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A NEW PROOF OF A THEOREM OF NARASIMHAN AND SESHADRI

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1. Introduction

In 1965 Narasimhan and Seshardri proved that the stable holomorphic vector bundles over a compact Riemann surface are precisely those arising from irreducible projective unitary representations of the fundamental group [5]. We shall give here a different, more direct, proof of this fact using the differential geometry of connections on holomorphic bundles. This complements, in a small way, the recent paper by Atiyah and Bott [1] in which the result of Narasimhan and Seshadri is used to calculate the cohomology of the moduli spaces of stable bundles, and which we take as a general reference for background and notation.

Let X be a compact Riemann surface with a Hermitian metric normalized to unit volume. If E is a vector bundle over X we write

$$\mu(E) = \text{degree}(\text{E})/\text{rank}(E),$$

where the degree is obtained by evaluating $c_1(E)$ on the fundamental cycle. A holomorphic bundle $\mathcal{E}$ is defined to be *indecomposable* if it cannot be written as a proper direct sum, and to be *stable* if for all proper holomorphic sub-bundles $\mathcal{F} < \mathcal{E}$,

$$\mu(\mathcal{F}) < \mu(\mathcal{E}), \text{ or equivalently } \mu(\mathcal{E}/\mathcal{F}) > \mu(\mathcal{E}).$$

Certainly a stable bundle is indecomposable. The theorem to be proved is:

Theorem. *An indecomposable holomorphic bundle $\mathcal{E}$ over X is stable if and only if there is a unitary connection on $\mathcal{E}$ having constant central curvature $*F = -2\pi i\mu(\mathcal{E})$. Such a connection is unique up to isomorphism.*

Note. If $\deg(\mathcal{E}) = 0$, these connections are flat and so are given by unitary representations of the fundamental group. In the general case it is easy to prove the equivalence of this form of the result with the statement of Narasimhan and Seshadri [1, §6].

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2. Definitions and notation, outline of proof

If E is a C^∞ Hermitian vector bundle over X, a unitary connection A on E gives an operator $d_A : \Omega^0(E) \to \Omega^1(E)$ which has a (0, 1) component $\bar\partial_A : \Omega^0(E) \to \Omega^{0,1}(E)$, and this defines a holomorphic structure $\mathcal{E}_A$ on E (see [1, §5] for a proof that there are sufficiently many local solutions of $\bar\partial_A s = 0$). Conversely if $\mathcal{E}$ is a holomorphic structure on E, there is a unique way to define a unitary connection A such that $\mathcal{E} = \mathcal{E}_A$.

A connection on E induces a connection on all associated bundles, in particular, on the bundle of endomorphisms End E. The "gauge group" $\mathcal{G}$ of unitary automorphisms of E acts as a symmetry group on the affine space $\mathcal{A}$ of all unitary connections on E by

$$u(A) = A - d_A u\, u^{-1}, \quad u \in \mathcal{G}, \quad A \in \mathcal{A}.$$

The action extends to the complexification $\mathcal{G}^{\mathbf{C}}$ = group of general linear automorphisms of E:

$$g(A) = A - (\bar\partial_A g)g^{-1} + ((\bar\partial_a g)g^{-1})^*, \quad g \in \mathcal{G}^{\mathbf{C}}, A \in \mathcal{A},$$

and connections define isomorphic holomorphic structures precisely when they lie in the same $\mathcal{G}^{\mathbf{C}}$ orbit. Thus the set of orbits parametrize all the holomorphic bundles of the same degree and rank as E (there are no further topological invariants of bundles over X). Given a holomorphic bundle $\mathcal{E}$, we write $0(\mathcal{E})$ for the corresponding orbit of connections on the appropriate C^∞ bundle.

Each connection A has a curvature $F(A) \in \Omega^2(\text{End } E)$, and $F(A + a) = F(A) + d_A a + a \wedge a$.

