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THE TWISTED INDEX PROBLEM FOR MANIFOLDS WITH BOUNDARY

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SECTION ZERO

0.1 Introduction

The Atiyah-Singer index theorem gives a formula in K-theory for the index of any elliptic operator. The Atiyah-Patodi-Singer twisted index theorem [2] is the suspension of the Atiyah-Singer index theorem and gives a measure of spectral flow and spectral asymmetry using the eta invariant with coefficients in a locally flat bundle. It is possible to recover the Atiyah-Singer theorem from the twisted index theorem using certain product formulas so the results of [2] can be viewed as a generalization of the ordinary index theorem as we shall see in §1.3.

It is well known that certain elliptic complexes (for example the signature complex) do not admit local boundary conditions. However, for those which do, the Atiyah-Bott theorem [1] provides a generalization of the index theorem to manifolds with boundary. In a similar fashion, not every twisted index problem admits local boundary conditions; the operator $*d \pm d*$ is one of those which does not as we shall see later. This paper is an effort to combine both the Atiyah-Bott index theorem and the Atiyah-Patodi-Singer twisted index theorem to derive a formula in terms of characteristic classes for the twisted index on a manifold with boundary given local boundary conditions. We are able to treat completely all the operators, arising naturally in Riemannian geometry, which admit local boundary conditions, but the general case is still incomplete although we have a number of strong results in that direction. This formula would contain both the Atiyah-Patodi-Singer twisted index

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theorem and the Atiyah-Bott index theorem (with strongly elliptic boundary conditions) as special cases.

In discussing boundary conditions, the natural boundary conditions to impose are much stronger than those considered by Atiyah-Bott. This is due in part to the more delicate nature of the eta invariant in contrast to the ordinary index and does pose technical difficulties we shall discuss in the third section.

The organization of this paper is as follows: the first section is divided into three subsections. §1.1 gives a brief review of secondary characteristic classes and the Chern character. In §1.2 we discuss Bott periodicity in the setting which we will need it. We also discuss suspensions and the relation of the index theorem to the twisted index theorem. In §1.3 we discuss the formula for the twisted index for manifolds without boundary and obtain equivalent formulas by suspension.

The second section is divided into four subsections. §2.1 gives notational conventions and a review of the definition of ellipticity. In §2.2 we discuss operators with leading symbols given by Clifford multiplication. In §2.3 we show that the eta function is regular at $s = 0$ for such operators with elliptic boundary conditions, while in §2.4 we derive a formula in K-theory for the twisted index of such operators.

The third section contains two subsections. §3.1 deals with deriving a suitable candidate in K-theory to generalize the formulas of Atiyah-Bott. In §3.2 we show that this formula in K-theory has the same functorial properties as the twisted index does.

The remainder of this introduction consists of a discussion of the eta invariant in the context which we shall need. This paper is quite topological in flavor and relies heavily on the results which we derived in [7] regarding the analytical facts about such operators.

In particular, the analysis of [7] requires in an essential fashion that certain bundles $\Pi_{\pm}(p)$ over the fiber spheres in $T^*(M)$ be topologically trivial in a very strong sense. We do not see at present how to remove this restriction in order still to obtain all the requisite analytic results. The referee has kindly informed us that the η-invariant for the signature operator in this context has been discussed by Cheeger (*Spectral geometry of singular Riemannian spaces*, to appear in J. Differential Geometry) in a context which can be viewed as equivalent to that of global boundary conditions. It is unclear to what extent such global boundary conditions would permit the analysis of [7] to be extended; in particular η can jump by a noninteger amount under perturbations of some boundary conditions as discussed in [7]. We would like to thank the referee for bringing this matter to our attention.

0.2 Analytical facts concerning the eta invariant

Let M be a compact manifold of dimension m with smooth boundary dM, and let V be a smooth vector bundle over M. Let P: $C^\infty(V) \to C^\infty(V)$ be a partial differential operator of order $d > 0$. If $dM \neq \varnothing$, we let B be a suitable boundary condition. Let P_B denote the operator P restricted to the space of smooth sections of V satisfying the boundary condition B. Let $y \in M$ and let (y, ξ) denote a point of the cotangent space T^*M. Let $p(y, \xi)$ be the leading symbol of P. Suppose that

$$\det(p(y, \xi) - it) \neq 0 \quad \text{for } \xi \in T^*M, t \in R, (\xi, t) \neq (0,0),$$

and say that p is elliptic with respect to the imaginary axis. If $dM \neq \varnothing$, we impose a stronger condition on p and the boundary condition which will be discussed in the second section.

Under these ellipticity conditions, the spectrum of P_B is discrete. Each generalized eigenspace is finite dimensional and consists of smooth sections to V satisfying the given boundary condition. Let $\{\lambda_\nu\}$ denote the eigenvalues of P_B repeated according to multiplicity; only a finite number of eigenvalues lie on the imaginary axis. We define

$$\eta(s, P, B) = \sum_{\mathrm{Re}(\lambda_\nu)>0} \lambda_\nu^{-s} - \sum_{\mathrm{Re}(\lambda_\nu)<0} (-\lambda_\nu)^{-s}$$

as a measure of the spectral asymmetry of the operator P_B. This is holomorphic in s for $\mathrm{Re}(s) \gg 0$. $\eta(s, P, B)$ has a meromorphic extension to $\mathbf{C}$ with isolated simple poles at $s = (m - n)/d$ for $n = 0, 1, 2, \cdots$. The residue of η at these poles is given by a local formula [7].

The pole at $s = 0$ is of particular interest. In [7] we showed that if $(P_\varepsilon, B_\varepsilon)$ is a smooth 1-parameter family of such operators and boundary conditions, then

$$\frac{d}{d\varepsilon} \mathrm{Re}\, s_{s=0}\eta(s, P_\varepsilon, B_\varepsilon) = 0,$$

which shows that the residue is a homotopy invariant. Let $V(iR)$ denote the finite dimensional vector space generated by the generalized eigensections corresponding to purely imaginary eigenvalues. We define

$$\tilde{\eta}(P, B) = \frac{1}{2}\left\{\eta(s, P, B) - \frac{1}{s}\, \mathrm{Re}\, s_{s=0}\eta(s, P, B) - \dim V(iR)\right\}_{s=0}, \quad \bmod Z,$$

in C/Z. $\tilde{\eta}(P_\varepsilon, B_\varepsilon)$ is differentiable in the parameter ε since we have corrected for the jumps which occur as eigenvalues cross the imaginary axis. We showed that $d\tilde{\eta}(P_\varepsilon, B_\varepsilon)/d\varepsilon$ is given by a local formula in the jets of the operators $(P_\varepsilon, B_\varepsilon, \dot{P}_\varepsilon, \dot{B}_\varepsilon)$.

If the boundary of M is empty, then $\eta(s, P)$ is regular at $s = 0$, [2], [6]. It is worth noting that this does not follow from a local calculation as the local formula for the residue at $s = 0$ does not vanish identically in general [5]. In the second section of this paper, we will show that $\eta(s, P, B)$ is regular at $s = 0$ for a suitable class of first order operators.

We can construct a nontrivial twisted index by taking coefficients in a locally flat bundle. Let ρ: $\pi_1(M) \to GL(k, C)$ be a representation of the fundamental group. If $\overline{M}$ is the universal cover of M, we define

$$V_\rho = \overline{M} \times C^k/\pi_1(M),$$

where $\pi_1(M)$ acts on $\overline{M}$ by deck transformations and on C^k by the representation ρ. The transition functions of V_ρ are locally constant. V_ρ inherits a connection ∇_ρ from $\overline{M} \times C^k$ with zero curvature. The holonomy of ∇_ρ is just ρ. Since the curvature of ∇_ρ is zero, the rational characteristic classes of V_ρ vanish. This implies that V_ρ is a torsion class in the reduced K-theory group of M; i.e., $nV_\rho \simeq 1^{nk}$ for some integer n.

Since the transition functions of V_ρ are locally constant, we can define P_ρ on $C^\infty(V \otimes V_\rho)$ and a boundary condition B_ρ to be locally isomorphic to k-copies of (P, B). (P_ρ, B_ρ) will also satisfy the given ellipticity conditions. The residue of the poles of η is given by a local formula. Since (P_ρ, B_ρ) and $k(P, B)$ are locally isomorphic, any local formulas will be the same, so the two local formulas cancel in the poles for $\eta(s, P_\rho, B_\rho) - k\eta(s, P, B)$. This shows that $\eta(s, P_\rho, B_\rho) - k\eta(s, P, B)$ defines an entire function of s.

Define

$$\operatorname{ind}(\rho, P, B) = \tilde{\eta}(P_\rho, B_\rho) - k\tilde{\eta}(P, B) \in C, \quad \operatorname{mod} Z.$$

If $(P_\varepsilon, B_\varepsilon)$ is a smooth 1-parameter family of such operators, the same cancellation of local formulas for locally isomorphic operators implies

$$\frac{d}{d\varepsilon} \operatorname{ind}(\rho, P_\varepsilon, B_\varepsilon) = 0,$$

which shows that $\operatorname{ind}(\rho, P, B)$ is a homotopy invariant of (P, B).

One cannot lift $\operatorname{ind}(\rho, P, B)$ from C mod Z to C consistently in general without imposing some additional structure. The bundle V_ρ is rationally trivial. We suppose henceforth that V_ρ is itself topologically trivial and choose a global frame $\vec{s}$ for V_ρ. This permits us to define $k(P, B)$ acting on $C^\infty(V \otimes V_\rho) \simeq C^\infty(V \otimes 1^k)$. The two operators (P_ρ, B_ρ) and $k(P, B)$ have the same leading symbol. Define $(P_\varepsilon, B_\varepsilon) = \varepsilon(P_\rho, B_\rho) + (1 - \varepsilon)(kP, kB)$. This 1-parameter family satisfies the ellipticity conditions and

$$\operatorname{ind}(\rho, P, B) = {}_0\!\int^1 \frac{d}{d\varepsilon} \tilde{\eta}(P_\varepsilon, B_\varepsilon)\, d\varepsilon.$$

Since the derivative is given by a local formula, this identity gives a lift of $\operatorname{ind}(\rho, P, B)$ from $C \bmod Z$ to C and shows that $\operatorname{ind}(\rho, P, B)$ is given by a local formula (which depends on the global frame $\vec{s}$ chosen). It should be noted that different choices of global frames will in general give rise to different liftings from $C \bmod Z$ to C.

If $dM = \varnothing$, then the twisted index theorem of Atiyah et al. [2] gives a formula for $\operatorname{ind}(\rho, P)$ in terms of secondary characteristic classes if m is odd; the corresponding generalization for even m can be found in [6]. There is also a formula in K-theoretic terms valid for general ρ. Since this formula is more complicated to explain, we shall restrict our attention to the case in which V_ρ is topologically trivial. We refer the reader to [2], [6] for examples.

Before discussing the Atiyah-Patodi-Singer formula for $\operatorname{ind}(\rho, P, B)$ which we will generalize, we first review the Atiyah-Bott and the Atiyah-Singer index theorems. We fix a Riemannian metric on M. Let $D(T^*M)$ denote the unit disk, and $S(T^*M)$ the unit sphere bundles of T^*M, i.e., let

$$D(T^*M) = \{\xi \in T^*M \colon |\xi| \leq 1\}, \quad S(T^*M) = \{\xi \in T^*M \colon |\xi| = 1\}.$$

If $dM = \varnothing$, then $S(T^*M)$ is the boundary of $D(T^*M)$.

Let $\Sigma(T^*M)$ be the fiber suspension of $S(T^*M)$. $\Sigma(T^*M)$ is the unit sphere bundle in $T^*M \oplus 1$. It can also be defined by taking two copies $D_\pm(T^*M)$ of the unit disk bundle and joining them along their common edge $S(T^*M)$. Let N and S be the north and south poles of $\Sigma(T^*M)$. N is the zero section to $D_+(T^*M)$, while S is the zero section to $D_-(T^*M)$.

We can describe the Atiyah-Singer index theorem using $\Sigma(T^*M)$. Let $Q\colon C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic operator with leading symbol q which defines a map $q\colon S(T^*M) \to \operatorname{END}(V_1, V_2)$ from the sphere bundle of T^*M to the bundle of maps from V_1 to V_2. Let $\Sigma(q)$ be the bundle over $\Sigma(T^*M)$,

$$V_{1|D_+(T^*M)} \cup V_{2|D_-(T^*M)},$$

where we use q to identify V_1 with V_2 over the edge $S(T^*M)$.

Let $\operatorname{TODD}(M)$ denote the real Todd class of M. This is a complicated polynomial in the Pontrjagin classes of T^*M. TODD is a multiplicative class in the sense that $\operatorname{TODD}(M_1 \times M_2) = \operatorname{TODD}(M_1) \wedge \operatorname{TODD}(M_2)$. We refer to [3] for details. Let $\operatorname{ch}(\Sigma q)$ denote the Chern character of the bundle Σq. Using a suitable orientation of ΣT^*M, which will be discussed in more detail later, the Atiyah-Singer formula becomes

$$\operatorname{index}(Q) = \int_{\Sigma(T^*M)} \operatorname{TODD}(M) \wedge \operatorname{ch}(\Sigma q).$$

(To be precise, there is an additional factor $(-1)^m$ because of the orientations chosen. We will discuss this in more detail in §2.3 and postpone until then a precise discussion of this formula.)

To extend this to the case in which $dM = \varnothing$, we must impose elliptic boundary conditions. If we restrict q to dM, then the boundary condition B defines an explicit homotopy q_B of the symbol to a symbol which is independent of the fiber coordinate. Define $M' = dM \times [-1,0] \cup M$ where we identify $dM \times \{0\}$ with the boundary of M; this sews on a collar. We use the homotopy to extend q to a symbol q_B defined on M' such that $q_B(y, \xi) = q_B(y)$ is independent of the fiber coordinate $\xi \in T^*M$ for $y \in dM^1$. If index(Q, B) is as defined in [1], then the Atiyah-Bott formula has the form

$$\operatorname{index}(Q, B) = \int_{\Sigma(T^*M')} \operatorname{TODD}(M) \wedge \operatorname{ch}(\Sigma q_B).$$

(It is worth noting that index(Q, B) is *not* $\dim(\operatorname{Ker} Q_B) - \dim \operatorname{Ker}(Q_B^*)$ in general.)

Since this is not the form in which the Atiyah-Bott formula is most commonly stated, it is worth digressing briefly to consider an alternate formulation. Suppose first that $dM = \varnothing$, and that $V_1 = V_2 = 1^j$ are trivial bundles. We can interpret q: $S(T^*M) \to GL(j, C)$. Let $\omega = dg \cdot g^{-1}$ be the Maurer-Cartan form and define:

$$\operatorname{Tch} = \sum_k c_k \operatorname{Tr}(\omega^{2k-1}) \quad \text{for } c_k = \left(\frac{i}{2\pi}\right)^k \frac{1}{(k-1)!} \int_0^1 (t - t^2)^{k-1}\, dt.$$

(We will discuss these normalizing constants further in §1.1. This is the transgression of the Chern character.) We pull-back Tch using q^* to define $q^*(\operatorname{Tch})$ as the sum of the odd-dimensional cohomology classes on $S(T^*M)$. An easy application of Stokes theorem converts the Atiyah-Singer formula into

$$\operatorname{index}(Q) = \int_{S(T^*M)} \operatorname{TODD}(M) \wedge q^*(\operatorname{Tch}).$$

If $dM \neq \varnothing$, $S(T^*M)$ is not the full boundary of $D(T^*M)$. The homotopy q_B defines an extension of q to the restriction of the unit disk bundle over dM:

$$q_B\colon D(T^*M)_{|dM} \to GL(j, C).$$

This defines q_B on all of the boundary of $D(T^*M)$, and the Atiyah-Bott formula can be expressed in the form

$$\operatorname{index}(Q, B) = \int_{d(D(T^*M))} \operatorname{TODD}(M) \wedge q_B^*(\operatorname{Tch}).$$

We now consider the twisted index.

Let $P\colon C^\infty(V) \to C^\infty(V)$ have symbol p and let $dM = \varnothing$. If P is elliptic with respect to the imaginary axis, let $\Pi_\pm(p)$ be the bundles over $S(T^*M)$ spanned by the generalized eigenvectors of p, which correspond to eigenvalues with positive/negative real part. (Since p has no purely imaginary eigenvalues, $\Pi_\pm(p)$ have constant rank and define smooth bundles.) In a suitable sense, P is determined by the bundles $\Pi_\pm(p)$ in much the same way that Q was determined by Σq; we refer to [2], [6] for a more precise description of this relationship. The virtual bundle $\Pi_+(p) - \Pi_-(p) \in K(S(T^*M))$ is an infinitesimal measure of the spectral asymmetry of P.

Let $\vec{s}$ be a global frame for V_ρ and let $\nabla_\rho(\vec{s}) = \omega \cdot \vec{s}$ define the connection 1-form $\omega \in C^\infty(T^*M \otimes \mathrm{END}(V))$. Let

$$\mathrm{Tch}(\omega) = \sum_k c_k \operatorname{Tr}(\omega^{2k-1}).$$

Then

$$\operatorname{ind}(\rho, P) = \frac{1}{2}\int_{S(T^*M)} \mathrm{Tch}(\rho) \wedge \mathrm{TODD}(M) \wedge \mathrm{ch}(\Pi_+(p) - \Pi_-(p)).$$

Since $\Pi_+(p) \oplus \Pi_-(p) = V$, $\mathrm{ch}(\Pi_+(p) - \Pi_-(p)) = 2\,\mathrm{ch}(\Pi_+(p)) - \mathrm{ch}(V)$. Since $\mathrm{ch}(V)$ does not depend on the fiber coordinate, it contributes nothing to the top dimensional $(2m-1)$-form which we integrate over $S(T^*M)$, so we can express

$$\operatorname{ind}(\rho, P) = \int_{S(T^*M)} \mathrm{Tch}(\rho) \wedge \mathrm{TODD}(M) \wedge \mathrm{ch}(\Pi_+(p)).$$

Unfortunately, this theorem does not generalize directly to the case of manifolds with boundary; a boundary condition does not define a homotopy of p to an operator with symbol independent of the fiber through symbols elliptic with respect to the imaginary axis. In §1.3 we will discuss a generalization of this formula which has the form

$$\operatorname{ind}(\rho, P) = \int_{\Sigma^{2j}(T^*M)} \mathrm{Tch}(\rho) \wedge \mathrm{TODD}(M) \wedge \mathrm{ch}\big(\Pi_+(\Sigma^{2j}p)\big),$$

where $j > 0$, and where Σ^ν will be defined in §1.1 and §1.2. We will use the elliptic boundary condition to define a homotopy of Σp, which we shall denote by Σp_B. We emphasize that this homotopy is not the suspension of a homotopty p_B in general. This will provide the context in which to generalize the Atiyah-Bott theorem.

In this paper we always work in the smooth category. Some of the homotopies which we will construct are only piecewise smooth; we always smooth out these continuous homotopies at the corners. To avoid complicating the exposition, we will usually not explicitly mention such smoothings.

SECTION ONE

1.1 The Chern character and secondary characteristic classes

We first recall the definition of the Chern character. Let W be a smooth vector bundle over some manifold N, and let ∇ be a connection on W. If $\vec{s}$ is a local frame for V, let $\nabla\vec{s} = \omega \cdot \vec{s}$ be the connection 1-form. The curvature is an invariantly defined section of $\Lambda^2(T^*M) \otimes \mathrm{END}(W)$ which is defined by $\Omega = d\omega - \omega \wedge \omega$. Define

$$\mathrm{ch}_k(\nabla) = \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \mathrm{Tr}(\Omega^k),$$

which is a closed $2k$-form independent of the frame $\vec{s}$ chosen. The total Chern character is given by

$$\mathrm{ch}(\nabla) = 1 + \mathrm{ch}_1(\nabla) + \cdots + \mathrm{ch}_k(\nabla) + \cdots \in H^{2*}(N; C).$$

If ∇_1 and ∇_0 are two connections on W, we form $\nabla_t = t\nabla_1 + (1-t)\nabla_0$. Let $\theta = \omega_1 - \omega_0$, then θ is tensorial. The connection 1-form of ∇_t is $\omega_t = t\theta + \omega_0$. If Ω_t is the curvature of the connection ∇_t, by using the identity

$$d\mathrm{Tr}\left(\theta\Omega_t^{k-1}\right) = \frac{1}{k}\frac{d}{dt}\mathrm{Tr}\left(\Omega_t^k\right)$$

we obtain (for further details see [4])

$$\mathrm{ch}_k(\nabla_1) - \mathrm{ch}_k(\nabla_0) = {}_0\!\int^1 \frac{d}{dt}\mathrm{ch}_k(\nabla_t)\,dt = d(\mathrm{Tch}_k(\nabla_1, \nabla_0)),$$

where

$$\mathrm{Tch}_k(\nabla_1, \nabla_0) = \left(\frac{i}{2\pi}\right)^k \cdot \frac{1}{(k-1)!}\mathrm{Tr}\left\{{}_0\!\int^1 \theta\Omega_t^{k-1}\,dt\right\}.$$

This shows that the difference $\mathrm{ch}_k(\nabla_1) - \mathrm{ch}_k(\nabla_0)$ is exact, so that $\mathrm{ch}(W) = \mathrm{ch}(\nabla) \in H^{2*}(N; C)$ is defined in cohomology independently of the connection chosen. It is immediate that

$$\mathrm{ch}(W_1 \oplus W_2) = \mathrm{ch}(W_1) + \mathrm{ch}(W_2), \quad \mathrm{ch}(W_1 \otimes W_2) = \mathrm{ch}(W_1)\mathrm{ch}(W_2).$$

The Chern character defines an isomorphism between K-theory with complex coefficients and the even dimensional cohomology on N with complex coefficients.

