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BOUNDARY REGULARITY AND THE DIRICHLET PROBLEM FOR HARMONIC MAPS

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In a previous paper [10] we developed an interior regularity theory for energy minimizing harmonic maps into Riemannian manifolds. In the first two sections of this paper we prove boundary regularity for energy minimizing maps with prescribed Dirichlet boundary condition. We show that such maps are regular in a full neighborhood of the boundary, assuming appropriate regularity on the manifolds, the boundary and the data. The reader may refer to Theorem 2.7 for a statement of the precise result. It is not surprising that the boundary regularity is actually stronger than the partial regularity we obtained for the interior. This is due to the fact that there are no nontrivial smooth harmonic maps from hemispheres S_+^{n-j} which map the boundary $S^{n-j-1} = \partial S_+^{n-j}$ to a point $(1 \leq j \leq n-2)$, and is analogous to the fact that in certain cases we were able to obtain complete regularity in the interior. Many authors have worked on boundary regularity for this general type of problem. We mention Hildebrandt and Widman [5] and Hamilton [4] as having obtained important results specifically for harmonic maps. Morrey had obtained the boundary regularity for domain dimension $n = 2$ in conjunction with his investigation of the Plateau problem in Riemannian manifolds [8].

In §3 of this paper, we observe that the direct method gives solvability of the Dirichlet problem under reasonable hypotheses on the manifolds. We give, as an application, an amusing proof of a theorem of Sacks and Uhlenbeck [9] on the existence of minimal 2-spheres representing the second homotopy group of a manifold. The same method gives smooth harmonic representations for $\pi_k(N)$ for a certain class of manifolds N. These are characterized by the nonexistence of lower dimensional harmonic spheres whose homogeneous extensions are minimal (see Proposition 3.4).

In the last section of the paper we discuss approximation of L_1^2 maps by smooth maps. We give a simple example of an L_1^2 map from the three-dimensional ball to the two-sphere which is not an L_1^2 limit of continuous maps. We

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also prove that L_1^2 maps from a two-dimensional manifold can be approximated by smooth maps. These approximation questions were explicitly posed by Eells-Lemaire [2].

J. Jost and M. Meier [6] have proven the boundary regularity for minima of a slightly larger class of functionals than those considered here. They need an additional restriction that the image of the map lie within a fixed uniformly Euclidean coordinate chart. The interior partial regularity in this setting had been previously developed by Giaquinta and Giusti [3].

1. Partial boundary regularity

We follow the notation of [10], letting M^n and N be Riemannian manifolds with $N \subset \mathbf{R}^k$ isometrically embedded. Let $L_1^2(M, N)$ be the subset of $L_1^2(M, \mathbf{R}^k)$ whose image lies in N a.e. For $u \in L_1^2(M, \mathbf{R}^k)$, the energy functional is given by

$$E(u) = \int_M \langle du, du \rangle \, dV = \int_M e(u),$$

where

$$e(u) = \sum_{\alpha, \beta} \sum_i g^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta} (\det g_{\gamma\delta})^{\frac{1}{2}} dx,$$

with $g_{\alpha\beta}$ being the metric tensor of M. The norm on $L_1^2(M, \mathbf{R}^k)$ is then given by

$$\|u\|_{1,2}^2 = E(u) + \int_M \sum_i (u^i(x))^2 \, dV.$$

As in [10] we will prove our results for slightly more general functionals $\tilde{E}(u) = E(u) + V(u)$ where $V(u)$ is of the form $V(u) = \int v(u) \, dV$ with

$$v(u) = \left[\sum_i \sum_\alpha \gamma_i^\alpha(x, u(x)) \frac{\partial u^i}{\partial x^\alpha}(x) + \Gamma(x, u(x)) \right] dV.$$

We assume throughout that the metric on M is C^2 and $\gamma, \Gamma \in C^r$ for $r \geqslant 2$. Let $L_{1,0}^2(M, \mathbf{R}^k)$ be the maps which are zero on ∂M. A map $u \in L_1^2(M, N)$ is $\tilde{E}$-minimizing provided $\tilde{E}(u) \leqslant \tilde{E}(v)$ for all $v \in L_1^2(M, N)$ with $u - v \in L_{1,0}^2(M, \mathbf{R}^k)$. We will be interested in this paper in the boundary regularity of $\tilde{E}$-minimizing maps, so we assume that ∂M is of class $C^{2,\alpha}$ and that u satisfies the Dirichlet boundary condition, that is, $u = v_0$ on ∂M where $v_0 \in C^{2,\alpha}(\partial M, N)$. Since the regularity question is local in M, we may choose coordinates x^α centered at a point $p_0 \in \partial M$ such that locally M is the upper

$\frac{1}{2}$-space $\mathbf{R}^n_+$. Thus we will deal with maps $u \in L^2_1(B^+_\sigma, N)$ which are $\tilde{E}$-minimizing. Clearly we may assume that $g_{\alpha\beta}(0) = \delta_{\alpha\beta}$, and also that the boundary data v_0 is defined on all of B_σ, i.e., $v_0 \in C^{2,\alpha}(B_\sigma, N)$.

Most of our arguments will be on B_1, so we define for $\Lambda > 0$ a class $\mathfrak{F}_\Lambda$ of functionals $\tilde{E}$ on B_1 satisfying $g_{\alpha\beta}(0) = \delta_{\alpha\beta}$ and such that for $x \in B^+_1$, $u \in N$

$$\sum_{\alpha,\beta,\tau} \left|\frac{\partial}{\partial x^\tau} g_{\alpha\beta}(x)\right| + |\gamma(x,u)| + |d_u\gamma(x,u)| + |\Gamma(x,u)|^{\frac{1}{2}} + |d_u\Gamma(x,u)|^{\frac{1}{2}} \leqslant \Lambda.$$

Denote by $\mathfrak{D}_\Lambda$ the class of boundary data $v_0 \in C^1(B_1, N)$ such that $v_0(0) = 0 \in \mathbf{R}^k$ where we assume without loss of generality that $0 \in N$. We assume for $v_0 \in \mathfrak{D}_\Lambda$, $x \in B_1$

$$|dv_0|(x) \leqslant \Lambda.$$

Throughout the paper we will assume N_0 is a fixed compact subset of N and our maps have image lying in N_0 a.e.. We use the notation

$$T_\sigma = \{x \in B_\sigma : x_n = 0\}.$$

Let $\mathcal{H}_\Lambda$ denote the space of maps $u \in L^2_1(B^+_1, N_0)$ such that u is $\tilde{E}$-minimizing for some $\tilde{E} \in \mathfrak{F}_\Lambda$ and $u = v_0$ on T_1 (in an L^2_1-sense) for some $u_0 \in \mathfrak{D}_\Lambda$. We observe the following lemma.