The plan of the proof is this: we suppose inductively that the result has been proved for bundles of lower rank (the case of the line bundles being an easy consequence of the Hodge theory), then we choose a minimizing sequence in $0(\mathcal{E})$ for a carefully constructed functional J of the curvature and extract a weakly convergent subsequence. Either the limiting connection is in $0(\mathcal{E})$ and we deduce the result by examining small variations within $0(\mathcal{E})$, or in another orbit $0(\mathcal{F})$ and we deduce that $\mathcal{E}$ is not stable. The main ingredient in this approach is a result of K. Uhlenbeck [6] on the weak compactness of the set of connections with L^2 bounded curvature.

In the intermediate stages of the argument we have to allow generalized connections of class L_1^2 (i.e., which differ from some fixed C^∞ connection by an element of the Hilbert space with norm $\|\alpha\|_{L_1^2}^2 = \|\alpha\|^2 + \|\nabla\alpha\|^2$) with curvature in L^2 and gauge transformations in L_2^2. As explained in ([6, §1], [1, §14]) the group actions and properties of curvature which we use extend without essential change (in particular $L_2^2 \Rightarrow C^0$, so the topology of the bundle is preserved), and it is proved in [1, Lemma 14.8] that each L_1^2 connection defines a holomorphic structure. For simplicity we shall work throughout with these more general objects with only occasional further comment.

Definition of the functional J. The "trace norm" is defined on $n \times n$ Hermitian matrices by

$$\nu(M) = \operatorname{Tr}(M^*M)^{1/2} = \sum_{i=1}^{n} |\lambda_i|,$$

where $\{\lambda_i\}$ are the eigenvalues of M, repeated according to multiplicity. We shall need to know that ν defines a norm and that if M is written in blocks: $M = \left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right)$, then $\nu(M) \geqslant |\operatorname{Tr} A| + |\operatorname{Tr} D|$. Both properties follow easily from the characterization:

$$\nu(M) = \max_{\{e_i\}} \sum_{i=1}^{n} |\langle Me_i, e_i \rangle|,$$

where $\{e_i\}$ runs over all orthonormal frames for $\mathbf{C}^n$. (There is a complete account of such convex invariant functions in [1, §12].)

Applying ν in each fibre we define, for any smooth self-adjoint section s in $\Omega^0(\operatorname{End} E)$,

$$N(s) = \left(\int_X \nu(s)^2 \right)^{1/2}.$$

Then N is a norm equivalent to the usual L^2 norm and so extends to the L^2 cross sections. For an L_1^2 connection A, set

$$J(A) = N\left(\frac{*F}{2\pi i} + \mu.1 \right),$$

where $\mu = \mu(E)$. Thus $J(A) = 0$ if and only if the connection is of the type required by the theorem. For bundles of rank 2 and degree 0, J is essentially the Yang-Mills functional $\|F\|_{L^2}$. For larger ranks the definition of J is chosen to make the inductive step (Lemma 3) carry through easily, although the connections we find in the end are of course Yang-Mills connections. Although J is not smooth, it does have the semi-continuity property: if $A_i \to A$ weakly in L_1^2, so $F(A_i) \to F(A)$ weakly in L^2, then $J(A) \leqslant \liminf J(A_i)$, because for each $\varepsilon > 0$ we can separate $*F(A)/2\pi i$ from the closed convex set $\{\alpha \mid N(\alpha + \mu 1) \leqslant J(A) - \varepsilon\}$ by a hyperplane.

3. Proof of main lemma

This section contains the main part of the argument for which we need to extract the following proposition from [6, Theorem 1.5].

