J. DIFFERENTIAL GEOMETRY
19 (1984) 207–219

HARMONIC MAPS FROM S^2 TO $G_{2,4}$

JAYAKUMAR RAMANATHAN

0. Intoduction

The energy of a smooth map $\phi\colon M \to N$ between two smooth Riemannian manifolds is given by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2\, d\mathrm{vol}_M, \tag{0.1}$$

where $d\phi\colon TM \to TN$ is the differential of the map ϕ and $d\mathrm{vol}_M$ is the volume element on M induced by the metric. The norm on $d\phi$ is given by

$$|d\phi|^2(x) = \sum_{i=1}^{m} \langle d\phi(e_i), d\phi(e_i)\rangle_N, \qquad x \in M, \tag{0.2}$$

where $e_1, \cdots, e_m$ is an orthonormal frame of TM_x. The map ϕ is harmonic if it is a critical point of this energy functional. This means that for any C^∞ one parameter variation $\phi_t\colon M \to N$ with $t \in (-\delta, \delta)$, $\delta > 0$ and $\phi_0 = \phi$,

$$\left.\frac{dE(\phi_t)}{dt}\right|_{t=0} = 0. \tag{0.3}$$

(See [4] and [5] for more information about the general theory of harmonic maps).

Eells and Wood [6] have given a classification theorem for harmonic maps $\phi\colon \mathbf{C}P^1 \to \mathbf{C}P^n$ which reduces their study to that of holomorphic maps of $\mathbf{C}P^1$ to $\mathbf{C}P^n$. In this note, we give a similar description of all harmonic maps $\phi\colon \mathbf{C}P^1 \to G_{2,4}$, $G_{2,4}$ being the Grassmann manifold of complex two planes in $\mathbf{C}^4$. This uses the results of Eeels and Wood in the special case of maps from $\mathbf{C}P^1$ to $\mathbf{C}P^3$. Our method involves a conservation law that holds for any harmonic map $\phi\colon \mathbf{C}P^1 \to G_{k,k+n}$, $G_{k,n+k}$ being the manifold of complex k planes in $\mathbf{C}^{n+k}$. The proof of this conservation law given here, suggested by John Rawnsley and Karen Uhlenbeck, is much simpler than the authors original computations.

Received July 20, 1983.

We conclude this section with some well-known results needed in the sequel. The references of this material are [4], [5], [6], and [9].

Let ϕ: $M \to N$ be a smooth map between two C^∞ compact Riemannian manifolds. Then there is a global C^∞ section of the vector bundle $\phi^*(TN)$ over M, denoted $\tau(\phi)$, satisfying the following property. Given a smooth one parameter variation ϕ_t: $M \to N$ with $t \in (-\delta, \delta)$, $\delta > 0$ and $\phi_0 = \phi$, we have

$$\frac{dE(\phi_t)}{dt}\bigg|_{t=0} = \int_M \left\langle \tau(f), \frac{d\phi_t}{dt}\bigg|_{t=0} \right\rangle_N d\,\mathrm{vol}_M. \tag{0.4}$$

Lemma 0.1. *Given a smooth global section of $\phi^*(TN)$, X, there is a smooth one parameter variation ϕ_t such that $X = (d\phi_t/dt)|_{t=0}$. Explicitly, the variation is given by $\phi_t(x) = \exp_{\phi(x)} tX$.*

Corollary 0.2. *The map ϕ is harmonic if and only if $\tau(\phi) = 0$.*

The section $\tau(\phi)$ is called the tension field of ϕ. Explicit formulae for $\tau(\phi)$ can be found in the references mentioned.

Theorem 0.3 [5]. *Let M and N be real analytic manifolds with real analytic metrics. Then a harmonic map ϕ: $M \to N$ is necessarily real analytic.*

Suppose now that M is a compact Riemann surface and N is a Kähler manifold. Also assume the metric on M is the real part of a Hermitian metric on M. (Hence the metric on M gives the same conformal structure as the complex structure on M). If ϕ: $M \to N$ is a given smooth map, the complexified differential $d\phi_{\mathbf{C}}$: $T_{\mathbf{C}}M \to T_{\mathbf{C}}N$ determines complex vector bundle maps $d\phi'$: $T^{1,0}M \to T^{1,0}N$ and $d\phi''$: $T^{0,1}M \to T^{1,0}N$. Define the ∂ energy of ϕ by

$$E'(\phi) = \int_M |d\phi'|^2\, d\,\mathrm{vol}_M \tag{0.5}$$

and the $\bar\partial$ energy of ϕ by

$$E''(\phi) = \int_M |d\phi''|^2\, d\,\mathrm{vol}_M. \tag{0.6}$$

The norms here are as in (0.2) with the distinction that a unitary frame on $T^{1,0}M$ $(T^{0,1}M)$ is used (as opposed to an orthonormal frame).

A straightforward computation (see [6]) gives

$$\begin{aligned} E'(\phi) &= \frac{1}{2}E(\phi) - \frac{1}{2}\int_M \phi^*(\omega_N), \\ E''(\phi) &= \frac{1}{2}E(\phi) + \frac{1}{2}\int_M \phi^*(\omega_N), \end{aligned} \tag{0.7}$$

where ω_N is the Kähler form of N.

Theorem 0.4 [6]. (a) *$E''(\phi) - E'(\phi) = \int_M \phi^*(\omega_N)$ depends only on the homotopy class of ϕ.*

(b) *The critical points of E, E' and E'' coincide.*

Denote by ∇ the pullback of the Hermitian connection on $T^{1,0}N$ to $\phi^*(T^{1,0}N)$. The Euler-Lagrange conditions for E' and E'' are given in the following result.

Theorem 0.5 [6]. (a) *The map ϕ is harmonic if and only if for any holomorphic chart (U, z) of M,*

$$\nabla_{\bar z} d\phi'\left(\frac{d}{dz}\right) \equiv 0 \tag{0.8}$$

on U, (where ∇_z is abbreviation for $\nabla_{d/dz}$ and $\nabla_{\bar z}$ abbreviates $\nabla_{d/d\bar z}$).

