J. DIFFERENTIAL GEOMETRY
18 (1983) 221–252

EXISTENCE OF CLOSED GEODESICS ON POSITIVELY CURVED MANIFOLDS

W. BALLMANN, G. THORBERGSSON & W. ZILLER

In [8] we examined stability properties of closed geodesics whose existence can be obtained by elementary methods. In this paper we apply Lusternik-Schnirelmann theory to obtain the existence of several closed geodesics below certain length levels. We will then examine stability properties of these closed geodesics.

Let g_0 be the standard metric on S^n of constant curvature 1. Using perturbation methods it follows that any metric on S^n, $n \geqslant 2$, sufficiently C^2 close to g_0, has at least as many closed geodesics of length approximately 2π as a function on the Grassmannian $G_{2,n-1}$ of unoriented two-planes in R^{n+1} has critical points. This in turn can be estimated from below by the so-called cup length which for $G_{2,n-1}$ is $g(n) = 2n - s - 1$, where $0 \leqslant s = n - 2^k < 2^k$. Hence there are at least $g(n)$ short closed geodesics for metrics on S^n sufficiently C^2 close to g_0. Note that $\frac{1}{2}(3n-1) \leqslant g(n) \leqslant 2n-1$.

Theorem A. *Suppose that M is homeomorphic to S^n and that $1/4 \leqslant \delta \leqslant K \leqslant 1$, where K denotes the sectional curvature of M.*

(i) *There exist at least $g(n)$ closed geodesics without self-intersections and with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,] \subset [2\pi, 4\pi]$. If all closed geodesics of length $\leqslant 4\pi$ are nondegenerate (an open and dense condition on the set of metrics with respect to the C^2 topology), then there exist at least $n(n+1)/2$ closed geodesics without self-intersections and with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,]$.*

(ii) *If the closed geodesics whose lengths lie in $[2\pi, 2\pi/\sqrt{\delta}\,]$ all have the same length l, then all geodesics are closed of length l. If the closed geodesics whose lengths lie in $[2\pi, 2\pi/\sqrt{\delta}\,]$ have only two different length values, then there exists a family of closed geodesics of equal length in $[2\pi, 2\pi/\sqrt{\delta}\,]$ such that every point of M lies in the image of some geodesic in the family.*

Received April 20, 1982, and, in revised forms, July 30 and October 5, 1982. The first author was supported by a research scholarship from the Deutsche Forschungsgemeinschaft. The last two authors were partially supported by the Sonderforschungsbereich Theoretische Mathematik at the University of Bonn. The second author did part of his work at the Institute of Pure and Applied Mathematics in Rio de Janeiro supported by GMD and CPNq. The third author was partially supported by the National Science Foundation.

It is well known that any simply connected manifold with $1/4 \leqslant K \leqslant 1$ is either homeomorphic to S^n or isometric to a symmetric space of rank 1. Hence the above theorem remains valid under the weaker assumption that M is simply connected and $1/4 \leqslant \delta \leqslant K \leqslant 1$.

Theorem B. *Suppose that $M = S^n$ and that the metric g satisfies $g \leqslant \alpha^2 g_0$, $1 \leqslant \alpha < 2$, and $l(M) \geqslant 2\pi$, where $l(M)$ denotes the length of a shortest geodesic loop of g. Then* (i) *and* (ii) *in Theorem* A *hold, where the lengths of the closed geodesics lie in* $[2\pi, 2\alpha\pi] \subset [2\pi, 4\pi)$.

Although Theorem A is better from a geometric point of view, it is worth noticing that the assumptions in Theorem B are C^1 conditions on the metric, whereas those in Theorem A are C^2. For example, Theorem B implies that every metric sufficiently C^1 close to the standard metric on S^n has $g(n)$ closed geodesics without self-intersections and with lengths approximately 2π.

Note also that, by the injectivity radius estimate, $l(M) \geqslant 2\pi$ is satisfied if n is even and $0 < K \leqslant 1$.

In the nondegenerate case the number $n(n+1)/2$ is optimal as shown by an n-dimensional ellipsoid with pairwise different axes close to 1: the $n(n+1)/2$ ellipses in the coordinate two-planes are closed geodesics without self-intersections and with lengths close to 2π. One can achieve that the lengths of all other closed geodesics are greater than any given number by choosing the axes sufficiently close to 1. In the general case it is not known whether the number $g(n)$ is also optimal. It is even not known whether $g(n)$ is the minimal number of critical points for a function on $G_{2,n-1}$. $G_{2,n-1}$ always admits a function with $2n-1$ critical points since $\dim G_{2,n-1} = 2n-2$. For $n = 3$ J. Milnor constructed an example (unpublished) of a function on $G_{2,2}$ with only $g(3) = 4$ critical points.

For $n = 2$ Lusternik and Schnirelmann [23] proved that any metric on S^2 has at least three closed geodesics without self-intersections (see also [6]).

Theorem B is a slightly extended version of a result of Alber [3], [4]. Alber's proof, however, relies on a topological result, the so-called lemma of Alber, which turned out to be false. See chapter 2 for a short discussion of this. Complete proofs of Alber's result were obtained by the authors, announced in [7], and independently by Anosov [5] and Hingston [17].

Theorem A and B are proved by applying Lusternik-Schnirelmann theory to the energy functional E on the space of unparametrized closed curves on M. If $\Lambda(M)$ denotes the space of curves $S^1 \to M$, then the space of unparametrized curves is the quotient $\Lambda(M)/O(2)$, where $O(2)$ acts by linear reparametrizations on S^1.

One of the difficulties in applying Lusternik-Schnirelmann theory is that different homology classes of $\Lambda(M)/O(2)$ can give rise to different critical points which are iterates of the same closed geodesic. One method to avoid this is to choose homology classes which have representatives below an energy level which is smaller then the energy of all iterated closed geodesics. Under the conditions of Theorem A or B this is achieved by working in $\Lambda^{8\pi^2-} = \{c \in \Lambda \mid E(c) < 8\pi^2\}$ which contains only prime closed geodesics.

A second difficulty is that $O(2)$ does not act freely on $\Lambda(M)$, which was the main source of errors in previous papers. It turns out that critical point theory can be applied to the space $P(M)$ of curves which no element of $O(2)$ except the identity leaves fixed. The necessary topological informations about the quotient $P(M)/O(2)$ can now be obtained using Gysin sequences and characteristic classes.

In the geometric part of the proof of Theorem A and B it is shown that the relevant topological cycles consisting of great and small circles on S^n can be deformed into $P^{8\pi^2-}/O(2)$. In Theorem B this is implied by the assumption $g \leqslant \alpha^2 g_0$, but in Theorem A this is the main difficulty and will be overcome by using a technique from [15] and [16] and a lemma, which for $\delta > 1/4$ gives a somewhat different proof of the sphere theorem.