Tch is the transgression of the Chern character and is independent of the frame chosen. Suppose for the moment that both ∇_1 and ∇_0 are flat so that $\Omega_1 = \Omega_0 = 0$. Choose a local frame so $\omega_0 \equiv 0$. Then $\omega_1 = \theta$ and $\Omega_1 = d\theta - \theta \wedge \theta = 0$.

Consequently $\omega_t = t\theta$ and $\Omega_t = td\theta - t^2\theta \wedge \theta = (t - t^2)\theta \wedge \theta$. As in the introduction, we define the constants

$$c_k = \left(\frac{i}{2\pi}\right)^k \frac{1}{(k-1)!} \int_0^1 (t - t^2)^{k-1}\, dt,$$

so that

$\mathrm{Tch}_k(\nabla_1, \nabla_0) = c_k \operatorname{Tr}(\theta^{2k-1})$.

We illustrate these ideas by establishing the equality of the two formulas which we have given for the index of an elliptic operator. Let $dM = \varnothing$ and let $V_1 = V_2 = 1^k$ be the trivial bundle. Let $Q\colon C^\infty(1^k) \to C^\infty(1^k)$ be an elliptic complex. Let $\vec{s}_\pm$ be global frames for Σq over a neighborhood of $D_\pm(T^*M)$ so that $\vec{s}_- = q\vec{s}_+$ on the overlap $S(T^*M)$. Choose connections $\nabla_\pm$ for Σq so $\nabla_\pm(\vec{s}_\pm) = 0$. Then $\Omega_\pm = 0$ on $D_\pm(T^*M)$. Thus

$$\operatorname{index}(Q) = \int_{\Sigma(T^*M)} \mathrm{TODD}(M) \wedge \mathrm{ch}(\nabla_-) = \int_{D_-(T^*M)} \mathrm{TODD}(M) \wedge \mathrm{ch}(\nabla_+).$$

On D_-, $\Omega_- = 0$ so $\mathrm{ch}(\nabla_+) = \mathrm{ch}(\nabla_+) - \mathrm{ch}(\nabla_-) = d\,\mathrm{Tch}(\nabla_+, \nabla_-)$. Stoke's theorem implies

$$\operatorname{index}(Q) = \int_{S(T^*M)} \mathrm{TODD}(M) \wedge \mathrm{Tch}(\nabla_+, \nabla_-).$$

Both connections have vanishing curvature near the equator $S(T^*M)$. The transition function is given by $\vec{s}_- = q\vec{s}_+$ so $\theta = dq \cdot q^{-1}$ and

$$\mathrm{Tch}(\nabla_+, \nabla_-) = q^*(\mathrm{Tch}) = \sum c_k \operatorname{Tr}\left(\left(dq \cdot q^{-1}\right)^{2k-1}\right),$$

which implies, as $S(T^*M) = -\mathrm{boundary}(D_-)$,

$$\operatorname{index}(Q) = \int_{S(T^*M)} \mathrm{TODD}(M) \wedge q^*(\mathrm{Tch}).$$

1.2 Bott periodicity

It will be helpful to have a brief review of Bott periodicity from a slightly nonstandard point of view to motivate the constructions we give in this and later sections. We adopt the notational conventions:

$[X, Y] = \{$homotopy classes of maps from X to $Y\}$.

$\mathrm{Vect}_k(X) = \{$isomorphism classes of k-dimensional complex vector bundles over $X\}$.

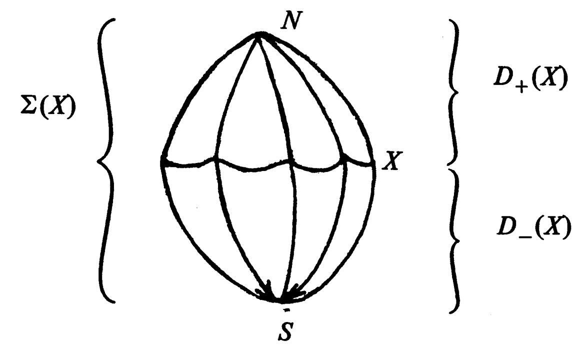
$GL(k, C) = \{k \times k$ invertible complex matrices$\}$.

$GL'(k, C) = \{k \times k$ invertible complex matrices without pure imaginary eigenvalues$\}$.

$U(k) = \{k \times k \text{ unitary matrices}\}$.
$U(*) =$ the limit of $U(k)$ under the inclusions $U(k) \to U(k+1) \to \cdots$.
$H(k) = \{k \times k \text{ Hermitian matrices } h \text{ with } h^2 = I\}$.
$H_0(2k) = \{h \in H(2k)\colon \mathrm{Tr}(h) = 0\}$.
$H_0(*) =$ the limit of $H_0(2k)$ under the natural inclusions
$H_0(2k) \to H_0(2k+2) \to \cdots$.

$U(k)$ is a deformation retract of $GL(k, C)$, and $H(k)$ is a deformation retract of $GL'(k, C)$. $U(k)$ and $H(k)$ are compact; $H_0(2k)$ is one of the connected components of $H(2k)$.

Let X be a compact simplicial complex. The suspension $\Sigma(X)$ is defined by identifying $X \times \{\pi/2\}$ to a single point N and $X \times \{-\pi/2\}$ into a single point S in the product $X \times [-\pi/2, \pi/2]$. Let $D_\pm(X)$ denote the northern and southern "hemispheres." Then $D_+(X) \cap D_-(X) = X$.



If W is a vector bundle, choose a fiber metric on W, and let ΣW be the fiberwise suspension of the sphere bundle $S(W)$. We may also identify ΣW with $S(W \oplus 1)$.

Let $2k > \dim(X)$ and identify $\tilde{K}_0(X)$ with $\mathrm{Vect}_k(X)$. If $W \in \mathrm{Vect}_k(X)$, there exists $W' \in \mathrm{Vect}_k(X)$ such that $W \oplus W' \simeq 1^{2k}$. W' is unique up to isomorphism, and we can choose the isomorphism $W \oplus W' \simeq 1^{2k}$ so that W and W' are orthogonal subbundles of 1^{2k}. If $x \in X$, let $\pi_\pm(x)$ be orthogonal projection on W and W' respectively in 1^{2k}, and let $p(x) = \pi_+(x) - \pi_-(x)$. This defines a map $p\colon X \to H_0(2k)$. Conversely, given $p\colon X \to H_0(2k)$, we can let $W = \Pi_+$ be the span of the positive eigenvectors of $p(x)$; W is the range of $\frac{1}{2}(1 + p(x)) = \pi_+(x)$. This identifies

$$\tilde{K}(X) = \mathrm{Vect}_k(X) = [X, H_0(2k)].$$

Similarly, if $q\colon X \to U(k)$, we let W be the bundle over ΣX which is defined by glueing $D_+(X) \times C^k$ to $D_-(X) \times C^k$ along the edge $X \times C^k$ using the clutching function q. This identifies $\mathrm{Vect}_k(\Sigma X) = [X, U(k)]$. We now define two suspension maps:

$$\Sigma\colon [X, U(k)] \to [\Sigma X, H_0(2k)], \quad \Sigma\colon [X, H_0(2k)] \to [\Sigma X, U(2k)],$$

which will be isomorphisms and represent Bott periodicity. Let $p\colon X \to H_0(2k)$ and let $q\colon X \to U(k)$. Define

$$\Sigma p(x, \theta) = \cos(\theta)p(x) - i\sin(\theta)I_{2k},$$

$$\Sigma q(x, \theta) = \begin{pmatrix} \sin(\theta)I_k & \cos(\theta)q^*(x) \\ \cos(\theta)q(x) & -\sin(\theta)I_k \end{pmatrix}.$$

It is immediate that these maps are well defined with the indicated ranges.

Lemma 1.2.1. *Let $p = \Sigma q$: $\Sigma X \to H_0(2k)$. Then the bundle $\Pi_+(p)$ is represented by the clutching function q.*

Proof. Let $a \in C^k$. Then $\frac{1}{2}(I + p)\binom{a}{0} \in \Pi_+(p)$. This is not $\binom{0}{0}$ for $a \neq 0$ and $\theta > -\pi/2$, and defines a frame for $\Pi_+(p)$ on $\Sigma_+(p)$ on ΣX minus the south pole S. Similarly, the map $\frac{1}{2}(I + p(S))$: $\Pi_+(p)(x, \theta) \to \Pi_+(p)(S)$ is an isomorphism for $(x, \theta) \neq N$. The composition of these two isomorphisms when restricted to the equator X gives the clutching function which sends

$$a \to \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(I + p)(x, 0)\begin{pmatrix} a \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ q(x)a \end{pmatrix}.$$

This map is homotopic to q, and our proof is complete.

In the introduction we defined Σq to be the bundle with clutching function q. To avoid notational confusion we replace that by $\Pi_+(\Sigma q)$ henceforth.

Σq is the element corresponding to q when we identify $[X, U(k)] = \mathrm{Vect}_k(\Sigma X) = [\Sigma X, H_0(2k)]$. We now compute the double suspension:

$$\Sigma^2 p(x, \phi, \theta) = \begin{pmatrix} \sin(\theta) & \cos(\theta)\{\cos(\phi)p(x) + i\sin(\phi)\} \\ \cos(\theta)\{\cos(\phi)p(x) - i\sin(\phi)\} & -\sin(\theta) \end{pmatrix},$$

$$\Sigma^2 q(x, \phi, \theta) = \begin{pmatrix} \cos(\theta)\sin(\phi) - i\sin(\theta) & \cos(\theta)\cos(\phi)q^*(x) \\ \cos(\theta)\cos(\phi)q(x) & -\cos(\theta)\sin(\phi) - i\sin(\theta) \end{pmatrix},$$

where ϕ is the variable of the first suspension, and θ is the variable of the second suspension. Bott periodicity is the assertion that the following two maps are isomorphisms in the stable range:

$$\Sigma^2\colon \mathrm{Vect}_k(X) = [X, H_0(2k)] \to \mathrm{Vect}_{2k}(\Sigma^2 X) = [\Sigma^2 X, H_0(4k)],$$
$$\Sigma^2\colon \mathrm{Vect}_k(\Sigma X) = [X, U(k)] \to \mathrm{Vect}_{2k}(\Sigma^3 X) = [\Sigma^2 X, U(2k)].$$

It is convenient for later work to extend the ranges and domains:

$$\Sigma\colon [X, GL'(k, C)] \to [\Sigma X, GL(k, C)],$$
$$\Sigma\colon [X, GL(k, C)] \to [\Sigma X, GL'(2k, C)].$$

If p: $X \to GL'(k, C)$, let $\Pi_{\pm}(p)$ be the subbundles of 1^k spanned by the generalized eigenvectors of p corresponding to eigenvalues with positive/negative real part. If q: $X \to GL(k, C)$, then q is the clutching function of the bundle $\Pi_+(\Sigma q)$.

To specialize to the case of a sphere $X = S^n$, we introduce coordinates $x = (x_1, \cdots, x_n) \in S^n$, $y = (x, x_{n+1}) \in S^{n+1}$, and $z = (y, x_{n+2}) \in S^{n+2}$, and extend p: $S^n \to GL'(k, C)$ and q: $S^n \to GL(k, C)$ to R^{n+1} to be homogeneous of degree 1 taking values in the space of $k \times k$ matrices. Then it is immediate that

$$\Sigma p(y) = p(x) - ix_{n+1},$$

$$\Sigma^2 p(z) = \begin{pmatrix} x_{n+2} & p(x) + ix_{n+1} \\ p(x) - ix_{n+1} & -x_{n+2} \end{pmatrix},$$

$$\Sigma q(y) = \begin{pmatrix} x_{n+1} & q^*(x) \\ q(x) & -x_{n+1} \end{pmatrix},$$

$$\Sigma^2 q(z) = \begin{pmatrix} x_{n+1} - ix_{n+2} & q^*(x) \\ q(x) & -x_{n+1} - ix_{n+2} \end{pmatrix}.$$

It is convenient to rewrite $\Sigma^2 p$ slightly. If we conjugate by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ then $\Sigma^2 p$ is replaced by the homotopic matrix:

$$\begin{pmatrix} p(x) & i(x_{n+1} + ix_{n+2}) \\ -i(x_{n+1} - ix_{n+2}) & -p(x) \end{pmatrix}$$
$$= \begin{pmatrix} p(x) & \exp(i(\theta + \pi/2)) \\ \exp(-i(\theta + \pi/2)) & -p(x) \end{pmatrix},$$

where $x_{n+1} + ix_{n+2} = \exp(i\theta)$. It is clear this matrix is homotopic to the matrix in which we replace $\exp(i(\theta + \pi/2))$ and $\exp(-i(\theta + \pi/2))$ by $\exp(i\theta)$ and $\exp(-i\theta)$ so $\Sigma^2 p$ is homotopic to the matrix:

$$\begin{pmatrix} p(x) & x_{n+1} + ix_{n+2} \\ x_{n+1} - ix_{n+2} & -p(x) \end{pmatrix}$$
$$= p(x) \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + x_{n+1} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + x_{n+2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

We suspend again to conclude:

Lemma 1.2.2. *Let p: $S^n \to GL'(k, C)$. Then the following hold.*

(a) $\Sigma p = p(x) - ix_{n+1}$.

(b)

$$\Sigma^2 p(z) = \begin{pmatrix} x_{n+2} & p(x) + ix_{n+1} \\ p(x) - ix_{n+1} & -x_{n+2} \end{pmatrix}.$$

This is homotopic to the matrix

$$p(x)\otimes\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}+x_{n+1}\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix}+x_{n+2}\begin{pmatrix} 0 & i\\ -i & 0\end{pmatrix}.$$

(c) $\Sigma^3 p(z, x_{n+3})$ *is homotopic to the matrix*

$$p(x)\otimes\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}+x_{n+1}\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix}+x_{n+2}\begin{pmatrix} 0 & i\\ -i & 0\end{pmatrix}+x_{n+3}\begin{pmatrix} -i & 0\\ 0 & -i\end{pmatrix}.$$

We can now describe Bott periodicity on spheres. Recall that:

$$\pi_n U(k)=\begin{cases} 0 & \text{if } n \text{ is even},\\ Z & \text{if } n \text{ is odd},\end{cases}$$

$$\operatorname{Vect}_k S^n=\begin{cases} Z & \text{if } n \text{ is even},\\ 0 & \text{if } n \text{ is odd},\end{cases}$$

provided that $2k > n$ is in the stable range. First suppose $n = 1$, and let $q(x) = x_0 + ix_1 = \exp(i\theta)$ generate $\pi_1(S^1) = Z$. Then we compute:

$$\Sigma^2 q(x)=\begin{pmatrix} x_2 - ix_3 & x_0 - ix_1\\ x_0 + ix_1 & -x_2 - ix_3\end{pmatrix}=\begin{pmatrix} \bar{w} & \bar{v}\\ v & -w\end{pmatrix},$$

for $w = x_2 + ix_3$ and $v = x_0 + ix_1$. If we multiply this matrix on the left by $\left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)$, we convert $\Sigma^2 q$ to the homotopic matrix $\left(\begin{smallmatrix} v & -w\\ \bar{w} & \bar{v}\end{smallmatrix}\right)$ which has values in $SU(2) = S^3$. The induced map $S^3 \to S^3$ is a diffeomorphism and consequently generates $\pi_3 S^3 = \pi_3 SU(2) = \pi_3 U(2) = Z$. This shows by explicit calculation that Σ^2: $\pi_1 S^1 \to \pi_3 U(2)$ and Σ^2: $\operatorname{Vect}_1(S^2) \to \operatorname{Vect}_2(S^4)$ are isomorphisms.

To generalize these isomorphisms, we introduce Clifford algebras. Fix $n = 2k$ and let $\{e_0,\cdots,e_n\}$ be a collection of $2^k \times 2^k$ self-adjoint matrices satisfying the relations $e_ie_j + e_je_i = 2\delta_{ij}$ where δ_{ij} is the Kronecker symbol. If $n = 2$, we can take $e_0 = \left(\begin{smallmatrix} 1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $e_1 = \left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)$, and $e_2 = \left(\begin{smallmatrix} 0 & i\\ -i & 0\end{smallmatrix}\right)$. More generally, we can take the matrices of the spin representation.

The dimension of the representation space is 2^k. Since this is the minimal dimension possible, the center $e_0 \cdots e_n = \pm i^k I$ is scalar.

Lemma 1.2.3. *Let* $\{e_0,\cdots,e_n\}$ *be Clifford matrices where* $2k = n$. *Define* $p(x) = x_0e_0 + \cdots + x_ne_n$: $S^n \to H_0(2^k)$. *Then* $\Pi_+ p$ *generates* $K_0(S^n) = Z$.

Proof. We remark that sometimes Clifford matrices are chosen to satisfy the commutation relations $e_je_k + e_ke_j = -2\delta_{jk}$. Such matrices are skew-symmetric and related to the convention which we have chosen by a factor of $\sqrt{-1}$. We also note that in general such maps $p(x)$ arise as the symbols of elliptic complexes as we shall see later.

It is clear $p(x)^2 = |x|^2 I = I$. Since $e_0e_1 + e_1e_0 = 0$, $\operatorname{Tr}(e_0) = 0$ so $\operatorname{Tr} p(x) = 0$. Thus in fact p: $S^n \to H_0(2^k)$ is trace free. $\Pi_+ \oplus \Pi_-$ is the trivial bundle on S^n. We project the flat connection on the trivial bundle to define connections $\nabla_\pm$ on $\Pi_\pm$. We wish to compute $\operatorname{ch}_k(\Pi_+)$. $SO(n+1)$ acts transitively on S^n and preserves the norm $|x|^2$. $p(x)$ is defined invariantly by the condition $p(x)^2 = |x|^2 I$ which is a coordinate free representation. Thus the group $SO(n+1)$ also acts on $\Pi_\pm$ to preserve all the structures. Since everything is equivariant, it suffices to compute at the point $A = (1, 0, \cdots, 0)$. Let $\vec{s}_0$ be a basis for $\Pi_+(A)$. Let $\vec{s}(x) = \pi_+(x)\vec{s}_0$. $\vec{s}(x)$ is a basis for $\Pi_+(x)$ for $x \neq -A$. We use the local frame $\vec{s}(x)$ to compute the curvature:

$$\nabla_+ \nabla_+ \vec{s}(x) = \pi_+ d(\pi_+ d(\pi_+ s_0)) = \pi_+ d\pi_+ d\pi_+ \vec{s}_0.$$

Since $\vec{s}(A) = \vec{s}_0$, this shows that the curvature at A is given by

$$\Omega(A) = \pi_+ d\pi_+ d\pi_+ = \tfrac{1}{8}(1 + e_0)(e_1 dx_1 + \cdots + e_n dx_n)^2.$$

Consequently

$$\begin{aligned}\Omega^k(A) &= n!2^{-n-1}(1 + e_0)(e_1 \cdots e_n)dx_1 \wedge \cdots \wedge dx_n \\ &= \pm i^k n! 2^{-n-1}(1 + e_0) d\,\mathrm{vol}(A).\end{aligned}$$

Since $\mathrm{Tr}(1 + e_0) = 2^k$, this shows that

$$\mathrm{ch}_k(\Omega) = \pm n!2^{-n-1}2^k(2\pi)^{-k}(k!)^{-1} d\,\mathrm{vol}(A) = \pm \mathrm{vol}(S^n)^{-1} \cdot d\,\mathrm{vol}(A).$$

Consequently

$$\int_{S^n} \mathrm{ch}_k(\Pi_+) = \pm 1.$$

We note for later use that if $\{e_0', \cdots, e_n'\}$ are self-adjoint matrices which obey the Clifford commutation relations and if $\mathrm{Tr}(e_0' \cdots e_n') \neq 0$, then Π_+ will be nontrivial. The computation is similar and therefore omitted.