(b) *The map ϕ is harmonic if and only if, for any holomorphic chart (U, z) on M,*

$$\nabla_z d\phi''\left(\frac{d}{d\bar z}\right) = 0 \tag{0.9}$$

on U.

1. The metric and connection on the Grassmannian

Let $G_{k,n+k}$ be the space of complex k planes k planes in $\mathbf{C}^{n+k}$. $G_{k,n+k}$ is a compact complex manifold of dimension kn. Let $E_{k,n+k}$ be the standard rank k holomorphic vector bundle over $G_{k,n+k}$. Explicitly, $E_{k,n+k} = \{(P, v) \in G_{k,n+k} \times C^{n+k}: v \in P\}$ with the first coordinate giving the projection map from $E_{k,n+k}$ to $G_{k,n+k}$. There is a vector bundle inclusion of $E_{k,n+k}$ into $G_{k,n+k} \times \mathbf{C}^{n+k}$, the trivial rank $n+k$ bundle over $G_{k,n+k}$. This trivial bundle has a standard Hermitial inner product and connection. Let $E^{\perp}_{k,n+k}$ be the orthogonal complement of $E_{k,n+k}$ in this trivial bundle. Hence $E^{\perp}_{k,n+k}$ is an antiholomorphic vector bundle of rank n over $G_{k,n+k}$.

Given an open set $U \subseteq G_{k,n+k}$, let $s \in \Gamma_U(E_{k,n+k})$ and $X \in \Gamma_U(T_{\mathbf{C}}G_{k,n+k})$ be sections over U. Then

$$X(s) = D_X s + B_X(s), \tag{1.1}$$

where $D_X s \in \Gamma_U(E_{k,n+k})$ and $B_X(s) \in \Gamma_U(E^{\perp}_{k,n+k})$. A standard argument shows B is a section of $\Lambda^1_{\mathbf{C}} G_{k,n+k} \otimes \operatorname{Hom}(E_{k,n+k}, E^{\perp}_{k,n+k})$, i.e. it is actually a tensor. (This well-known fact follows from an argument analogous to that showing the 2nd fundamental form of a submanifold is a tensor; see [9].) Similarly, for $s \in \Gamma_U(E^{\perp}_{k,n+k})$ and $X \in \Gamma_U(T_{\mathbf{C}}G_{k,n+k})$,

$$X(s) = D^*_X s + B^*_X s, \tag{1.2}$$

where $D^*_X s \in \Gamma_U(E_{k,n+k})$ and B^* is a section of $\Lambda^1_{\mathbf{C}} G_{k,n+k} \otimes \operatorname{Hom}(E^{\perp}_{k,n+k}, E_{k,n+k})$.

The Hermitian structure on $E_{k,n+k}$ and $E^{\perp}_{k,n+k}$ induce a Hermitian structure on the bundle $\operatorname{Hom}(E_{k,n+k}, E^{\perp}_{k,n+k})$. Moreover the connections D and D^*

induce a connection on $\mathrm{Hom}(E_{k,n+k}, E_{k,n+k}^{\perp})$ that is compatable with this Hermitian structure. Denote this induced connection by $\tilde{D}$. If $T \in \Gamma_U(\mathrm{Hom}(E_{k,n+k}, E_{k,n+k}^{\perp}))$ and $s \in \Gamma_U(E_{k,n+k})$ on some open set of U,

$$(\tilde{D}_X T)(s) = D_X^*(Ts) - T(D_X s). \tag{1.3}$$

The metric we use on $G_{k,n+k}$ is the Fubini-Study metric. Our task is to identify the holomorphic tangent space, $T^{1,0}G_{k,n+k}$, with $\mathrm{Hom}(E_{k,n+k}, E_{k,n+k}^{\perp})$ as Hermitian vector bundles with connections.

Proposition 1.1. *The vector bundle map I: $T^{1,0}G_{k,n+k} \to \mathrm{Hom}(E_{k,n+k}, E_{k,n+k}^{\perp})$ given by $X \mapsto B_X$ is an isomorphism of vector bundles.*

Proof. Let $U(k, n+k)$ be the space of matrices Z with k columns and $n+k$ rows such that ${}^t\bar{Z}Z = \mathrm{Id}$. In other words $U(k, n+k)$ is the space of unitary k frames in $\mathbf{C}^{n+k}$. The map Π: $U(k, n+k) \to G_{k,n+k}$, which takes $Z \in U(k, n+k)$ to the k plane spanned by the column vectors of Z, makes $U(k, n+k)$ into a fibre bundle over $G_{k,n+k}$ with fibres isomorphic to $U(k)$. We have that

$$T_Z U(k, n+k) = \{A \in M(k, n+k) \colon {}^t\bar{A}Z + {}^t\bar{Z}A = 0\}, \tag{1.4}$$

where $M(k, n+k)$ is the space of all complex matrices with k columns and $n+k$ rows. An inner product on $U(k, n+k)$ is given by

$$(A_1, A_2) = \mathrm{Re}\big(\mathrm{Tr}({}^t\bar{A}_2 A_1)\big) \tag{1.5}$$

for $A_i \in T_Z U(k, n+k)$. The vector bundle V consisting of those vectors in $TU(k, n+k)$ tangent to the fibres of Π is given by

$$V_Z = \{ZA \colon A \in u(k)\}, \tag{1.6}$$

where $u(k)$ is the Lie algebra of $U(k)$. Let H be the orthogonal complement of V in $TU(k, n+k)$. The bundle H has a natural complex structure that makes the inner product (1.5) into the real part of a Hermitian metric on H. This inner product projects via Π to the Fubini-Study metric on $G_{k,n+k}$ (see [9] for more information).