Lemma 1.1. *Given $p \in \partial M$, we can choose coordinates centered at p on M, and the origin in $\mathbf{R}^k$ so that the boundary values $v_0 \in C^1(\partial M, N)$ satisfy $v_0(0) = 0$, and so that we have $u \in \mathcal{H}_\Lambda$. Then the map $u_\lambda(x) = u(\lambda x)$ satisfies $u_\lambda \in \mathcal{H}_{\lambda\Lambda}$ for any $\lambda \in (0, 1]$.*

This lemma reduces the regularity question to the study of maps $u \in \mathcal{H}_\Lambda$ where Λ is arbitrarily small. We next observe that [10, Lemma 2.3] carries over directly to our setting to yield

Lemma 1.2. *If Λ is sufficiently small and $u \in \mathcal{H}_\Lambda$, then there exists a constant $c = c(n)$ such that for $\sigma \in (0, 1]$*

$$E^+_\sigma(u) \leqslant (1 + c\Lambda\sigma)E^+_\sigma(w) + c\Lambda\sigma^{n-1}$$

for any $w \in L^2_1(B^+_1, N)$ with $w = u$ on ∂B^+_σ, where we have used E^+_σ to denote energy taken on B^+_σ.

As in [10], a suitable scaling inequality plays an important role in our proof. To derive such an inequality, we first consider approximate reflection of maps $u \in \mathcal{H}_\Lambda$. We define a map $\hat{u} \in L^2_1(B_1, \mathbf{R}^k)$ by

$$\hat{u}(x', x_n) = -\left(u(x', -x_n) - v_0(x', -x_n)\right),$$

where $x' = (x_1, \ldots, x_{n-1})$ denotes the first $(n-1)$-coordinates, and the above definition is for $x_n \leqslant 0$. We take $\hat{u} = u - v_0$ in B_1^+. We use E_σ^x to denote energy taken on $B_\sigma(x)$, and we now derive our scaling inequality.

Lemma 1.3. *Given $u \in \mathcal{H}_\Lambda$ for Λ sufficiently small, we have for $x \in B_{\frac{1}{2}}^+$, $0 < \sigma \leqslant \rho \leqslant \frac{1}{2}$*

$$\sigma^{2-n} E_\sigma^x(\hat{u}) \leqslant c\left[\rho^{2-n} E_\rho^x(\hat{u}) + \Lambda\rho\right].$$

Proof. Our proof involves consideration of two cases. First, if $\rho \leqslant x_n$ we can apply [10, Proposition 2.4] to assert $\sigma^{2-n}E_\sigma^x(u) \leqslant c[\rho^{2-n}E_\rho^x(u) + \Lambda\rho]$. Since $\hat{u} = u - v_0$ in B_1^+ and $B_\rho(x) \subset B_1^+$ we have by the triangle inequality

$$\begin{aligned} \sigma^{2-n} E_\sigma^x(\hat{u}) &\leqslant 2\sigma^{2-n} E_\sigma^x(u) + c\Lambda^2\sigma^2 \\ &\leqslant c\left[\rho^{2-n} E_\rho^x(u) + \Lambda\rho\right] \leqslant c\left[\rho^{2-n} E_\rho^x(\hat{u}) + \Lambda\rho\right], \end{aligned}$$

where we have used the fact that $v_0 \in \mathcal{D}_\Lambda$, and not bothered to distinguish between constants. This gives the required results for $\rho \leqslant x_n$. For $\rho > x_n$, we first note that if $\sigma \leqslant x_n$ the above argument gives

$$\sigma^{2-n} E_\sigma^x(\hat{u}) \leqslant c\left[x_n^{2-n} E_{x_n}^x(\hat{u}) + \Lambda x_n\right]. \tag{1.1}$$

Thus if $\bar{\sigma} = \max\{\sigma, x_n\}$, it suffices to prove

$$\bar{\sigma}^{2-n} E_{\bar{\sigma}}^x(\hat{u}) \leqslant c\left[\rho^{2-n} E_\rho^x(\hat{u}) + \Lambda\rho\right], \tag{1.2}$$

because if $\sigma < x_n$, then $\bar{\sigma} = x_n$, and (1.1) together with (1.2) gives the desired conclusion. We may assume without loss of generality that $2x_n < \rho$ and $\bar{\sigma} < \rho/4$, for if $\rho \leqslant 2x_n$, then (1.1) already implies our conclusion, while if $\bar{\sigma} \geqslant \rho/4$, inequality (1.2) is automatic. Thus if $x = (x', x_n)$, we have the inclusions

$$B_{\bar{\sigma}}(x) \subset B_{2\bar{\sigma}}(x') \subset B_{\rho/2}(x') \subset B_\rho(x).$$

Thus in proving (1.2) we can work with balls centered at $(x', 0) \in T_1$. Therefore without loss of generality we can assume $x_n = 0$ and $x \in T_1$. We prove (1.2) under this assumption. By a linear change of coordinates we can assume $x = 0$ and $g_{\alpha\beta}(x) = \delta_{\alpha\beta}$. Since $\hat{u}$ is an odd mapping with respect to reflection about T_1, inequality (1.2) is equivalent to

$$\bar{\sigma}^{2-n} E_\sigma^+(\hat{u}) \leqslant c\left[\rho^{2-n} E_\rho^+(\hat{u}) + \Lambda\rho\right], \tag{1.3}$$

where E_σ^+ denotes energy taken over B_σ^+. Inequality (1.3) is proven as in [10, Proposition 2.4]. For $\sigma \in (0, \rho]$ one considers the comparison map v_σ on B_ρ^+ given by

$$\begin{aligned} v_\sigma(x) &= u(x), \quad |x| \geqslant \sigma, \\ v_\sigma(x) &= \hat{u}\left(\sigma \frac{x}{|x|}\right) + v_0(x), \quad |x| \leqslant \sigma. \end{aligned}$$

We can calculate as in [10]

$$(1.4)\qquad E_\sigma^+\left(\hat{u}\left(\sigma\frac{x}{|x|}\right)\right) = \frac{1}{n-2}\left(\sigma\frac{d}{d\sigma}E_\sigma^+(\hat{u}) - \int_{\partial B_\sigma^+}\left|\frac{\partial\hat{u}}{\partial r}\right|^2 d\xi\right).$$