We also prove the following theorems:

Theorem C. *Suppose M^n is compact, simply connected, and a Z_2 homology sphere. Assume that $1/p^2 < \delta \leqslant K \leqslant 1$ for some integer $p \geqslant 2$ and that $i(M) \geqslant \pi$ where $i(M)$ is the injectivity radius. Then there exists at least $](n-1)/(p-1)[$ closed geodesics on M with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,] \subset [2\pi, 2p\pi)$, where $]x[$ denotes the smallest integer $\geqslant x$. In particular, if $1/(n-1)^2 < K \leqslant 1$ and $i(M) \geqslant \pi$, then M has at least two such closed geodesics.*

Theorem D. *Suppose M^n is compact, simply connected, and not a Z_2 homology sphere. Assume that $1/16 < \delta \leqslant K \leqslant 1$, $i(M) \geqslant \pi$, and let $k = \min\{i > 0 \mid H_i(M, Z_2) \neq 0\}$. Then there exists at least k closed geodesics without self-intersections and with lengths in $[2\pi, \pi/\sqrt{\delta}\,] \subset [2\pi, 4\pi)$. If $3k < n+2$, then there exist at least $g(k)$ such closed geodesics.*

In particular, for 1/16-pinched metrics on P^mC and P^mH, $m \geqslant 3$, one obtains at least three, resp. seven closed geodesics. 1/16-pinched metrics on P^2C, P^2H, P^2Ca have at least two, four, resp. eight closed geodesics.

We also prove a theorem similar to Theorem B for these projective spaces. If g_0 denotes the standard metric on these spaces with $1/4 \leqslant K \leqslant 1$, then any metric with $g < 4g_0$ and $l(M) \geqslant 2\pi$ has at least three, seven, resp. fifteen closed geodesics. Note, however, that one could expect more closed geodesics from perturbation theory.

In Theorem C and D the Morse-Schoenberg comparison theorem and Gysin sequences are used to bring the relevant cycles under the appropriate energy levels.

Finally we examine the stability properties of these closed geodesics. The Morse indices of the closed geodesics in Theorem A lie in $[n-1, 3(n-1)]$. Unless the Morse index is $2(n-1)$ a closed geodesic turns out to be nonhyperbolic if the metric is sufficiently pinched. In fact, if $\operatorname{ind}(c) \leqslant k$ and $n-1 \leqslant k < 2(n-1)$, or if $\operatorname{ind}_0(c) = \operatorname{ind}(c) + \operatorname{null}(c) \geqslant k$ and $2(n-1) < k \leqslant 3(n-1)$, then the linearized Poincaré map has at least $2(n-1) - k$ resp. $k - 2(n-1)$ Jordan blocks with eigenvalues on the unit circle if the metric is sufficiently pinched. In particular, if $\operatorname{ind}(c) \leqslant n-1$ or $\operatorname{ind}_0(c) \geqslant 3(n-1)$, then c is of elliptic-parabolic type if the metric is sufficiently pinched. We obtain

Theorem E. *If $((2n-2)/(2n-1))^2 \leqslant \delta \leqslant K \leqslant 1$, then there exist at least $g(n) - 1$ nonhyperbolic closed geodesics without self-intersections and with lengths in $[2\pi, 2\pi/\sqrt{\delta}]$. If $n \neq 2^k$, then there exist at least $g(n)$ such nonhyperbolic closed geodesics. If all closed geodesics of length $\leqslant 4\pi$ are nondegenerate, then there exist at least $]n^2/2[$ such nonhyperbolic closed geodesics.*

In §1 we explain Lusternik-Schnirelemann theory in a setting which suits our purposes. We then show that this theory can be applied to $P(M)$. In §2 we prove topological results which are needed in later sections. In §3 we prove Theorems B, C, and D. In §4 we prove Theorem A, and in §5 we discuss properties of the linearized Poincaré map and prove Theorem E.

Throughout the paper the coefficient ring of all homology and cohomology groups is Z_2.

We are grateful to D. Epstein for helpful discussions, and would like to thank the University of Pennsylvania and the University of Bonn for their hospitality.

Some of the results in this paper were announced in [7].

1. Critical point theory

Closed geodesics and point curves of a Riemannian manifold M are the critical points of the energy functional

$$E(c) = \frac{1}{2}\int_0^1 \langle \dot{c}, \dot{c} \rangle$$

defined on the Hilbert manifold

$$\Lambda = \Lambda(M) = \left\{ c : S^1 \to M \mid c \text{ is absolutely continuous and } \int_0^1 \langle \dot{c}, \dot{c} \rangle < \infty \right\},$$

where $S^1 = R/Z = [0,1]/\{0,1\}$. Critical point theory, therefore, is an important tool in estimating the number of closed geodesics.

Lusternik-Schnirelmann theory

Suppose X is a metric space, $F: X \to r$ a continuous function bounded from below, and $\phi: X \times [0, \infty) \to X$ a continuous deformation such that $F(\phi(x, t)) \leqslant F(x)$ for all $x \in X$ and $t \geqslant 0$. For any subset Y of X and $\kappa \in R$ set $Y^\kappa = \{x \in Y \mid F(x) \leqslant \kappa\}$ and $Y^{\kappa-} = \{x \in Y \mid F(x) < \kappa\}$. The set $K = \{x \in X \mid \phi(x, t) = x$ for all $t \geqslant 0\}$ will be called the set of *critical points*. Assume that

$$(*)\qquad \begin{array}{l}\text{for any } \kappa \text{ and for any neighborhood } U \text{ of } \{x \in K \mid F(x) = \kappa\}\\ \text{there exists a constant } \varepsilon > 0 \text{ such that } F(x) - F(\phi(x, 1)) \geqslant 2\varepsilon\\ \text{for all } x \in X^{\kappa+\varepsilon} - (X^{\kappa-\varepsilon} \cup U).\end{array}$$

Let $Z \subset Y \subset X$ and assume Y and Z are invariant under the deformation ϕ. Suppose $h \in H_*(Y, Z)$ is nonzero. Set

$$(1.1)\qquad \kappa(h) = \inf_{z \in h} \max_{x \in |z|} F(x).$$

$\kappa(h)$ is called the critical level of h.

Note that we do not demand that $K \subset Y$. This will be used in the applications in §3. For many applications, e.g., in the proofs of Theorems A and B, it is sufficient to assume $K \subset Y$.

(1.2) Let $h \in H_*(Y, Z)$ be nonzero. Then for any neighborhood U of $\{x \in K \mid F(x) = \kappa(h)\}$ in X there exist a $z \in h$ and an $\varepsilon > 0$ such that $|z| \subset Y^{\kappa(h)-\varepsilon} \cup (U \cap Y)$. In particular, there exists a critical point x with $F(x) = \kappa(h)$.

Proof. By continuity of the deformation ϕ there exists a neighborhood V of $\{x \in K \mid F(x) = \kappa(h)\}$ such that $\phi(x, 1) \in U$ for all $x \in V$. By $(*)$ there exists an $\varepsilon > 0$ such that $F(x) - F(\phi(x, 1)) \geqslant 2\varepsilon$ for all $x \in X^{\kappa(h)+\varepsilon} - (X^{\kappa(h)-\varepsilon} \cup V)$. The definition of $\kappa(h)$ implies that there exists a $z_0 \in h$ with $|z| \subset Y^{\kappa(h)+\varepsilon}$. Since ϕ is continuous and Y and Z are invariant, $z = \phi(z_0, 1)$ is also a cycle in h, and we have $|z| \subset Y^{\kappa(h)-\varepsilon} \cup (U \cap Y)$. q.e.d.

It is not true in general that linearly independent homology classes give rise to different critical points. There exists for example a differentiable function on the two-torus with only three critical points, although the sum of Betti numbers is equal to four.