Given $X \in T_P^{1,0}G$, it is enough to show that B_X has the same norm as X. Without loss of generality we may assume P is the k plane of those vectors in $\mathbf{C}^{n+k}$ whose last n coordinates are all zero. Let $s = (s_1, \cdots, s_k)$ be a lift from a neighborhood of P to $U(k, n+k)$ such that $s(P) = \binom{\mathrm{Id}}{0}$ (Id is the unit $k \times k$ matrix). Then $ds(X)$ is the matrix $(X(s_1), \cdots, X(s_k))$. The vertical vectors at $s(P)$ are of the form $\binom{A}{0}$, when $A \in u(k)$ and 0 is the $n \times k$ zero matrix. Hence $(B_X(s_1), \cdots, B_X(s_k))$ gives the component of $ds(X)$ orthogonal to the fibres of Π. The result follows.

Proposition 1.2. *The connection $\hat{D}$ on $\mathrm{Hom}(E_{k,n+k}, E_{k,n+k}^{\perp})$ pulls back via I to the Levi-Cevita connection on $T^{1,0}G_{k,n+k}$.*

Proof. Since $I^*(\tilde{D})$ is a Hermitian connection on $T^{1,0}G_{k,n+k}$ it suffices to check that $I^*(\tilde{D})$ is of type $(1,0)$. This automatically forces $I^*(\tilde{D})$ to be the Levi-Cevita connection. Let X be a local holomorphic vector field on $G_{k,n+k}$. We need to check that $(I^*\tilde{D})_Y(X) = \tilde{D}_Y(B_X) = 0$ for any $(0,1)$ vector Y. If s is a holomorphic section of $E_{k,n+k}$,

$$\tilde{D}_Y(B_X)(s) = D_Y^*(B_X(s)) - B_X(D_Y(s)).$$

$D_Y s = 0$ so $\tilde{D}_Y(B_X(s)) = D_Y^*(B_X(s))$. Now $X(s) = D_Y(s) + B_Y(s)$. Differentiating by the $(0,1)$ vector Y and noting that $D_X(s)$ is holomorphic gives

$$0 = Y(X(s)) = Y(D_X(s)) + Y(B_X(s)) = Y(B_X(s)).$$

It follows that $D_Y^*(B_X(s)) = 0$. Hence $\tilde{D}_Y(B_X) = 0$. Hence $\tilde{D}_Y(B_X) = 0$. This finishes the proof.

2. A conservation law

Let $\phi: M \to G_{k,n+k}$ be a smooth map from a compact Riemann surface. Write $E = \psi^* E_{k,n+k}$ and $E^\perp = \phi^* E^\perp_{k,n+k}$. The connections D and D^* pull-back to connections ∇ and ∇^* on E and $E^\perp$, respectively. Note that $M \times \mathbf{C}^{n+k} \cong E \oplus E^\perp$ as Hermitian vector bundles. The connections ∇ and ∇^* are precisely the restrictions of the trivial connection on $M \times \mathbf{C}^{n+k}$ to E and $E^\perp$, respectively. On an open set $U \subseteq M$, suppose $X \in \Gamma_U(T_{\mathbf{C}}M)$ and $s \in \Gamma_U(E)$. Then as in equation (1.1),

$$X(s) = \nabla_X s + \beta_X(s), \tag{2.1}$$

where β is a global section of $\Lambda^1 M \otimes \operatorname{Hom}(E, E^\perp)$. If $s \in \Gamma_U(E)$ then

$$X(s) = \nabla_X^* s + \beta_X^*(s), \tag{2.2}$$

where β^* is a global section of $\Lambda^1 M \otimes \operatorname{Hom}(E^\perp, E)$. Since $\operatorname{Hom}(E, E^\perp) = \phi^*\tilde{D}$ coincides with the connection induced by ∇ and ∇^* on $\operatorname{Hom}(E, E^\perp)$.

Proposition 2.1. (a) *The map ϕ is harmonic if and only if, on each holomorphic chart (U, z) with a unitary frame $e_1, \cdots, e_k$ of E,*

$$\nabla_{\bar{z}}^* \beta_z(e_i) = \sum_{j=1}^{k} \left\langle \frac{de_i}{d\bar{z}}, e_j \right\rangle \beta_z(e_j), \qquad 1 \leqslant i \leqslant k. \tag{2.3}$$

(b) *The map ϕ is harmonic if and only if on each holomorphic chart (U, z) with a unitary frame $e_1, \cdots, e_k$ of E,*

$$\nabla_z \beta_{\bar{z}}^*(e_i) = \sum_{j=1}^{k} \left\langle \frac{de_i}{dz}, e_j \right\rangle \beta_{\bar{z}}^*(e_j), \qquad 1 \leqslant i \leqslant k. \tag{2.4}$$

(*Here, we have abbreviated $\beta_{d/dz}$ by β_z, etc.*)

Proof. By the identification given in Propositions 1.1 and 1.2, we have

$$(2.5) \qquad I(d\phi'(d/dz)) = B_{d\phi'(d/dz)}, \qquad I(d\phi''(d/d\bar{z})) = B_{d\phi''(d/d\bar{z})}.$$

Pulling back via ϕ one has

$$(2.6) \qquad \phi^*\big(B_{d\phi'(d/dz)}\big) = B_z, \quad \phi^*\big(B_{d\phi''(d/d\bar{z})}\big) = B_{\bar{z}}.$$

Equations (0.8) and (0.9) can then be written as

$$(2.7) \qquad (\phi^*\tilde{D})_{d/d\bar{z}}(\beta_z) \equiv 0, \quad (\phi^*\tilde{D})_{d/dz}(\beta_{\bar{z}}) \equiv 0 \quad \text{on } U.$$

Evaluating on the section e_i gives

$$(2.8) \qquad \begin{aligned} (\phi^*\tilde{D})_{\bar{z}}(\beta_z)(e_i) &= \nabla_{\bar{z}}^*\beta_z(e_i) - \beta_z(\nabla_{\bar{z}}e_i), \\ (\phi^*\tilde{D})_z(\beta_{\bar{z}})(e_i) &= \nabla_z^*\beta_{\bar{z}}(e_i) - \beta_{\bar{z}}(\nabla_z e_i). \end{aligned}$$

Now since

$$(2.9) \qquad \nabla_{\bar{z}}e_i = \sum_{j=1}^{k} \left\langle \frac{de_i}{d\bar{z}}, e_j \right\rangle e_j, \qquad \nabla_z e_i = \sum_{j=1}^{k} \left\langle \frac{de_i}{dz}, e_j \right\rangle e_j,$$

equations (2.3) and (2.4) immediately follow.