This shows that Π_+ is nontrivial. If W is any vector bundle over S^n, then the Atiyah-Singer index theorem for the spin complex with coefficients in W shows that

$$\mathrm{index}(\mathrm{spin}_W) = \int_{S^n} \mathrm{ch}_k(W) \in Z.$$

Thus the map ch_k: $K_0(S^n) \to H^n(S^n) = Z$ is an isomorphism which shows that Π_+ is the required generator. Hence the proof of the lemma is complete.

Given $\{e_0, \cdots, e_n\}$ we define

$$e_\nu' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes e_\nu \quad \text{for } \nu \leq n,$$

$$e_{n+1}' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \otimes I_{2^k}, \qquad e_{n+2}' = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes I_{2^k}.$$

The $\{e_0', \cdots, e_{n+2}'\}$ are $2^{k+1} \times 2^{k+1}$ Clifford matrices. It is immediate that $\Sigma^2 p(z) = x_0 e_0' + \cdots + x_n e_n' + x' e_{n+1}' + x'' e_{n+2}'$. This implies that Σ^2 takes a generator for $K_0(S^n)$ to a generator for $K_0(S^{n+2})$ and proves

Lemma 1.2.4. *Σ^2: $K_0(S^n) \to K_0(S^{n+2})$ and Σ^2: $\pi_{n-1}U(*) \to \pi_{n+1}U(*)$ are isomorphisms.*

We constructed explicit generators for K_0S^n in Lemma 1.2.3. By suspending these generators, we construct generators for the homotopy groups of the unitary group as follows.

Lemma 1.2.5. *Let $2k = n$, and let $\{e_0, \cdots, e_n\}$ be $2^k \times 2^k$ Clifford matrices. Set $q(x) = x_0e_0 + , \cdots, +x_ne_n - ix_{n+1}$. Then q generates $\pi_{n+1}U(2^k)$.*

We remark in general that given a linear map $p(x)$ with $p(x)^2 = |x|^2I$ that the bundles $\Pi_{\pm}(p)$ are defined and $\int \mathrm{ch}_k \Pi_+ p \in Z$. The natural connections on these bundles actually have harmonic curvatures as are discussed in [8].

1.3 Orientation of T^*M, $\Sigma^\nu T^*M$, and formulas for $\mathrm{ind}(\rho, P)$ if $dM = \varnothing$

Let $y = (y_1, \cdots, y_m)$ be local coordinates on M, and let $\xi = (\xi_1, \cdots, \xi_m)$ be the corresponding dual fiber coordinates on T^*M. It is customary to orient T^*M using the simplectic orientation on T^*M. Let

$$\omega_{2m} = dy_1 \wedge d\xi_1 \wedge \cdots \wedge dy_m \wedge d\xi_m$$

define the orientation of T^*M. If $\vec{N}$ is the outward pointing normal on $S(T^*M)$, we orient $S(T^*M)$ by taking the orientation of Stokes theorem;

$$\vec{N} \wedge \omega^s_{2m-1} = \omega_{2m}.$$

For example, if $m = 1$ and $M = S^1$, then $S(T^*M) = M \times \{1\} \cup M \times \{-1\}$ has the orientation $d\theta$ on $M \times \{-1\}$ and $-d\theta$ on $M \times \{1\}$.

Let $u = (u_1, \cdots, u_k)$ be the natural coordinates on R^k. Let $\omega_{2m+k} = \omega_{2m} \wedge du_1 \wedge \cdots \wedge du_k$ define the orientation on $T^*M \oplus 1^k$, and orient $\Sigma^k T^*M = S(T^*M \oplus 1^k)$ so that $\vec{N} \wedge \omega^s_{2m+k-1} = \omega_{2m+k}$. Unfortunately, the simplectic orientation on T^*M is not really the correct orientation from the point of view of the index theorem, and it would probably be preferable to take the orientation given by the negative simplectic structure. Let Q: $C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic complex over M with symbol q. Define

$$\Sigma q(\xi, u) = \begin{pmatrix} u & q^* \\ q & -u \end{pmatrix} : T^*M \oplus 1 \to \mathrm{END}(V_1 \oplus V_2).$$

Then the transition function of $\Pi_+ \Sigma q$ is q, and the Atiyah-Singer index theorem becomes

$$\mathrm{index}(Q) = (-1)^m \int_{\Sigma T^*M} \mathrm{Todd}(M) \wedge \mathrm{ch}(\Pi_+ \Sigma q).$$

We suppressed this sign in the introduction, because we had not discussed the orientation conventions at that point.

We illustrate this formula with a specific example which we shall need later.

Example 1.3.1. Let $M = T_2$ be the flat torus, and U a holomorphic line bundle of Chern class 1. Identify $\Lambda^{0,0}(T^*M) = \Lambda^{0,1}(T^*M)$ with the trivial line bundle. Let $Q\colon C^\infty(U) \to C^\infty(U)$ be the Dolbeault complex with coefficients in U; $Q = \bar\partial_U$. The index of Q is 1. Since M is flat, $\mathrm{TODD}(M) = 1$. Modulo a constant factor, the symbol of Q is multiplication by $(\xi_1 + i\xi_2)$. Let

$$p(\xi, u) = \begin{pmatrix} u & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -u \end{pmatrix} = u\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \xi_1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \xi_2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Then $\Sigma q(\xi, u) = p(\xi, u) \otimes 1_U$. We compute

$$\int_{\Sigma(T^*M)} \mathrm{TODD}(M) \wedge \mathrm{ch}(\Pi_+(\Sigma q)) = \int_{\Sigma(T^*M)} \mathrm{ch}(\Pi_+ p) \wedge \mathrm{ch}(U).$$

Topologically, $\Sigma(T^*M) = T_2 \times S^2$. The orientation on $T^*M \oplus 1$ is $dy_1 \wedge d\xi_1 \wedge dy_2 \wedge d\xi_2 \wedge du = -dy_1 \wedge dy_2 \wedge d\xi_1 \wedge d\xi_2 \wedge du$, so this identification reverses the orientation. Since the integral of $\mathrm{ch}(U)$ over T_2 is 1, we must compute $-\int_{S^2} \mathrm{ch}(\Pi_+ p)$. We argue as in the first section that this is just

$$-\frac{i}{2\pi} \cdot \mathrm{vol}(S^2) \cdot \frac{2}{8} \mathrm{Tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = 1,$$

which is correct.

The Atiyah-Patodi-Singer formula expresses:

$$\mathrm{ind}(\rho, P) = (-1)^m \int_{S(T^*M)} \mathrm{Tch}(\rho) \wedge \mathrm{TODD}(M) \wedge \mathrm{ch}(\Pi_+ p).$$

Again, it is helpful to illustrate this formula with a specific example:

Example 1.3.2. Let $m = 1$ and let $P = -i\partial/\partial\theta$ on the circle $S^1 = [0, 2\pi]$. Let ρ_ε be the representation of π_1 given on the generator by $e^{2\pi i\varepsilon}$. Since the locally flat section defining ∇_ρ is given by $s(\theta) = e^{i\varepsilon\theta}$, we compute $\nabla_\rho(1) = -i\varepsilon d\theta$ and consequently $\mathrm{Tch}(\rho) = \varepsilon d\theta/2\pi$. $S(T^*M) = S^1 \times \{1\} \cup S^1 \times \{-1\}$. The symbol of P is just multiplication by the dual variable ξ, so $\mathrm{ch}(\Pi_+ p) = 1$ on $S^1 \times \{1\}$ and 0 on $S^1 \times \{-1\}$. The orientation on $S \times \{1\}$ is $-d\theta$, so multiplying by $(-1)^m = -1$ cancels this sign and gives

$$\mathrm{ind}(\rho, P) = \int_{S^1 \times \{1\}} \mathrm{Tch}(\rho) = \int_1 \frac{\varepsilon}{2\pi} d\theta = \varepsilon.$$

We can also compute this index directly. $P_\varepsilon = e^{i\varepsilon\theta} P e^{-i\varepsilon\theta} = P - \varepsilon$. Thus

$$\eta(s, P_\varepsilon) = \sum_{n \in Z} \mathrm{sign}(n - \varepsilon)|n - \varepsilon|^{-s} = \sum_{n > 0} \{(n - \varepsilon)^{-s} - (n + \varepsilon)^{-s}\} + \varepsilon^{-s},$$

provided that ε is small and positive. Differentiating this we get

$$\frac{d}{d\varepsilon}\eta(s, P_\varepsilon) = s\left\{\sum_{n>0} (n-\varepsilon)^{-s-1} + (n+\varepsilon)^{-s-1} - \varepsilon^{-s-1}\right\}.$$

Comparing this with the ordinary zeta function gives

$$\frac{d}{d\varepsilon}\eta(0, P_\varepsilon) = 2.$$

Since $\eta(s, P_0) = 0$, this implies $\eta(0, P_\varepsilon) = 2\varepsilon$ and $\operatorname{ind}(\rho, P) = \varepsilon$.

The following combinatorial formula relates the twisted index formula and the index formula.

Lemma 1.3.1. *Let M_1 and M_2 be manifolds without boundary. Let V_1 and V_2 be vector bundles over M_1, and let V_3 be a vector bundle over M_2. Let*

$$q\colon S(T^*M_1) \to \operatorname{END}(V_1, V_2) \quad \text{and} \quad p\colon \Sigma^j(T^*M_2) \to \operatorname{END}(V_3)$$

be symbols for $j \geqslant 0$. Assume that q is elliptic and that p has no purely imaginary eigenvalues. Extend q and p to T^*M_1 and $T^*M_2 \oplus 1^j$ to be homogeneous of orders $\nu_i > 0$. Define the symbol

$$r = \begin{pmatrix} p & q^* \\ q & -p \end{pmatrix}\colon T^*(M_1 \times M_2) \oplus 1^j \to \operatorname{END}((V_1 \oplus V_2) \otimes V_3),$$

which is elliptic with respect to the imaginary axis on $\Sigma^j(T^*(M_1 \times M_2))$. Let ρ be a representation of $\pi_1(M_2)$ such that V_ρ is topologically trivial. Extend ρ to $\pi_1(M_1 \times M_2)$ so that V_ρ is independent of the first coordinate and such that the global frame is independent of the first coordinate. Then

$$\int_{\Sigma^j(T^*(M_1\times M_2))} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M_1 \times M_2) \wedge \operatorname{ch}(\Pi_+(r))$$

$$= \int_{\Sigma(T^*M_1)} \operatorname{TODD}(M_1) \wedge \operatorname{ch}(\Pi_+(\Sigma q))$$

$$\int_{\Sigma^j(T^*M_2)} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M_2) \wedge \operatorname{ch}(\Pi_+(p)).$$

Proof. If $j = 0$ and if q and p are polynomials of the same degree, then this is just the assertion that $\operatorname{ind}(\rho, R)(-1)^m = \operatorname{ind}(\rho, P)(-1)^{m_1}\operatorname{index}(Q)(-1)^{m_2}$. This follows from the identity $\zeta(s, R) = \zeta(s, P)\operatorname{index}(Q)$ and $\zeta(s, R_\rho) = \zeta(s, P_\rho)\operatorname{index}(Q)$ which was discussed in [2], [6].

It is clear that the degrees of homogeneity do not matter, so we take $\nu_1 = \nu_2 = 1$. We smooth off the extension to be identically zero near T^*M_i so that everything is smooth. We also note that $\operatorname{TODD}(M_1 \times M_2) = \operatorname{TODD}(M_1) \wedge \operatorname{TODD}(M_2)$. We shall give a combinatorial proof of this lemma rather than attempting to extend the proof given above for differential operators to the pseudo-differential case to keep the present discussion as self-contained as possible. The orientations are crucial in our discussion, so we pay unusually careful attention to them in what follows.

We choose local coordinates

$$(x_1,\cdots,x_n,\xi_1,\cdots,\xi_n)\in T^*(M_1),$$
$$(y_1,\cdots,y_m,\zeta_1,\cdots,\zeta_m,u_1,\cdots,u_j)\in T^*(M_2)\oplus 1^j.$$

If $\zeta_1>0$, then $d\zeta_1$ points outwards, so the orientation of $\Sigma^j(T^*(M_1\times M_2))$ is given by

$$-dx_1\wedge d\xi_1\wedge\cdots\wedge dx_n\wedge d\xi_n\wedge dy_1\wedge dy_2\wedge d\zeta_2\wedge\cdots\wedge dy_m\wedge d\zeta_m \wedge du_1\wedge\cdots\wedge du_j.$$

We parametrize $\Sigma^j(T^*(M_1\times M_2))=S(T^*M_1)\times[0,\pi/2]\times\Sigma^j(T^*M_2)$ in the form

$$(\cos(\theta)\cdot\xi,\sin(\theta)\cdot\zeta,\sin(\theta)\cdot u),$$

where $|\xi|^2=|\zeta|^2+u^2=1$. At the point $\theta=\pi/4$, $\xi=(1,0,\cdots,0)$, $\zeta=(1,0,\cdots,0)$; this gives the orientation

$$dx_1\wedge d\theta\wedge dx_2\wedge d\xi_2\wedge\cdots\wedge dx_n\wedge d\xi_n\wedge dy_1\wedge dy_2\wedge d\zeta_2 \wedge\cdots\wedge dy_m\wedge d\zeta_m\wedge du_1\wedge\cdots\wedge du_j.$$

We define

$$r=\begin{pmatrix}\sin(\theta) & \cos(\theta)q^*\\ \cos(\theta)q & -\sin(\theta)\end{pmatrix}\otimes p=\Sigma q\otimes p.$$

Consequently r has no purely imaginary eigenvalues, and

$$\Pi_+(r)=\Pi_+(\Sigma q)\otimes\Pi_+(p)\oplus\Pi_-(\Sigma q)\otimes\Pi_-(p).$$
$$\operatorname{ch}(\Pi_+(p))+\operatorname{ch}(\Pi_-(p))=\operatorname{ch}(\Pi_+(p)\oplus\Pi_-(p))=\operatorname{ch}(V_3).$$

Since V_3 does not depend on the fiber coordinates (ξ,u), we may replace $\operatorname{ch}(\Pi_-(p))$ by $-\operatorname{ch}(\Pi_+(p))$ without changing the integral. Similarly, we may replace $\operatorname{ch}(\Pi_-(\Sigma q))$ by $-\operatorname{ch}(\Pi_+(\Sigma q))$ without changing the integral. This expresses

$$\int_{\Sigma^j(T^*(M_1\times M_2))}\operatorname{Tch}(\rho)\wedge\operatorname{TODD}(M_1\times M_2)\wedge\operatorname{ch}(\Pi_+(r))$$
$$=2\int_{S(T^*M_1)\times[0,\pi/2]}\operatorname{TODD}(M_1)\wedge\operatorname{ch}(\Pi_+(\Sigma q))$$
$$\cdot\int_{\Sigma^j(T^*M_2)}\operatorname{Tch}(\rho)\wedge\operatorname{TODD}(M_2)\wedge\operatorname{ch}(\Pi_+p),$$

where we use the orientations on $S(T^*M_1) \times [0, \pi/2]$, and $\Sigma^j T^*M_2$, which agree with

$$-dx_1 \wedge d\theta \wedge dx_2 \wedge d\xi_2 \wedge \cdots \wedge dx_n \wedge d\xi_n,$$
$$-dy_1 \wedge dy_2 \wedge d\zeta_2 \wedge \cdots \wedge dy_{m1} d\zeta_m \wedge du \wedge \cdots \wedge du_m,$$

at $\theta = \pi/4$, $\zeta = (1, 0, \cdots, 0)$ and $\zeta = (1, 0, \cdots, 0)$.

We wish to extend this integral to range over $[-\pi/2, \pi/2]$. If we replace θ by $-\theta$, we change the orientation and replace Σq by

$$\begin{pmatrix} -\sin(\theta) & \cos(\theta) q^* \\ \cos(\theta) q & \sin(\theta) \end{pmatrix}.$$

Conjugating this with the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ we find that this is equivalent to

$$\begin{pmatrix} -\sin(\theta) & -\cos(\theta) q^* \\ -\cos(\theta) q & \sin(\theta) \end{pmatrix} = -r.$$

If we replace r by $-r$, then we interchange the roles of Π_+ and Π_-. If we replace Π_- by Π_+ again, we must change the sign. This sign change takes care of the change of orientation, so the integral over $[-\pi/2, 0]$ is equal to the integral over $[0, \pi/2]$. Therefore

$$2 \int_{S(T^*M_1) \times [0, \pi/2]} \mathrm{TODD}(M_1) \wedge \mathrm{ch}(\Pi_+(\Sigma q))$$
$$= \int_{S(T^*M_1) \times [-\pi/2, \pi/2]} \mathrm{TODD}(M_1) \wedge \mathrm{ch}(\Pi_+(\Sigma q)).$$

We identify $S(T^*M_1) \times [-\pi/2, \pi/2]$ with ΣT^*M_1 by setting $(\xi, u_1) = (\cos(\theta)\xi, \sin(\theta))$. This gives the orientation

$$-dx_1 \wedge dx_2 \wedge d\xi_2 \wedge \cdots \wedge dx_n \wedge d\xi_n \wedge du_1$$

which is the orientation for ΣT^*M_1 to that we have chosen. This completes the proof of the lemma.

This doubling argument in extending the parameter range will play an important role in the second section, when we discuss $\mathrm{ind}(\rho, P, B)$ for manifolds with boundary. Unfortunately, the formula

$$\mathrm{ind}(\rho, P) = (-1)^m \int_{S(T^*M)} \mathrm{Tch}(\rho) \wedge \mathrm{TODD}(M) \wedge \mathrm{ch}(\Pi_+(p))$$

does not generalize to manifolds with boundary. We must suspend this formula in order to obtain a suitable extension if $dM \neq \varnothing$.

Theorem 1.3.2. *Let j be a nonnegative integer. Let $dM = \varnothing$. Let ρ be a representation of $\pi_1(M)$ such that V_ρ is topologically trivial. Let P be a pseudo-differential operator on $C^\infty(V)$, which is elliptic with respect to the imaginary* *axis. Define*

$$\operatorname{ind}_j(\rho, P) = (-1)^m \int_{\Sigma^{2j}(T^*M)} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M) \wedge \operatorname{ch}\big(\Pi_+(\Sigma^{2j}p)\big).$$

Then $\operatorname{ind}(\rho, P) = \operatorname{ind}_j(\rho, P)$.

Proof. If $j = 0$, this is the formula proved in [3], [6]. We proceed by induction on j. Apply Lemma 1.3.1 to $M_1 = T_2$ and $M_2 = M$, where T_2 is the flat torus. Let

$$r = \begin{pmatrix} \Sigma^{2j-2}p & q^* \\ q & -\Sigma^{2j-2}p \end{pmatrix},$$

where q is the symbol of the Dolbeault complex on T_2 with coefficients in a holomorphic line bundle of Chern class 1 as discussed in Example 1.3.1.

Lemma 1.3.1 implies that

$$\operatorname{ind}_{j-1}(\rho, R) = \operatorname{index}(Q)\operatorname{ind}_{j-1}(\rho, P) = \operatorname{ind}(\rho, P)$$

by induction. We compute directly. Let (x_1, x_2, v_1, v_2) be coordinates on T^*T_2 and let $(y_1,\cdots,y_m, \xi_1,\cdots,\xi_m, u_1,\cdots,u_{2j-2})$ be coordinates on $T^*M \oplus 1^{2j-2}$. Then

$$r = \begin{pmatrix} \Sigma^{2j-2}p & v_1 - iv_2 \\ v_1 + iv_2 & -\Sigma^{2j-2}p \end{pmatrix} \otimes 1_U.$$

Topologically, $T^*(T_2 \times M) \oplus 1^{2j-2} = (T^*M \oplus 1^{2j}) \times T_2$. However, the orientations

$$dx_1 \wedge dv_1 \wedge dx_2 \wedge dv_2 \wedge dy_1 \wedge d\xi_1 \wedge \cdots \wedge dy_m \wedge d\xi_m \wedge du_1 \wedge \cdots \wedge du_{2j-2},$$

$$dy_1 \wedge d\xi_1 \wedge \cdots \wedge dy_m \wedge d\xi_m \wedge du_1 \wedge \cdots \wedge du_{2j-2} \wedge dv_1 \wedge dv_2 \wedge dx_1 \wedge dx_2$$

do not agree. We replace v_2 by $-v_2$ to take care of the flip in orientation so that

$$r = \begin{pmatrix} \Sigma^{2j-2}p & v_1 + iv_2 \\ v_1 - iv_2 & -\Sigma^{2j-2}p \end{pmatrix} \otimes 1_U = \Sigma^{2j}p \otimes 1_U.$$

Therefore we compute

$$\begin{aligned} \operatorname{ind}(\rho, p) &= \operatorname{ind}_{j-1}(\rho, p) = \operatorname{ind}_{j-1}(\rho, R) \\ &= (-1)^m \int_{\Sigma^{2j}(T^*M)} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M) \wedge \operatorname{ch}\big(\Pi_+(\Sigma^{2j}p)\big) \cdot \int_{T_2} \operatorname{ch}(U), \\ &= \operatorname{ind}_j(\rho, P) \end{aligned}$$

which completes the proof.