Since $|dv_0| \leq c\Lambda$, we get by the Schwartz inequality

$$(1.5)\qquad E_\sigma^+(v_\sigma) \leq (1 + c\Lambda\sigma)E_\sigma^+\left(\hat{u}\left(\sigma\frac{x}{|x|}\right)\right) + c\Lambda\sigma^{n-1}.$$

We cannot directly use v_σ as a comparison map even though $v_\sigma = u$ on ∂B_σ^+ because v_σ does not have image in N. To remedy this problem we observe that since $v_0(0) \in N$ and $|dv_0| \leq c\Lambda$, the distance from $v_\sigma(x)$ to N is at most $c\Lambda\sigma$. Thus if Λ is small, we can use the projection $\pi\colon \mathcal{O} \to N$ to push $v_\sigma(x)$ onto N; that is, we use $\pi \circ v_\sigma$ as a comparison. First observe that since $\operatorname{dist}(v_\sigma, N) \leq c\Lambda\sigma$,

$$(1.6)\qquad E_\sigma^+(\pi \circ v_\sigma) \leq (1 + c\Lambda\sigma)E_\sigma^+(v_\sigma).$$

Now combining Lemma 1.3 with (1.4), (1.5), and (1.6) we have

$$E_\sigma^+(\hat{u}) \leq \frac{\sigma(1 + c\Lambda\sigma)}{n-2}\left[\frac{d}{d\sigma}E_\sigma^+(\hat{u}) - \int_{\partial B_\sigma^+}\left|\frac{\partial\hat{u}}{\partial r}\right|^2\right] + c\Lambda\sigma^{n-1}.$$

This implies

$$(1.7)\qquad 0 \leq \sigma^{2-n}\int_{\partial B_\sigma^+}\left|\frac{\partial\hat{u}}{\partial r}\right|^2 \leq \frac{d}{d\sigma}\left[(1 + c\Lambda\sigma)^{n-2}\sigma^{2-n}E_\sigma^+(\hat{u})\right] + c\Lambda.$$

Since $E_\sigma^+(\hat{u})$ is a nondecreasing function, we can integrate this inequality from $\bar{\sigma}$ to ρ and discard the radial derivative term to get

$$(1 + c\Lambda\bar{\sigma})^{n-2}\bar{\sigma}^{2-n}E_\sigma^+(\hat{u}) \leq (1 + c\Lambda\rho)^{n-2}\rho^{2-n}E_\rho^+(\hat{u}) + c\Lambda(\rho - \bar{\sigma}).$$

This implies (1.3) which concludes the proof of Lemma 1.3.

Taking the radial term into consideration we now get the following result whose proof we omit since it is the same as the proof of [10, Lemma 2.5].

Lemma 1.4. *Let $u \in \mathcal{K}_\Lambda$. There is a sequence $\lambda(i) \to 0$, $\lambda(i) \in (0, 1]$ such that $u_{\lambda(i)}$ converges weakly in $L_1^2(B_1^+, N)$ to a limiting map $u_0 \in L_1^2(B_1, N)$. The map u_0 is a map satisfying $\partial u_0/\partial r = 0$ a.e. in B_1^+, and also satisfying $u_0 = 0$ on T_1.*

We now prove the initial regularity theorem under smallness assumptions on the energy.

Regularity Estimate 1.5. *There exists $\bar{\varepsilon} > 0$ depending only on n and $N_0 \subseteq N$ such that if $u \in \mathcal{K}_\Lambda$, $\Lambda \leq \bar{\varepsilon}$ and $E_1^+(u) \leq \bar{\varepsilon}$, then u is Hölder continuous on $B_{\frac{1}{2}}^+$ and satisfies $|u(x) - u(y)| \leq c|x - y|^\alpha$ for $x, y \in B_{\frac{1}{2}}^+$ where $c, \alpha > 0$ depend only on n, N_0.*

Proof. We give the necessary modifications of [10, §3]. Let $\hat{u}$ be the odd map used in Lemma 1.3, and let φ be a symmetric mollifier, $\varphi > 0$, support $\varphi \subset B_1$, $\varphi(x) = \varphi(|x|)$, $\int \varphi = 1$. Applying Lemma 1.3 as in [10] gives us the inequality

$$(1.8) \qquad h^{-n}\int_{B_h(x)} \hat{u}^{(h)}(x) - |\hat{u}(y)|^2\, dy \leq c_1\bar{\varepsilon},$$

where $\hat{u}^{(h)}(x) = \int \varphi^{(h)}(x-z)\hat{u}(z)\,dz$. Note that by (1.2) and the fact that $v_0(0) = 0 \in N$ we have for a.e. $y \in B_1^+$

$$(1.9) \qquad \operatorname{dist}(\hat{u}(y), N_0) \leq c_2\bar{\varepsilon}.$$

Let $\mathcal{O}$ denote a normal neighborhood of N in $\mathbf{R}^k$. Since N_0 is compact, $\mathcal{O}$ contains a uniform neighborhood of N_0. Thus we see immediately from (1.8), (1.9)

$$(1.10) \qquad \operatorname{dist}(\hat{u}^{(h)}(x), N_0) \leq c_3\bar{\varepsilon}^{\frac{1}{2}}$$

for $x \in B^+_{\frac{1}{2}}$, $0 < h \leq \frac{1}{4}$. In particular, if $\bar{\varepsilon}$ is small we can set $\hat{u}_h(x) = \pi \circ \hat{u}^{(h)}(x)$ to get comparison maps into N. Since $\hat{u}$ is an odd map and φ is a symmetric mollifier, we also have

$$(1.11) \qquad \hat{u}^{(h)}(x) = 0 \qquad \text{for } x \in T_{\frac{1}{2}},\ h \in (0, \tfrac{1}{4}].$$

(Recall $T_\sigma = \{x \in B_\sigma \colon x_n = 0\}$.) We now proceed with the following lemma which is Lemma 3.2 in [10].

Lemma 1.5. *Let $\bar{h} = \bar{\varepsilon}^{-\frac{1}{8}}$, and suppose $h \in (0, \bar{h}]$. Then we have*

$$\int_{B^+_{1/2}} |d\hat{u}^{(h)}|^2\, dx \leq c_4 E_1^+(\hat{u}),$$

$$\sup_{x \in B^+_{1/2}} \left|\hat{u}^{(\bar{h})}(x) - \hat{u}^{(\bar{h})}(0)\right|^2 \leq c_4\bar{\varepsilon}^{\frac{3}{4}}.$$

We now choose $h = h(x)$ where $h(x) = h(r)$, $r = |x|$. Let $\tau = \varepsilon^{\frac{1}{16}}$ and suppose $\varepsilon \in (\tau, \frac{1}{4})$. Choose $h(r)$ to be a nonincreasing function of r satisfying

$$(1.12) \qquad h(r) = \bar{h} \quad \text{for } r \leq \theta,\ h(\theta + \tau) = 0,\ |h'(r)| \leq 2\bar{\varepsilon}^{\frac{1}{16}}.$$

We then set for $x \in B_{\frac{1}{2}}$

$$\hat{u}^{(h(x))}(x) = \int \varphi^{(h(x))}(x - y)\hat{u}(y)\,dy.$$

The following result is immediate from [10, Lemma 3.3], (1.2), and (1.11).