The following theorem is a special case of general results of Koszul and Malgrange [8].

Theorem 2.2. *Given a C^∞ complex vector bundle B over a Riemann surface M, a complex connection on B induces a unique holomorphic structure on B whose $\bar{\partial}$ operator is the* (0, 1) *part of the connection.*

It follows that the (0, 1) *part of the connection $\phi^*\tilde{D}$ induces a complex structure on* $\mathrm{Hom}(E, E^\perp)$.

Proposition 2.3. *The map ϕ is harmonic if and only if, on each holomorphic chart (U, z), the section β_z is a holomorphic section of* $\mathrm{Hom}(E, E^\perp)$.

Proof. As was implicit in the proof of Proposition 2.1, ϕ is harmonic if and only if each holomorphic chart (U, z),

$$(\phi^*\tilde{D})_{\bar{z}}\beta_z = 0$$

(see (2.7)). By Theorem 2.2 this is just what it means for β_z to be homomorphic.

Let $\beta = \beta' + \beta''$ where β' is a global section of $\Lambda^{0,1}M \otimes \mathrm{Hom}(E, E^\perp)$ and β'' is a global section of $\Lambda^{0,1}M \otimes \mathrm{Hom}(E, E^\perp)$. The previous result shows that ϕ is harmonic if and only if β' is a holomorphic section of $\Lambda^{1,0}M \otimes \mathrm{Hom}(E, E^\perp)$. Let $\phi^\perp : M \to G_{k,n+k}$ be given by $\phi^\perp(p) = (\phi(p))^\perp$. Then $\phi^\perp$ is harmonic if and only if $(\beta^*)'$ is a holomorphic section of $\Lambda^{1,0}M \otimes \mathrm{Hom}(E^\perp, E)$. Since $G_{k,n+k}$ and $G_{n,n+k}$ are isometric, ϕ is harmonic if and only if $\phi^\perp$ is. The following lemma then follows.

Lemma 2.4. *If ϕ is a harmonic map then the section $(\beta^*)' \circ \beta'$ of $\Lambda^{1,0}M \otimes \Lambda^{1,0}M \otimes \mathrm{Hom}(E, E)$ is holomorphic. In holomorphic coordinates (U, z), $(\beta^*)' \circ \beta'$ has the form $dz \otimes dz \otimes \beta_z^* \circ \beta_z$.*

Fix a holomorphic chart (U, z) of M. Let $s_1, \cdots, s_k$ be the elementary symmetric functions of $\beta_z^* \circ \beta_z$. They are clearly holomorphic on U. Moreover, it can be easily checked that the forms

$$S_1 = s_1\, dz \otimes dz, \quad S_2 = s_2\, dz^4, \ldots, S_k = s_k\, dz^{2k} \tag{2.10}$$

are globally defined on M. Hence we have the following theorem.

Theorem 2.5. *If $\phi\colon M \to G_{k,n+k}$ is a harmonic map, the differential forms in* (2.10) *are all holomorphic.*

Corollary 2.6. *If $\phi\colon \mathbf{C}P^1 \to G_{k,n+k}$ is harmonic then the forms $S_1, \cdots, S_k$ vanish identically.*

3. Harmonic maps to $G_{2,4}$

We first recall the main theorem in [6] and [7]. Let $h\colon M \to \mathbf{C}P^n$ be a full holomorphic map (i.e. $\mathrm{Im}\,\phi$ lies in no proper projective subspaces of $\mathbf{C}P^n$). Let (U, z) be a holomorphic chart of M and $\tilde{h}\colon U \to S^{2n+1}$ a lift of h. Define

$$\mathcal{O}_k(z) = \mathrm{span}\{d^j\tilde{h}/dz^j\colon 0 \leqslant j \leqslant k\}, \qquad k \leqslant n. \tag{3.1}$$

It can easily be checked that $\mathcal{O}_k$ is independent of the choice of coordinate and lift. Moreover $\dim \mathcal{O}_k = k + 1$ everywhere except possibly at a discrete set of points. As it turns out, $\mathcal{O}_k$ can be extended uniquely to a globally defined holomorphic map $\mathcal{O}_k\colon M \to G_{k+1,n+1}$. Put $\mathcal{O}_{-1} = \{0\}$ and define

$$h_k(z) = \mathcal{O}_k(z) \cap \mathcal{O}_{k-1}^{\perp}(z) \tag{3.2}$$

for $k = 0, 1, \cdots, n$. The map $h_k\colon M \to \mathbf{C}P^n$ is harmonic for each of the above values of k. $h_0 = h$ is holomorphic and h_n is antiholomorphic.

The procedure can be reversed when $M = \mathbf{C}P^1$.

Theorem 3.1 [7]. *Let $\phi\colon \mathbf{C}P^1 \to \mathbf{C}P^n$ be a full harmonic map. Then there is a unique full holomorphic map $h\colon \mathbf{C}P^1 \to \mathbf{C}P^n$ and an integer k, $0 \leqslant k \leqslant n$, such that $\phi = h_k$.*

This theorem was discovered by the physicists, Glaser and Stora [7]. Eells and Wood [6] have given a mathematically rigorous proof of this theorem. This theorem was also proved by Burns [1]. More recently, Jon Wolfson has given a proof from the point of view of moving frames [11].

The following technical lemma will be needed in later sections. It is implicit in the papers mentioned.

