu_S|^2 \geqslant \eta$, and since $H^1(\text{reg}\, S) = 0$, $(\pi \nu_S)^{\#}\omega$ is exact. Moreover, arguing via [6, 4.5.17] and the maximum principle (as in the second part of the proof of [6, 5.4.15]), our task is then further reduced; we need only consider the case $S = \partial(\mathbf{E}^N \llcorner M) \llcorner U$ for some measurable $M \subset \mathbf{R}^N$. But in this case, we have $S \in \mathcal{Q}(N, \pi, R)$.

Consequently, letting C be any oriented tangent cone to S at $0 \in \text{spt}(S) \sim \text{spt}(\partial S)$, we infer from Lemma 2 (our closure lemma), that $C \in \mathcal{Q}(N, \pi, \infty)$. By Theorem 2 of §4, C is then a hyperplane. According to [6, 5.4.6], S is therefore regular at 0. q.e.d.

In the following, $\kappa_1, \cdots, \kappa_{N-1}$ denote the principal curvature functions of the smooth hypersurface M.

Theorem 4. *Let $M \subset \mathbf{R}^N$ be a smooth area-minimizing hypersurface. Suppose $H^1(M) = 0$ and* Gauss(M) *omits a neighborhood U of some great S^{N-3} in*

S^{N-1}. *Then there exists a constant* $C = C(N, U)$ *such that for all* $p \in M$,

$$\sum_{i=1}^{N-1} \kappa_i(p)^2 \leqslant \frac{C}{\operatorname{dist}(p, \partial M)^2}.$$

Proof. As in the proof of Theorem 3, we choose a two-dimensional subspace $\pi \subset \mathbf{R}^N$, orthogonal to a great $S^{N-3} \subset U \subset S^{N-1}$. Also, we define

$$\eta = \eta(U) > 0, \quad \text{via } \eta = \operatorname{dist}(S^{N-1} \sim U, \ker \pi).$$

It is then not difficult to see that the *failure* of Theorem 4 would imply the following:

There exists a sequence of smooth hypersurfaces $\{S_n\} \subset \mathscr{Q}(N, \pi, 1)$, with principal curvature functions $\kappa_{n,i}$, such that for each $n = 1, 2, 3, \cdots$,

$$0 \in \operatorname{spt}(S)_n, \quad \sum_{i=1}^{N-1} \kappa_{n,i}(0)^2 > n,$$

with 0 the origin of $\mathbf{R}^N$.

But this statement leads to a contradiction. Indeed, if it were true, one could deduce, using the argument of L. Simon in [14, Theorem 1], the existence of a subsequence of $\{S_n\}$ converging to a limit $S \in \mathbf{I}_{N-1}(U(0, 1))$, with $0 \in \operatorname{sing} S$. On the other hand, it follows from our closure lemma (Lemma 2) that $S \in \mathscr{Q}(N, \pi, 1)$, hence satisfies the hypotheses of Theorem 3. Therefore $0 \in \operatorname{reg} S$, a contradiction, and Theorem 4 is established.

Theorem 5. *Let S be an area-minimizing hypersurface in* $\mathbf{R}^N$ *with* $\partial S = 0$. *If* $H^1(\operatorname{reg} S) = 0$ *and* $\operatorname{Gauss}(S)$ *omits a thickened great* S^{N-3} *in* S^{N-1}, *then each component of* $\operatorname{spt}(S)$ *is a hyperplane*.

Proof. Applying Theorem 3 with $U = \mathbf{R}^N$, we see that $\operatorname{sing} S = \varnothing$, hence $\operatorname{spt}(S)$ is a smooth hypersurface. Applying Theorem 4 to any component M of $\operatorname{spt}(S)$, in conjunction with our assumption that $\partial S = 0$, we see that M is totally geodesic. This proves Theorem 5.

6. A class of examples

We conclude with some examples. Our purpose is thereby to demonstrate that the hypothesis "$H^1 = 0$", so characteristic of our results, is necessitated by considerations of truth, as opposed to proof.