Suppose that $h_1, h_2 \in H_*(Y, Z)$ are nonzero, and that there exists a cohomology class $\omega \in H^*(Y)$, $* \geqslant 1$, such that $\omega \cap h_2 = h_1$. We say that h_1 *is subordinate to* h_2.

(1.3) If $\omega \cap h_2 = h_1$, then $\kappa(h_2) \geqslant \kappa(h_1)$, and equality implies that the restriction of ω to $U \cap Y$ is nonzero, where U is any neighborhood of $\{x \in K \mid F(x) = \kappa(h_2) = \kappa(h_1)\}$ in X.

Proof. $\kappa(h_2) \geqslant \kappa(h_1)$ since any cycle in h_2 contains a cycle of h_1 by the definition of the cap product, see [25, p. 254]. Suppose $\kappa(h_2) = \kappa(h_1) = \kappa$. Let $i: U \cap Y \to Y$ and $j: Y \to (Y, U \cap Y)$ be the inclusions. If $i^*(\omega) = 0$, then $\omega = j^*(\eta)$. This implies that there is a cocycle Z in ω such that $Z(\sigma) = 0$ for all simplices σ with $|\sigma| \subset U \cap Y$. By (1.2) there exists a cycle $z \in h_2$ such that $|z| \subset Y^{\kappa-\varepsilon} \cup (U \cap Y)$ for some $\varepsilon > 0$. Choose a subdivision z' of z such that any simplex of z' is either contained in $Y^{\kappa-}$ or in $U \cap Y$. Now $Z \cap z'$ is a cycle in h_1 with support $|Z \cap z'| \subset Y^{\kappa-}$. This is a contradiction to the definition of $\kappa = \kappa(h_1)$. q.e.d.

A sequence of homology classes $h_1, \cdots, h_k$ of (Y, Z) is called a *chain of subordinate homology classes* if h_i is subordinate to h_{i+1}, $1 \leqslant i \leqslant k-1$; k is called the length of the chain. We say that (Y, Z) *contains k subordinate homology classes* if there exists a chain of subordinate homology classes of (Y, Z) of length k.

We want to point out that in our version of Lusternik-Schnirelmann theory the equality $\kappa(h_i) \equiv \kappa(h_{i+1})$ of critical levels does not in general imply the existence of infinitely many critical points on this level. It may happen that there exists only one critical point x and the restriction of the corresponding cohomology class to $U \cap Y$ is nonzero for any neighborhood U of x, compare (2.5). However, if Y is locally contractible and $K \subset Y$, then $\omega \mid U \cap Y = 0$ for any sufficiently small neighborhood U of $x \in K$. From (1.2) and (1.3) we get immediately the usual consequence of Lusternik-Schnirelmann theory:

(1.4) If Y is locally contractible and $K \subset Y$, and if (Y, Z) contains k subordinate homology classes, then there exist at least k critical points.

Applications to the space of closed curves

$\Lambda(M)$ is a Hilbert manifold with a natural Riemannian metric. The energy functional E on $\Lambda(M)$, M compact, is C^∞ and satisfies condition (C) of Palais and Smale. As a consequence, E and the flow ϕ generated by $-\text{grad } E$ satisfy condition $(*)$. Here K corresponds to the critical points of E in the usual sense, i.e., closed geodesics and the point curves. For general facts about $\Lambda(M)$ see [14] and [20].

$O(2)$ acts on $\Lambda(M)$: for, $c \in \Lambda(M)$ and $\chi \in O(2)$ set $(\chi c)(t) = c(\chi t)$, where $O(2)$ acts on S^1 as usual. This action of $O(2)$ is only continuous, but each element of $O(2)$ acts by an isometry. Hence E and grad E are invariant under $O(2)$, and for any subgroup $\Delta \subset O(2)$ we get an induced function $E/\Delta: \Lambda/\Delta \to R$ and an induced flow ϕ/Δ which satisfies condition $(*)$ of Lusternik-Schnirelmann theory, since $(*)$ is satisfied by E and ϕ. Λ/Δ is also locally contractible for any Δ in $O(2)$, but Λ/Δ is not a Hilbert manifold since the action of $O(2)$ on $\Lambda(M)$ is neither free nor smooth. The fact that $O(2)$ does not act freely is the main difficulty in computations of $H_*(\Lambda/\Delta)$. In fact, $H_*(\Lambda/O(2))$ is unknown, even for $M = S^n$. For any subset $X \subset \Lambda(M)$ we will denote the quotient $X/O(2)$ by $\overline{X}$, in particular $\overline{\Lambda} = \Lambda/O(2)$.

For each $c \in \Lambda$ we define the iterates c^q, $q \in Z$, by $c^q(t) = c(qt)$. c is called prime if $c = d^q$ implies $q = \pm 1$. The main difficulty in the applications of critical point theory to the energy functional arises from the fact that c^q is a critical point if c is, but is considered geometrically the same as c. Thus (1.4) cannot be used directly to show the existence of more than one closed geodesic.

Let $c \in \Lambda(M)$ be a curve which is fixed under some nontrivial element $\chi \in O(2)$, i.e., $\chi c = c$. If c is not a point curve, and $\chi \in SO(2)$, then c is an iterate $c = d^q$, $q > 1$. If $\chi \in O(2) \setminus SO(2)$, then c satisfies $c = c^{-1}$ or some reparameterization $\bar{\chi} c$, $\bar{\chi} \in SO(2)$, satisfies $(\bar{\chi} c)^{-1} = \bar{\chi} c$.

Let $P = P(M)$ be the set of curves $c \in \Lambda(M)$ such that no element of $O(2)$, except the identity, keeps c fixed. Let $V = V(M) = \{c \in \Lambda(M) \mid \lim_{t\to\infty} E(\phi_t c) = 0\}$. P and V are invariant under the gradient flow of the energy functional and the action of $O(2)$. Both P and V are also open, and hence P/Δ and V/Δ are locally contractible for any subgroup Δ of $O(2)$. Note that the energy flow retracts V/Δ into V^κ/Δ for any $\kappa \geqslant 0$. In the next section we will prove that $(\overline{P}, \overline{V} \cap \overline{P})$ contains the appropriate number of subordinate homology classes.

1.5. Lemma. *Let L_0 be the length of a shortest closed geodesic on M. Assume that $(\overline{P}, \overline{V} \cap \overline{P})$ contains a chain $h_1, \cdots, h_k$ of subordinate homology classes with $\omega_i \cap h_{i+1} = h_i$, $\omega_i \in H^*(\overline{P})$.*

(i) *If for every closed geodesic $c \in \overline{\Lambda}$ and any sufficiently small neighborhood U of c in $\overline{\Lambda}$ we have $\omega_i \mid U \cap \overline{P} = 0$, then there exist at least k/m geometrically different closed geodesics on M with lengths in $[L_0, \sqrt{2\kappa(h_k)}\,]$, where $m \leqslant \sqrt{2\kappa(h_k)}\,/L_0 < m+1$.*

(ii) *If $\kappa(h_k) < 2L_0^2$, then there exist at least k geometrically different closed geodesics on M with lengths in $[L_0, \sqrt{2\kappa(h_k)}\,]$. These closed geodesics do not have self-intersections if $\kappa(h_k) < 2l(M)^2$, where $l(M)$ is the length of a shortest geodesic loop.*

Proof. (i) If $c \in \overline{\Lambda}$ has energy $< L_0^2/2$, then $c \in \overline{V}$ since there is no closed geodesic of energy $< L_0^2/2$. Hence $\kappa(h_i) \geqslant L_0^2/2$ since the h_i are nonzero homology classes. If $\kappa(h_1) < \kappa(h_2) < \cdots < \kappa(h_k)$, then there are k distinct critical points of energy $\kappa(h_1), \cdots, \kappa(h_k)$. These critical points are prime or iterate closed geodesics. Since a closed geodesic has length $\geqslant L_0$ and $(m+1)L_0 > \sqrt{2\kappa(h_k)}$, at least k/m of these are geometrically different. If $\kappa(h_i) = \kappa(h_{i+1})$ for some i, then there exist infinitely many closed geodesics of energy $\kappa(h_i)$. Otherwise there would exist a neighborhood U of $\{c \in \overline{\Lambda} \mid c \text{ is a closed geodesic with } E(c) = \kappa(h_i)\}$ in $\overline{\Lambda}$ such that $\omega_i \mid U \cap \overline{P} = 0$ by the assumption made on ω_i. But this would contradict (1.3).