Proposition. *Suppose that $A_i \in \mathcal{Q}$ is a sequence of L_1^2 connections with $\|F(A_i)\|_{L^2}$ bounded. Then there are a subsequence $\{i'\} \subset \{i\}$ and L_2^2 gauge transformations $u_{i'}$ such that $u_{i'}(A_{i'})$ converges weakly in L_1^2.*

From this we deduce:

Lemma 1. *Let $\mathcal{E}$ be a holomorphic bundle over X. Then either* $\inf J|_{0(\mathcal{E})}$ *is attained in* $0(\mathcal{E})$ *or there is a holomorphic bundle $\mathcal{F} \not\cong \mathcal{E}$ of the same degree and rank as $\mathcal{E}$ and with* $\inf J|_{0(\mathcal{F})} \leqslant \inf J|_{0(\mathcal{E})}$; $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$.

Proof. Pick a minimizing sequence A_i for $J|_{0(\mathcal{E})}$. Since N is equivalent to the L^2 norm, we have $\|F(A_i)\|_{L^2}$ bounded and can apply the proposition to deduce that, without loss of generality, $A_i \to B$ weakly in L_1^2 and

$$J(B) \leqslant \liminf J(A_i) = \inf J|_{0(\mathcal{E})}.$$

Since B defines a holomorphic structure $\mathcal{E}_B$, the lemma will follow if we show $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_B) \neq 0$. (The two alternatives holding as $\mathcal{E} \cong \mathcal{E}_B$ or not.) To see this, define for any $A, A' \in \mathcal{Q}$ a connection $d_{AA'}$ on the bundle $\mathrm{Hom}(E, E) = E^* \otimes E$ built from the connection A on the left hand factor and A' on the right, with a corresponding

$$\bar{\partial}_{AA'} : \Omega^0(\mathrm{Hom}(E, E)) \to \Omega^{0,1}(\mathrm{Hom}(E, E)).$$

Thus solutions of $\bar{\partial}_{AA'}s = 0$ correspond exactly to elements of $\mathrm{Hom}(\mathcal{E}_A, \mathcal{E}_{A'})$.

If in fact $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_B) = 0$, then $\bar{\partial}_{A_0B}$ has no kernel, and since it is a first order elliptic operator we have

$$\|\bar{\partial}_{A_0B}s\|_{L^2} \geqslant C\|s\|_{L_1^2} \quad \text{for some } C, \text{ all } s.$$

By the Sobolev inequality $\|s\|_{L_1^2} \geqslant \mathrm{Const}\|s\|_{L^4}$ so

$$\|\bar{\partial}_{A_0B}s\|_{L^2} \geqslant C_1\|s\|_{L^4} \text{ say.}$$

On the other hand $L_1^2 \Rightarrow L^4$ is compact, so $A_i \to B$ in L^4 norm and $\bar{\partial}_{A_0B} - \bar{\partial}_{A_0A_i}$ is the algebraic operator $s \to (B - A_i)_{0,1}.s$. Thus

$$\left\|\left(\bar{\partial}_{A_0B} - \bar{\partial}_{A_0A_i}\right)s\right\|_{L^2} \leqslant C_2\|A_i - B\|_{L^4}\|s\|_{L^4},$$

so for each i and all s, $\|\bar{\partial}_{A_0A_i}s\|_{L^2} \geqslant (C_1 - C_2\|A_i - B\|_{L^4})\|s\|_{L^4}$.

Since $A_i \to B$ in L^4 norm, this implies that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_{A_i}) = 0$ for large enough i, contradicting $\mathcal{E}_{A_i} \cong \mathcal{E}$.

4. Curvature and holomorphic extensions

We have to show that if $\mathcal{E}$ is stable, the second alternative of Lemma 1 does not occur. In general if $\alpha : \mathcal{E} \to \mathcal{F}$ is a (nonzero) holomorphic map of bundles over X, there are proper extensions and a factorization (cf. [5, §4]):

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} & & \\ 0 & \longleftarrow & \mathcal{N} & \longleftarrow & \mathcal{F} & \longleftarrow & \mathcal{M} & \longleftarrow & 0 \end{array}$$

With rows exact, rank $\mathcal{Q} =$ rank $\mathcal{M}$, $\det \beta \not\equiv 0$, $\deg \mathcal{Q} \leqslant \deg \mathcal{M}$.