We note that if V is topologically trivial, then the integral can also be rewritten using Stokes theorem as

$$-\int_{\Sigma^{2j-1}(T^*M)} \mathrm{Tch}(\rho) \wedge \mathrm{TODD}(M) \wedge (\Sigma^{2j-1}p)^*(\mathrm{Tch}).$$

In the introduction we remarked that the twisted index theorem implies the Atiyah-Singer index theorem as a special case. Deriving the index theorem from the twisted index theorem is in a sense circular since the index theorem was used to prove the twisted index theorem in the first instance. We present the derivation nevertheless because it is instructive and illustrates the results of this subsection.

Let Q: $C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic pseudo-differential operator over a compact Riemannian manifold M_1 without boundary. We may assume without loss of generality that Q is first order. Suppose for the moment that Q is in fact differential. Let $P = -i\partial/\partial\theta$ on $C^\infty(S^1)$. Let $M = M_1 \times S^1$ and let $R = \left(\begin{smallmatrix} P & Q^* \\ Q & -P \end{smallmatrix}\right)$ be the twisted operator over M defined earlier. Let g generate $\pi_1(S^1) = Z$ and let $\rho(g) = \exp(2\pi i\varepsilon)$ for ε real. We extend ρ to be trivial on $\pi_1(M_1)$. Then

$$\mathrm{ind}(\rho, R) = \mathrm{ind}(\rho, P)\mathrm{index}(Q) = \varepsilon\,\mathrm{index}(Q)$$

by Example 1.3.2. Furthermore

$$\begin{aligned}
\mathrm{ind}(\rho, R) &= \int_{S(T^*M)} \mathrm{TODD}(M) \wedge \mathrm{Tch}(\rho) \wedge \mathrm{ch}(\Pi_+ r) \\
&= \int_{\Sigma(T^*M_1)} \mathrm{TODD}(M_1) \wedge \mathrm{ch}(\Pi_+ \Sigma q) \cdot \int_{S(T^*S^1)} \mathrm{Tch}(\rho) \wedge \mathrm{ch}(\Pi_+ (p)) \\
&= \varepsilon \cdot \int_{\Sigma(T^*M_1)} \mathrm{TODD}(M_1) \wedge \mathrm{ch}(\Pi_+ \Sigma q) = \varepsilon \cdot \mathrm{index}(Q)
\end{aligned}$$

by the combinatorial argument given in the proof of Lemma 1.3.1. This identity true (mod Z) for all values of ε implies

$$\mathrm{index}(Q) = \int_{\Sigma(T^*M_1)} \mathrm{TODD}(M_1) \wedge \mathrm{ch}(\Pi_+ \Sigma Q),$$

which is the Atiyah-Singer index formula.

If Q is not differential, then R is not a pseudo-differential operator on $M_1 \times S^1$. There is a technical trick to handle this case, and we refer to [6] for details to avoid unduely complicating the exposition. The Atiyah-Singer theorem can be viewed as a map

$$\mathrm{ind}\colon K(\Sigma(S(T^*M))) \to Z,$$

and the twisted index can be interpreted as a map

$$\operatorname{ind}(\rho,*): K(S(T^*M)) \to R/Z.$$

By using Bott periodicity, one can relate $K(S(T^*M))$ to $K(\Sigma^2(S(T^*M)))$. By using product formulas as discussed, it is possible, in a certain sense we shall not make precise, to regard the one formula as the suspension of the other.

SECTION TWO

We now return to the case of $dM \neq \varnothing$. Although we shall be primarily concerned with the first order case in this section and shall postpone a detailed treatment of the higher order case until the third section, we first review briefly the definition of ellipticity that we shall be using.

2.1 Notational conventions

Let M be a compact Riemannian manifold of dimension m, and let $y = (y_1, \cdots, y_m)$ be a system of local coordinates on M. Let dM denote the smooth boundary of M. Near dM, we choose coordinates $y = (x, r)$ for $x = (x_1, \cdots, x_{m-1})$ so $M = \{y: r(y) \geqslant 0\}$. We further normalize the choice of coordinates by requiring that $\partial/\partial r$ is the inward unit normal on dM and that the curves $y(r) = (x_0, r)$ are unit speed geodesics for any $x_0 \in dM$. If we use the inward geodesic flow to identify a neighborhood of dM in M with $dM \times [0, r_0)$, we define a splitting $T(M) = T(dM) \oplus T(R)$ and a dual splitting $T^*(M) = T^*(dM) \oplus T^*(R)$. Let $\xi = (\zeta, z)$ for $\zeta = (\zeta_1, \cdots, \zeta_{m-1}) \in T^*(dM)$. If P: $C^\infty(V) \to C^\infty(V)$ is a differential operator, let $p(y, \xi) = p(x, r, \zeta, z)$ be the leading symbol of P.

If $dM = \varnothing$, it suffices to assume $\det(p - it) \neq 0$ for $(\xi, t) \neq (0,0)$ as an ellipticity condition. For manifolds with boundary, however, the corresponding analysis is much harder, and it was convenient to work with the heat equation in our earlier paper [7]. Consequently, we must impose a stronger condition of ellipticity in this case. Let

$$\mathcal{C} = \{\lambda \in C: |\mathrm{Im}(\lambda)| \geqslant |\mathrm{Re}(\lambda)|;$$

this is a 45° cone about the imaginary axis. We say that P is elliptic with repsect to $\mathcal{C}$ if $\det(p - \lambda) \neq 0$ for $(\xi, \lambda) \neq (0,0)$ and $\xi \in T^*M, \lambda \in \mathcal{C}$.

A graded vector bundle U over dM is a bundle U together with a fixed decomposition into bundles of the form $U = U_0 \oplus \cdots \oplus U_{d-1}$. We permit $U_j = \{0\}$ in this decomposition. Let W be the bundle of Cauchy data over dM. W consists of d-copies of the restriction of V to dM and inherits a natural grading where $W_j = V_{|dM}$ represents normal derivatives of order j. Let $D_r = -i\partial/\partial r$, and $\phi_j = (D_r)^j\phi_{|dM}$ for $\phi \in C^\infty(V)$. Define the natural map γ: $C^\infty(V) \to C^\infty(W)$ by $\gamma\phi = (\phi_0,\cdots,\phi_{d-1})$.

Let W' be an auxiliary graded vector bundle over dM, and let B: $C^\infty(W) \to C^\infty(W')$ be a tangential differential operator. Decompose $B = B_{ij}$ for B_{ij}: $C^\infty(W_i) \to C^\infty(W_j)$. B is of *graded order* ν if ν is the smallest integer such that $\operatorname{ord}(B_{ij}) \leqslant \nu + j - i$ for all (i, j). We define the graded leading symbol of B by

$$\sigma^g(B)_{ij} = b_{ij} = \begin{cases} 0 & \text{if } \operatorname{ord}(B_{ij}) < \nu + j - i, \\ \sigma(B_{ij}) & \text{if } \operatorname{ord}(B_{ij}) = \nu + j - i. \end{cases}$$

We assume henceforth that $\dim(W) = d \cdot \dim(U)$ is even. Let W' be an auxiliary graded vector bundle over dM of dimension $\frac{1}{2}\dim(W)$. Let B: $C^\infty(W) \to C^\infty(W')$ be a tangential differential operator of graded order 0. Consider the ordinary differential equation

$$\text{(2.1)} \qquad \begin{gathered} P(x,0,\zeta,D_r)\phi(r) = \lambda\phi(r) \quad \text{and} \quad \lim_{r\to\infty}\phi(r) = 0, \\ \zeta \in T^*(dM), \quad \lambda \in \mathcal{C}, \quad (\zeta,\lambda) \neq (0,0). \end{gathered}$$

We say that the pair (P, B) is elliptic with respect to $\mathcal{C}$ if P is elliptic with respect to $\mathcal{C}$ and if for every such (ζ, λ) and $\psi' \in W'$ there is a unique solution ϕ to (2.1) satisfying $\sigma^g(B)(x, \zeta)\gamma\phi = \psi'$.

There is an alternate formulation of this condition of ellipticity which is purely algebraic in nature and will prove useful in what follows. Let $W_\pm(\zeta, \lambda)$ be the subsets of W corresponding to Cauchy data of solutions of (2.1) vanishing as $r \to \pm\infty$. Decompose $p(x,0,\zeta,z) = \Sigma_j p_{d-j}(x,\zeta)z^j$ where p_j is homogeneous of order j in ζ. We rewrite the ordinary differential equation of (2.1) as a first order system in the form

$$-i(\partial/\partial r + \tau(\zeta,\lambda))\phi = 0,$$

where τ is the $d \times d$ matrix

$$\tau = i\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ p_0^{-1}(p_d-\lambda) & p_0^{-1}p_{d-1} & p_0^{-1}p_{d-2} & \cdots & p_0^{-1}p_2 & p_0^{-1}p_1 \end{pmatrix}.$$

If $\tau\phi = -iz\phi$, then $\Sigma_j p_{d-j}(x,\zeta)z^j\phi_{d-1} = \lambda\phi_{d-1}$. This implies that τ does not have any purely-imaginary eigenvalues for $(\zeta, \lambda) \neq (0,0)$ if P is elliptic with respect to $\mathcal{C}$. It is clear from this description that $W_\pm(\zeta, \lambda)$ are the span of the generalized eigenvectors of τ, which correspond to eigenvalues with positive/negative real part, and therefore that $W_\pm(\zeta, \lambda)$ define vector bundles over $T^*(dM) \times \mathcal{C} - (0,0)$. Let $\lambda = it$ and let $\Pi_\pm(\tau)(\zeta, t) = W_\pm(\zeta, it)$ define bundles over $\Sigma(T^*dM)$; these bundles will be important in what follows.

The ellipticity condition can be rephrased in this language as the condition

$$\sigma^g B(x, \zeta)\colon W_+(\zeta, \lambda) \to W' \text{ is an isomorphism}$$
$$\text{for } (\zeta, \lambda) \neq (0,0) \zeta \in T^*(dM), \lambda \in \mathcal{C}.$$

In particular, the existence of such boundary conditions implies that the bundle $\Pi_+(\tau)$ over $\Sigma(T^*dM)$ is topologically trivial. We will construct operators in the next subsection, for which this is not true and which therefore do not admit such boundary conditions.

2.2 Operators with symbol given by Clifford multiplication

We restrict for the remainder of the section two to first order operators. B is a 0^{th} order boundary condition and $\tau = ip_0^{-1}(p(x, \zeta) - \lambda)$. Choose a Riemannian metric on M, and let $|\xi|$ be the length of $\xi \in T^*M$. We say that the leading symbol of P is given by Clifford multiplication if

$p(y, \xi)^2 = |\xi|^2 I$, i.e., $P^2 = -g^{ij}\partial^2/\partial x_i \partial x_j \cdot I +$ lower order terms,

where we adopt the convention of summing over repeated indices. Such an operator is automatically elliptic with respect to $\mathcal{C}$ since the eigenvalues of p are $\pm|\xi|$. Equivalently, let $\{e_1, \cdots, e_m\}$ be a local orthonormal frame for T^*M. If M is oriented, we shall suppose $e_1 \wedge \cdots \wedge e_m = \omega_m$ is the orientation form on M. Expand $\xi = \xi_i e_i \in T^*M$ and let $p(y, \xi) = \xi_i p_i(y)$. The leading symbol of P is given by Clifford multiplication if $p_i p_j + p_j p_i = 2\delta_{ij}$, or equivalently if $\{p_1, \cdots, p_m\}$ is a set of Clifford matrices.

Such operators arise naturally in differential geometry.

Example 2.2.1. Let $V = \oplus_p \Lambda^p T^*M$ be the bundle of all differential forms and let $P = (d + \delta)$.

Example 2.2.2. Let M be a holomorphic manifold, and let $V = \oplus_p \Lambda^{0,p}(T^*M)$ and let $P = \sqrt{2}(\bar{\partial} + \delta'')$.

Example 2.2.3. Let M be a spin manifold, $V = \Delta(M)$ be the total spin bundle, and P be the Dirac operator.

Example 2.2.4. Let M be an oriented *odd* dimensional manifold, and let $*$ be the Hodge operator. Let $V = \oplus_p \Lambda^{2p}(T^*M)$ be the bundle of even differential forms. Let $P = \oplus_p i^{(m-1)/2}(-1)^{p+1}(*d - d*)$; this is the operator which appears in the Atiyah-Patodi-Singer signature theorem for manifolds with boundary.

Example 2.2.5. Let W be a coefficient bundle, and let $P\colon C^\infty(V) \to C^\infty(V)$ have leading symbol given by Clifford multiplication. Let $P_W\colon C^\infty(V \otimes W) \to C^\infty(V \otimes W)$ have leading symbol $p \otimes 1_W$. P_W is well defined modulo 0^{th} order terms.

Let $\operatorname{ext}(\xi)$ denote exterior multiplication, and $\operatorname{int}(\xi)$ the dual map interior multiplication. Let $c(\xi) = i(\operatorname{ext}(\xi) - \operatorname{int}(\xi))$ be *Clifford multiplication*; this is the leading symbol of $(d + \delta)$. Let $\operatorname{Clif}(T^*M)$ be the universal tensor bundle generated by T^*M subject to the relations

$$\xi_1 \cdot \xi_2 + \xi_2 \cdot \xi_1 = 2(\xi_1, \xi_2).$$

Since $c(\xi)^2 = |\xi|^2 I$, it extends to an algebra morphism $c\colon \operatorname{Clif}(T^*M) \to \operatorname{END}(\Lambda T^*M)$. If we send $\theta \to c(\theta) \cdot 1$, we define a vector space isomorphism between $\operatorname{Clif}(T^*M)$ and $\Lambda(T^*M)$; this is not, of course, an algebra morphism.

If the leading symbol of P is given by Clifford multiplication, then p extends to an algebra morphism $p\colon \operatorname{Clif}(T^*M) \to \operatorname{END}(V)$. Conversely, given such an algebra morphism or representation, we can construct an operator P with symbol p. If we fix a connection ∇ on V, then P can be defined using the diagram:

$$P\colon C^\infty(V) \to C^\infty(T^*M \otimes V) \xrightarrow{-ip} C^\infty(V).$$

It is worth noting that in this situation we can always choose an inner product on V, so $p(y, \xi)$ is unitary and Hermitian for $|\xi| = 1$, and consequently we can always find a formally self-adjoint operator P with symbol p.

Next dM we choose a local orthonormal frame, so $e_m = dr$ is the normal covector. We expand $p(y, \xi) = p(x, r, \zeta, z) = \Sigma_{i=1}^{m-1} \zeta_i p_i + z p_m$. Recall that

$$\tau(\zeta, \lambda) = ip_m(p(\zeta) - \lambda) \quad \text{for } \lambda \in \mathcal{C}, \zeta \in T^*(dM), (\zeta, \lambda) \neq (0,0).$$

$\Pi_\pm(\zeta, \lambda)$ are the complementary subbundles of V, which are spanned by the eigenvectors of τ corresponding to eigenvalues with positive/negative real part. (We will also denote these bundles by $\Pi_\pm(\tau)$ when it is necessary to distinguish them from other similarly defined bundles.)

Since P is a first order operator, a boundary condition B is just a 0^{th} order map $B\colon V \to W'$ where $\dim(V) = 2\dim(W')$ such that the null space $N(B)$ does not intersect Π_+. Suppose $\hat{p} \in \operatorname{END}(V)$ satisfies

$$\hat{p}^2 = 1, \hat{p}p(\xi) + p(\xi)\hat{p} = 0.$$

This is equivalent to assuming $\{\hat{p}, p_1, \cdots, p_m\}$ forms a set of Clifford matrices. Define $q = ip_m\hat{p}$, and note that $\{q, p_m, ip_m p_1, \cdots, ip_m p_{m-1}\}$ forms a set of Clifford matrices, so that q anticommutes with τ. Let $B_\pm = \frac{1}{2}(I \pm q)$ be the projection on the ± 1 eigenspace of q. $B_\pm$ will be said to be a *Clifford* boundary condition.

Lemma 2.2.1. *Let the leading symbol of P be given by Clifford multiplication and let B be a Clifford boundary condition. Then (P, B) is elliptic with respect to the cone $\mathcal{C}$.*

Proof. We recall that $\tau(\zeta, \lambda) = ip_m p(\zeta) + i\lambda p_m$ since $p_m^{-1} = p_m$. Since τ anticommutes with q, these two operators do not have any common nonzero generalized eigenvector. Assume $B = B_+$ for the sake of definiteness. $N(B) = \Pi_-(q)$ and $\Pi_-(q) \cap \Pi_+(\tau) = \{0\}$, so B is injective from $\Pi_+(\tau)$ to $\Pi_+(q)$ and hence bijective as both these spaces have dimension $\frac{1}{2}\dim(V)$. We note that if P is formally self-adjoint, then (P, B) is self-adjoint if $\hat{p}$ (or q) is self-adjoint.

We can now state a basic existence result.

Lemma 2.2.2. *Let the leading symbol of P be given by Clifford multiplication.*

(a) *If* $\dim(M) = m$ *is even, and M is orientable, then there always exist Clifford boundary conditions for P.*

(b) *If* $\dim(M) = m$ *is odd and if* $\mathrm{Tr}(p_1 \cdots p_m) \neq 0$, *then there do not exist any boundary conditions so that (P, B) is elliptic with respect to the cone $\mathcal{C}$. In particular, there do not exist such boundary conditions for the operator of Example* 2.2.4.

Proof. Suppose first that m is even, and let ω_m be the orientation form on M. Let $p(\omega_m) = p_1 \cdots p_m$. It is immediate that $\{(-i)^{m/2}p(\omega_m), p_1, \cdots, p_m\}$ is a set of Clifford matrices, and thus (a) is proved.

We suppose next that m is odd, and let $2k = m - 1$. We fix $y \in dM$, and let S^{2k} be the unit sphere in $\Sigma(T^*dM)$. We regard $\Pi_+(\tau)$ as a bundle over S^{2k} by setting $\lambda = it$ so $\tau(\zeta, t) = ip_m p(\zeta) - tp_m$. We computed in the first section that modulo some universal constant $a_k \neq 0$,

$$\begin{aligned}\int_{S^{2k}} \mathrm{ch}_k(\Pi_+(\tau)) &= a_k \mathrm{Tr}(ip_m p_1 \cdots ip_m p_{m-1} \cdot p_m) = a'_k \mathrm{Tr}(p_1 \cdots p_{m-1} p_m) \\ &= a'_k \mathrm{Tr}(p_1 \cdots p_m) = a'_k \mathrm{Tr}(p(\omega_m)).\end{aligned}$$

Consequently, if this trace is nonzero, the bundle $\Pi_+(\tau)$ is topologically nontrivial on each fiber sphere. This implies that there cannot exist a map B: $\Pi_+(\tau) \to W'$, and consequently there exists no good boundary conditions. It is an easy verification that $\mathrm{Tr}(p_1 \cdots p_m) \neq 0$ for Example 2.2.4, so this operator does not admit boundary conditions of the sort which we are considering.

If m is even, set $p_0 = (-i)^{m/2}p(\omega_m)$. We replace P by $\frac{1}{2}(P - p_0 P p_0)$ without changing the leading symbol of P to assume P anticommutes with p_0. Decompose

$$P = P_\pm : C^\infty(\Pi_\pm(p_0)) \to C^\infty(\Pi_\mp(p_0))$$

to define a two-term elliptic complex. This yields the signature, Dolbeault, and spin complexes from Examples 2.2.1–2.2.3. We note that the boundary condition corresponding to p_0 in these examples does *not* define an elliptic boundary condition for these complexes; these three elliptic complexes do not admit local elliptic boundary conditions.