Lemma 1.6. *For $\theta \in (\tau, \frac{1}{4}]$, the map u_h given by*

$$u_h = \pi \circ \left(\hat{u}^{(h)} + v_0\right) \in L_1^2\left(B_{\frac{1}{2}}^+, N\right)$$

satisfies $u_h = u$ on $(B^+_{\frac{1}{2}} \sim B^+_{\theta+\tau}) \cup T$. Moreover, we have

$$\int_{B^+_{\theta+\tau} \sim B^+_\theta} |du_h|^2\, dx \leqslant c_5\left[\int_{B^+_{\theta+2\tau} \sim B_{\theta-\tau}} |du|^2\, dx + \bar{\varepsilon}\theta^n\right].$$

As in [10, (3.4)] we will be finished if we can prove the following result.

Proposition 1.7. *There exists $\bar{\varepsilon} = \bar{\varepsilon}(n, N_0) > 0$ such that if $u \in \mathcal{H}_\Lambda$, $\Lambda \leqslant \bar{\varepsilon}$, $E_1^+(u) \leqslant \bar{\varepsilon}$, then we have*

$$\bar{\theta}^{2-n} E^+_{\bar{\theta}}(u) \leqslant \tfrac{1}{2}\left(E_1^+(u) + \Lambda\right)$$

for some $\bar{\theta} = \bar{\theta}(n, N_0) \in (0, 1)$.

The proof of the regularity estimate, given Proposition 1.7, is identical to the proof in the interior given in [10], so we proceed with the proof of Proposition 1.7. Let v be the solution of the linear Dirichlet problem

$$\Delta v = 0 \quad \text{in } B^+_{\frac{1}{2}}, \qquad v = \hat{u}^{(\bar{h})} \quad \text{on } \partial B^+_{\frac{1}{2}}.$$

As in [10] we prove the following inequalities:

$$\sup_{B^+_{1/2}} |v - \hat{u}^{(h)}| \leqslant c_6 \bar{\varepsilon}^{\frac{3}{4}}, \tag{1.13}$$

$$\sup_{B^+_{1/4}} |dv|^2 \leqslant c_6\left(E_1^+(u) + \Lambda\right). \tag{1.14}$$

For $\theta \in (0, \frac{1}{4}]$ we get, setting $u_{\bar{h}} = \pi \circ (\hat{u}^{(\bar{h})} + v_0)$,

$$\begin{aligned} \theta^{2-n} E^+_\theta(u_{\bar{h}}) &\leqslant c_7\left[\theta^{2-n} E^+_\theta(\hat{u}^{(h)}) + \theta^2\Lambda\right] \\ &\leqslant 2c_7\left[\theta^{2-n}\int_{B^+_\theta}\left\{\left|d\left(\hat{u}^{(\bar{h})} - v\right)\right|^2 + |dv|^2\right\} dx + \theta^2\Lambda\right]. \end{aligned} \tag{1.15}$$

Integrating by parts gives

$$\int_{B^+_{1/2}} \left|d\left(\hat{u}^{(\bar{h})} - v\right)\right|^2 = -\int_{B^+_{1/2}} \left(\hat{u}^{(\bar{h})} - v\right)\Delta \hat{u}^{(\bar{h})} - v).$$

By (1.13) and the harmonic property of v we get

$$\int_{B^+_{1/2}} \left|d\left(\hat{u}^{(\bar{h})} - v\right)\right|^2 \leqslant c_8 \bar{\varepsilon}^{\frac{3}{8}} \int_{B^+_{1/2}} |\Delta \hat{u}^{(\bar{h})}|. \tag{1.16}$$

From the fact that $\hat{u}(x^*) = -\hat{u}(x)$ for $x^* = (x', -x_n)$ when $x = (x', x_n)$ it follows that

$$\Delta \hat{u}^{(\bar{h})}(x) = \int_{\mathbf{R}^n} \left[\Delta_x \varphi^{(\bar{h})}(x-y)\right] \hat{u}(y)\, dy$$
$$= \int_{\mathbf{R}^n_+} \Delta_x\left(\varphi^{(\bar{h})}(x-y) - \varphi^{(\bar{h})}(x-y^*)\right) \hat{u}(y)\, dy.$$

Since $\varphi^{(\bar{h})}$ is an even function, we may write

$$\Delta \hat{u}^{(\bar{h})}(x) = -\int_{\mathbf{R}^n_+} \left[\Delta_y \zeta_x(y)\right] \hat{u}(y)\, dy,$$

where $\zeta_x(y) = \varphi^{(\bar{h})}(x-y) - \varphi^{(\bar{h})}(x^* - y)$. Note that $\zeta_x = 0$ on $\partial \mathbf{R}^n_+$, and $0 \leq \zeta_x(y) \leq \varphi^{(h)}(x-y)$ for $x, y \in \mathbf{R}^n_+$. Since $\hat{u} = 0$ on $\partial \mathbf{R}^n_+$ we have

$$\Delta \hat{u}^{(h)}(x) = \int_{\mathbf{R}^n_+} d\zeta_x \cdot d(u - v_0)\, dy.$$

By the Euler-Lagrange equation for u [10, Lemma 2.1] and by (1.2) this gives

$$\left|\Delta \hat{u}^{(\bar{h})}(x)\right| \leq c_9 \left\{ \int_{\mathbf{R}^n_+} \varphi^{(\bar{h})}(x-y)\left[|du|^2 + \Lambda\right] dy + \Lambda h^{-1} \right\},$$

where we have used the obvious fact $|d\zeta_x| \leq c(\bar{h})^{-n-1} \chi_{B_{\bar{h}}(x)}$ where χ_E denotes the characteristic function of E. Integrating over $x \in B^+_{\frac{1}{2}}$ we have

$$\int_{B^+_{1/2}} |\Delta \hat{u}^{(\bar{h})}| \leq c_{10}\left[\left(E^+_1(u) + \Lambda\right) + \Lambda(\varepsilon)^{-\frac{1}{8}}\right].$$

Using this in (1.16) and combining with (1.14), (1.15) yield

$$\theta^{2-n} E^+_\theta(u_{\bar{h}}) \leq c_{11}\left(\theta^{2-n} \bar{\varepsilon}^{\frac{1}{4}} + \theta^2\right)\left(E^+_1(u) + \Lambda\right) \tag{1.17}$$

for any $\theta \in (0, \frac{1}{4})$.