Lemma 3.2. *Let h: $M \to \mathbf{C}P^n$ be a full holomorphic map. Suppose $\tilde{h}_k$: $U \to S^{2n+1}$ are lifts of h_k on a holomorphic chart (U, z) for $0 \leqslant k \leqslant n$. Then for a fixed i,*

$$\left\langle d\tilde{h}_i/dz, \tilde{h}_k \right\rangle = 0 \tag{3.3}$$

for $k > i + 1$ or $k < i$. Note also that $\langle \tilde{h}_i, d\tilde{h}_k/d\bar{z} \rangle = \langle d\tilde{h}_i/dz, \tilde{h}_k \rangle = 0$ for the same range of indices.

Proof. $d\tilde{h}_i/dz$ lies in $\mathcal{O}_{i+1}(z)$ for each $z \in U$. By definition $\tilde{h}_k(z)$ is orthogonal to $\mathcal{O}_{i+1}(z)$ for $k > i + 1$. Suppose $k < i$. Now $h_0 = h$ is holomorphic. This implies that $d\tilde{h}_0/d\bar{z}$ is a scalar function times h_0. It follows from (3.1) that

$$\frac{d\tilde{h}_k}{d\bar{z}} = \frac{d}{d\bar{z}}\left(\sum_{j=1}^{k} \alpha_j \frac{d^j \tilde{h}_0}{dz^j} \right) = \sum_{j=0}^{k} \alpha_j \frac{d^j}{dz^j}\left(\frac{d\tilde{h}_0}{d\bar{z}} \right) + \frac{d\alpha_j}{d\bar{z}} \frac{d^j \tilde{h}_0}{dz^j} \tag{3.4}$$

is in $\mathcal{O}_k$. Hence $\langle dh_i/dz, \tilde{h}_k \rangle = \langle \tilde{h}_i, dh_k/d\bar{z} \rangle = 0$ on U.

We now consider harmonic maps ϕ: $\mathbf{C}P^1 \to G_{2,4}$. It is convenient to make the following definition.

Definition 3.3. Given a smooth ϕ: $M \to G_{k,n+k}$, an adapted chart, $\{(U, z), (e_i), (f_j)\}$, for ϕ at p is:

(a) a holomorphic chart (U, z) containing p,

(b) maps e_i: $U \to \mathbf{C}^{n+k}$ $(1 \leqslant i \leqslant k)$ and f_j: $U \to \mathbf{C}^{n+k}$ $(1 \leqslant j \leqslant n)$ such that at each $z \in U$, $e_1, \cdots, e_k, f_1, \cdots, f_n$ gives a unitary frame of $\mathbf{C}^{n+k}$ and

(c) $e_1, \cdots, e_k$ spans the bundle $\phi^*(E_{k,n+k})$ over U.

Lemma 3.4. *Given a harmonic map ϕ: $\mathbf{C}P^1 \to G_{2,4}$ at least one of the tensors β' or $(\beta^*)'$ have to be of rank strictly less than 2.*

Proof. Let $\{(U, z), (e_i), (f_j)\}$ be an adpated chart for ϕ. Corollary 2.6 implies that $\beta_z^* \circ \beta_z$ is nilpotent on U. Hence there is an open subset $V \subseteq U$ such that at least one of β_z or β_z^* has less than maximal rank. But ϕ is real analytic (Theorem 0.3), so both β' and $(\beta^*)'$ are real analytic objects. Hence if either one has less than maximal rank on some open set of M, it must be so on all of M.

By replacing ϕ with $\phi^\perp$ if necessary, we may henceforth assume that β' is of rank less than two everywhere.

Lemma 3.5. *If β' is not identically zero then it has only isolated zeros.*

Proof. On any holomorphic chart (U, z) the tensor β_z is a holomorphic section of $\mathrm{Hom}(E, E^\perp)$. Since β' is given locally as $dz \otimes \beta_z$, β' is a holomorphic section of $\Lambda^{1,0}\mathbf{C}P^1 \otimes \mathrm{Hom}(E, E^\perp)$. The result follows.

If β' is identically zero then ϕ is antiholomorphic. Henceforth in this section we assume that ϕ is not antiholomorphic.

Lemma 3.6. *Let ϕ: $\mathbf{C}P^1 \to G_{2,4}$ be a harmonic map with $\beta' \not\equiv 0$ and* rank β' < 2 *everywhere. Given a point $p \in \mathbf{C}P^1$, there exists an adapted holomorphic chart $\{(U, z), (e_i), (f_j)\}$ for ϕ at p such that*:

(a) $\beta_z(e_1) = 0$ *and* $\beta_{\bar z}^*(f_1) = 0$;

(b) $\langle de_1/dz, f_1 \rangle = 0$ *and* $\langle df_2/d\bar z, e_1 \rangle = 0$.

Proof. We have that β' is a holomorphic section of the bundle $\Lambda^{1,0}\mathbf{C}P^1 \otimes \mathrm{Hom}(E, E^\perp)$. On any holomorphic chart (U, z) of M, β_z is a nonzero holomorphic section of $\mathrm{Hom}(E, E^\perp)$. β_z has only isolated zeros in U and where nonzero is always of rank < 2. If β_z vanishes at some point $p \in U$, then β_z can be expressed locally as $(z - z(p))^m \gamma$ where γ is a nonzero holomorphic section of $\mathrm{Hom}(E, E^\perp)$ in a neighborhood of p. Clearly γ is also of rank < 2. It follows that $\mathrm{Ker}\,\beta_z$ and $\mathrm{Im}\,\beta_z$ give holomorphic line bundles in E and $E^\perp$ respectively. Let (e_1, e_2) be a unitary frame of E, on a perhaps small open set of U, such that e_1 spans $\mathrm{Ker}\,\beta_z$. Let (f_1, f_2) be a unitary frame of $E^\perp$ such that f_2 spans $\mathrm{Im}\,\beta_z$. Since e_1 spans $\mathrm{Ker}\,\beta_z$ we have that $\beta_z(e_1) = 0$. Now $\langle \beta_z(e_i), f_1 \rangle = 0$ for $i = 1, 2$. By adjointness $\langle \beta_z(e_i), f_1 \rangle = -\langle e_i, \beta_{\bar z}^*(f_1) \rangle$. Hence $\beta_{\bar z}^*(f_1) = 0$. Also $\langle \beta_z(e_2), f_1 \rangle = \langle de_2/dz, f_1 \rangle = 0$ and

$$\langle e_1, df_2/d\bar z \rangle = \langle e_1, \beta_{\bar z}^\perp(f_2) \rangle = \langle \beta_z(e_1), f_2 \rangle = 0.$$

Hence both equations in (b) are true. This finishes the proof.