The de Rham complex does not fit this pattern since it does not depend on the orientation of M. It is related to another Clifford boundary condition for the operator $(d + \delta)$. Near dM, we decompose any form into tangential and normal components as

$$\theta = \theta_1 + \theta_2 \wedge e_m \quad \text{for } \theta_i \in \Lambda(T^*dM).$$

Define $q(\theta_1 + \theta_2 \wedge e_m) = \theta_1 - \theta_2 \wedge e_m$ and let $B_\pm = \frac{1}{2}(1 \pm q)$. This gives Clifford boundary conditions for $(d + \delta)$; it also defines relative/absolute boundary conditions for the de Rham complex.

Not every boundary condition is a Clifford boundary condition. However, it is possible to find a normal form for elliptic boundary conditions. Let (P, B) be elliptic with respect to the cone $\mathcal{C}$ with the leading symbol of P given by Clifford multiplication. Let $\phi \in N(B) \cap p_m N(B)$. We decompose $\phi = \phi_+ + \phi_-$ so that $p_m\phi = \phi_+ - \phi_-$. Then $\phi_\pm = \frac{1}{2}(I \pm p_m\phi) \in N(B)$. However $\phi_+ \in \Pi_+(\tau)(0, -i)$ and $\phi_- \in \Pi_+(\tau)(0, +i)$, so the ellipticity condition implies $\phi_+ = \phi_- = 0$.

Since $N(B) \cap p_m N(B) = \{0\}$, these two subspaces are complementary. We define q to be $+1$ on $p_m N(B)$ and -1 on $N(B)$. If $B' = \frac{1}{2}(1 + q)$, then $N(B') = N(B)$ so (P, B') is equivalent to the problem (P, B) and is elliptic with respect to $\mathcal{C}$. Since $qp_m + p_m q = 0$, this proves

Lemma 2.2.3. *Let the leading symbol of P be given by Clifford multiplication, and let (P, B) be elliptic with respect to the cone $\mathcal{C}$. Then B is equivalent to a boundary condition B' of the form $B' = \frac{1}{2}(1 + q)$ where $q^2 = 1$ and $qp_m + p_m q = 0$.*

2.3 $\operatorname{Res}_{s=0} \eta(s, P, B)$

We shall construct a sequence of homotopies for later use. We work only with the leading symbol, and damp out any homotopy away from the boundary. We choose a connection ∇ for V, and let (P, B) be elliptic with respect to the cone $\mathcal{C}$. We choose a Riemannian metric on M, and let $3r_0$ be the radius of normal coordinates on dM. We identify $dM \times [0, 3r_0)$ with a neighborhood of dM in M using geodesic normal coordinates.

Using parallel translation along the geodesic normal rays in dM, we may identify the fiber of V at any point (x, r) with the fiber of V at $(x, 0)$ for $r < 3r_0$. In the first homotopy, we replace the original metric by a product metric near the boundary, and we replace P by an operator whose symbol is covariant constant in the normal direction near dM.

Homotopy 2.3.1. Let $f(t, r)$ be a smooth function so that

$$f(t, r) = r \quad \text{for } r > 2r_0,\ 0 \leqslant f(t, r) \leqslant r,$$
$$f(0, r) = r, \quad f(1, r) = r_0 \quad \text{for } r < r_0.$$

Define $p_t(y, \xi) = p_t(x, r, \xi) = p(x, f(t, r), \xi)$. Since $p(x, 0, \xi) = p_t(x, 0, \xi)$, (P_t, B) is elliptic with respect to the cone $\mathcal{C}$. $p_1(x, r, \xi) = p(x, 0, \xi)$, so the symbol is covariant constant in the normal direction for $r < r_0$. If we apply this construction to the operator $(d + \delta)$ and square the resulting symbol, we get a 1-parameter family of metrics connecting the original metric to a metric which is product near the boundary.

We note that if the leading symbol of P is given by Clifford multiplication, then the leading symbol of P_t is still given by Clifford multiplication with respect to a perturbed metric. The metric at $t = 1$ is product near the boundary. This homotopy is valid equally well for higher order operators.

Let the symbol of P be given by Clifford multiplication. Apply Lemma 2.2.3 to assume $B = \frac{1}{2}(1 + q)$ for $q^2 = 1$ and $qp_m + p_m q = 0$. Extend q to be covariant constant in the normal direction, so $q^2 = 1$ and $qp_m + p_m q = 0$. Decompose $V = \Pi_-(q) \oplus \Pi_+(q)$ into the ± 1 eigenspaces of q. We assume the connection on V is chosen, so it splits under this decomposition. Let $\vec{s}_-$ be a local frame for $\Pi_-(q)$, which is covariant constant in the normal direction, and let $\vec{s}_+ = p_m \vec{s}_-$ be a local frame for $\Pi_+(q)$ which is covariant constant in the normal direction.

Assume that P is as constructed in Homotopy 2.3.1, and near dM decompose

$$P = -i\frac{\partial}{\partial r}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + P_T, \quad B\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \beta, \quad q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where P_T is a tangential partial differential operator with coefficients independent of r. The sign of q, so B is projection on the *second* factor, is chosen to make later sign conventions work out correctly.

Let $p_T = p(x, 0, \zeta, 0)$ be the symbol of P_T. The identities $p_m p_T + p_T p_m = 0$ and $p_T^2 = |\zeta|^2 I$ imply that p_T must have the form

$$p_T = a(x, \zeta)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + b(x, \zeta)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

where

$$a(x,\zeta)^2 + b(x,\zeta)^2 = |\zeta|^2 I, \quad a(x,\zeta)b(x,\zeta)a(x,\zeta) = b(x,\zeta)a(x,\zeta).$$

The next lemma gives a useful criteria for ellipticity.

Lemma 2.3.1. *Let P have symbol p which is in the form*

$$p(x,r,\zeta,z) = z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a(x,\zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b(x,\zeta)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

where $a(x,\zeta)$ and $b(x,\zeta)$ are matrices, which are linear in ζ and commute. Let B be the projection on the second factor. Then (P, B) is elliptic with respect to the cone $\mathcal{C}$ if and only if the matrices $a(x,\zeta)^2$ and $a(x,\zeta)^2 + b(x,\zeta)^2$ have no eigenvalues μ with $\mathrm{Re}(\mu) \leq 0$ *for* $\zeta \neq 0$.

Proof. It is clear that

$$p(x,r,\zeta,z)^2 = \{z^2 + a^2(x,\zeta) + b^2(x,\zeta)\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We suppose first that (P, B) is elliptic with respect to $\mathcal{C}$. Then p^2 has no eigenvalues with $\mathrm{Re}(\mu) \leq 0$, so $a^2(x,\zeta) + b^2(x,\zeta)$ has no eigenvalues with $\mathrm{Re}(\mu) \leq 0$. We compute

$$\tau(x,\zeta,\lambda) = \begin{pmatrix} b(x,\zeta) & -i\lambda - ia(x,\zeta) \\ -i\lambda + ia(x,\zeta) & -b(x,\zeta) \end{pmatrix}.$$

Suppose a^2 has an eigenvalue with $\mathrm{Re} \leq 0$. Then a must have an eigenvalue $\lambda \in \mathcal{C}$. Since a and b commute, b preserves the eigenspaces of a, so we can find α such that $a\alpha = \lambda\alpha$ and $b\alpha = \mu\alpha$. Thus

$$\tau(x,\zeta,\lambda)\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \mu\alpha \\ 0 \end{pmatrix}.$$

If $\mathrm{Re}(\mu) > 0$, then $\binom{\alpha}{0} \in \Pi_+(\tau)(\zeta,\lambda)$, while if $\mathrm{Re}(\mu) < 0$, then $\binom{\alpha}{0} \in \Pi_-(\tau)(\zeta,\lambda) = \Pi_+(\tau)(-\zeta,-\lambda)$. Since $B\binom{\alpha}{0} = 0$, this contradicts the assumed ellipticity.

Next we suppose that a^2 and $a^2 + b^2$ have no eigenvalues with nonpositive real part for $\zeta \neq 0$. This implies p^2 has no eigenvalues with nonpositive real part for $(z,\zeta) \neq (0,0)$, and consequently that p has no eigenvalues in $\mathcal{C}$. To study the ellipticity of the boundary condition, we decompose a into Jordan blocks which are preserved by b. By taking the direct sum of two copies of these Jordan blocks, we obtain a subspace which is invariant under $\{p, \tau\}$. If $B\phi = 0$, we decompose $\phi = \Sigma\phi_\nu$ for the ϕ_ν in distinct Jordan blocks. Then $B\phi_\nu = 0$. Consequently, it suffices to verify ellipticity if we assume a has a single Jordan block.

Suppose first that a is a 1×1 block with eigenvalue a, so that τ is a 2×2 matrix of the form

$$\tau = \begin{pmatrix} b & -i\lambda - ia \\ -i\lambda + ia & -b \end{pmatrix},$$

where a and b are scalars. τ has two distinct eigenvalues, so $\Pi_{\pm}(\tau)$ are one-dimensional and consist of eigenvectors. Since $-i\lambda + ia \neq 0$, $\binom{1}{0}$ is *not* an eigenvector. Thus B is injective and the proof is complete.

Next we study a 2×2 Jordan block; the general case is similar and is therefore omitted. τ is a 4×4 matrix with two distinct eigenvalues of multiplicity two. Choose a Jordan basis for the matrix a so that $a\alpha_1 = \mathfrak{a}\alpha_1$, $a\alpha_2 = \mathfrak{a}\alpha_2 + \alpha_1$. Since a and b commute, b preserves this subspace and $b\alpha_1 = \mathfrak{b}\alpha_1$. The two-dimensional space of all vectors of the form $\phi = \binom{c}{d}\alpha_1$ is τ-invariant. The restriction of τ to this subspace has two distinct eigenvalues, and $\binom{1}{0}\alpha_1$ is not one of them. Suppose $N(B) \cap \Pi_+(\tau) \neq \{0\}$, and choose a basis for $\Pi_+(\tau)$ in the form

$$\phi_1 = \begin{pmatrix} * \\ 1 \end{pmatrix}\alpha_1, \quad \phi_2 = \begin{pmatrix} * \\ 0 \end{pmatrix}\alpha_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\alpha_2,$$

where $*$ indicates some complex number. We compute

$$\tau\phi_2 = \begin{pmatrix} * \\ * \end{pmatrix}\alpha_1 + \begin{pmatrix} * \\ -i\lambda + i\mathfrak{a} \end{pmatrix}\phi_2.$$

Since $-i\lambda + i\mathfrak{a} \neq 0$, this is not in the span of $\{\phi_1, \phi_2\}$ which contradicts the fact that $\Pi_+(\tau)$ is a τ-invariant subspace. Thus the proof is complete.

We use this lemma to construct a homotopy in which we replace the symbol p by a new symbol which anticommutes with $p_0 = -ip_m q$.

Homotopy 2.3.2. Let P have symbol p of the form

$$p(x, r, \zeta, z) = z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a(x, \zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b(x, \zeta)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

which is covariant constant in the normal direction for $r < 3r_0$ for some r_0. Assume that $a(x, \zeta)$ and $b(x, \zeta)$ are linear in ζ and commute. We also assume that (P, B) is elliptic with respect to the cone $\mathcal{C}$ where B denotes the projection on the second factor. Let $f(t, r)$ be smooth such that

$$0 \leqslant f(t, r) \leqslant 1, \quad f(t, r) = 1 \quad \text{for } r \geqslant 2r_0,$$
$$f(0, r) = 1, \quad f(1, r) = 0 \quad \text{for } r < r_0.$$

Let P_t have symbol

$$p_t(x, r, \zeta, z) = z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a(x, \zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + f(t, r)b(x, \zeta)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

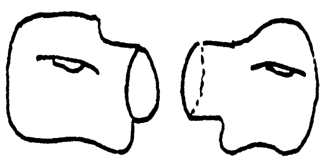
We assumed that a and b commute. Using the ellipticity condition, a^2 and $a^2 + b^2$ have no eigenvalues with nonpositive real part. It follows that a^2 and $(1 - f^2)a^2 + f^2(a^2 + b^2)$ have no eigenvalues with nonpositive real part, and consequently (P_t, B) is elliptic for $t \in [0, 1]$ by Lemma 2.3.1.

Let (P, B) be elliptic with respect to the cone $\mathcal{C}$ with the leading symbol of P given by Clifford multiplication. Using Homotopies 2.3.1 and 2.3.2 we may replace P by a homotopic operator of the form

$$P = -i\frac{\partial}{\partial r}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + A_T\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad B\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_2$$

near dM where A_T is a tangential differential operator with coefficients independent of the normal parameter r which is elliptic with respect to the cone $\mathcal{C}$. Such an operator will be said to *split* near dM.

Let $\overline{M}$ be the double of M. $\overline{M}$ is constructed by taking two copies M_1 and M_2 of M and glueing them along the common boundary dM. On the first copy M_1 we take a neighborhood of dM of the form $dM \times (-r_0, 0]$, and on the second copy M_2 we take a neighborhood of dM of the form $dM \times [0, r_0)$. Since the metric is product near dM, it extends smoothly to the double.



We take two copies P_i of the operator P on M_i. Near dM these have the form

$$P_1 = i\frac{\partial}{\partial r}\cdot\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + A_T\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

$$P_2 = -i\frac{\partial}{\partial r}\cdot\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + A_T\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where the difference in sign in the coefficient of $\partial/\partial r$ is caused by the difference between the inward and the outward normal. Consequently we cannot patch together these two operators directly.

Let $\overline{V}$ be the bundle over $\overline{M}$ consisting of two copies V_i of V over M_i which are patched together near dM using the transition function $\left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. In other words, if $v_i \in V_i$ near dM, we decompose $v_i = v_i^+ + v_i^-$ using the decomposition of $V_i = \Pi_+(q) \oplus \Pi_-(q)$, and then identify $v_1^+ = v_2^+$ and $v_1^- = -v_2^-$. The identity

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} P_1 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = P_2$$

implies that the P_i patch together smoothly to define an operator $\bar{P}$: $C^\infty(\bar{V}) \to C^\infty(\bar{V})$ over $\bar{M}$.

Let M_0 be the manifold $dM \times [0, r_0]$ which we regard as a submanifold of M. Let P_0: $C^\infty(V_0) \to C^\infty(V_0)$ be the restriction of P to M_0, and let B_0 be the boundary condition:

$$B_0\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = 0 \quad \text{implies} \quad \alpha_2(x,0) = \alpha_2(x, r_0) = 0.$$

At $r = 0$, this is just the original boundary condition. At $r = r_0$, we conjugate (P_0, B_0) by the endomorphism $\left(\begin{smallmatrix} I & 0\\ 0 & -I\end{smallmatrix}\right)$ to take care of the change in orientation to see that this is isomorphic to the original boundary condition. Thus (P_0, B_0) is elliptic with respect to the cone $\mathcal{C}$.

If ρ is a representation of $\pi_1(M)$, we double V_ρ to define $\bar{V}_\rho$ over $\bar{M}$ and restrict V_ρ to define V_ρ^0 over M_0. The following lemma relates these three operators.

Lemma 2.3.2.

(a) $\operatorname{Res}_{s=0} \eta(s, P, B) = \frac{1}{2}\{\operatorname{Res}_{s=0} \eta(s, P_0, B_0) + \operatorname{Res}_{s=0} \eta(s, \bar{P})\}$.

(b) $\operatorname{Ind}(\rho, P, B) = \frac{1}{2}\{\operatorname{Ind}(\rho, P_0, B_0) + \operatorname{Ind}(\rho, \bar{P})\}$.

Proof. We proved in [7] that $\operatorname{Res}_{s=0} \eta(s, P, B)$ can be computed in terms of a local formula:

$$\operatorname{Res}_{s=0} \eta(s, P, B) = \int_M a(y, P)d\operatorname{vol}(y) + \int_{dM} a(x, P, B)d\operatorname{vol}(x),$$

where $d\operatorname{vol}(y)$ and $d\operatorname{vol}(x)$ denote the Riemannian measures on M and dM, and $a(y, P)$ and $a(x, P, B)$ are smooth local invariants of the jets of the total symbols of the operators involved. Since $\bar{P}$ is locally isomorphic to P, we have

$$\operatorname{Res}_{s=0} \eta(s, \bar{P}) = \int_{\bar{M}} a(\bar{y}, \bar{P})d\operatorname{vol}(\bar{y}) = 2\int_M a(y, P)d\operatorname{vol}(y).$$

Similarly

$$\operatorname{Res}_{s=0} \eta(s, P_0, B_0) = \int_{M_0} a(y, P)d\operatorname{vol}(y) + 2\int_{dM} a(x, P, B)d\operatorname{vol}(x).$$

However, this residue is independent under perturbations. $a(y, P)$ is not dependent on the normal parameter r so

$$\int_{M_0} a(y, P)d\operatorname{vol}(y) = r_0 \cdot \int_{dM} a(x,0, P)d\operatorname{vol}(x).$$

Since this is independent of r_0, it must vanish. We add up the two local formulas to prove the first assertion; the proof of the second follows similarly and is therefore omitted.

The next lemma lets us compute $\eta(s, P_0, B_0)$.

Lemma 2.3.3. *Let* $P_0 = -i\partial/\partial r\begin{pmatrix}0&1\\1&0\end{pmatrix} + A_T\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ *on* $dM \times [0, r_0]$ *with boundary condition B given by the projection on the second factor. Let* A_T *be a tangential partial differential operator with coefficients independent of r. If* (P_0, B) *is elliptic with respect to the cone* $\mathcal{C}$, *then* $\eta(s, P_0, B) = \eta(s, A_T)$.

Proof. This is Theorem 3.4 of [7] which is based on the identity $\eta(s, P_0, B) = \eta(s, A_T) \cdot \operatorname{index}(-i\partial/\partial r, B)$; the index is 1 in this setting.

We say that (P, B) is homotopic to an operator which splits near dM if there is a 1-parameter family of operators (P_t, B) which are elliptic with respect to $\mathcal{C}$ such that $P_0 = P$ and that P_1 splits near dM. In particular, Homotopies 2.3.1 and 2.3.2 give

Lemma 2.3.4. *Let* (P, B) *be elliptic with respect to the cone* $\mathcal{C}$, *and let the leading symbol of P be given by Clifford multiplication. Then* (P, B) *is homotopic to an operator which splits near dM.*

The basic regularity result of this paper is the following.

Theorem 2.3.5. *Let* (P, B) *be a first order operator elliptic with respect to the cone* $\mathcal{C}$ *and homotopic to an operator which splits near dM. Then* $\operatorname{Res}_{s=0} \eta(s, P, B) = 0$.

Proof. Using the invariance of the residue under homotopy, we may assume that P splits near the boundary. We then apply Lemmas 2.3.2 and 2.3.3 to compute

$$\operatorname{Res}_{s=0} \eta(s, P, B) = \tfrac{1}{2}\left\{\operatorname{Res}_{s=0} \eta(s, \bar{P}) + \operatorname{Res}_{s=0} \eta(s, A_T)\right\}.$$

Both $\bar{P}$ and A_T are defined on manifolds without boundary, so the right-hand side vanishes by [2], [6]. We also note that the formula for $\operatorname{ind}(\cdot, \cdot)$ on manifolds without boundary could be used to derive a corresponding cohomological formula in this case using these techniques. Rather than doing this directly, we state instead the relevant formula and then prove it is correct by showing it has the necessary universal properties.

We note that in particular this theorem applies if the leading symbol of P is given by Clifford multiplication by Lemma 2.3.4.

2.4 $\operatorname{Ind}_1(\rho, P, B)$ for first order operators

We will use the boundary condition to extend p to a collared neighborhood of M so that the extension depends on $|t|$ and not on t near the boundary. The following lemma will be used to show that the resulting integral is independent of the choices made.

Lemma 2.4.1. *Let* $\bar{V}$ *be a vector bundle over M. Let* $ds^2(\varepsilon)$ *be a* 1-*parameter family of Riemannian metrics on M, and* $\nabla(\varepsilon)$ *be a* 1-*parameter family of connections on* $\bar{V}$. *Let* ρ *be a representation of* $\pi_1(M)$, *and* $g(\varepsilon)$: $\Sigma(T^*M) \to \operatorname{END}(\bar{V}, \bar{V})$ *be a* 1-*parameter family of elliptic endomorphisms. Assume that*

$g_\varepsilon(y, \xi, t) = g_\varepsilon(y, \xi, -t)$ *near* dM. *Then*

$$\int_{\Sigma^2(T^*M)} \mathrm{Tch}(\rho) \wedge \mathrm{TODD}(M) \wedge \mathrm{ch}(\Pi_+(\Sigma g(\varepsilon)))$$

is independent of the parameter ε.