The next two lemmas give bounds on the curvature of bundles expressed as extensions; the first exactly as in [1, §8], the second a little stronger, exploiting the special properties of J.

First some generalities: if we have any exact sequence of holomorphic bundles $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$, then any unitary connection A on $\mathcal{S}$ has the shape:

$$A = \begin{pmatrix} A_s & \beta \\ -\beta^* & A_{\mathcal{U}} \end{pmatrix} \quad \text{with } A_{\mathcal{S}}, A_u \text{ connections}$$

on $\mathcal{S}$, $\mathcal{U}$ and β in $\Omega^{0,1}(\mathcal{U}^* \otimes \mathcal{S})$ because $\mathcal{S}$ is a holomorphic subbundle (β is a representative of the extension class in $H^1(\mathcal{U}^* \otimes \mathcal{S})$). In the corresponding curvature matrix

$$F(A) = \begin{pmatrix} F(A_s) - \beta \wedge \beta^* & d\beta \\ -d\beta^* & F(A_u) - \beta^* \wedge \beta \end{pmatrix},$$

where $d\colon \Omega^1(\mathcal{U}^* \otimes \mathcal{S}) \to \Omega^2(\mathcal{U}^* \otimes \mathcal{S})$ is built from $A_{\mathcal{U}}, A_{\mathcal{S}}$, the quadratic terms have a definite sign (this is the principle that curvature decreases in holomorphic subbundles and increases in quotients). For convenience normalize so that $*\mathrm{Tr}(\beta^* \wedge \beta) = 2\pi i |\beta|^2$.

Conversely, connections on $\mathcal{S}$, $\mathcal{U}$ and a representative β of the extension class define a unique connection on $\mathcal{T}$, and any nonzero multiple of β also gives a bundle isomorphic to $\mathcal{T}$ (although a different extension class).

Lemma 2. *If $\mathcal{F}$ is a holomorphic bundle over X which can be expressed as an extension*: $0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{N} \to 0$, *and if* $\mu(\mathcal{M}) \geqslant \mu(\mathcal{F})$ (*so also* $\mu(\mathcal{F}) \geqslant \mu(\mathcal{N})$), *then for any unitary connection A on $\mathcal{F}$*,

$$J(A) \geqslant \mathrm{rk}\,\mathcal{M}(\mu(\mathcal{M}) - \mu(\mathcal{F})) + \mathrm{rk}\,\mathcal{N}(\mu(\mathcal{F}) - \mu(\mathcal{N})), = J_0 \textit{ say},$$

with equality occurring only if the extension splits.

Proof. Using the property of ν on block matrices mentioned in §3 and the notation of the discussion above we have

$$\nu\left(\frac{*F(A)}{2\pi i} + \mu.1\right) \geqslant \left|\mathrm{Tr}\left(\frac{F(A_m) - \beta \wedge \beta^*}{2\pi i} + \mu.1\right)\right| + \left|\mathrm{Tr}\left(\frac{F(A_n) - \beta^* \wedge \beta}{2\pi i} + \mu.1\right)\right|,$$

where $\mu = \mu(\mathcal{F})$. So

$$\begin{aligned}J(A) &\geqslant \int_X \nu\left(\frac{*F}{2\pi i} + \mu.1\right)\\ &\geqslant \left|\int_X \left\{\operatorname{Tr}\left(\frac{*F(A_{\mathcal{M}})}{2\pi i} + \mu.1\right) - |\beta|^2\right\}\right|\\ &\quad + \left|\int_X \left\{\operatorname{Tr}\left(\frac{*F(A_{\mathcal{N}})}{2\pi i} + \mu.1\right) + |\beta|^2\right\}\right|.\end{aligned}$$

But $\int_X \operatorname{Tr}(*F(A_M)/2\pi i) = -\deg \mathcal{M} \leqslant \mu \int_X \operatorname{Tr} 1_{\mathcal{M}}$ by hypothesis, so the first term on the right is $\operatorname{rk}(\mathcal{M})(\mu(\mathcal{M}) - \mu(\mathcal{F})) + \|\beta\|^2$. Similarly for the second term.