Moreover, it is clear from the discussion that the elements in the above frame are uniquely determined up to multiplication by scalars with values in $S^1 \subseteq \mathbf{C}^*$. Hence the elements in the above frame give globally defined maps of $\mathbf{C}P^1$ to $\mathbf{C}P^3$ with

$$\begin{aligned} &\Phi_i\colon \mathbf{C}P^1 \to \mathbf{C}P^3 \quad \text{corresponding to } e_i,\ i = 1, 2, \text{ and} \\ &\Psi_i\colon \mathbf{C}P^1 \to \mathbf{C}P^3 \quad \text{corresponding to } f_i,\ 1 = 1, 2. \end{aligned} \tag{3.5}$$

Theorem 3.7. *Let ϕ: $\mathbf{C}P^1 \to G_{2,4}$ be a harmonic map with β' not identically zero and* rank $\beta' < 2$ *everywhere. Then there is a nonconstant holomorphic map h: $\mathbf{C}P^1 \to \mathbf{C}P^3$ with $\Phi_2 = h_{m-1}$ and $\Psi_2 = h_m$ for some m, $1 \leqslant m \leqslant 3$.*

Proof. Let $\{(U, z), (e_i), (f_j)\}$ be an adapted chart for ϕ given by Lemma 3.6. Then equations (2.3) and (2.4) give

$$\begin{aligned} \nabla_{\bar z}^*\beta_z(e_2) &= \langle de_2/d\bar z, e_1 \rangle \beta_z(e_1) + \langle de_2/d\bar z, e_2 \rangle \beta_z(e_2) \\ &= \langle de_2/d\bar z, e_2 \rangle \beta_z(e_2), \\ \nabla_z \beta_{\bar z}^*(f_2) &= \langle df_2/dz, f_2 \rangle \beta_{\bar z}^*(f_2) + \langle df_2/dz, f_1 \rangle \beta_{\bar z}^*(f_1) \\ &= \langle df_2/dz, f_2 \rangle \beta_{\bar z}(e_2). \end{aligned} \tag{3.6}$$

Now $\beta_{\bar{z}}^{*}(f_2)$ is some nonzero scalar function times e_2. The second equation in (3.6) then implies that $\langle de_2/dz, e_1\rangle = 0$ on U. This means that

$$(3.7) \qquad \beta_{z,\phi}(e_2) = \beta_{z,\Phi_2}(e_2).$$

We also have that $\beta_{z,\phi}(e_2)$ is a scalar function times f_2. Since $\langle df_2/d\bar{z}, e_1\rangle = 0$,

$$(3.8) \qquad \nabla_{\bar{z},\phi}^{*}\beta_{z,\phi}(e_2) = \nabla_{\bar{z},\Phi_2}^{*}\beta_{z,\Phi_2}(e_2)$$

The first equation in (3.6) can then be written as

$$(3.9) \qquad \nabla_{\bar{z},\Phi_2}^{*}\beta_{z,\Phi_2}(e_2) = \langle de_2/d\bar{z}, e_2\rangle \beta_{z,\Phi_2}(e_2).$$

This is just (2.3) for the map Φ_2: $\mathbf{C}P^1 \to \mathbf{C}P^3$. A similar argument works for Ψ_2: $\mathbf{C}P^1 \to \mathbf{C}P^3$. Moreover, since $\beta_{z,\Phi_2}(e_2)$ is a scalar multiple of f_2, it follows that Φ_2 and Ψ_2 are consecutive harmonic maps generated by some holomorphic map h: $\mathbf{C}P^1 \to \mathbf{C}P^3$.

Let H: $\mathbf{C}P^1 \to G_{2,4}$ be given by $H(p) = (h_{m-1}(p) \oplus h_m(p))$ for all $p \in \mathbf{C}P^1$. A map F: $\mathbf{C}P^1 \to \mathbf{C}P^3$ such that $F(p) \perp h_{m-1}(p) \oplus h_m(p)$ for all $p \in \mathbf{C}P^1$ determines a unique subbundle of rank 1 in $H^*E_{2,4}$. Conversely, any subbundle of rank 1 in $H^*E_{2,4}$ determines a map from $\mathbf{C}P^1$ to $\mathbf{C}P^3$ with this property. The trivial bundle $\mathbf{C}P^1 \times \mathbf{C}^4$ induces a connection on the subbundle $H^*E_{2,4}$. The $(1,0)$ part of this connection induces an antiholomorphic structure on $H^*E_{2,4}$ (see Theorem 2.2).

Theorem 3.8. *Let ϕ: $\mathbf{C}P^1 \to G_{2,4}$ be as in Theorem 3.7. The complex line bundle determined by Φ_1 in $H^*E_{2,4}$ is antiholomorphic.*

Proof. Using the frame described in Lemma 3.6, we have $\langle de_1/dz, f_i\rangle \equiv 0$ for $i = 1,2$. It follows that the line bundle $\Phi_1^*E_{1,4}$ is lift fixed by the $(1,0)$ part of the connection on $H^*E_{2,4}$. The conclusion follows.

We now prove a converse to this theorem.