(ii) If $c \in \overline{\Lambda}$ is a closed geodesic with $E(c) < 2L_0^2$, then the length of c is $< 2L_0$. It follows that c is contained in $\overline{P}$. Since $\overline{P}$ is open and locally contractible, the first statement in (ii) follows from (1.4). Since a closed geodesic with a self-intersection is the union of two geodesic loops, and since a geodesic loop has length $\geqslant l(M)$, the second statement is clear. q.e.d.

If c is a closed geodesic, $O(2) \cdot c$ is a submanifold of $\Lambda(M)$. c is called *nondegenerate* if its orbit $O(2) \cdot c$ is a nondegenerate critical submanifold of $E: \Lambda(M) \to R$. The index of c (resp. nullity of c), denoted by $\operatorname{ind}(c)$ (resp. $\operatorname{null}(c)$) is equal to the index (resp. nullity $-$ 1) of c as a critical point of $E: \Lambda \to R$. We also set $\operatorname{ind}_0(c) = \operatorname{ind}(c) + \operatorname{null}(c)$.

1.6. Lemma. *Let L_0 be the length of a shortest closed geodesic on M, and let κ satisfy $\kappa \leqslant 2L_0^2$. Assume that all closed geodesics of energy $< \kappa$ are nondegenerate. If there exist k linearly independent homology class $h_1, \cdots, h_k$ in $H_*(\overline{P}^{\kappa-}, \overline{V} \cap \overline{P}^{\kappa-})$ of dimensions $r_1, \cdots, r_k$ respectively, then there exist k distinct closed geodesics $c_1, \cdots, c_k$ with lengths in $[L_0, \sqrt{2\kappa})$, and* $\operatorname{ind}(c_i) = r_i$. *These closed geodesics do not have self-intersections if $\kappa \leqslant 2l(M)^2$.*

Proof. Since $P^{\kappa-}$ contains only nondegenerate critical circles, there are only finitely many critical levels which we denote by α_i and on each level only finitely many critical circles. Hence there exists an $\varepsilon > 0$ such that all critical points in $P^{\alpha_i+\varepsilon} - P^{(\alpha_i-\varepsilon)-}$ have energy α_i. Let $O(2) \cdot \gamma_{ij}$ be the finitely many critical orbits of energy α_i and let T_{ij} be a submanifold of Λ transversal to $O(2) \cdot \gamma_{ij}$ at γ_{ij}. Then γ_{ij} is a nondegenerate critical point of $E \mid T_{ij}$. Let U_{ij} be a Morse chart for $E \mid T_{ij}$ around γ_{ij}. Then we have

$$H_m\left(U_{ij}^{\alpha_i+\varepsilon}, U_{ij}^{\alpha_i-\varepsilon}\right) = \begin{cases} 0, & m \neq \operatorname{ind}(\gamma_{ij}), \\ Z_2, & m = \operatorname{ind}(\gamma_{ij}). \end{cases}$$

But since all closed geodesics in $P^{\kappa-}$ are prime, the projection $U_{ij} \to \overline{U}_{ij}$ is a homeomorphism onto a neighborhood of $\overline{\gamma}_{ij}$ in $\overline{P}^{\kappa-}$ if U_{ij} is sufficiently small. $\kappa \leqslant 2L_0^2$ implies as in (1.5) that a closed geodesic in $\Lambda^{\kappa-}$ is contained in $P^{\kappa-}$. Hence $(*)$ is satisfied on $X = P^{\kappa-}$ and therefore also on $\overline{P}^{\kappa-}$. Since the gradient flow on $P^{\kappa-}$ induces a flow on $\overline{P}^{\kappa-}$, this can be used to show that

$$H_m(\overline{P}^{\alpha_i+\varepsilon}, \overline{P}^{\alpha_i-\varepsilon}) = \bigoplus_j H_m\left(\overline{U}_{ij}^{\alpha_i+\varepsilon}, \overline{U}_{ij}^{\alpha_i-\varepsilon}\right).$$

Using the exact homology sequence of a triple as in [24, §5], it follows that

$$\dim H_m(\overline{P}^{\kappa-}, \overline{P}^{\varepsilon}) \leqslant \sum_{i,j} \dim H_m\left(\overline{U}_{ij}^{\alpha_i+\varepsilon}, \overline{U}_{ij}^{\alpha_i-\varepsilon}\right).$$

It follows from the definition of V that the gradient flow retracts $\bar{V} \cap \bar{P}^{\kappa-}$ onto $\bar{P}^{\varepsilon}$. This proves (1.6).

§2. Topological results

The space of biangles on the sphere

Let S^n be the n-sphere with the metric of constant curvature 1. We assume that S^n is embedded in R^{n+1} in the usual way, i.e., $S^n = \{x \in R^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\}$. A great circle is a closed geodesic of length 2π. The set G of all great circles is a closed subset of $\Lambda(S^n)$ and invariant under the action of $O(2)$.

$\bar{G} = G/O(2) \subset \bar{\Lambda}(S^n)$ is called the set of unparametrized great circles. A great circle $c \in G$ is determined by the orthonormal two-frame $(c(0), c(1/4))$ in R^{n+1}. With respect to this identification $\bar{G}$ is isomorphic to the Grassmannian of unoriented two-planes in R^{n+1}. The homology and cohomology of $\bar{G}$ with Z_2 coefficients can be described as follows:

For any pair of integers a, b such that $0 \leqslant a \leqslant b \leqslant n-1$, the set of unparametrized great circles contained in $S^{b+1} = \{x \in S^n \mid x_{b+2} = \cdots = x_n = 0\}$ which meet the subsphere $S^a = \{x \in S^n \mid x_{a+1} = \cdots = x_n = 0\}$ is the carrier of a cycle $[a, b]$ of dimension $a + b$. The homology class of $[a, b]$ is also denoted by $[a, b]$. The set of all $[a, b]$, $0 \leqslant a \leqslant b \leqslant n-1$, $a + b = k$, is a basis of $H_k(\bar{G})$. The dual class of $[a, b]$ with respect to this basis is denoted by (a, b). Note that the number of linearly independent homology classes is $n(n+1)/2$.