Lemma 3. *Suppose that $\mathcal{E}$ is a stable holomorphic bundle and make the inductive hypothesis that the main theorem has been proved for bundles of lower rank. If $\mathcal{E}$ can be expressed as an extension $0 \to \mathcal{P} \to \mathcal{E} \to \mathcal{Q} \to 0$ (so from the definition of stability $\mu(\mathcal{P}) < \mu(\mathcal{E}) < \mu(\mathcal{Q})$), then there is a connection A on $\mathcal{E}$ with*

$$\begin{aligned}J(A) &< \operatorname{rk} \mathcal{P}(\mu(\mathcal{E}) - \mu(\mathcal{P})) + \operatorname{rk} \mathcal{Q}(\mu(\mathcal{Q}) - \mu(\mathcal{E}))\\ &= J_1 \textit{ say}.\end{aligned}$$

Proof. Observe first that for a general extension $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$, the connections in $0(\mathcal{T})$ given by triples $(A_s, A_{\mathcal{U}}, t\beta)$ converge in C^∞ as $t \to 0$ to a connection in $0(\mathcal{S} \oplus \mathcal{U})$. Now it is not hard to prove ([1, §7], [2]) that any holomorphic bundle $\mathcal{P}$ has a canonical "semi-stable filtration"

$$0 = \mathcal{P}_0 < \mathcal{P}_1 < \cdots < \mathcal{P}_k = \mathcal{P}$$

with $\mathcal{P}_i/\mathcal{P}_{i-1}$ semi-stable and $\mu(\mathcal{P}_i/\mathcal{P}_{i-1})$ decreasing with i.

Each quotient $\mathcal{P}_{i/\mathcal{P}_{i-1}}$ has in turn a filtration [2] with associated quotients $\mathcal{C}_{ij}$ stable and $\mu(\mathcal{C}_{ij}) = \mu(\mathcal{P}_{i/\mathcal{P}_{i-1}}) < \mu(\mathcal{P}_1)$. If $\mathcal{P} < \mathcal{E}$ are the bundles in the statement of the lemma, we must have $\mu(\mathcal{C}_{ij}) < \mu(\mathcal{E})$ since $\mathcal{E}$ is stable. Also rank $\mathcal{C}_{ij} <$ rank $\mathcal{E}$, so we can suppose inductively that each $\mathcal{C}_{ij}$ has a connection with constant central curvature. Applying the observation to each of the steps in the filtrations we find connections $A^t_{\mathcal{P}} \in 0(\mathcal{P})(t \neq 0)$ converging as $t \to 0$ to $A^0_{\mathcal{P}} \in 0(\oplus_{ij}\mathcal{C}_{ij})$, and $*F(A^0_{\mathcal{P}}) + 2\pi i \Lambda_{\mathcal{P}}$ where $\Lambda_{\mathcal{P}}$ is the constant diagonal matrix with entries the $\mu(\mathcal{C}_{ij}) < \mu(\mathcal{E})$.

Similarly there are $A^t_{\mathcal{Q}} \to A^0_{\mathcal{Q}}$, $*F(A^0_{\mathcal{Q}}) = 2\pi i \Lambda_{\mathcal{Q}}$, and the entries of $\Lambda_{\mathcal{Q}}$ are greater than $\mu(\mathcal{E})$.