Theorem 3.9. *Let $\Phi_2 = h_{m-1}$ and $\Psi = h_m$ where h: $\mathbf{C}P^1 \to \mathbf{C}P^3$ is a nonconstant holomorphic map. Defining H as before, we have that any antiholomorphic subbundle of rank 1 in $H^*E_{2,4}$ gives rise to a map Φ_1: $\mathbf{C}P^1 \to \mathbf{C}P^3$ with $\Phi_1 \oplus \Phi_2$: $\mathbf{C}P^1 \to G_{2,4}$ harmonic. Moreover $\phi = \Phi_1 \oplus \Phi_2$ has $\beta' \not\equiv 0$ and* rank β' *< 2 everywhere.*

Proof. Let (U, z) be a holomorphic chart on $\mathbf{C}P^1$ with lifts e_i: $U \to S^7 \subseteq C^4$, $i = 1,2$, of Φ_i: $\mathbf{C}P^1 \to \mathbf{C}P^3$, $i = 1,2$. Assume also that f: $U \to S^7$ gives a lift of Ψ on U. Then since $\langle e_1, e_2\rangle = 0$ and $\langle e_1, f\rangle = 0$, $\langle de_1/dz, f\rangle = -\langle e_1, df/d\bar{z}\rangle = 0$. (Lemma 3.2 implies that $df/d\bar{z}$ is always in the span of e_2 and f.) Since Φ_1 gives an antiholomorphic line bundle in $H^*E_{2,4}$, it follows that $\beta_{z,\phi}(e_1) = 0$ on U. Since Φ_2 is harmonic,

$$\nabla_{\bar{z},\Phi_2}^{*}\beta_{z,\Phi_2}(e_2) = \langle de_2/d\bar{z}, e_2\rangle \beta_{z,\Phi_2}(e_2).$$

But, by Lemma 3.2, $\beta_{z,\Phi_2}(e_2) = \beta_{z,\phi}(e_2)$ and $\nabla^*_{\bar z,\phi}\beta_{z,\phi}(e_2) = \nabla^*_{\bar z,\Phi_2}\beta_{z,\Phi_2}(e_2)$. The harmonic equations for Φ_2 can then be rewritten as

$$\nabla^*_{\bar z\phi}\beta_{z,\phi}(e_2) = \langle de_2/d\bar z, e_2\rangle \beta_{z,\phi}(e_2).$$

Since $\beta_{z,\phi}(e_2) = 0$ it follows that (e_1, e_2) satisfies the system (2.3). Hence ϕ is harmonic. Also $\beta_z(e_2)$ is a nontrivial scalar function times f. So β' is nonzero and rank $\beta' < 2$.

We now write out more explicitly the maps in Theorem 3.9 when the holomorphic map h: $\mathbf{C}P^1 \to \mathbf{C}P^3$ is full.

Example 3.10. Let h: $\mathbf{C}P^1 \to \mathbf{C}P^3$ be a full holomorphic map. Then let h_0, h_1, h_2, h_3 be the harmonic maps generated by h as in (1.14) ($h_0 = h$).

Case I. $\Phi_2 = h_0$, $\Psi = h_1$. In this case Φ_1: $\mathbf{C}P^1 \to \mathbf{C}P^3$ is forced to be antiholomorphic map such that $\Phi_1(p) \subseteq h_2(p) \oplus h_3(p)$ for all $p \in \mathbf{C}P^1$. Any such choice of map Φ_1 yields a harmonic map $\phi = \Phi_1 \oplus \Phi_2$: $\mathbf{C}P^1 \to G_{2,4}$. This is a special case of the examples of Din and Zakrzewski [3] of harmonic maps into general Grassmannians.

Case II. $\Phi_2 = h_2$, $\Psi = h_3$. In this case $H = h_0 \oplus h_1$ and Φ_1 is forced to come from an antiholomorphic bundle of rank 1 in $H^*E_{2,4}$.

Case III. $\Phi_2 = h_1$, $\Psi = h_2$. In this case, $H = h_0 \oplus h_3$, and Φ_1: $\mathbf{C}P^1 \to \mathbf{C}P^3$ comes from an antiholomorphic line bundle in $H^*E_{2,4}$.

4. An example in the higher rank case

In this section we give some examples of harmonic maps into higher rank Grassmannians that are suggested by Theorem 3.9.

Let h: $\mathbf{C}P^1 \to \mathbf{C}P^N$ be nondegenerate holomorphic. By the method in §1, h generates $N+1$ harmonic maps $h_0, \cdots, h_N$ with $h_0 = h$.

Lemma 4.1. *Let $S = \{i_1, \cdots, i_j\} \subseteq \{0, \cdots, N\}$ be an arbitrary nonempty subset. Then $\phi = h_{i_1} \oplus \cdots \oplus h_{i_k}$ is a harmonic map from $\mathbf{C}P^1$ to $G_{k,N+1}$.*

Proof. Given a point $p \in \mathbf{C}P^1$, let $e_0, \cdots, e_N$ be a unitary frame defined on a holomorphic chart (U, z), $p \in U$, such that e_i: $U \to S^{2N+1}$ is a lift of h_i: $\mathbf{C}P^1 \to \mathbf{C}P^N$. We want to show that $(e_{i_1}, \cdots, e_{i_k})$ satisfies the system (2.18). Since each h_i is harmonic we have

$$\tilde\nabla^*_{z,h_i}\beta_{z,h_i}(e_i) = \langle de_i/d\bar z, e_i\rangle \beta_{z,h_i}(e_i) \tag{4.1}$$

for $i = 0, \cdots, N$. Lemma 3.2 shows that $\langle de_{i_m}/d\bar z, e_{i_1}\rangle \not\equiv 0$ only if $m = 1$ or $i_1 = i_m - 1$. If $i_1 = i_m - 1$, $\beta_z(e_{i_1}) = 0$. Hence the system we have to verify is

$$\tilde\nabla^*_{\bar z,\phi}\beta_{z,\phi}(e_{i_m}) = \langle de_{i_m}/d\bar z, e_{i_m}\rangle \beta_{z,\phi}(e_{i_m}) \tag{4.2}$$

for $m = 1, \cdots, k$. If $i_m + 1 \in S$, $\beta_{z,\phi}(e_{i_m}) = 0$ and equation (4.2) is valid. Suppose $i_m + 1 \notin S$. Then using Lemma 3.2, $\beta_{z,\phi}(e_{i_m}) = \beta_{z,h_{i_m}}(e_{i_m})$ and $\tilde{\nabla}^*_{\bar{z},\phi}\beta_{z,\phi}(e_{i_m}) = \tilde{\nabla}^*_{\bar{z},h_{i_m}}(e_{i_m})$. Equation (4.1) then implies equation (4.2) for such m.