Proof. Choose two values a and b of the parameter ε, and let ε range from a to b. Without loss of generality we may assume that $ds^2(\varepsilon)$, $\nabla(\varepsilon)$, and $g(\varepsilon)$ are independent of ε near $\varepsilon = a$ and $\varepsilon = b$. Let $N = M \times [a, b]$ with the metric $ds^2(\varepsilon) + d\varepsilon^2$. We extend $\bar{V}$ to N with connection $\nabla(\varepsilon)$. $\mathrm{TODD}(N) = \mathrm{TODD}(M)$. We apply Stokes theorem to the closed differential form

$$\mathrm{Tch}(\rho) \wedge \mathrm{TODD}(N) \wedge \mathrm{ch}(\Pi_+(\Sigma g(\varepsilon)))$$

to conclude that the integral over $d\{\Sigma^2(T^*M) \times [a, b]\}$ is zero.

This boundary consists of two pieces: $\Sigma^2(T^*M) \times d\{[a, b]\}$ and $d\{\Sigma^2 T^*(M)\} \times [a, b]$. We complete the proof of the lemma by showing the integral over this second piece is zero. By hypothesis, g is invariant under the orientation reversing map $(y, \xi, t) \to (y, \xi, -t)$. Therefore this differential form is invariant as well. This implies that the corresponding integral must be zero.

Let (P, B) be a first order operator elliptic with respect to the cone $\mathcal{C}$. We do not necessarily assume the symbol of P is given by Clifford multiplication. We can assume the range $R(B)$ is a subspace of V by replacing B by the projection on some subspace complementary to the null space $N(B)$. We choose a metric on V so that $N(B)$ and $R(B)$ are orthogonal, and replace B by the orthogonal projection on B. Let $q = 2B - I$. Then $R(B) = \Pi_+(q)$ and $N(B) = \Pi_-(q)$.

We suppress dependence on $x \in dM$ for notational convenience, and define

$$\tau(\zeta, \lambda) = ip_m^{-1}(p(\zeta) - \lambda) \quad \text{for } (\zeta, \lambda) \neq (0,0) \in T^*(dM) \times \mathcal{C},$$
$$\Sigma p(\xi, t) = p(\xi) - it\colon \Sigma(T^*M) \to GL(V) \quad \text{for } (\xi, t) \in \Sigma(T^*M).$$

We will construct a sequence of homotopies to deform τ to q through matrices with no purely imaginary eigenvalues, and then multiply by $-ip_m$ to get a homotopy of Σp to $-ip_m q$.

In the first homotopy, we replace τ by τ_1 to be defined below, supress dependence on (ζ, λ) for notational convenience, let λ be pure imaginary, and suppose $|\zeta|^2 + |\lambda|^2 = 1$.

Homotopy 2.4.1. Let $\pi_\pm(\tau)$ denote the projection on $\Pi_\pm(\tau)$ relative to the splitting $V = \Pi_+(\tau) \oplus \Pi_-(\tau)$. Let $\tau_1 = \pi_+(\tau) - \pi_-(\tau)$ and $\tau_u = u\tau_1 + (1 - u)\tau$ for $u \in [0, 1]$. If μ is an eigenvalue of τ, then $(1 - u)\mu \pm u$ is the corresponding eigenvalue of τ_u, where we select $\pm$ as $\mathrm{Re}(\mu) > 0$ or $\mathrm{Re}(\mu) < 0$. Consequently, τ_u has no purely imaginary eigenvalue.

Let $\pi_+^g(\tau)$ be the orthogonal projection on $\Pi_+(\tau)$ and let $\pi_-^g(\tau) = I - \pi_+^g(\tau)$. In the next homotopy, we replace τ_1 by $\tau_2 = 2\pi_+^g(\tau) - I$ which is unitary Hermitian.

Homotopy 2.4.2. Let $\Pi_+(\tau_u) = \Pi_+(\tau)$ and let $\Pi_-(\tau_u) = \{v_u = (u-1)\pi_-^g(\tau)v + (2-u)v$ for $v \in \Pi_-(\tau)\}$ and $u \in [1,2]$. It is clear that $\pi_-^g(v_u) = \pi_-^g(v)$. If this vanishes, then $v \in \Pi_+(\tau)$ so $v = 0$. This implies $\dim(\Pi_-(\tau_u)) = \dim \Pi_-(\tau) = \frac{1}{2}\dim(V)$ and also that $\Pi_-(\tau_u)$ does not intersect $\Pi_+(\tau_u)$ so that $V = \Pi_+(\tau_u) \oplus \Pi_-(\tau_u)$. Define $\tau_u = \pm 1$ on the appropriate subspaces.

We now use the boundary condition to construct the final homotopy.

Homotopy 2.4.3. Define subspaces $V_u = \{v_u = (u-2)Bv + (3-u)v$ for $v \in \Pi_+(\tau)\}$ and $u \in [2,3]$. If $Bv_u = 0$ then $Bv = 0$, so $v \in N(B)$ and $v = 0$ by the assumed ellipticity. Consequently the V_u have constant rank. Let π_u be the orthogonal projection on V_u and let $\tau_u = 2\pi_u - I$.

We reparametrize the interval and connect the three homotopies to construct τ_u for $u \in [-1,0]$ with $\tau_{-1} = q$ and $\tau_0 = \tau$; τ_u has no purely imaginary eigenvalues.

If P splits near dM, it is possible to give an equivalent formulation of this homotopy which is more useful for computational purposes.

Homotopy 2.4.4. Let P split near dM. Then

$$p(x,0,\zeta,z) = z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a(x,\zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\tau(x,\zeta,\lambda) = \begin{pmatrix} 0 & -i\lambda - ia \\ -i\lambda + ia & 0 \end{pmatrix},$$

where a has no purely imaginary eigenvalues. We define

$$\tau(\theta,x,\zeta,\lambda) = \sin(\theta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \cos(\theta)\tau(x,\zeta,\lambda), \qquad \theta \in \left[\frac{-\pi}{2}, 0\right].$$

It is immediate that $\tau^2 = \{\sin^2(\theta) + \cos^2(\theta)(a^2 - \lambda^2)\}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so that τ has no purely imaginary eigenvalues. Thus $\tau_{-\pi/2} = q$ and $\tau_0 = \tau$.

Lemma 2.4.2. *Let P split near dM. Let $\theta = u \cdot \pi/2$ and let $\tau'(u,\cdot)$ be the homotopy given by Homotopy* 2.4.4 *joining q to τ for $u \in [-1,0]$. Let $\tau(u,\cdot)$ be the homotopy given by Homotopies* 2.4.1 *through* 2.4.3. *Then these two homotopies are equivalent; i.e., there exists $T(s,u,\cdot)$ with no purely imaginary eigenvalue such that*

$$T(0,u,\cdot) = \tau(u,\cdot), \quad T(1,u,\cdot) = \tau'(u,\cdot),$$
$$T(s,-1,\cdot) = q, \qquad T(s,0,\cdot) = \tau(\cdot).$$

Proof. Expand

$$p(x,r,\zeta,z) = z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a(x,\zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By Lemma 2.3.1 we know a has no eigenvalues in $\mathcal{C}$. Use the homotopies constructed in Homotopies 2.4.1 through 2.4.3 to construct a 1-parameter family a_S joining a to a_1 where $a_1^2 = |\zeta|^2$, and a_1 is Hermitian.

Since the a_s have no eigenvalues in $\mathcal{C}$, the corresponding p_s define operators P_s such that (P_s, B) is elliptic with respect to $\mathcal{C}$ by Lemma 2.3.1. Now apply the Homotopies 2.4.1 through 2.4.3 and 2.4.4 to this family to construct the two-parameter families $\tau(s, u, \cdot)$ and $\tau'(s, u, \cdot)$.

This reduces the proof of Lemma 2.4.2 to the case in which $a^2 = |\zeta|^2$, a Hermitian. Decompose $\Pi_-(\tau)$ into $\pm$ eigenspaces of a, and set $w = i\lambda \pm i|\zeta|$ for λ pure imaginary and $|w| = 1$. This reduces the proof to the case in which $\tau = \left(\begin{smallmatrix} 0 & \bar{w} \\ w & 0 \end{smallmatrix}\right)$. Homotopies 2.4.1 and 2.4.2 do not change τ at all; Homotopies 2.4.3 and 2.4.4 are clearly equivalent rotations of the relevant eigenspaces involved.

Suppose P is covariant constant near dM in M by applying Homotopy 2.3.1, and also assume the metric on M is product near dM. Let $\tilde{M} = dM \times [-1, 0] \cup M$ joined along the edge $dM \times 0 = dM$, and extend V_ρ and V over $\tilde{M}$ to be independent of the normal parameter r. We smooth out Homotopies 2.4.1 through 2.4.3 to assume that τ_u is identically q near $u = -1$ and identically τ near $u = 0$. We extend τ_u from $\Sigma(T^*dM) = S(T^*M \oplus 1)$ to $T^*M \oplus 1$ so that $\tau_u(y, a\xi, a\lambda) = f(a)\tau_u(y, \xi, \lambda)$ where f: $[0, 1] \to [0, 1]$ is a smooth monotonic map which is identically 0 near 0 and identically 1 near 1. (If we just extend τ_u to be homogeneous of degree 1, it will be continuous but not smooth at $(\xi, \lambda) = (0, 0)$ since τ_u is not linear in general. This step can be avoided if we use Homotopy 2.4.4 as τ_u is linear in this case.)

Define the smooth symbol Σp_B on $\Sigma T^*(\tilde{M})$ by

$$\Sigma p_B = \begin{cases} p(x, r, \zeta, z) - it & \text{for } r \geqslant 0, \\ -ip_m(\tau_r(x, \zeta, it) + iz) & \text{for } r \leqslant 0. \end{cases}$$

We emphasize that the notation Σp_B is *not* the suspension of p_B but rather the extension of Σp using the boundary condition B. The whole point of the discussion in the first section was to work with $\Sigma^2 p$ as Σp_B does not in general desuspend. By an abuse of notation, we will let $\Sigma^\nu p_B = \Sigma^{\nu-1}(\Sigma p_B)$.

We choose a fixed connection on V and the Levi-Civita connection on T^*M. Let $\Pi_+(\Sigma^2 p_B)$ be the bundle over $\Sigma^2 T^*(\tilde{M})$ with clutching function Σp_B discussed in the first section. This bundle inherits a natural connection which is the projection of the connection on $V = \Pi_+ \oplus \Pi_-$. We take the component of the differential form

$$\mathrm{Tch}(\rho) \wedge \mathrm{TODD}(\tilde{M}) \wedge \mathrm{ch}(\Pi_+(\Sigma^2 p_B)),$$

and integrate it to define

$$\operatorname{ind}_1(\rho, P, B) = (-1)^m \int_{\Sigma^2 T^*(\tilde{M})} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M) \wedge \operatorname{ch}\big(\Pi_+(\Sigma^2 p_B)\big).$$

Near the boundary of $\Sigma^2(T^*\tilde{M})$,

$$\Sigma p_B(x, r, \zeta, z, t) = -ip_m\Big\{f\big(|\zeta|^2 + t^2\big)q + iz\Big\}.$$

We could make a further homotopy to change this to a symbol which is independent of (ζ, z, t) but this is not necessary. Since this depends only on $|\zeta|^2 + t^2$, Σp_B factors through $\tilde{M} \times 1^2$, so the $2m+1$ differential form defining ind_1 vanishes identically near $r = -1$.

Lemma 2.4.3. *Let (P, B) be a first order operator elliptic with respect to the cone $\mathcal{C}$. Then the following hold.*

(a) $\operatorname{ind}_1(\rho, P, B)$ *is a homotopy invariant of (P, B) independent of the metric on M and the connection on V.*

(b) *There are local formulas $a(y, \rho, P)$ and $a(x, \rho, P, B)$ which depend functorially on the jets of the metric, the jets of the connection on V, the connection 1-form of ∇_ρ, and the jets of the total symbols of (P) and (P, B) such that*

$$\operatorname{ind}_1(\rho, P, B) = \int_M a(y, \rho, P)\,d\operatorname{vol}(y) + \int_{dM} a(x, \rho, P, B)\,d\operatorname{vol}(x).$$

Proof. (a) follows directly from Lemma 2.4.1 since Σp_B depends only on t^2 and not on t near $d\tilde{M}$. Construct local formulas by integrating over the fibers of $\Sigma^2(T^*M)$ to define $a(y, \rho, P)$ for $y \in M$, and integrate over both the fibers and the normal variable on $dM \times [-1, 0]$ to define $a(x, \rho, P, B)$. q.e.d.

The manifold $\tilde{M}$ is diffeomorphic to M where we simply slide the collar inside M using the geodesic normal flow suitably damped away from the boundary. Thus we can regard $\Sigma^2 p_B$ as being defined on $\Sigma^2(T^*M)$ if we like; this is done by performing the homotopies inside M instead of on a collared neighborhood.

We can now prove the basic formula of this paper.

Theorem 2.4.4. *Let (P, B) be a first order operator elliptic with respect to the cone $\mathcal{C}$ and homotopic to an operator which splits near the boundary. Let ρ be a representation of the fundamental group such that V_ρ is topologically trivial. Then*

$$\operatorname{ind}(\rho, P, B) = \operatorname{ind}_1(\rho, P, B).$$

Proof. Both ind and ind_1 are homotopy invariants given by local formulas. Without loss of generality we assume (P, B) splits near dM. We know

$$\operatorname{ind}(\rho, P, B) = \tfrac{1}{2}\operatorname{ind}(\rho, P_0, B_0) + \tfrac{1}{2}\operatorname{ind}(\rho, \bar{P})$$

by Lemma 2.3.2. A similar argument shows the same is true for ind_1. Therefore it suffices to prove that

$$\operatorname{ind}(\rho, \bar{P}) = \operatorname{ind}_1(\rho, \bar{P}), \quad \operatorname{ind}(\rho, P_0, B^0) = \operatorname{ind}_1(\rho, P_0, B_0),$$

where $\bar{P}$ and (P_0, B_0) are as defined in Lemma 2.3.2. Since $\overline{M}$ has no boundary, the first equality is the statement of Lemma 1.3.2. We may therefore restrict our attention to the case where

$$M = dM \times [0,1], \quad P = -i\frac{\partial}{\partial r}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

written in block form. We assume the boundary condition is the projection on the second factor and that A is a tangential first order differential operator elliptic with respect to the cone $\mathcal{C}$ whose coefficients are independent of the normal parameter r.

Since Σp is independent of r over M, the integral vanishes over M, so we may restrict attention to $\tilde{M} - M$. We study first the portion over $dM \times 0$ and use Homotopy 2.4.4 to define Σp_B:

$$\begin{aligned}\tau(x, r, \zeta, it) = \sin(r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &+ \cos(r)a(x,\zeta)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ &+ \cos(r)\cdot t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ & x \in dM, r \in \left[\frac{-\pi}{2}, 0\right], \zeta \in T^*(dM), t \in R.\end{aligned}$$

If we set $\Sigma p_B = -ip_m(\tau + iz)$, this defines

$$\begin{aligned}\Sigma p_B(x, r, \zeta, t) = \sin(r)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} &+ \cos(r)a(x,\zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &+ \cos(r)\cdot t\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

This gives the contribution over $dM \times [-\pi/2, 0]$ for the part of $\tilde{M} - M$, which is near the left-hand edge. The right-hand edge is isomorphic to the left-hand edge if we replace r by $-r$ and conjugate by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. It is tempting to compute the full integral by simply doubling this contribution. We do this in a way which will extend Σp_B to $dM \times [-\pi/2, \pi/2]$

We replace r by $-r$ and z by $-z$. This preserves the orientation and transforms Σp_B to the form

$$\begin{aligned}-\sin(r)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} &+ \cos(r)a(x,\zeta)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \cos(r)\cdot t\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} - z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

We conjugate this with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to transform this back to Σp_B. This permits us to regard $\tilde{M} - M = dM \times [-\pi/2, \pi/2]$ where Σp_B is defined by

$$\Sigma p_B = \sin(r)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \cos(r)a(x,\zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$+\cos(r)\cdot t\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Define $\omega_{2m-2} = dx_1 \wedge d\zeta_1 \wedge dx_{m-1} \wedge d\zeta_{m-1}$ so that the orientation is given by $\omega_{2m+1} = \omega_{2m-2} \wedge dr \wedge dz \wedge dt$ on $T^*M \oplus 1$. Introduce new parameters

$$u_1 = z, \quad u_2 = \sin(r), \quad u_3 = \cos(r)\cdot t,$$

and replace ζ by $\cos(r)\zeta$. This changes the orientation and replaces Σp_B by

$$u_1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + u_2\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + u_3\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} + a(x,\zeta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The relation $t^2 + z^2 + |\zeta|^2 = 1$ becomes the relation $|\zeta|^2 + |u|^2 = 1$, so the new domain of integration is $\Sigma^3 T^*(dM)$. By Lemma 1.2.2(c) we can replace the symbol by $\Sigma^3 a$. Since the orientation has been reversed,

$$\operatorname{ind}_1(\rho, P, B)$$
$$= (-1)^{m-1}\int_{\Sigma^4 T^*(dM)} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(dM) \wedge \operatorname{ch}(\Pi_+ \Sigma^4 a).$$

Use Theorem 1.3.2 to evaluate this integral as ind(ρ, A). Since ind(ρ, A) = ind(ρ, P, B), the proof of the theorem is complete.

SECTION THREE

3.1 Definition of $\operatorname{ind}_1(\rho, P, B)$ if $dM \neq \emptyset$ and $d \geqslant 1$

It is convenient to work with a larger class of symbols in defining the homotopies which we will work with. Supress dependence on $x \in dM$. Let $(\zeta, z, t) \in T^*(M) \oplus 1$ and let

$$q(\zeta, z, t) = \Sigma_j q_j(\zeta, t) z^{d-j}\colon T^*(M) \oplus 1 \to \operatorname{END}(V, V)$$

be invertible for $(\zeta, z, t) \neq (0,0,0)$. Suppose the q_j are continuous and homogeneous of order j in (ζ, t). Then

$$q_j(c\zeta, c^d t) = c^j q_j(\zeta, t) \quad \text{for } c \geqslant 0,$$

which implies $q_0(\zeta, t) = q_0$ is independent of (ζ, t). Define τ as in §2.1. The ellipticity of q implies τ has no purely imaginary eigenvalues. Let B: $C^\infty(W) \to C^\infty(W')$ be a differential boundary condition, and assume that $\sigma^g(B)$: $\Pi_+(\tau) \to W'$ is an isomorphism for $(\zeta, t) \neq (0,0)$. If (P, B) is elliptic with respect to the cone $\mathcal{C}$, then $q = \Sigma p$ satisfies these conditions.

If $d > 1$, then q and τ do not act on the same bundle, and τ is not homogeneous. Consequently the construction of §2.4 does not generalize, so we use instead the Atiyah-Bott homotopy of [1]. We review their construction in the context we shall be using since some of the technical details and notation differ from their paper owing to the presence of the parameter t.

Let $V_j = V \otimes 1_j$ denote the direct sum of j-copies of V. W is the restriction of V_d to the boundary. Let $\mathrm{END}(V, V)$ act on V_j in block form. Let $S^j(q) = q \oplus 1_{j-1}$ on V_j. This process of adding trivial factors is called stabilization. It is clear $\Sigma(S^j q) = \Sigma q \oplus \Sigma(1_{j-1})$. Since $\Sigma(1_{j-1})$ does not depend on (ζ, z, t), it will not affect the $2d + 1$ component of the differential form defining ind_1. We stabilize as often as necessary without affecting ind_1.

The bundle W' does not extend over M in general. This causes certain technical problems which we correct as follows. $\sigma^g(B)(0)$: $W \to W'$ is surjective. We split this surjection to express $W = W' \oplus W''$, where W'' is the null space of $\sigma^g(B)(0)$, and W' is identified as the orthogonal complement of W'' in W. We will use the boundary condition to construct a homotopy q_u which joins $S^{2d+1}(q)$ to $S^{2d+1}(q_0)$ which does not depend on (z, ζ, t). If we replace q by $q_0^{-1}q$, we do not change the ellipticity conditions. We then replace $(q_0^{-1}q)_u$ by $S^{2d+1}(q_0)(q_0^{-1}q)_u$ to construct a homotopy joining $S^{2d+1}(q)$ to $S^{2d+1}(q_0)$. Consequently we shall assume without loss of generality that $q_0 = I$.