For each t, $A^t_{\mathcal{P}}$ and $A^t_{\mathcal{Q}}$ give an operator d_t on the forms with values in the C^∞ bundle corresponding to $\mathcal{Q}^* \otimes \mathcal{P}$. For $t \neq 0$, choose the harmonic representative β_t of the extension class corresponding to $\mathcal{E}$, i.e., $d_t\beta_t = 0$, scaled to $\|\beta_t\|_{L^2} = 1$.

Since $d_t \to d_0$ as $t \to 0$ and the d_t are elliptic on $\Omega^{0,1}$, there is a uniform bound $\|\beta_t\|_{C^0} <$ Const (because for each t we have the usual elliptic bounds

$$\|\alpha\|_{L^2_{K+1}} \leq C_t\big(\|d_t\alpha\|_{L^2_K} + \|\alpha\|_{L^2_K}\big),$$

and the C_t can be uniformly bounded since the d_t converge).

By our general discussion the triples $(A^t_{\mathcal{P}}, A^t_{\mathcal{Q}}, s\beta_t)$ $(s, t > 0)$ give connections $A(s, t) \in 0(\mathcal{E})$ with curvature

$$F(s, t) = \begin{pmatrix} F(A^t_{\mathcal{P}}) - s^2\beta_\tau \wedge \beta^*_\tau & 0 \\ 0 & F(A^t_{\mathcal{Q}}) - s^2\beta^*_\tau \wedge \beta_\tau \end{pmatrix}.$$

It is clear that $J(A(s, t)) \to J_1$ as $s, t \to 0$. We have to check that for suitably chosen small s, t, $J(A(s, t)) < J_1$.

Since $\Lambda_{\mathcal{P}} - \mu.1_{\mathcal{P}}$ has all its eigenvalues negative, the same is true for nearby matrices, and for such matrices $\nu() = -\mathrm{Tr}()$. Using the uniform bound on the β_t and $F(A^t_{\mathcal{P}}) \to \Lambda_{\mathcal{P}}$, together with the corresponding facts for the other component, we have for small s, t,

$$\nu\left(\frac{*F(s, t)}{2\pi i} + \mu.1\right) = J_1 - 2s^2|\beta_t|^2 + \varepsilon(t),$$

where $\varepsilon(t) \to 0$ with t. So

$$J(A(s, t))^2 = \int_X \big(J_1^2 - 2s^2|\beta_t|^2 + \varepsilon(t)\big)^2.$$

Choosing s so small that $s^4\int_X|\beta_t|^4$ is much less than $s^2\int_X|\beta_t|^2 = s^2$ (using the uniform bound again), and then t small enough for the terms in ε to be negligible gives $J(A(s, t)) < J_1$.

5. Proof of the theorem

First, the final clause of Lemma 2 shows that if $\mathcal{E}$ is an indecomposable bundle with a connection of the type required by the theorem (i.e., $J = 0$), then $\mathcal{E}$ must be stable.

Conversely, if $\mathcal{E}$ is stable and the theorem has been proved for bundles of lower rank, then $\inf J|_{0(\mathcal{E})}$ is attained in $0(\mathcal{E})$. For if not, Lemma 1 constructs a bundle $\mathcal{F}$ with $\deg \mathcal{F} = \deg \mathcal{E}$, $\operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{E}$, $\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$ and

$\inf J|_{0(\mathcal{E})} \geqslant \inf J|_{0(\mathcal{F})}$. In the corresponding diagram (*) we have

$$\mu(\mathcal{M}) \geqslant \mu(\mathcal{Q}) > \mu(\mathcal{E}) = \mu(\mathcal{F}).$$

So we can apply Lemma 2 to the bottom row of (*) to deduce

$$\inf J|_{0(\mathcal{F})} \geqslant J_0,$$

and Lemma 3 to the top row to deduce

$$\inf J|_{0(\mathcal{E})} < J_1.$$

But $\operatorname{rk} \mathcal{Q} = \operatorname{rk} \mathcal{M}$, $\operatorname{rk} \mathcal{P} = \operatorname{rk} \mathcal{N}$, $\deg \mathcal{Q} > \deg \mathcal{M}$, $\deg \mathcal{P} < \deg \mathcal{N}$ implies $J_1 \leqslant J_0$ and we obtain a contradiction.