Suppose that subset S of $\{0, \cdots, N\}$ is small enough so that the set $S' = \{j \in \mathbf{Z}\colon 0 \leqslant j \leqslant N, j \notin S, j - 1 \notin S\}$ is nonempty. Let the cardinality of S' be given by k'. Then define the map $H\colon \mathbf{C}P^1 \to G_{k',N+1}$ by $H = h_{j_1} \oplus \cdots \oplus h_{j_k}$, where $S' = \{j_1, \cdots, j_{k'}\}$. $H^*E_{k',N+1}$ has a complex connection $\tilde{D}$ gotten by restricting the standard connection on $\mathbf{C}P^1 \times \mathbf{C}^{N+1}$. By Theorem 4.2, $H^*E_{k',N+1}$ has a unique antiholomorphic structure induced by the $(1,0)$ part of $\tilde{D}$. Let V be a fixed antiholomorphic subbundle of $H^*E_{k',N+1}$ of rank r. V induces a map $\psi\colon \mathbf{C}P^1 \to G_{r,N+1}$ such that $\psi(p) \perp \phi(p)$ for all $p \in \mathbf{C}P^1$. With this set-up we have the following.

Proposition 4.2. *$\phi \oplus \psi\colon \mathbf{C}P^1 \to G_{r+k,N+1}$ gives a harmonic map.*

Proof. Let (U, z) be a holomorphic chart on $\mathbf{C}P^1$ with lifts $e_0, \cdots, e_N$ of $h_0, \cdots, h_N$ respectively. So $\langle e_i, e_j \rangle = \delta_{ij}$ on U. Assume also that maps of U to S^{2N+1}, $f_1, \cdots, f_r$, are given that produce a unitary frame for V on U. We can now easily verify that the unitary frame $e_{i_1}, \cdots, e_{i_k}, f_1, \cdots, f_r$ satisfies system (2.18). (Here the connection and second fundamental forms involved are with respect to the map $\phi \oplus \psi$.) By Lemma 4.8 and the definition of S', system (4.2) still holds with β and ∇^* now being induced by the map $\phi \oplus \psi$. Since $\beta_z(f_j) = 0$ for $1 \leqslant j \leqslant r$, (4.2) gives

$$
\begin{aligned}
\nabla^*_{\bar{z}}\beta_z(e_{i_m}) &= \langle de_{i_m}/d\bar{z}, e_{i_m} \rangle \beta_z(e_{i_m}) \\
&= \sum_{j=1}^{k} \langle de_{i_m}/d\bar{z}, e_{i_j} \rangle \beta_z(e_{i_j}) \\
&\quad + \sum_{j=1}^{k} \langle de_{i_m}/d\bar{z}, f_j \rangle \beta_z(f_j).
\end{aligned} \tag{4.3}
$$

By Lemma 4.8 and the definition of S', $\langle df_j/d\bar{z}, e_{i_m} \rangle \equiv 0$ on U for all $1 \leqslant j \leqslant r$ and $1 \leqslant m \leqslant k$. It follows that

$$
\begin{aligned}
\nabla^*_{\bar{z}}\beta_z(f_j) = 0 = &\sum_{m=1}^{k} \langle df_j/d\bar{z}, e_{i_m} \rangle \beta_z(e_{i_m}) \\
&+ \sum_{m=1}^{r} \langle df_j/d\bar{z}, f_m \rangle \beta_z(f_m).
\end{aligned} \tag{4.4}
$$

Equations (4.3) and (4.4) together give system (2.18) for the map $\phi \oplus \psi$.

References

[1] D. Burns, *Harmonic maps from* $\mathbf{C}P^1$ *to* $\mathbf{C}P^n$, Lecture Notes in Math. Vol. 949, Springer, New York, 1982, 48–55.

[2] S. S. Chern, *Complex manifolds without potential theory*, 2nd ed., Springer, New York, 1979.

[3] A. M. Din & W. J. Zakrzewski, *General classical solutions in the* $\mathbf{C}P^{n-1}$ *model*, Nuclear Phys. B **174** (1980) 397–406.

[4] J. Eells & L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978) 1–68.

[5] J. Eells & J. H. Sampson, *Harmonic mapping of Riemannian manifolds*, Amer. J. Math. **86** (1964) 109–160.

[6] J. Eells & J. Wood, *Harmonic maps from surfaces to complex projective space*, Advances in Math. to appear.

[7] V. Glaser & R. Stora, *Regular solutions of the* $\mathbf{C}P^n$ *models and further generalizations*, preprint, 1980.

[8] J. L. Koszul & B. Malgrange, *Sur certaines structures fibreés complexes*, Arch. Math. (Basel) **9** (1958) 102–109.

[9] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. II, Interscience, New York, 1966.

[10] B. Malgrange, *Lectures on the theory of several complex variables*, Tata Institute, Bombay, 1958.

[11] J. Wolfson, *Minimal surfaces in complex manifolds*, Thesis, Berkeley, 1982.

[12] J. Wood & S. Erdem, *On the construction of harmonic maps into a Grassmannian*, preprint, 1982.

[13] H. H. Wu, *Equidistribution theory of holomorphic curves*, Annals of Math. Studies No. 64, Princeton University Press, Princeton, 1970.

UNIVERSITY OF CHICAGO