Following [10] let $\gamma_n \in (0, \frac{1}{32}]$ be a number to be chosen depending only on n, and let $\bar{\theta} = \bar{\varepsilon}^{\gamma_n}$. Let ρ be the greatest integer less than or equal to $\bar{\theta}/(3\tau)$ where $\tau = \bar{\varepsilon}^{\frac{1}{16}}$ and write

$$\left[\bar{\theta}, \bar{\theta} + 3\rho\tau\right] = \bigcup_{i=1}^{p} I_i, \quad |I_i| = 3\tau,$$

where each I_i is a closed interval. Since $\gamma_n \leq \frac{1}{32}$, we have $p \geq \frac{1}{3}(\bar{\varepsilon})^{-1/32} - 1$. We choose an interval I_j for some j with $1 \leq j \leq p$ such that

$$\int_{r \in I_j} |du|^2\, dx \leq p^{-1} E^+_1(u) \leq c_{12} \bar{\varepsilon}^{1/32} E^+_1(u). \tag{1.18}$$

Let θ be the number such that $I_j = [\theta - \tau, \theta + 2\tau]$, and let $h(x)$ be as in Lemma 1.6. Thus by Lemma 1.2, Lemma 1.6, (1.17), and (1.18) we have

$$\begin{aligned}\bar{\theta}^{2-n}E_{\bar{\theta}}^{+}(u) &\leq c_{13}\left[\theta^{2-n}E_{\theta}^{+}(u_{\bar{h}}) + \bar{\varepsilon}^{\frac{1}{32}}\bar{\theta}^{2-n}E_1^{+}(u) + \bar{\theta}\Lambda\right]\\ &\leq c_{14}(\bar{\theta}^{2-n}\bar{\varepsilon}^{\frac{1}{32}} + \bar{\theta})(E_1^{+}(u) + \Lambda),\end{aligned}$$

since $\theta \in [\bar{\theta}, 2\bar{\theta}]$. Choosing γ_n small now finishes the proof of the regularity estimate.

We get the following corollary concerning partial boundary regularity. First, we say that a point $p \in \overline{M}$ is a regular point for u if u is continuous in a neighborhood of p in M. The neighborhood may be taken as a ball if $p \in \operatorname{Int} M$ or as a half ball if $p \in \partial M$. The singular set $\mathbb{S}$ of u is then the complement of the regular set. Note that if $v_0 \in C^{2,\alpha}(\partial M, N)$ and u is $\tilde{E}$-minimizing with $u = v_0$ on ∂M, then u is $C^{2,\alpha}$ in a half ball centered at $p \in \partial M$ provided u is continuous in this half ball.

Corollary 1.8. *If $u \in L_1^2(M, N)$ is $\tilde{E}$-minimizing ($u \in N_0$ a.e.) with $u = v_0$ on ∂M where $v_0 \in C^{2,\alpha}(\partial M, N)$, then $\mathcal{H}^{n-2}(\mathbb{S}) = 0$. More generally, if $u \in \mathcal{H}_\Lambda$ then $\mathcal{H}^{n-2}(\mathbb{S} \cap (B_1^+ \cup T_1)) = 0$.*

The proof of the corollary is the same as [10, Corollary 2.7], so we omit the argument.

2. A geometric lemma and complete boundary regularity

We first reduce the boundary regularity problem to a question about the existence of certain harmonic maps of hemispheres which take a constant value on the boundary. Most of the work pertaining to this reduction has already been done in [10], so we indicate the necessary modifications. For a fixed point $u^* \in \mathbf{R}^k$, we use the notation

$$W_\sigma^{+}(u) = \int_{B_\sigma^+} |u - u^*|^2\, dx.$$

We observe first that our main extension lemma works also in half balls.

Lemma 2.1. *For $n \geq 2$ there exist $\delta = \delta(n, N_0)$ and a constant $q = q(n)$ such that if $\varepsilon \in (0,1)$ is given, and $u \in L_1^2(\partial B_\sigma^+, N_0)$ satisfies $\sigma^{4-2n}E(u)W(u) \leq \delta^2\varepsilon^q$, then there exists $\bar{u} \in L_1^2(B_\sigma^+, N_0)$, $\bar{u}\,|_{\partial B_\sigma^+} = u$ such that*

$$\begin{aligned}E_\sigma^{+}(\bar{u}) &\leq c(\varepsilon\sigma E(u) + \varepsilon^{-q}\sigma^{-1}W(u)),\\ W_\sigma^{+}(\bar{u}) &\leq c\varepsilon^{-q}W(u).\end{aligned}$$

Proof. By rescaling we may assume $\sigma = 1$. The lemma now follows from the corresponding result on B_1 given in [10, Lemma 4.3] by the fact that B_1^+ is

Lipschitz equivalent to B_1. Note that the hypotheses and conclusions are invariant (up to constants) under bi-Lipschitz transformations of the domain.

We get the following strengthening of the regularity estimate whose proof is an obvious modification of the proof of [10, Proposition 4.5].

Proposition 2.2. *Given $B > 0$ there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, N_0, B)$ such that if $u \in \mathcal{K}_\Lambda$, $\Lambda < \varepsilon_0$, $E_1^+(u) \leqslant B$, and $W_1^+(u) \leqslant \varepsilon_0$, then u is Hölder continuous on $B_{\frac{1}{2}}^+$, and $|u(x) - u(y)| \leqslant c|x - y|^\alpha$ for $x, y \in B_{\frac{1}{2}}^+$, where $\alpha = \alpha(n) > 0$ and $c = c(n, N_0)$.*

The following compactness theorem is proved as in [10, Proposition 4.6].

Proposition 2.3. *Let $\{u_i\} \subset \mathcal{K}_\Lambda$ be a weakly convergent (in L_1^2) sequence with limit u_0 such that $E_1^+(u_i) \leqslant B$ for some $B > 0$. Then u_0 is locally Hölder continuous outside a closed set $\mathcal{S}_0 \subset \overline{M}$ with $\mathcal{K}^{n-2}(\mathcal{S}_0) = 0$. Moreover, u_i converges to u_0 in L_1^2 norm on $B_{\frac{1}{2}}^+$ and uniformly on compact subsets of $\overline{B_{\frac{1}{2}}^+} \sim \mathcal{S}_0$.*

At this point the proof of the boundary regularity becomes somewhat simpler than the interior proof. The reason is that we will be able to rule out all nontrivial boundary tangent maps in some generality. We can now strengthen Lemma 1.4.