We ignore smoothness questions for the moment and work with continuous symbols and homotopies. In computing the degree of homogeneity, we consider t as a variable of order d and ξ as a variable of order 1. Since $|\xi|^2 + t^2$ is not homogeneous, it is more convenient to work with the homogeneous function $|\xi|^2 + |t|^{2/d}$. We define

$$\Sigma(T^*M)_d = \{(\xi, t) \in T^*M \oplus 1 : |\xi|^2 + |t|^{2/d} = 1\}.$$

Radial projection defines homeomorphisms between $\Sigma(T^*M)$ and $\Sigma(T^*M)_d$.

We parametrize $\Sigma(T^*M)_d$ by setting

$$z = -\cos(\theta), \quad \zeta = \sin(\theta)\tilde{\zeta}, \quad t = \sin^d(\theta)\tilde{t},$$

for $0 \leqslant \theta \leqslant \pi$ and $(\tilde{\zeta}, \tilde{t}) \in \Sigma(T^*dM)_d$. Let $\alpha = \cos(\theta) + i\sin(\theta)$ Then

$$q = \left(\frac{-1}{2}\right)^d \Sigma_j i^j q_j(\tilde{\zeta}, \tilde{t})\left(\alpha - \frac{1}{\alpha}\right)^j \left(\alpha + \frac{1}{\alpha}\right)^{d-j}$$

The parameter α ranges over a half-circle. Let $\beta = \alpha^2$ range over the whole circle. Then

$$(-\alpha)^d q = \tilde{q}(\tilde{\xi}, \tilde{t}, \beta) = \Sigma_j \tilde{q}_j(\tilde{\xi}, \tilde{t})\beta^{d-j},$$

where the $\tilde{q}_j$ are linear combinations of the q_j. For example, if $d = 2$,

$$\begin{aligned} q &= z^2 + q_1 z + q_2, \\ 4\tilde{q} &= (1 + iq_1 - q_2)\beta^2 + (2 + 2q_2)\beta + (1 - iq_1 - q_2). \end{aligned}$$

Homotopy 3.1.1. Replace q by $\tilde{q}$ by multiplying q by $\exp(-du\,\mathrm{LOG}(-2\alpha))$ for $0 \leqslant u \leqslant 1$. The multiplicative factor only depends on $(z, |\xi|, |t|)$.

It is clear $\tilde{q}(\tilde{\xi}, \tilde{t}, 1) = 1$. We will construct a homotopy $\tilde{q}_u$ so that $\tilde{q}_u(\tilde{\xi}, \tilde{t}, 1) = 1$ for all u. This homotopy will project back to define a continuous homotopy on $\Sigma(T^*M)_d$. We restrict henceforth to the parameter space $S^1 \times \Sigma(T^*dM)_d$. The second step is to reduce the problem to the first order. We define

$$L^d(\tilde{q}) = \begin{vmatrix} \tilde{q}_0 & \tilde{q}_1 & \cdots & \tilde{q}_{d-1} & 1 \\ -\beta & 1 & \cdots & 0 & 0 \\ 0 & -\beta & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\beta & 1 \end{vmatrix} : S^1 \times \Sigma(T^*dM)_d$$

$$\to \mathrm{END}(V_{d+1}, V_{d+1}).$$

This is invertible for all $(\tilde{\xi}, \tilde{t}, \beta)$. The two matrices $L^d(\tilde{q})$ and $S^{d+1}(\tilde{q})$ are related by the identity

$$L^d(\tilde{q}) \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ \beta & 1 & \cdots & 0 & 0 \\ \beta^2 & \beta & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta^d & \beta^{d-1} & \cdots & \beta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \bar{q}_1 & \cdots & \bar{q}_{d-1} & \bar{q}_d \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} S^{d+1}(\tilde{q}),$$

where $\bar{q}_1 = (\tilde{q}(\beta) - \tilde{q}(0))/\beta$ and $\bar{q}_i = (\bar{q}_{i-1}(\beta) - q_{i-1}(0))/\beta$ for $i > 1$. This gives an identity of the form

$$L^d(\tilde{q}) = H_1(\tilde{\xi}, \tilde{t}, \beta) S^{d+1}(\tilde{q}) H_2(\beta),$$

where H_1 is an upper triangular matrix, and H_2 is a lower triangular matrix.

Homotopy 3.1.2. We first construct a homotopy H_u' connecting $S^{d+1}(\tilde{q})$ with $L^d(\tilde{q})$ by homotoping H_i through triangular matrices to the identity, and then set $H_u(\tilde{\xi}, \tilde{t}, \beta) = (H_u')^{-1}(\tilde{\xi}, \tilde{t}, 1) \cdot H_u'(\tilde{\xi}, \tilde{t}, \beta)$ to ensure that $H_u(\tilde{\xi}, \tilde{t}, 1) = 1$ for all u. This connects $S^{d+1}(\tilde{q})$ with an elliptic endomorphism $a(\tilde{\xi}, \tilde{t})\beta + b(\tilde{\xi}, \tilde{t})$ which is first order in the parameter β; $a(\tilde{\xi}, \tilde{t}) + b(\tilde{\xi}, \tilde{t}) = I$.

For the next homotopies, we define the projection

$$\pi_1 = \frac{1}{2\pi i} \int_{|\beta|=1} (a\beta + b)^{-1} d(a\beta + b) = \frac{1}{2\pi i} \int_{|\beta|=1} (\beta + a^{-1}b)^{-1} \, d\beta.$$

Homotopy 3.1.3. Consider $(a\beta + ub)\pi_1 + (au\beta + b)(1 - \pi_1)$ for $0 \leqslant u \leqslant 1$.

Homotopy 3.1.4. Consider $(a + ub)\beta\pi_1 + (au + b)(1 - \pi_1)$ for $0 \leqslant u \leqslant 1$. We refer to [1] for a proof that these endomorphisms are elliptic. This connects $(a\beta + b)$ to $(a + b)\beta\pi_1 + (a + b)(1 - \pi_1) = \beta\pi_1 + (1 - \pi_1)$. We adjust the homotopy as above to ensure that it is always I at $\beta = 1$.

We use the boundary condition for the final homotopy. There is a natural identification of range π_1 with $\Pi_+(\tau)$ discussed in [1]. We stabilize again to consider $(\beta\pi_1 + (1 - \pi_1)) \oplus 1_{W'} \oplus 1_{W'''}$ on V_{2d+1}.

Homotopy 3.1.5. We use the boundary condition to identify range(π_1) with W'. We rotate these two subspaces to transform this operator through a homotopy to $\pi_1 + (1 - \pi_1) \oplus \beta 1_{W'} \oplus 1_{W'''}$. On $\Sigma(T^*M)_d$, $\beta = \alpha^2$ so we can eliminate this last factor of β in a homotopy as was done in Homotopy 3.1.1.

We connect these homotopies to define $q_B(u)$ joining $S^{2d+1}(q)$ to $S^{2d+1}(q_0)$. The process which assigns to an elliptic pair (q, B) the homotopy $q_B(u)$ has certain functorial properties.

Definition 3.1.1.(a) Such a process is said to be *invariant* if it is coordinate free. Let a be an endomorphism independent of (ξ, t), and let $A_j = a \oplus \cdots \oplus a$ j-times on V_j. The symbol aqa^{-1} is elliptic with respect to the boundary condition $A_d B A_d^{-1} = \bar{B}$. We require that

$$(aqa^{-1})_{B'}(u) = A_{2d+1} q_B(u) A_{2d+1}^{-1} \quad \text{for all } u.$$

(b) Such a processs is said to be *continuous* if it depends continuously on parameters. Let $(q(v), B(v))$ be a continuous 1-parameter elliptic family. We require that $(q(v))_{B(v)}(u)$ depends continuously on all variables.

(c) Such a process is said to *depend locally on* (ζ, t) if there is no global information required. Let $(\zeta_0, t_0) \in (T^*dM)_d$ be given, and (q^i, B^i) be elliptic for $i = 1, 2$. Assume that

$$q^1(\zeta_0, z, t_0) = q^2(\zeta_0, z, t_0) \quad \text{for all } z,$$
$$\sigma^g(B^1)(\zeta_0) = \sigma^g(B^2)(\zeta_0).$$

We require that

$$q^1_{B^1}(u)(\zeta_0, z, t_0) = q^2_{B^2}(u)(\zeta_0, z, t_0) \quad \text{for all } z, u.$$

The following is an immediate consequence of the construction given.

Lemma 3.1.1. *The process which associates to an elliptic pair (q, B) a homotopy $q_B(u)$ joining $S^{2d+1}(q)$ to $S^{2d+1}(q_0)$ is invariant and continuous, and depends locally on (ζ, t).*

In §2.4 we defined an extension using a different process. It is clear that that processs is invariant, continuous, and local in (ζ, t). We show that these two processes are equivalent by proving that these three properties essentially characterize such a process of $d = 1$.

Lemma 3.1.2. *Suppose we are given two processes which associate to an elliptic (q, B) a homotopy $q^i_B(u)$ joining $S^{2d+1}(q)$ to an endomorphism depending on $(|\zeta|, z, |t|)$ for $u = 1$. Assume the processes are invariant and continuous, and depend locally on (ζ, t) for $i = 0, 1$. Then we can construct a 2-parameter family $q_B(u, v)$ joining $q^0_B(u)$ to $q^1_B(u)$, which is invariant, continuous, and local in (ζ, t) and such that $q_B(1, v)$ depends only on $(|\zeta|, z, |t|)$; i.e., the defining condition is preserved.*

Proof. Without loss of generality we assumed that the process $i = 1$ is given by the stabilization of the homotopy of §2.4. Let $q = q_0(z - i\tau)$, and let $\tau(v)$ be the 1-parameter family joining τ to an endomorphism which depends on $(|\zeta|, z, |t|)$. Let $q(v) = q_0(z - i\tau(v))$. In Homotopies 2.4.1 and 2.4.2, we do not change $\Pi_+(\tau)$. In Homotopy 2.4.3 we rotate $\Pi_+(\tau)$ to W'. Therefore $(q(v), B)$ is elliptic. We define $q(u, v) = q(uv)$ to define a homotopy joining $q = q(0)$ to $q(v)$ for $u \in [0, 1]$. We apply the other process with starting condition $q(v)$ as $u \in [1, 2]$ to construct the 2-parameter family $q_B(u, v)$. The homotopy $q_B(u, 0)$ is equivalent to the application of the other process to q. The homotopy $q(u)(1)$ is the homotopy of §2.4 with another homotopy glued on for $u \in [\frac{1}{2}, 1]$. Since $q(1)$ only depends on $(|\zeta|, z, |t|)$, the local nature implies that $q(u, 1)$ only depends on $(|\zeta|, z, |t|)$. We just undo this additional homotopy to construct a homotopy from $q(u, 1)$ to the homotopy of §2.4 to complete the proof.

We use the homotopy $q_B(u)$ to define an extension of $S^{2d+1}(q)$ over $dM \times [-1, 0]\cup M$ which agrees with $S^{2d+1}(q_0)$ on $dM \times \{-1\}$. We use the geodesic flow to identify $dM \times [-1,0]uM$ with M so this extension becomes defined over M. When this construction is applied to $q = \Sigma p$, we denote the resulting extension by $(\Sigma p)_B = \Sigma p_B$ supressing the stabilizations involved in the interests of notational simplicity. By an abuse of notation we let $\Sigma^j p_B = \Sigma^{j-1}(\Sigma p)_B$. We emphasize that in general this does not desuspend; $(\sigma p)_B$ is *not* the suspension of some extension p_B. We define

$$\operatorname{ind}_1(\rho, P, B) = (-1)^m \int_{\Sigma^2(T^*M)} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M) \wedge \operatorname{ch}\big(\Pi_+ (\Sigma^2 p_B)\big).$$

By Lemmas 3.1.2 and 2.4.1, this agrees with the definition given in the second section if $d = 1$.

3.2 Functorial properties of ind_1

In this section we will verify that $\operatorname{ind}(\rho, P, B)$ and $\operatorname{ind}_1(\rho, P, B)$ have the same functorial properties. We assume that (P, B) is a dth order differential operator (not pseudo-differential) which is elliptic with respect to the cone $\mathcal{C}$.

Lemma 3.2.1. *If (P, B) is elliptic with respect to the cone $\mathcal{C}$, then the following hold.*

(a) $\operatorname{ind}_1(\rho, P, B)$ *is a homotopy invariant of* (P, B).

(b) *There are local formulas $a(y, \rho, p)$ and $a(x, \rho, P, B)$ which depend functorially on the jets of the metric, the jets of the connection on V, the connection 1-form of ∇_ρ, and the jets of the total symbols of (P) and (P, B) such that*

$$\operatorname{ind}_1(\rho, P, B) = \int_M a(y, \rho, P)\,d\operatorname{vol}(y) + \int_{dM} a(x, \rho, P, B)\,d\operatorname{vol}(x).$$

Proof. The proof is exactly the same as that given for Lemma 2.4.3, and is therefore omitted.

In Lemma 1.3.1 we considered a twisted product formula relating the twisted index formula and the index formula. In that lemma, we supposed $dM_1 = dM_2 = \varnothing$. We now consider the generalization to the case $dM_1 = \varnothing$, $dM_2 \neq \varnothing$. We will consider the other case $dM_1 \neq \varnothing$, $dM_2 = \varnothing$ later.

Lemma 3.2.2. *Let $dM_1 = \varnothing$, and let Q: $C^\infty(V_1) \to C^\infty(V_2)$ be a dth order elliptic complex over M_1. Let P: $C^\infty(V_3) \to C^\infty(V_3)$ be a dth order operator over M_2, and let B be a boundary condition such that (P, B) is elliptic with respect to the cone $\mathcal{C}$. Over $M_1 \times M_2$ we define*

$$R = \begin{pmatrix} P & Q^* \\ Q & -P \end{pmatrix} : C^\infty((V_1 \oplus V_2) \otimes V_3) \to C^\infty((V_1 \oplus V_2) \otimes V_3)$$

with boundary condition $B' = B \oplus B$. Then

(a) (R, B') *is elliptic with respect to the cone* $\mathcal{C}$,

(b) $\operatorname{ind}(\rho, R, B') = \operatorname{ind}(\rho, P, B) \cdot \operatorname{index}(Q)$,

(c) $\operatorname{ind}_1(\rho, R, B') = \operatorname{ind}_1(\rho, P, B) \cdot \operatorname{index}(Q)$.

Proof. (a) and (b) follow from Theorem 3.4(a) of [7]. (c) is proved by making a calculation similar to that made for the proof of Lemma 1.3.1. Let $e_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ act in block form on $(V_1 \oplus V_2) \otimes V_3$. We can construct a homotopy connecting e_0 to the identity by replacing -1 by $\exp(\pi i u)$ as $0 \leqslant u \leqslant \pi$. We may apply Lemma 2.4.1 to replace the symbol Σr by $e_0 \Sigma r$ in computing ind_1.

Choose local coordinates (y^i, ξ^i) for $T^*(M_i)$. Let (t, v) be the real parameters of $T^*(M_1 \times M_2) \oplus 1^2$. We supress the dependence of our symbols on $y = (y_1, y_2)$ for notational convenience. We defined ind_1 for symbols which were pseudo-differential in (ζ, t). We perform a homotopy to replace q by a symbol such that

$$q^*q(\xi^1) = |\xi^1|^{2d} I \quad \text{on } V_1, \qquad qq^*(\xi^1) = |\xi^1|^{2d} I \quad \text{on } V_2.$$

This does not affect the ellipticity condition. We compute

$$e_0 \Sigma r = p(\xi^2) I - i \begin{pmatrix} t & iq^* \\ -iq & -t \end{pmatrix} (\xi^1) = p(\xi^2) - i\Sigma(-iq)(\xi^1, t).$$

We change notation to replace $-iq$ by q without changing $\operatorname{index}(Q)$. Let $c \geqslant 0$ be the parameter $|\xi^1|^{2d} + t^2$. Then $\Sigma(q)^2 = c^2 I$. Let $\pi_\pm(\Sigma q)$ denote the projection on the $\pm c$ eigenspaces $\Pi_\pm(\Sigma q)$. Then for $c > 0$

$$e_0 \Sigma r = p(\xi^2, c) \otimes \pi_+(\Sigma q)(\xi^1, t) \oplus p(\xi^2, -c) \otimes \pi_-(\Sigma q)(\xi^1, t).$$

The boundary condition B' commutes with the projections $\pi_\pm(\Sigma q)$. Consequently, the homotopies defined in §3.1 respect this decomposition, and we can express $e_0 \Sigma r_{B'}$ in terms of Σp_B and $\pi_\pm(\Sigma q)$. Since $\pi_\pm(\Sigma q)$ are projections, they commute with suspension so that

$$\begin{aligned} \operatorname{ch}(\Pi_+ \Sigma(e_0 \Sigma r_{B'})) &= \operatorname{ch}(\Pi_+(\Sigma^2 p_B))(\xi^2, c, v) \cdot \operatorname{ch}(\Pi_+(\Sigma q))(\xi^1, t) \\ &\quad + \operatorname{ch}(\Pi_+(\Sigma^2 p_B))(\xi^2, -c, v) \cdot \operatorname{ch}(\Pi_-(\Sigma q))(\xi^1, t). \end{aligned}$$

We replace the region $|\xi|^2 = t^2 + v^2 = 1$ by the region $|\zeta|^{2d} + z^2 + t^2 + v^2 = 1$, and parametrize this region in the form: $|\zeta|^{2d} + t^2 = c^2$, $z^2 + v^2 + c^2 = 1$, $c > 0$. After performing the integral over $\Sigma(T^* M_1)_d$ parametrized by (y^1, ξ^1, t) and taking into account the induced orientations, we compute

$$\int_{\Sigma(T^* M_1)_d} \operatorname{TODD}(M_1) \wedge \operatorname{ch}(\Pi_+(\Sigma q)) = \operatorname{index}(Q) \cdot (-1)^m,$$

$$\int_{\Sigma(T^* M_1)_d} \operatorname{TODD}(M_1) \wedge \operatorname{ch}(\Pi_-(\Sigma q)) = -\operatorname{index}(Q) \cdot (-1)^m,$$

which implies that

$$\operatorname{ind}_1(\rho, R, B')$$

$$= \operatorname{index}(Q)\int_{\Sigma^2 T^* M_2} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M_2) \wedge \operatorname{ch}\big(\Pi_+(\Sigma^2 p_B)\big)(\xi^2, c, v)(-1)^{m_2}$$

$$-\operatorname{index}(Q)\int_{\Sigma^2 T^* M_2} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M_2) \wedge \operatorname{ch}\big(\Pi_+(\Sigma^2 p_b)\big)(\xi^2, -c, v)(-1)^{m_2},$$

where the integral is restricted to range over $c > 0$. When we combine the second integral and take into consideration the change in orientation imposed by replacing c by $-c$, this yields $\operatorname{index}(Q) \cdot \operatorname{ind}_1(\rho, P, B)$ which completes the proof.

Lemmas 3.2.1 and 3.2.2 are simple formula consequences of the fact that the process involved in defining the extension Σp_B is invariant, continuous, and local in (ζ, t).

Before we consider the other generalization of Lemma 1.3.1 to the case $dM_1 \neq \varnothing$ and $dM_2 = \varnothing$, we review the Atiyah-Bott index theorem. Let Q: $C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic complex over M_1, and let B: $C^\infty(V_1) \to C^\infty(W')$ be a boundary condition. We omit the parameter λ and define τ as in the second section. We say that (Q, B) is elliptic with respect to $\{0\}$ if the symbol of Q is elliptic for $\xi \neq 0$, and $\sigma^g(B)(\zeta)$: $\Pi_+(\tau)(\zeta) \to W'$ is an isomorphism for $\zeta \neq 0$. We consider the operator

$$Q \oplus B\colon C^\infty(V_1) \to C^\infty(V_2) \oplus C^\infty(W'),$$

and define $\operatorname{index}(Q, B) = \dim\ker(Q \oplus B) - \dim\operatorname{coker}(Q \oplus B)$. In general, this is not $\operatorname{index}(Q_B) = \dim\ker(Q_B) - \dim\operatorname{coker}(Q_B)$ since B need not be surjective. We use the boundary condition to define an extension q_B of $S^{2d+1}(q)$ to a symbol which agrees with $S^{2d+1}(q_0)$ near dM. The Atiyah-Bott formula [1] expresses

$$\operatorname{index}(Q, B) = \int_{\Sigma(T^*M)} \operatorname{TODD}(M) \wedge \operatorname{ch}(\Pi_+(\Sigma q_B))(-1)^m.$$

This is too general a setting for our purposes so we specialize. Let

$$R = \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix}\colon C^\infty(V_1 \oplus V_2) \to C^\infty(V_1 \oplus V_2),$$

and let $B = B_1 \oplus B_2$ be a boundary condition. We assume that (R, B) is self-adjoint and that (R, B) is elliptic with respect to the cone $\mathcal{C}$. The self-adjointness condition is equivalent to assuming that (Q, B_1) and (Q^*, B_2) are adjoints. The ellipticity with respect to the cone $\mathcal{C}$ implies that both (Q, B_1) and (Q, B_2) are elliptic with respect to $\{0\}$, but it is a much stronger condition.