Define $s \geqslant 0$ by $2^k + s = n < 2^{k+1}$, and set $g(n) = 2n - s - 1 = n + 2^k - 1$. Using formulas of Chern [12] for the cup product of $\bar{G}$, Alber showed that $(0,1)^{2n-2s-2} \cup (1,1)^s = (n-1, n-1)$, see e.g., [20, p. 50]. Hence the sequence of cohomology classes ω_i, $1 \leqslant i \leqslant g(n) - 1$, such that $\omega_i = (0,1)$ for $1 \leqslant i \leqslant 2n - 2s - 2$ and $\omega_i = (1,1)$ otherwise, gives rise to a chain of subordinate homology classes $h_1, \cdots, h_{g(n)}$, inductively defined by $h_{g(n)} = [n-1, n-1]$, $\omega_i \cap h_{i+1} = h_i$. Since $H^1(\bar{G}) = Z_2$ and $(0,1)^{2n-2s-1} = 0$, any chain of subordinate homology classes has length $\leqslant g(n)$. The chain of the h_i's can be altered by changing the ordering of the ω_i's.

Set $\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2)$. We denote both the action of θ on Λ and its restriction to G by θ. Notice that $\theta c = c^{-1}$. The S^1 bundle $G/\theta \to \bar{G}$ is the sphere bundle of the canonical two-dimensional vector bundle over $\bar{G}$. Hence we obtain the following which will be crucial later on:

(2.1) The Stiefel-Whitney classes of the S^1 bundle $G/\theta \to \bar{G}$ generate the cohomology ring of $\bar{G}$. The first Stiefel-Whitney class is $(0, 1)$, and the second is $(1, 1)$; see [12].

A circle on S^n is an injective curve $c: S^1 \to S^n$, parametrized proportional to arc-length, such that $\mathrm{im}(c) = S^n \cap \sigma$, where σ is a two-plane in R^{n+1} with $d(\sigma, 0) < 1$. The set C of all circles is a subset of $\Lambda(S^n)$ which is invariant under $O(2)$ and contains G. $\overline{C} = C/O(2)$ is called the space of unparametrized circles.

For any two-plane σ in R^{n+1} there is a unique parallel two-plane σ' such that $0 \in \sigma'$. This defines for any circle c on S^n a unique great circle $\alpha(c)$ on S^n parallel to c. $\alpha: C \to G$ is $O(2)$-equivariant and an open $(n-1)$-disc bundle. For any $\varepsilon \in (0, 2)$ let C^ε be the subset of C consisting of circles c such that $\| c(0) - c(1/2) \| \leqslant \varepsilon$. $\bar{\alpha}: \overline{C} \to \overline{G}$ is an open $(n-1)$-disc bundle, and one can use the Thom isomorphism T to compute the homology and cohomology of $(\overline{C}, \overline{C}^\varepsilon)$. Define $\{a, b\}$ to be the unique homology class in $H_*(\overline{C}, \overline{C}^\varepsilon)$ such that $T\{a, b\} = [a, b]$, and set $(a, b)' = \bar{\alpha}^*(a, b)$. From the definition of T (see [25, p. 259]) follows $(a, b)' \cap \{c, d\} = \Sigma \{e_i, f_i\}$ if and only if $(a, b) \cap [c, d] = \Sigma [e_i, f_i]$. Hence $(\overline{C}, \overline{C}^\varepsilon)$ contains a chain of $g(n)$ subordinate homology class whose dimensions lie in $[n-1, 3(n-1)]$. It follows from (2.1) that $(0, 1)'$ and $(1, 1)'$ are the Stiefel-Whitney classes of the S^1-bundle $C/\theta \to \overline{C}$.

Remark. Let $i: (\overline{C}, \overline{C}^\varepsilon) \to (\overline{\Lambda} - \overline{\Lambda}^0, \overline{\Lambda}^\delta - \overline{\Lambda}^0)$ and $j: \overline{C} \to \overline{\Lambda} - \overline{\Lambda}^0$ be the inclusions, where $\overline{\Lambda} = \overline{\Lambda}(S^n)$, $\overline{\Lambda}^\delta$ is defined with respect to the energy of some metric, and $\varepsilon, \delta > 0$ are sufficiently small. The lemma of Alber states that i_* is injective and carries subordinate classes into subordinate classes, see [2] and (2.3.5) in [20]. This is equivalent to the surjectivity of j^* and $i_*\{0, 0\} \neq 0$. On the other hand, it follows as in the proof of (2.2) below that $\Lambda - \Lambda^0 \to \Lambda$ is a weak homotopy equivalence. Hence $\pi_i(\Lambda - \Lambda^0) \simeq \pi_i(\Lambda)$. The canonical fibration $\Lambda \to M$ has a section and hence $\pi_i(\Lambda) \simeq \pi_i(M) \oplus \pi_{i+1}(M)$. Therefore $\pi_1(\Lambda - \Lambda^0) = 0$ if $n > 3$. Using (6.3) in [10, p. 91], it follows that $\pi_1(\overline{\Lambda} - \overline{\Lambda}^0) = 0$ if $n > 3$. This contradicts the surjectivity of j^* since $(0, 1)$ cannot be in the image of j^*. One can also show that $(1, 1)$ is not in the image of j^* if $n \geqslant 4$. The lemma of Alber becomes true if we replace $\overline{\Lambda} - \overline{\Lambda}^0$ by $\overline{P}$ and $\overline{\Lambda}^\delta - \overline{\Lambda}^0$ by $\overline{P} \cap \overline{\Lambda}^\delta$ or $\overline{P} \cap \overline{V}$. The proof is similar to the proofs of (2.3) and (2.4) below.

In the proofs of Theorems A, C, and D it will be essential to replace the small circles by biangles. This was already used in [19] to push the cycles $\{0, i\}$ under the $8\pi^2$ energy level. We will only consider the space of great circles and biangles with initial point on the equator. It will be possible to control the lengths of these curves in the geometric constructions in Chapter 4. This space is only invariant under the subgroup Γ of $O(2)$ generated by $\theta = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and $\eta = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$, which turns out to be sufficient for our topological considerations. Notice that $\theta c = c^{-1}$ and $(\eta c)(t) = c(t + 1/2)$.

A closed curve c on S^n is called a biangle if $c \,|\, [0, 1/2]$ and $c \,|\, [1/2, 1]$ are half great circles and if $c(1/4) \neq c(3/4)$. Let B be the set of all biangles. B is invariant under the subgroup Γ of $O(2)$ defined above. B contains the space G of great circles. Let B_e be the set of all biangles c such that $c(0)$ lies on the equator $S^{n-1} = \{x \in S^n \mid x_n = 0\}$ and let G_e be the set of great circles in B_e. B_e and G_e are invariant under Γ and Γ operates freely on both spaces. Hence G_e/Γ is a compact manifold since G_e is. We have $\dim G_e/\Gamma = 2(n-1) = \dim \overline{G}$, where $\overline{G} = G/O(2)$ as in Chapter 1. The preimage of an unparametrized great circle in $\overline{G}$ transversal to S^{n-1} with respect to the projection $p : G_e/\Gamma \to \overline{G}$ consists of one point. Hence p has degree 1 mod 2, i.e., p maps the fundamental class $[n-1, n-1]_\Gamma$ of G_e/Γ on the fundamental class $[n-1, n-1]$ of $\overline{G}$.