Proposition 2.4. *Given $u \in \mathcal{K}_\Lambda$, there is a sequence $\lambda(i) \to 0$ such that the scaled maps $u_{\lambda(i)}$ given by $u_{\lambda(i)}(x) = u(\lambda(i)x)$ converge in L_1^2 norm to a radially independent harmonic map $u_0 \in L_1^2(B_1^+, N)$ with $u_0 = 0$ on T_1.*

Proof. This is immediate from Proposition 2.3 and Lemma 1.4.

The norm convergence in the above result insures that u_0 is nontrivial if $0 \in \mathcal{S}$. We will now show that no such maps u_0 can exist. This fact is related to the geometry of the sphere, in particular to conformal transformations. For this reason we express the metric on S^n in conformally flat form. With the north pole as center of polar coordinates, the spherical metric takes the form $ds^2 = 4(1 + \rho^2)^{-2}(d\rho^2 + \rho^2 d\xi^2)$ where $\xi \in S^{n-1}$. The equator of S^n in these coordinates is the set $\{\rho = 1\}$, and $S_+^n = \{\rho < 1\}$. Given a map $u \in L_1^2(S_+^n, N)$ with $u = u^*$ on ∂S_+^n some $u^* \in N$, we rescale u in these coordinates by setting u_β for $\beta > 1$

$$\begin{aligned} u_\beta(\rho, \xi) &= u(\beta\rho, \xi) && \text{for } \rho \leqslant \beta^{-1}, \\ u_\beta(\rho, \xi) &= u(1, \xi) = u^* && \text{for } \beta^{-1} \leqslant \rho < 1. \end{aligned}$$

We compute the spherical energy of u_β by

$$E_1(u_\beta) = \int_{\{\rho \leqslant \beta^{-1}\}} \left(\frac{2}{1 + \rho^2}\right)^{n-2} \rho^{n-1} \cdot \left(\beta^2 \left|\frac{\partial u}{\partial \rho}\right|^2 (\beta\rho, \xi) + \rho^{-2} \left|\frac{\partial u}{\partial \xi}\right|^2 (\beta\rho, \xi)\right) d\rho\, d\xi.$$

By changing variables to $\sigma = \beta\rho$, this becomes

$$E_1(u_\beta) = \int_{S^n_+} \beta^{2-n}\left[\frac{2}{1+(\sigma/\beta)^2}\right]^{n-2} \sigma^{n-1}|d_0u|^2(\sigma,\xi)\,d\sigma\,d\xi,$$

where $|d_0u|^2(\rho,\xi) = |\partial u/\partial\rho|^2 + \rho^{-2}|\partial u/\partial\xi|^2$. If $e(u)$ denotes the spherical energy density, i.e.,

$$e(u) = \sigma^{n-1}\left(\frac{2}{1+\sigma^2}\right)^{n-2}|d_0u|^2\,d\sigma\,d\xi,$$

then the above expression takes the form

$$E_1(u_\beta) = \int_{S^n_+}\left[\frac{\beta(1+\rho^2)}{\beta^2+\rho^2}\right]^{n-2} e(u)(\rho,\xi). \tag{2.1}$$

This expression is valid for $n \geqslant 2$, and for $n = 2$ it simply expresses the conformal invariance of energy. The following result can now be stated. (See also J. Wood [11].)

Geometric Lemma 2.5. *A smooth harmonic map u_0: $S^n_+ \to N$ taking a constant value on ∂S^n_+ is itself constant for $n \geqslant 2$.*

Proof. The result for $n = 2$ was proven by Lemaire [7]. Lemaire's result applies when the domain is the unit disk in the complex plane, but since energy is invariant under conformal transformations for $n = 2$, this implies the result we are after. For $n > 2$, since u is assumed to be smooth, the family u_β represents a valid variation of u, so we must have $(d/d\beta)E_1(u_\beta) = 0$ at $\beta = 1$. Thus by (2.1) we get

$$(2-n)\int_{S^n_+}(1-\rho)\cdot e(u)(\rho,\xi) = 0,$$

which implies $e(u) \equiv 0$ in S^n_+ and u_0 is constant, thereby establishing the geometric lemma.

The above proof encounters difficulties if u_0 is not assumed *a priori* smooth because it is not clear that u_β represents an admissible variation of u_0 in that case. For the maps u_0 arising as limits of $u_{\lambda(i)}$ for $u \in \mathcal{H}_\Lambda$ we can prove the result for singular maps.

Proposition 2.6. *A map $u_0 \in L^2_1(B^+_1, N)$ arising as a limit of $u_{\lambda(i)}$ for $u \in \mathcal{H}_\Lambda$ (see Proposition 2.4) is constant.*

Proof. First observe that for $n = 3$ we must have that u_0 is smooth away from the origin since $\mathcal{H}^{n-2}(\mathbb{S}_0) = 0$. Therefore this case follows from Lemaire's result [7]. Thus we assume $n \geqslant 4$. Let $r = |x|$ on $\mathbf{R}^n$ so that (r, ρ, ξ) represent coordinates for $\mathbf{R}^n_+$, ρ and ξ as above. For $\bar{\beta} \in (1, \frac{3}{2})$ let $\beta(r)$ be a smooth nonincreasing function satisfying $\beta(r) = 1$ for $r \geq 1$, $\beta(r) = \bar{\beta}$ for $r \leq \frac{1}{2}$, $|\beta'(r)| \leq 4(\beta - 1)$. Let $\Omega_\beta \subset \mathbf{R}^n_+$ be given by $\Omega_\beta = \{(r, \rho, \xi)\colon \rho < \beta(r)^{-1}\}$, and consider the diffeomorphism $F\colon \Omega_\beta \to \mathbf{R}^n_+$ defined by $F(r, \rho, \xi) = (r, \beta(r)\rho, \xi)$. Now for each i, consider the comparison map w_i given by $w_i = u_{\lambda(i)} \circ F$ on Ω_β, $w_i(r, \rho, \xi) = u_{\lambda(i)}(r, 1, \xi)$ for $(r, \rho, \xi) \in \mathbf{R}^n_+ \sim \Omega_\beta$. Since F is the identity outside B_1^+, we clearly have $w_i = u_{\lambda(i)}$ on ∂B_1^+, and hence by Lemma 1.2 we have

$$E_1^+(u_{\lambda(i)}) \leq (1 + c\lambda(i)\Lambda)E_1^+(w_i) + c\lambda(i)\Lambda.$$