First, let

$$S^p(r) = \mathbf{R}^{p+1} \cap \{x : |x| = r\},$$

and recall that whenever $0 < p \in \mathbf{Z}$, there is a smooth minimal hypersurface $M_{p,1} \subset S^{p+2}$, obtained by setting

$$M_{p,1} = S^p(p/p+1)^{1/2} \times S^1(1/p+1)^{1/2} \subset S^{p+2} \subset \mathbf{R}^{p+1} \times \mathbf{R}^2.$$

It is then easily determined that for certain constants α, $\beta > 0$ with $\alpha^2 + \beta^2 = 1$, we have

$$\mathrm{Gauss}(M_{p,1}) = S^p(\alpha) \times S^1(\beta).$$

This set manifestly omits $S^p(1) \times (0,0)$, which is a great S^p in S^{p+2}. Thus, except for the obvious fact that $H^1(M_{p,1}) \neq 0$, the hypotheses of our spherical Bernstein result, Theorem 1, all obtain. Yet $M_{p,1}$ is *not* an equator. In other words, *if the hypothesis "$H^1(M) = 0$" is deleted from Theorem* 1, *the resulting statement becomes false*.

A similar situation exists with regard to the results of §5. Indeed, according to Lawson [10], the cone

$$C_{p,1} = 0 \divideontimes M_{p,1} \subset \mathbf{R}^{p+3}$$

is, for sufficiently large $p \in \mathbf{Z}$, an area-minimizing hypersurface. The same reasoning used above then shows: *if the hypothesis "$H^1 = 0$" is deleted from any of Theorems* 3, 4, *or* 5, *the resulting statements become false*. In fact, *for Theorems* 4 *and* 5, *this holds even within the class of smooth hypersurfaces*. To see the latter fact, we first recall the proof that $C_{p,1}$ is minimizing. In that proof, the existence of a homothetically invariant foliation of $\mathbf{R}^{p+3} \sim \mathrm{spt}(C_{p,1})$ by *smooth* area-minimizing hypersurfaces is established. Each leaf of this foliation is asymptotic to $C_{p,1}$ at infinity, and if M is a leaf in the component of $\mathbf{R}^{p+3} \sim C_{p,1}$ containing $(0,0,\cdots,0,1)$, it is not difficult to see that M is diffeomorphic to $\mathbf{R}^{p+1} \times S^1$. At the same time, we must have that $\mathrm{Gauss}(M)$ is precisely the component of $S^{p+2} \sim \mathrm{Gauss}(M_{p,1})$ which omits $S^p(1) \times (0,0)$. Thus, if not for the fact that $H^1(M) \neq 0$, M would present a smooth counterexample to our last two theorems.

References

[1] F. J. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. **84** (1966) 277–292.

[2] E. Bombieri, E. DeGiorgi & E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969) 243–268.

[3] E. Bombieri & E. Giusti, *Harnack's inequality for elliptic differential equations on minimal surfaces*, Invent. Math. **15** (1972) 24–46.

[4] S. S. Chern, *Minimal surfaces in an Euclidean space of N dimensions*, Differential and Combinatorial Topology: A Symposium in Honor of Marston Morse, Princeton University Press, Princeton, NJ, 1965.

[5] S. S. Chern & R. Osserman, *Complete minimal surfaces in Euclidean n-space*, J. Analyse Math. **19** (1967) 15–34.

[6] H. Federer, *Geometric measure theory*, Springer, Berlin, 1969.

[7] ———, *The singular sets of area-minimizing rectifiable currents with codimension one and area-minimizing flat chains mod two with arbitrary codimension*, Bull. Amer. Math. Soc. **76** (1970) 767–771.

[8] W. H. Fleming, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo (2) **11** (1962) 69–90.

[9] W. Y. Hsiang, *New examples of minimal embeddings of S^{n-1} into $S^n(1)$–the spherical Bernstein problem for $n = 4, 5, 6$*, Bull. Amer. Math. Soc. **7** (1982) 377–379.

[10] H. B. Lawson, Jr., *The equivariant Plateau problem and interior regularity*, Trans. Amer. Math. Soc. **173** (1972) 231–249.

[11] R. Osserman, *Proof of a conjecture of Nirenberg*, Comm. Pure Appl. Math. **12** (1959) 229–232.

[12] ———, *Global properties of minimal surfaces in $\mathbf{E}^3$ and $\mathbf{E}^n$*, Ann. of Math. (2) **80** (1964) 340–364.

[13] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. **88** (1968) 62–105.

[14] L. Simon, *Remarks on curvature estimates for minimal hypersurfaces*, Duke Math. J. **43** (1976) 545–553.

[15] S. T. Yau, *Problem section*, *Seminar on differential geometry*, Annals of Math. Studies, No. 102, Princeton University Press, Princeton, NJ, 1982, 664–706.

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