For any $[a, b]$ there exists a unique cohomology class ω such that $\omega \cap [n-1, n-1] = [a, b]$. Set $[a, b]_\Gamma = p^*\omega \cap [n-1, n-1]_\Gamma$ and $(a, b)_\Gamma = p^*(a, b)$. Then $p_*[a, b]_\Gamma = [a, b]$. In particular, p_* is surjective and p^* is injective. From the naturality of cup and cap product it follows that $(a, b)_\Gamma \cap [c, d]_\Gamma = \Sigma[e_i, f_i]_\Gamma$ if and only if $(a, b) \cap [c, d] = \Sigma[e_i, f_i]$.

For $\varepsilon \geqslant 0$ set $B_e^\varepsilon = \{c \in B_e \mid \|c(1/4) - (3/4)\| \leqslant \varepsilon\}$. A biangle $c \in B_e$ is determined by the triple $(c(0), c(1/4), c(3/4))$ and $c \in G_e$ is determined by $(c(0), c(1/4))$. The map

$$(c(0), c(1/4), c(3/4)) \to \left(c(0), \frac{c(1/4) - c(3/4)}{\|c(1/4) \to c(3/4)\|} \right),$$

defines a Γ equivariant projection $\gamma : B_e \to G_e$. $\gamma/\Gamma : B_e/\Gamma \to G_e/\Gamma$ is an open $(n-1)$-disc bundle. Thus we may use the Thom isomorphism T to identify the homology of $(B_e, B_e^\varepsilon)/\Gamma$ with the homology of G_e/Γ for $0 < \varepsilon < 2$. There is a unique homology class $\{a, b\}_B$ of $(B_e, B_e^\varepsilon)/\Gamma$ such that $T\{a, b\}_B = \{a, b\}_\Gamma$. Set $(a, b)_B = (\gamma/\Gamma)^*(a, b)_\Gamma$. From the definition of the Thom isomorphism and properties of cup and cap product it follows immediately that $(a, b)_B \cap \{c, d\}_B = \Sigma\{e_i, f_i\}_B$ if and only if $(a, b)_\Gamma \cap [c, d]_\Gamma = \Sigma[e_i, f_i]_\Gamma$. Hence $(B_e, B_e^\varepsilon)/\Gamma$ contains a chain of $g(n)$ subordinate homology classes whose dimensions lie in $[n-1, 3(n-1)]$.

Remark. Let $i : (B_e, B_e^\varepsilon)/\Gamma \to (\overline{P}, \overline{V} \cap \overline{P})$ be the inclusion. Our next step will be to show that i_* is injective on the subspace generated by the $\{a, b\}_B$. It is clear that $i_*\{a, b\}_B = j_*\{a, b\}$ where $j : (\overline{C}, \overline{C}^\varepsilon) \to (\overline{P}, \overline{V} \cap \overline{P})$ is the inclusion. In particular, it will follow that j_* is injective and maps subordinate classes into subordinate ones.

On the topology of the space of closed curves

For any subgroup $\Delta \subset O(2)$ set $\Lambda_\Delta = \Lambda_\Delta(M) = \{c \in \Lambda \mid \exists \chi \in \Delta, \chi \neq \mathrm{id}$ such that $\chi c = c\}$. As in section 1, $P = P(M) = \Lambda(M) - \Lambda_{O(2)}(M)$.

2.2. Lemma. *Suppose $U \subset \Lambda$ is open. Then the inclusion $U - \Lambda_\Delta \to U$ is a weak homotopy equivalence for every subgroup Δ of $O(2)$.*

Proof. We have to show that the inclusion induces isomorphisms on all homotopy groups. Suppose $f\colon S^m \to U$ is a continuous map. Since $\operatorname{im}(f)$ is compact, it is contained in some $\Lambda^{\kappa -}$, $\kappa < \infty$.

Let k be an integer such that $k > 2\kappa/i(M)^2$. There is a deformation retraction $r_u(k)\colon \Lambda^{\kappa-} \to \Lambda^{\kappa-}$, $0 \leqslant u \leqslant 1$, which deforms $\Lambda^{\kappa-}$ into the finite dimensional approximation $\Lambda(k, \kappa)$, see [24, §16]. Here $\Lambda(k, \kappa)$ consists of geodesic polygons $c\colon S^1 \to M$ with breaks in i/k, $0 \leqslant i \leqslant k$, such that $E(c) < \kappa$. For all $0 \leqslant u \leqslant 1$, $\lim_{k\to\infty} r_u(k) \circ f = f$ by the definition of r, hence $\operatorname{im}(r_u(k) \circ f) \subset U$ independently of u for k sufficiently large. Then $r_1(k) \circ f\colon S^m \to U$ is homotopic to f and $\operatorname{im}(r_1(k) \circ f) \subset \Lambda(k, \kappa)$.

We will now show that $\Lambda_\Delta \cap \Lambda(k, \kappa)$ is the union of finitely many submanifolds of codimension $\geqslant k\dim(M)/2$ if k is chosen appropriately. Since every closed geodesic has length $\geqslant 2i(M)$, we can choose k such that the multiplicity of every closed geodesic in $\Lambda(k, \kappa)$ divides k. Then, if $c \in \Lambda_\Delta \cap \Lambda(k, \kappa)$, there exists a $\chi \in \Delta$ with $\chi c = c$ and $\chi \in \Delta \cap \mathbf{Z}_k$ or $\chi \in \Delta \cap \theta\mathbf{Z}_k$. Therefore $\chi(\Lambda(k, \kappa)) \subset \Lambda(k, \kappa)$, and the fixed point set of $\chi \mid \Lambda(k, \kappa)$ is a submanifold of codimension $\geqslant k\dim(M)/2$. We can now choose k such that this codimension is also greater than $m + 1$. It then follows from transversality arguments that $r_1(k) \circ f$ can be deformed away from Λ_Δ. This shows that the inclusion $U - \Lambda_\Delta \to U$ induces surjective maps on the homotopy groups. Similarly one shows that it induces injections. q.e.d.

A C^∞ map $f\colon S^k \to M$ induces a continuous map $B(f)\colon B \to \Lambda$. If $B(f)(B_e) \subset P$, then we obtain a continuous map

$$B(f)\colon (B_e, B_e^\varepsilon) \to (P, V \cap P)$$

for ε sufficiently small, and a map

$$B(f)/\Gamma\colon (B_e, B_e^\varepsilon)/\Gamma \to (P, V \cap P)/\Gamma.$$

If $\pi\colon B/\Gamma \to \bar{P}$ is the projection, we denote $\pi \circ B(f)/\Gamma$ by $\bar{B}(f)\colon (B_e, B_e^\varepsilon)/\Gamma \to (\bar{P}, \bar{V} \cap \bar{P})$.

2.3. Lemma. *Suppose M is simply connected and $H_i(M) = 0$ for $0 < i < k$. If $f\colon S^k \to M$ is differentiable, Z_2-homologically nontrivial, and $B(f)(B_e) \subset P$, then $\bar{B}(f)_*\{0, 0\}_B \neq 0$.*

Remark. One can always approximate a map $g\colon S^k \to M$ by a C^∞ map f such that $f \simeq g$ and such that f is injective on an open set, which, after a reparameterization, contains an open hemisphere of S^k. Then $B(f)(B_e) \subset P$ is satisfied.