By Proposition 2.4, $u_{\lambda(i)}$ converges in norm to u_0, so it follows directly that

$$E_1^+(u_0) \leq E_1^+(w_0), \tag{2.2}$$

where $w_0 = u_0 \circ F$ on Ω_β, $w_0 = 0$ on $\mathbf{R}^n_+ \sim \Omega_\beta$. We will show that (2.2) implies u_0 is constant. We write $E_1^+(w_0) = E^r + E^t$ where E^r is the radial part in $\mathbf{R}^n$ and E^t the spherical part. From (2.1) we easily get

$$E^t = \int_0^1 r^{n-3}\left[\int_{S_+^{n-1}} \eta(\beta(r), \rho)e^t(u_0)(r, \rho, \xi)\right] dr,$$

where

$$\eta(\beta, \rho) = \left(\frac{\beta(1 + \rho^2)}{\beta^2 + \rho^2}\right)^{n-3}.$$

By elementary calculus we have

$$\eta(\beta, \rho) \leq 1 - (\beta - 1)(1 - \rho).$$

Therefore we get, since $\partial u_0/\partial r = 0$,

$$E^t \leq E_1^+(u_0) - (\bar{\beta} - 1)\int_{B_{1/2}^+} (1 - \rho)|du_0|^2. \tag{2.3}$$

On the other hand we compute directly

$$E^r \leq c(\bar{\beta} - 1)^2 E_1^+(u_0),$$

which combined with (2.3) gives

$$E_1^+(w_0) \leq E_1^+(u_0) - (\bar{\beta} - 1)\left[\int_{B_{1/2}^+} (1 - \rho)|du_0|^2 - c(\bar{\beta} - 1)E_1^+(u_0)\right].$$

If u_0 is not constant, we can choose $(\bar{\beta} - 1)$ sufficiently small to contradict (2.2). This completes the proof of Proposition 2.6.

The above result combined with Proposition 2.4 and Regularity Estimate 1.5 gives us the following theorem.

Regularity Theorem 2.7. *Let M be a compact manifold with $C^{2,\alpha}$ boundary. Suppose $u \in L_1^2(M, N)$ is $\tilde{E}$-minimizing and satisfies $u(x) \in N_0$ a.e. for a compact subset $N_0 \subset N$. Suppose $v \in C^{2,\alpha}(\partial M, N_0)$ and $u = v$ on ∂M. Then the singular set $\mathcal{S}$ of u is a compact subset of the interior of M; in particular, u is $C^{2,\alpha}$ in a full neighborhood of ∂M.*

3. The Dirichlet problem and harmonic spheres

In this section we observe that the direct method gives solvability of the Dirichlet problem, and we give, as an application, a proof of the result of Sacks-Uhlenbeck [9] that $\pi_2(N)$ can be represented by harmonic (minimal) maps of S^2 into N. Let M be a compact manifold with (possibly empty) $C^{2,\alpha}$ boundary, and let $\tilde{E}$ be a functional of the type which we are considering. Suppose N is compact with or without boundary.

Proposition 3.1. *Suppose ∂N is empty and let $v \in C^{2,\alpha}(\partial M, N)$ be given. Suppose v extends to a map $v \in L_1^2(M, N)$. There exists a map $u \in L_1^2(M, N)$ with $u = v$ on ∂M which is $\tilde{E}$-minimizing over all L_1^2 extensions of v. The map u is $C^{2,\alpha}$ near ∂M and has possibly a singular set $\mathcal{S}$ of Hausdorff dimension at most $n - 3$ in the interior of M. The interior regularity theory of* [10] *applies to u.*

Proof. Let u_i be a minimizing sequence of extensions of v, and observe that $E(u_i) \leq K$ for some constant K. Thus a subsequence converges weakly to $u \in L_1^2(M, N)$ which is also an extension of v. Applying the interior and boundary regularity theorems then gives the conclusions.

Remark. If $\partial M = \varnothing$, Proposition 3.1 asserts the existence of a map which is $\tilde{E}$-minimizing over all competing maps from M to N. If $\tilde{E}$ is the ordinary energy functional, this minimizing map is obviously constant.

We next observe that for the energy functional E we can weaken the requirement that $\partial N = \varnothing$, and merely require that ∂N is locally convex with respect to N.

Corollary 3.2. *Suppose ∂N is locally convex with respect to N and $v \in C^{2,\alpha}(\partial M, N_0)$ where N_0 is a compact subset of the interior of N. There exists a map $u \in L_1^2(M, N)$ which minimizes the energy E over all extensions of v in $L_1^2(M, N)$. The map u has image contained in a compact subset N_1 of the interior of N and enjoys the regularity properties described in Proposition* 3.1.

Proof. We need only show that we can find a minimizing sequence $\hat{u}_i$ with image contained in a suitable compact subset N_1 of the interior of N. To do this let u_i be any minimizing sequence, and let N_1 be the set of points of N a distance at least ε_0 from ∂N. Since ∂N is convex, for ε_0 small the set $\{p \in N: d(p, \partial N) = \varepsilon\}$ is locally convex for $\varepsilon \in (0, \varepsilon_0]$. Also suppose ε_0 is so small that

$\operatorname{dist}(N_0, \partial N) > \varepsilon_0$. Let F be the Lipschitz map given by $F(p) = p$ for $p \in N_1$, and $F(p)$ is the point of ∂N_1 nearest to p for $p \in N \sim N_1$. From the convexity hypothesis we clearly have that F is distance nonincreasing. Thus if we set $\hat{u}_i = F \circ u_i$, then $E(\hat{u}_i) \leq E(u_i)$, and since $\operatorname{dist}(N_0, \partial N) > \varepsilon_0$, we have $\hat{u}_i = v$ on ∂M. Thus $\hat{u}_i$ is a minimizing sequence with image in $\overline{N}_1$. This proves Corollary 3.2.

We now give an application of our results to prove existence of harmonic spheres.

Proposition 3.3. *Suppose N is compact without boundary or compact with convex boundary. Any smooth map v: $S^2 \to N$ which does not extend continuously to B^3 is homotopic to a sum of smooth harmonic (hence minimal) maps u_j: $S^2 \to N, j = 1,\cdots,P$.*

Proof. Since v is smooth, it has finite energy and hence the map $\bar{v}(x) = v(x/|x|)$ is a finite energy extension of v to B^3. Thus there exists a least energy extension $u \in L_1^2(B^3, N)$ of v. By the previous results u is smooth near $\partial B^3 = S^2$ and has isolated singular points $x_1,\cdots,x_p$ in B^3, ($p \geq 1$ since v does not extend continuously to B^3). By the results of [10] each x_j is associated to a minimizing tangent map (MTM) hence a smooth harmonic u_j: $S^2 \to N$. This proves the required result.