Under these conditions, it is immediate that

$$\operatorname{index}(Q_{B_1}) = \operatorname{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{-tR^2 B}\right),$$

so this index is given by a local formula.

We noted that index$(Q, B) \neq$ index(Q_B) in general. However, in this more restricted situation, we can prove

Lemma 3.2.3. *Let (R, B) be self-adjoint and elliptic with respect to $\mathcal{C}$, and assume $R = \left(\begin{smallmatrix} 0 & Q^* \\ Q & 0 \end{smallmatrix}\right)$ and $B = B_1 \oplus B_2$. Then*

(a) *there is a map $A\colon C^\infty(W') \to C^\infty(V_1 \oplus V_2)$ such that $BA = 1_{W'}$,*

(b) index$(Q, B_1) =$ index(Q_{B_1}).

Proof. We prove (a) by induction on d. We first suppose $d = 1$ so $B = B_{00}$: $V \to W'$ is simply an endomorphism. The ellipticity implies that B is surjective, so we can choose A such that $AB = 1_{W'}$. Let $A(\psi)(x, r) = f(r)A\psi(x)$ where $f(r)$ is a smooth function which is identically 1 near $r = 0$ and identically 0 away from dM. Then $A\colon C^\infty(W') \to C^\infty(W') \to C^\infty(V)$ and $BA\psi = \psi$. Next we suppose $d = 2$ so

$$B = (B_{00}) \oplus (B_{11}D_r + B_{10})\colon C^\infty(V) \to C^\infty(W_0' \oplus W_1').$$

B_{00} and B_{11} are endomorphisms. B_{10} is a first order tangential operator. The ellipticity with respect to $\mathcal{C}$ implies that $B_{00} \oplus B_{11}\colon \Pi_+(\tau)(0, i) \to W'$ is an isomorphism so in particular B_{00} and B_{11} are surjective. Define A_{00} and A_{11} such that $B_{00}A_{00} = 1_{W_0'}$ and $B_{11}A_{11} = 1_{W_1'}$. We define

$$A(\psi_0, \psi_1) = f(r)\{A_{00}\psi_0 + irA_{11}(\psi_1 - B_{10}\psi_0)\},$$

and verify $BA = 1_{W'}$. The general case is completely similar and is therefore omitted. This proves that B is surjective, which yields (b).

We can twist two index problems to get another index problem. Let Q: $C^\infty(V_1) \to C^\infty(V_2)$ over M_1 and let $\tilde{Q}$: $C^\infty(\tilde{V}_1) \to C^\infty(\tilde{V}_2)$ over M_2 be two elliptic complexes. We form

$$\tilde{R} = \begin{pmatrix} \tilde{Q} & Q^* \\ Q & -\tilde{Q}^* \end{pmatrix}\colon C^\infty(V_1 \otimes \tilde{V}_1 \oplus V_2 \otimes \tilde{V}_2) \to C^\infty(V_1 \otimes \tilde{V}_2 \oplus V_2 \otimes \tilde{V}_1)$$

over $M = M_1 \times M_2$. The symbol of $\tilde{R}$ is elliptic, and if $dM = \varnothing$ it is immediate that index$(\tilde{R}) =$ index(Q)index$(\tilde{Q})$. We refer to [2] for details. If $dM_1 \neq \varnothing$, we must impose boundary conditions.

Lemma 3.2.4. *Let $R = \left(\begin{smallmatrix} 0 & Q^* \\ Q & 0 \end{smallmatrix}\right)$ with boundary condition $B = B_1 \oplus B_2$ over M_2. Assume (R, B) is elliptic with respect to the cone $\mathcal{C}$ and that R_B is self-adjoint. Let $dM_2 = \varnothing$ and let $\tilde{Q}$ be an elliptic operator over M_2. Let*

$$\tilde{R} = \begin{pmatrix} \tilde{Q} & Q^* \\ Q & -\tilde{Q}^* \end{pmatrix}, \qquad \tilde{B} = B_1 \otimes 1 \oplus B_2 \otimes 1.$$

Then $(\tilde{R}, \tilde{B})$ *is elliptic with respect to* $\{0\}$ *and*

$$\operatorname{index}(\tilde{R}, \tilde{B}) = \operatorname{index}(Q, B)\operatorname{index}(\tilde{Q}).$$

Proof. Introduce fiber coordinates (ζ^1, z) over M_1, and ξ^2 over M_2. The ellipticity of the symbol $\tilde{r}$ is immediate since

$$\tilde{r}^*\tilde{r} = \begin{pmatrix} \tilde{q}^*\tilde{q} + q^*q & 0 \\ 0 & \tilde{q}\tilde{q}^* + qq^* \end{pmatrix}.$$

If p is a symbol, let $p_n = p(x, 0, \zeta, D_r)$. We diagonalize $\tilde{q}^*\tilde{q}$ to assume $\tilde{q}^*\tilde{q} = a^2$. After choosing a suitable basis for $\tilde{V}_1 \oplus \tilde{V}_2$ we can assume $\tilde{r}$ has the form

$$\begin{pmatrix} a & q^* \\ q & -a \end{pmatrix} \quad \text{for } a \geqslant 0.$$

If $\xi^2 = 0$, the ellipticity is clear so we may assume $a > 0$. We solve the equations

$$r_n\phi_\pm = \pm ia\phi_\pm, \quad \lim_{r\to\infty}\phi_\pm(r) = 0, \quad \sigma^g B(\zeta^1)\phi_\pm = \psi.$$

If $\phi_\pm = (\phi^1_\pm, \phi^2_\pm)$, we define $\Phi_\pm = (\phi^1_\pm, \pm i\phi^2_\pm)$, then $\tilde{r}_n\Phi_\pm = 0$. Let $\Phi = a\Phi_+ + b\Phi_-$. Then this has the boundary values $((a+b)\psi_1, i(a-b)\psi_2)$. We solve $a + b = 1$, $a - b = -i$ to find Φ with the desired boundary values. This proves the ellipticity. The multiplicative property of the index is a formal computation which is exactly the same as that given in [2] if $dM = \varnothing$, is therefore omitted.

We specialize to the following case. Let $M_2 = T_2$ be the flat torus, and let $\tilde{V}_1 = \tilde{V}_2 = U$ be a holomorphic line bundle with Chern character 1. Let $\tilde{Q} = (d/d\bar{z})^d$: $C^\infty(U) \to C^\infty(U)$, which has index d; the case $d = 1$ was discussed in Example 1.3.1. Let (R, B) be as given in Lemma 3.2.4 and form $\tilde{R}$. If $\alpha = \xi_1^2 + i\xi_2^2$, then the symbol of $\tilde{R}$ is given by

$$\tilde{r} = \begin{pmatrix} \alpha^d & q^* \\ q & -\bar{\alpha}^d \end{pmatrix} \otimes 1_U.$$

Decompose $\Sigma(T^*(M_1 \times M_2)) = \Sigma^3(M_1) \times M_2$, and integrate $\operatorname{ch}(U)$ over M_2 to get 1. When the change of orientation is taken into account, we conclude

$$d \cdot \operatorname{index}(Q, B) = \int_{\Sigma^3(T^*M_1)} \operatorname{TODD}(M_1) \wedge \operatorname{ch}\left(\Pi_+\left(\Sigma\begin{pmatrix} \alpha^d & q^* \\ q & \bar{\alpha}^d \end{pmatrix}_B\right)\right) \cdot (-1)^{m_1}.$$

We let $\beta = \alpha^d$. This defines a d-fold branched cover of $\Sigma^3(T^*M_1)$ so

$$\operatorname{index}(Q, B) = \int_{\Sigma^3(T^*M_1)} \operatorname{TODD}(M_1) \wedge \operatorname{ch}\left(\Pi_+\left(\Sigma\begin{pmatrix} \beta & q^* \\ q & \beta \end{pmatrix}_B\right)\right) \cdot (-1)^{m_1}.$$

If we replace β by $-\bar{\beta}$, we reverse the orientation and recognize the resulting matrix as $\Sigma^2 q$. Therefore we have

Lemma 3.2.5. *Let $R = \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix}$ with boundary condition $B = B_1 \oplus B_2$ over M. Assume that (R, B) is elliptic with respect to the cone $\mathcal{C}$ and that R_B is self-adjoint. Then*

$$\begin{aligned}\operatorname{index}(Q, B_1) &= \operatorname{index}(Q_{B_1}) \\ &= \int_{\Sigma^3(T^*M)} \operatorname{TODD}(M) \wedge \operatorname{ch}\left(\Pi_+\left(\Sigma(\Sigma^2 q)_B\right)\right) \cdot (-1)^m.\end{aligned}$$

This lemma is the generalization of Theorem 1.3.2 involving multiple suspensions. We can use this lemma to prove

Lemma 3.2.6. *Let $R = \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix}$ with boundary condition $B = B_1 \oplus B_2$ over M_1. Assume that (R, B) is elliptic with respect to the cone $\mathcal{C}$ and that R_B is self-adjoint. Let P be elliptic with respect to the cone $\mathcal{C}$ over M_2 with $dM_2 = \varnothing$, and let ρ be a representation of $\pi_1(M_2)$ such that V_ρ is topologically trivial. We extend V_ρ to $M_1 \times M_2$ to be independent of the first factor. Let $\tilde{R} = \begin{pmatrix} P & Q^* \\ Q & -P \end{pmatrix}$ and $\tilde{B} = B \otimes 1$. Then $(\tilde{R}, \tilde{B})$ is elliptic with respect to the cone $\mathcal{C}$ and*

$$\operatorname{ind}(\rho, \tilde{R}, \tilde{B}) = \operatorname{index}(Q_B) \cdot \operatorname{ind}(\rho, P) = \operatorname{ind}_1(\rho, \tilde{R}, \tilde{B}).$$

Proof. The ellipticity of the symbol with respect to the cone $\mathcal{C}$ is immediate. We suppose for the sake of simplicity that the symbol of p is diagonalizable; the general case follows using Jordan normal form. We study the symbol $\begin{pmatrix} a & q^* \\ q & -a \end{pmatrix}$ for scalar a. We solve the equations

$$r_n \phi_\pm = \pm\sqrt{\lambda^2 - a^2}\,\phi_\pm = \pm\mu\phi_\pm, \quad \lim_{r\to\infty} \phi_\pm(r) = 0, \quad \sigma^g(B)(\zeta^1)\phi_\pm = \psi.$$

Since the matrix $\begin{pmatrix} a & \pm\mu \\ \pm\mu & -a \end{pmatrix}$ has eigenvalue λ, we can choose $\Phi_\pm = (x_\pm \phi^1, y_\pm \phi^2)$ so that $\tilde{r}_n \Phi_\pm = \lambda \Phi_\pm$. If $(a, \lambda) \neq (0,0)$, then the vectors (x_+, y_+) and (x_-, y_-) are linearly independent. Thus we can choose c_+ and c_- so that $\sigma^g(B)(\zeta^1)(c_+\Phi_+ + c_-\Phi_-) = \psi$. This proves the ellipticity. The first equality is a purely formal calculation, and we refer to [2] for details.

Because $\operatorname{ind}(\rho, P)$ and $\operatorname{ind}_1(\rho, \tilde{R}, \tilde{B})$ are homotopy invariants, we can perform a homotopy to replace P by a pseudo-differential operator with $p^2 = |\xi^2|^d I$. This does not affect the ellipticity. This replaces $\tilde{r}$ by

$$\begin{aligned}\Sigma\tilde{r} = &\begin{pmatrix} |\xi^2|^d - it & q^* \\ q & -|\xi^2| - it \end{pmatrix} \otimes \pi_+(p) \\ \oplus &\begin{pmatrix} -|\xi^2|^d - it & q^* \\ q & |\xi^2| - it \end{pmatrix} \otimes \pi_-(p).\end{aligned}$$

We now argue exactly as in the proof of Lemma 2.3.2 that this implies

$$\operatorname{ind}_1(\rho, \tilde{R}, \tilde{B}) = \int_{S(T^*M_2)} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M_2) \wedge \operatorname{ch}(\Pi_+ p)(-1)^{m_2}$$

$$\cdot \int_{\Sigma^3(T^*M_1)} \operatorname{TODD}(M_1) \wedge \operatorname{ch}(\Pi_+ \Sigma(\Sigma^2 q)_b)(-1)^{m_1}.$$

We now use Theorem 1.3.2 and Lemma 3.2.5 to evaluate this formula as $\operatorname{ind}(\rho, P) \cdot \operatorname{index}(Q_B)$.

The next functorial property we will study is related to the process of taking powers. Let j be an odd positive integer and let $\mathcal{C}_j = \{\lambda\colon \lambda^j \in \mathcal{C}\}$. This is the complement of a narrow cone about $R - \{0\}$.

Lemma 3.2.7. *Let (P, B) be elliptic with respect to the cone $\mathcal{C}_j$ and let $B_j = B \oplus BP \oplus \cdots \oplus BP^{j-1}$. Then*

(a) *(P^j, B_j) is elliptic with respect to the cone $\mathcal{C}$,*

(b) $\operatorname{ind}(\rho, P^j, B_j) = \operatorname{ind}(\rho, P, B)$,

(c) $\operatorname{ind}_1(\rho, P^j, B_j) = \operatorname{ind}_1(\rho, P, B)$.

Proof. There is of course a similar ellipticity statement which is omitted here as we shall not need it. If $\xi \neq 0$, the spectrum of p is contained in the complement of $\mathcal{C}_j$. This implies that the spectrum of p^j is contained in the complement of $\mathcal{C}$ which verifies interior ellipticity. Let $(\zeta, \lambda) \neq (0,0)$ and let $\{\lambda_1, \cdots, \lambda_j\}$ be the distinct jth roots of λ. Decompose $p_n^j - \lambda = (p_n - \lambda_1) \cdots (p_n - \lambda_j)$. Suppose Φ is given with

$$(p_n^j - \lambda)\Phi = 0, \quad \operatorname*{Lim}_{r \to \infty} \Phi(r) = 0, \quad \sigma^g(B_j)(\zeta)\Phi = 0.$$

We verify the ellipticity of the boundary condition by checking that this implies $\Phi = 0$. Define $\phi_i = (p_n - \lambda_{i=1}) \cdots (p_n - \lambda_j)$. Then

$$(p_n - \lambda_i)\phi_i = \phi_{i-1}, \quad \operatorname*{Lim}_{r \to \infty} \phi_i(r) = 0, \quad \sigma^g(B)(\zeta)\phi_i = 0.$$

Since $(p_n - \lambda_1)\phi_1 = (p_n^j - \lambda)\Phi = 0$, the ellipticity of (P, B) implies $\phi_1 = 0$. Since $(p_n - \lambda_2)\phi_2 = \phi_1 = 0$, we apply the same argument to conclude $\phi_2 = 0$. By induction this implies $\phi_j = \Phi = 0$ which completes the proof of (a).

If j is an odd integer, since the spectrum of P_B lies in a cone near the real axis, $\{\operatorname{sign} \operatorname{Re}(\lambda)\} |\lambda|^{-js} = \{\operatorname{sign} \operatorname{Re}(\lambda^j)\} |\lambda|^{-js}$ except for a finite number of λ. This implies $\eta(s, P^j, B_j) = \eta(js, P, B)$ which proves (b).

We prove (c) as follows. Let $\{t_1, \cdots, t_j\}$ be the jth roots of it and let $\wp_i = p - t_i$. We apply the construction of §3.1 to define $\tilde{q}(\zeta, t, \lambda)$ and $\wp_i(\zeta, t, \beta)$ corresponding to $q = \Sigma(p^j)$ and $\wp_i$ respectively. The identity $q = \wp_1 \cdots \wp_j$ implies $\tilde{q} = \tilde{\wp}_1 \cdots \tilde{\wp}_j$. Define a homotopy from $S^j(\tilde{\wp}_i) = \tilde{\wp}_i \oplus I_{j-1}$ to $I_{i-1} \oplus \tilde{\wp}_i \oplus I_{j-i}$ by rotating the relevant subspaces using block elements of $GL(j, C)$.

Multiply these homotopies to construct a homotopy from $S^j(\tilde{q})$ to $\tilde{\wp}_1 \oplus \cdots \oplus \tilde{\wp}_j$, and apply Homotopies 3.1.2 through 3.1.5 to this homotopy to create a 1-parameter family of homotopies. We treat $(\wp_1 \oplus \cdots \oplus \wp_j)$ as though it were an operator of order $j \cdot d$. The resulting homotopty is equivalent to the homotopy defined by treating this symbol as though it were of order d. Therefore

$$\operatorname{ind}_1\big(\rho, P^j, B_j\big) = \Sigma_i \int_{\Sigma^2(T^*M)} \operatorname{Tch}(\rho) \wedge \operatorname{TODD}(M) \wedge \operatorname{ch}\Big(\Pi_+\big(\Sigma(\wp_i)_B\big)\Big)(-1)^m.$$

Let $2k + 1 = j$. There are $k + 1$ branches of the jth root function which lie in the same half-plane of $C - R$ as does it. These define functions $\wp_i$ which are homotopic to Σp. The remaining k branches define functions $\wp_i$ which are homotopic to $p + it$. Since this corresponds to a reversed orientation, these cancel off to give $\operatorname{ind}_1(\rho, P, B)$ which completes the proof.

The remaining functorial properties are much easier to prove. We summarize them in the following.

Lemma 3.2.8. (a) *Let* (P, B) *be elliptic with respect to* $\mathcal{C}$. *Then* $\operatorname{ind}(\rho, P, B) = -\operatorname{ind}(\rho, -P, B)$ *and* $\operatorname{ind}_1(\rho, P, B) = -\operatorname{ind}_1(\rho, -P, B)$.

(b) *Let* (P_i, B_i) *be elliptic with respect to* $\mathcal{C}$ *and of the same order. Then*

$$\operatorname{ind}(\rho, P_1 \oplus P_2, B_1 \oplus B_2) = \operatorname{ind}(\rho, P_2, B_2),$$

$$\operatorname{ind}_1(\rho, P_1 \oplus P_2, B_1 \oplus B_2) = \operatorname{ind}_1(\rho, P_1, B_1) \oplus \operatorname{ind}_1(\rho, P_2, B_2).$$

(c) *Let* $\mathcal{C}_0$ *be the complement of the* $45°$ *cone about* R^+. *If* (P, B) *is elliptic with respect to* $\mathcal{C}_0$, *then* $\operatorname{ind}(\rho, P, B) = \operatorname{ind}_1(\rho, P, B) = 0$.

Proof. Assertions (a)–(c) for ind were proved in [7]. We note $\Sigma(-p)(\xi, t) = -\Sigma(p)(\xi, -t)$. $-\Sigma$ is homotopic to Σ where we replace (-1) by $\exp(\pi i \varepsilon)$ for the homotopy. Thus we can compute ind_1 for $-P$ by replacing t by $-t$. This reverses the orientation and changes the sign as claimed. Assertion (b) is immediate since our constructions are additive over direct sums and the Chern character is additive. If (P, B) is elliptic with respect to $\mathcal{C}$, then we can define $\Sigma(p)_B(\xi, t)$ for t in the upper half-plane. By homotoping $t < 0$ to $t > 0$ through the imaginary axis, this replaces $\Sigma(p)_B$ by a symbol which only depends on $|t|$. The integral vanishes for such a symbol; the proof is thus complete.

These functorial properties suggest that $\operatorname{ind} = \operatorname{ind}_1$ in general, but we have not been able to establish this fact in general. Unfortunately, there are formidable technical difficulties which prevent a straightforward application of the methods of [2] which are related to the fact that it is not possible to introduce pseudo-differential boundary conditions in defining the eta invariant. The authors hope to deal with the higher order case in a later paper using the functorial properties established in this section.

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