Proof. Under the projection $(B_e, B_e^\varepsilon) \to (B_e, B_e^\varepsilon)/\Gamma$, $\{0, 0\}_B$ is the image of a homology class Z which is represented by the fundamental class of (F, F^ε), where F is a fibre of the $(n-1)$-disc bundle $\gamma: B_e \to G_e$ defined above. Let $S^k = I^k/\partial I^k$, where $I = [0, 1]$. To the C^∞ map f we associate a continuous map $F_f: (I^{k-1}, \partial I^{k-1}) \to (\Lambda, p)$ by $F_f(x)(t) = f(x, t)$. We also denote by $F_f \in H_{k-1}(\Lambda)$ the image of F_f under the Hurewicz homomorphism. By (2.2) the inclusions $P \to \Lambda$ and $V \cap P \to V$ are weak homotopy equivalences, and under the isomorphisms $H_{k-1}(\Lambda) \xrightarrow{\sim} H_{k-1}(\Lambda, V) \xleftarrow{\sim} H_{k-1}(P, V \cap P)$, F_f represents $B(f)_* Z$.

The S^1 action on Λ gives rise to natural homomorphism $H_*(A) \to H_{*+1}(A)$ for any $O(2)$ invariant subset A of Λ. The image of a homology class h is denoted by $S^1 h$. Let $\rho: \Lambda \to \Lambda/\theta$ be the projection and $e: \Lambda/\theta \to M$ the evaluation $c \to c(0)$. Since $e_* \rho_* S^1 F_f$ is equal to the homology class of f, it follows that $\rho_* S^1 F_f \neq 0$ in $H_k(\Lambda/\theta)$.

Since $H_{k-1}(P) \xrightarrow{\sim} H_{k-1}(P, V \cap P)$, there exists a homology class X in $H_{k-1}(P)$ which represents $B(f)_* Z$. Let $p: P \to \bar{P}$ be the projection and $\tau: H_*(\bar{P}) \to H_{*+1}(P/\theta)$ be the transfer of the Gysin sequence of the S^1 bundle $q: P/\theta \to \bar{P}$. It follows from the definition of τ that $\tau p_* X = \rho_* S^1 X$. Since $\rho_* S^1 X$ is not zero in $H_k(\Lambda/\theta)$, it is a fortiori not zero in $H_k(P/\theta)$. Hence $p_* X \neq 0$.

To prove $\bar{B}(f)_* \{0, 0\}_B \neq 0$ in $H_{k-1}(\bar{P}, \bar{V} \cap \bar{P})$ it now suffices to show that $p_* X$ is not in the image of $H_{k-1}(\bar{V} \cap \bar{P}) \to H_{k-1}(\bar{P})$. Now $H_i(V \cap P) \xrightarrow{\sim} H_i(V) \xleftarrow{\sim} H_i(\Lambda^0)$, and since $\Lambda^0 \simeq M$ we get $H_i(V \cap P) = 0$ for $0 < i < k$. Let Q be the set of all $c \in V \cap P$ such that $\lim_{t \to \infty} \phi_t c$ is a point curve in some fixed open convex ball in M. By (2.2) we have $H_*(Q) = 0$ for $* \geqslant 1$, $H_0(Q) = \mathbf{Z}_2$. Q is invariant under the gradient flow of E and the action of $O(2)$. From the Gysin sequences associated to $Q \to Q/\theta$, $P \to P/\theta$ and $Q/\theta \to \bar{Q}$, $P/\theta \to \bar{P}$ it follows that $H_i(\bar{Q}) \to H_i(\bar{V} \cap \bar{P})$ is an isomorphism for $i \leqslant k-1$. Hence $p_* X$ is in the image of $H_{k-1}(\bar{Q}) \to H_{k-1}(\bar{V} \cap \bar{P})$ if it is in the image of $H_{k-1}(\bar{V} \cap \bar{P}) \to H_{k-1}(\bar{P})$. But then $\tau p_* X = \rho_* S^1 X$ is in the image of $H_k(Q/\theta) \to H_k(P/\theta)$ by the naturality of τ. On the other hand, the composition $H_k(Q/\theta) \to H_k(P/\theta) \to H_k(\Lambda/\theta)$ is obviously zero, whereas $\rho_* S^1 X$ is not zero in $H_k(\Lambda/\theta)$.

2.4. Theorem. *Suppose M is simply connected and $H_i(M) = 0$ for $0 < i < k$. If $f: S^k \to M$ is C^∞. Z_2 homologically nontrivial, and $B(f)(B_e) \subset P$, then $\bar{B}(f)_*$ maps the subspace of $H_*((B_e, B_e^\varepsilon)/\Gamma)$ generated by the $\{a, b\}_B$, $0 \leqslant a \leqslant b \leqslant k-1$, injectively, and the image of $\bar{B}(f)^*: H^*(\bar{P}) \to H^*(B_e/\Gamma)$ contains the subspace generated by the $(a, b)_B$, $0 \leqslant a \leqslant b \leqslant k-1$. In particular, $(\bar{P}, \bar{V} \cap \bar{P})$ contains $g(k)$ subordinate homology classes.*

Proof. Since $B(f)(G_e) \subset P$ we have $B(f)(G) \subset P$, and hence f induces a map $\bar{G}(f): \bar{G} \to \bar{P}$. The pullback of the S^1 bundle $P/\theta \to \bar{P}$ is the S^1 bundle

$G/\theta \to \overline{G}$. Since the total Stiefel-Whitney class of the latter bundle generates $H^*(\overline{G})$ by (2.1), $\overline{G}(f)^*$ is surjective by naturality. From the definition of $(a, b)_\Gamma$ it follows that the subspace generated by these is contained in the image of $(\pi \circ G_e(f)/\Gamma)^*$, where $G_e(f): G_e \to P$ is the map induced by f and $\pi: P/\Gamma \to \overline{P}$ is the projection. Since $(\gamma/\Gamma)^*$ is an isomorphism, im $\overline{B}(f)^*$ contains the subspace generated by the $(a, b)_B$. $\overline{B}(f)_*\{0, 0\}_B \neq 0$ by (2.3). Let x be an element in the subspace of $H_*((B_e, B_e^\varepsilon)/\Gamma)$ generated by the $\{a, b\}_B$. It follows from the relation we have with the cap product of elements in $H^*(\overline{G})$ and $H_*(\overline{G})$ that there is a cohomology class ω in the subspace of $H^*(B_e/\Gamma)$ generated by the $(a, b)_B$ such that $\omega \cap x = \{0, 0\}_B$. There is a class $\eta \in H^*(\overline{P})$ such that $\overline{B}(f)^*\eta = \omega$ by what we proved above. Hence we have $\overline{B}(f)_*\{0, 0\}_B = \overline{B}(f)_*(\omega \cap x) = \eta \cap \overline{B}(f)_* x$, and therefore $\overline{B}(f)_* x \neq 0$. The same argument shows that $\overline{B}(f)_*$ maps subordinate classes into subordinate classes. q.e.d.

Let $\omega_1 \in H^1(\overline{P})$ and $\omega_2 \in H^2(\overline{P})$ be the Stiefel-Whitney classes of the S^1 bundle $q: P/\theta \to \overline{P}$. We saw in the proof of (2.4) that the pullbacks $\overline{B}(f)^*\omega_1$ and $\overline{B}(F)^*\omega_2$ generate the subspace of $H_*(B_e/\Gamma)$ generated by the $(a, b)_B$. Therefore, if homology classes h_1 and h_2 in the subspace of $H_*((B_e, B_e^\varepsilon)/\Gamma)$ generated by the $\{a, b\}_B$ are subordinate with respect to a cohomology class in the subspace of $H^*(B_e/\Gamma)$ generated by the $(a, b)_B$, then $\overline{B}(f)_*(h_1)$ and $\overline{B}(f)_*(h_2)$ are subordinate with respect to some polynomial in ω_1 and ω_2. In order to apply (1.5)(i) we have to investigate whether a closed geodesic $\bar{c} \in \overline{\Lambda}$ has a neighborhood $\overline{U}$ such that $\omega_i \mid \overline{U} \cap \overline{P} = 0$.