If one attempts to extend this result to higher dimensions, one encounters the problem that singularities are not necessarily isolated, and hence the blown-up map at a singular point may itself have singularities. For $k \geq 2$, we say that a simply connected manifold N is *geometrically k-connected* if every MTM from $\mathbf{R}^j \to N$ is constant for $j = 3,\cdots,k + 1$. The following theorem is a direct consequence of [10] and the proof of Proposition 3.3.

Proposition 3.4. *Suppose N is compact without boundary and suppose the universal cover $\tilde{N}$ of N is geometrically $(k - 1)$-connected for some $k \geq 3$. Then each class in $\pi_k(N)$ can be represented by a sum of harmonic maps of $S^k \to N$.*

Remark. It seems to be quite difficult to check whether a given manifold is geometrically k-connected.

4. Remarks on approximations of L_1^2 maps

We have been working in the space $L_1^2(M, N)$ which we have defined as the maps in $L_1^2(M, \mathbf{R}^k)$ whose values lie almost everywhere in N. Another definition which one might consider is to define $L_1^2(M, N)$ to be the closure of $C^\infty(M, N)$ in the $L_1^2(M, \mathbf{R}^k)$ norm. The following example shows that for $n \geq 3$ the two spaces are not the same.

EXAMPLE. The map $u \in L_1^2(B_1^3, S^2)$ given by $u(x) = x/|x|$ is not an L_1^2 limit of a sequence $u_i \in C^\infty(B_1^3, S^2)$. To see this one can simply observe that if such a sequence u_i did exist, then for almost every $r \in (\frac{1}{2}, 1)$ we would have $L_1^2(\partial B_r^3, S^2)$ convergence of u_i to the map x/r. In particular, we would have a sequence $v_i \in C^\infty(S^2, S^2)$ (say $v_i(x) = u_i(rx)$), each v_i having degree zero, converging to the identity map of S^2. By taking a subsequence we could assume dv_i converges pointwise a.e. to the identity. Thus in particular the Jacobian $J(v_i) \to 1$ a.e. on S^2. Since $|J(v_i)| \leq \frac{1}{2} |dv_i|^2$ and $|dv_i|^2$ converges in L_1 norm to a limit, the dominated convergence theorem implies that

$$\lim_{i \to \infty} \int_{S^2} J(v_i) = 4\pi.$$

The fact that each v_i has degree zero implies that the integral of $J(v_i)$ is zero for each i, a contradiction.

Our next result shows that for $n = 2$ an L_1^2 map is a limit of smooth maps. The method we employ is essentially the same as our method of comparison construction in the proof of the partial regularity theorem.

Proposition. *Let M^2 be a compact surface with possibly empty C^1 boundary. Let N be a compact manifold without boundary. Then $C^\infty(M, N)$ is dense in $L_1^2(M, N)$.*

Proof. By standard extension theorems (see [8, Theorem 3.43]) we may assume that M is compactly contained in M_1 and that any given $u \in L_1^2(M, N)$ is the restriction of a map $\bar{u} \in L_1^2(M_1, \mathbf{R}^k)$ to M. We assume M_1 is isometrically embedded in $\mathbf{R}^n$. Let $\mathcal{U}$ be a normal neighborhood of M_1 in $\mathbf{R}^n$, and $\mathcal{O}$ be a normal neighborhood of N in $\mathbf{R}^k$. For $\varepsilon < \operatorname{dist}(M, \partial M_1)$ the function $G_\varepsilon(x) = E_{B_\varepsilon^2(x)}(\bar{u})$ is a continuous function of $x \in M$. The function G_ε clearly decreases when ε decreases, and $\lim_{\varepsilon \downarrow 0} G_\varepsilon(x) = 0$ for all $x \in M$. Therefore it follows that G_ε converges uniformly to zero in M. Now extend $\bar{u}$ to a map $\bar{\bar{u}} \in L_1^2(\mathcal{U}, \mathbf{R}^k)$ by setting $\bar{\bar{u}}(x) = \bar{u}(\underline{\underline{P}}(x))$ where $\underline{\underline{P}}\colon \mathcal{U} \to M_1$ denotes nearest point projection. Since the metric on $\mathcal{U}$ is uniformly equivalent to a product metric on $M_1 \times B^{n-2}$, we clearly have for $x \in \underline{\underline{P}}^{-1}(M)$

$$E_{B_\varepsilon^n(x)}(\bar{\bar{u}}) \leq c\varepsilon^{n-2} E_{B_\varepsilon^2(\underline{\underline{P}}(x))}(\bar{u}) = c\varepsilon^{n-2} G_\varepsilon(x). \tag{4.1}$$

Note also that $\bar{\bar{u}}(x) \in N$ a.e. $x \in \underline{\underline{P}}^{-1}(M)$. Let $\varphi(x)$ be a mollifier on B_1^n, and set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, and

$$\bar{\bar{u}}_\varepsilon(x) = \int \varphi_\varepsilon(x-y)\bar{\bar{u}}(y)\,dy$$

for $x \in M$. By the Poincaré inequality and (4.1)

$$\varepsilon^{-n} \int_{B_\varepsilon^n(x)} \left|\bar{\bar{u}}(y) - \bar{\bar{u}}_\varepsilon(x)\right|^2 dy \leq c\varepsilon^{2-n} E_{B_\varepsilon^n(x)}(\bar{\bar{u}}) \leq cG_\varepsilon(x). \tag{4.2}$$

Since $\bar{\bar{u}}: \underline{\underline{P}}^{-1}(M) \to N$, this inequality implies that for all $x \in M$ we have

$$\text{dist}\left(\bar{\bar{u}}_\varepsilon(x), N\right) \leq cG_\varepsilon^{\frac{1}{2}}(x). \tag{4.3}$$

Let $\pi: \mathcal{O} \to N$ be the nearest point projection map, and observe that by (4.3) and the fact that G_ε converges uniformly to zero, we can define a smooth map $v_\varepsilon: M \to N$ by setting $v_\varepsilon(x) = \pi \circ \bar{\bar{u}}_\varepsilon(x)$. It is quite easy to see that $\lim_{\varepsilon \downarrow 0} \| v_\varepsilon - u \|_{1,2,M} = 0$. This completes the proof of the proposition.

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