So suppose $\inf J|_{0(\mathcal{E})}$ is attained at $A \in 0(\mathcal{E})$. The operator $d_A^* d_A$ acting on L_2^2 self-adjoint sections of End E has kernel the constant scalars (because any other element of the kernel would have eigenspaces that would decompose $\mathcal{E}$ holomorphically), and the projection of $*F/(2\pi i)$ onto the scalars is $\mu(\mathcal{E})$. Thus by the Hodge theory (in a version [4] adapted to the case where the coefficients need not be smooth), there is a self-adjoint section $h \in L_2^2$ such that

$$d_A^* d_A h = 2\pi\mu - i^*F(A).$$

For small t, $1 + th = g_t \in \mathcal{G}^{\mathbf{C}}$. Let $A_t = g_t(A) \in 0(\mathcal{E})$. Then we can compute the curvature

$$\begin{aligned} F(A_t) &= F(A) - \partial_A\big((\bar\partial_A g)g^{-1}\big) + \bar\partial_A\big(g^{-1}(\partial_A g)\big) \\ &\quad - \bar\partial_A g g^{-2} \partial_A g + g^{-1}\partial_A g \bar\partial_A g g^{-1} \\ &= F(A) - t\big(\partial_A\bar\partial_A - \bar\partial_A\partial_A\big)h + q(t, h) \text{ say,} \end{aligned}$$

with $\|q(t, h)\|_{L^2} \leqslant C(\|h\|_{L_2^2}).t^2$ for small t. Since $d_A^* d_A = i^*(\bar\partial_A\partial_A - \partial_A\bar\partial_A)$ we get

$$N\left(\frac{*F(A_t)}{2\pi i} + \mu\right) = N\left(\frac{*F(A)}{2\pi i} + \mu\right).(1 - t) + O(t^2).$$

And if $J(A_t)$ is to be a minimum at $t = 0$ we must have $*F(A)/2\pi i = -\mu$ as required.

Referring again to the paper by Uhlenbeck [6, Corollary 14.] we find that in some gauge the solution is smooth (i.e., there is $u \in \mathcal{G}$ with $u(A)$ smooth). Thus in each stable orbit $0(\mathcal{E})$ there is at least one connection of the required type.

Finally, to see that A is unique up to the action of $\mathcal{G}$ on $0(\mathcal{E})$, recall that any $\tilde g \in \mathcal{G}^{\mathbf{C}}$ can be factored $\tilde g = g.u$ with $g = g^*$, $u \in \mathcal{G}$, so if A, B are distinct solutions we can suppose $B = g(A)$ $g = g^*$. One computes the formula

$$F(A) = F(B) = \mu.1 \Rightarrow \partial_A\bar\partial_A g^2 = -\big\{(\bar\partial_A g^2)g^{-1}\big\}\big\{(\bar\partial_A g^2)g^{-1}\big\}^*.$$

Taking the trace $\tau = \mathrm{Tr}(g^2)$ of this gives $\Delta\tau \leqslant 0$ with equality if and only if $\bar{\partial}_A g^2 = 0$. By the maximum principle the only possibliity is $\Delta\tau = 0$ everywhere and $\bar{\partial}_A g^2 = \partial_A g^2 = 0$. Again since $\mathcal{E}$ is indecomposable, we must have g a constant scalar and $A = B$. This completes the proof of the theorem.

Remark. One may ask whether a holomorphic vector bundle over a compact Kähler manifold has a distinguished connection. In [3] a natural condition is suggested, which reduces in complex dimension one to the condition of the theorem above. In a future article we shall prove that for certain complex surfaces this condition too is precisely related to the algebro-geometric condition of stability.

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