2.5. Proposition. *Suppose $\bar{c} \in \overline{\Lambda}$ is a closed geodesic.*

(i) *If $\overline{U}$ is a sufficiently small neighborhood of $\bar{c}$ in $\overline{\Lambda}$, then $\omega_1 \mid \overline{U} \cap \overline{P} = 0$*

(ii) *There exists a neighborhood $\overline{U}$ of $\bar{c}$ such that $\omega_2 \mid \overline{U} \cap \overline{P} = 0$ if and only if the multiplicity of c is odd.*

Proof. Let $\overline{U}$ be a neighborhood of a closed geodesic $\bar{c}$ in $\overline{\Lambda}$ which lifts to a tubular neighborhood U of $O(2) \cdot c$ in Λ. U has two connected components W and θW. The S^1 bundle $(U \cap P)/\theta \to \overline{U} \cap \overline{P}$ is equivalent to the oriented S^1 bundle $W \cap P \to \overline{U} \cap \overline{P}$. Hence $\omega_1 \mid \overline{U} \cap \overline{P} = 0$.

It follows from (2.2) that $H^0((U \cap P)/\theta) = H^1((U \cap P)/\theta) = \mathbf{Z}_2$. Hence also $H^0(\overline{U} \cap \overline{P}) = \mathbf{Z}_2$. We consider the Gysin sequence

$$\cdots \to H_2(\overline{U} \cap \overline{P}) \xrightarrow{\omega_2 \cap} H_0(\overline{U} \cap \overline{P}) \xrightarrow{\tau} H_1(W \cap P) \to H_1(\overline{U} \cap \overline{P}) \to 0.$$

Let m be the multiplicity of c. Then $\{\chi c \mid 0 \leqslant \chi \leqslant 1/m\}$ is homeomorphic to S^1 and represents a generator of $H_1(W)$. Hence the S^1 orbit of any $y \in W \cap P$

represents m times a generator of $H_1(W \cap P)$. Since the transfer $\tau: H_0(\overline{U} \cap \overline{P}) \to H_1(W \cap P)$ maps the point class to the fundamental class of the fibre, it follows that τ is an isomorphism if m is odd and zero if m is even. Hence $H_*(\overline{U} \cap \overline{P}) = 0$ if m is odd and $* \geqslant 1$. Therefore also $H^*(\overline{U} \cap \overline{P}) = 0$ in these cases. For m even we obtain $H_*(\overline{U} \cap \overline{P}) = 0$ if $*$ is odd and $H_*(\overline{U} \cap \overline{P}) = \mathbf{Z}_2$ if $*$ is even and $\geqslant 0$. It follows that $H^*(\overline{U} \cap \overline{P}) = \mathbf{Z}_2$ if $*$ is even and $\geqslant 0$, and ω_2^i is a generator of $H^{2i}(\overline{U} \cap \overline{P})$.

Remark. Using similar arguments it is also possible to calculate the integer homology and cohomology of $\overline{U} \cap \overline{P}$. In fact, it is not hard to see that $\overline{U} \cap \overline{P}$ has the homotopy type of a classifying space $B_{\mathbf{Z}_m}$ where m is the multiplicity of c. Similarly $\overline{V} \cap \overline{P}$ has the homotopy type of $M \times B_{O(2)}$, and $\overline{P}$ has the homotopy type of the homotopy quotient $\Lambda \times_{O(2)} E$. This shows that our theory is analogous to the critical point theory on the homotopy quotient, compare [17].

3. Some existence theorems for closed geodesics

3.1. Theorem. *Let (M, g) be a simply connected, compact Riemannian manifold with $l(M) \geqslant 2\pi$, where $l(M)$ is the length of a shortest geodesic loop, and $H_i(M) = 0$ for $0 < i < k$. Suppose there is an injective differentiable map $f: S^k \to M$ which is not $\mathbf{Z}_2$ nullhomologous and satisfies $f^*(g) \leqslant \alpha^2 g_0$ for an $\alpha \in [1, 2)$, where g_0 is the metric of constant curvature 1 on S^k. Then (M, g) has at least $g(k)$ closed geodesics without self-intersections and with lengths in $[2\pi, 2\alpha\pi] \subset [2\pi, 4\pi)$. If all closed geodesics of length $\leqslant 4\pi$ are nondegenerate, then there exist at least $k(k+1)/2$ such closed geodesics.*

If $M = S^n$ we can apply (3.1) to $\mathrm{id}: S^n \to S^n$. If $M = P^mC$, P^mH, of P^2Ca respectively, and g_0 is the canonical metric on M with maximal sectional curvature 1, then M contains totally geodesically embedded spheres S^k of constant curvature 1 which represent a generator of $H_k(M)$, where $k = 2, 4$, and 8 respectively. Since $H_i(M) = 0$ for $0 < i < k$ we get:

3.2. Corollary. *Suppose $M = S^n$, P^mC, P^mH, or P^2Ca respectively. If g is a metric on M such that $l(M) \geqslant 2\pi$ and $g \leqslant \alpha^2 g_0$ for an $\alpha \in [1, 2)$, where g_0 is the standard metric on M with maximal sectional curvature 1, then (M, g) has at least $g(k)$ closed geodesics without self-intersections and with lengths in $[2\pi, 2\alpha\pi]$, where $k = n, 2, 4, 8$ respectively.*

Remarks. (a) The $\mathbf{Z}_2$ Hurewicz theorem [18, p. 305], implies that there exists a map $f: S^k \to M$ which is not $\mathbf{Z}_2$ nullhomologous for $k = \min\{i > 0 \mid H_i(M) \neq 0\}$. f can be chosen to be smooth, but not necessarily injective.

(b) The length of a shortest geodesic loop is a lower-semicontinuous function on the space of metrics with the C^1 topology, and hence an appropriate renormalization of g_0 lies in the C^1 interior of metrics satisfying $g < 4g_0$ and $l(M) \geqslant 2\pi$. Notice that $i(M)$ is only upper-semicontinuous in the C^1 topology (but continuous in the C^2 topology).

Proof of (3.1). The assumptions on f guarantee that $\bar{B}(f): (B_e, B_e^\varepsilon)/\Gamma \to (\bar{P}, \bar{V} \cap \bar{P})$ maps $g(k)$ subordinate classes into subordinate ones by (2.4). Since $f^*(g) \leqslant \alpha^2 g_0$, the critical values of the homology classes $\bar{B}(f)_*\{a, b\}_B$ are $\leqslant 2\alpha^2\pi^2$. The first claim in (3.1) now follows from (1.5)(ii). Since the $k(k+1)/2$ classes $\bar{B}(f)_*\{a, b\}$ are linearly independent homology classes in $(\bar{P}^{2\alpha^2\pi^2}, \bar{V} \cap \bar{P}^{2\alpha^2\pi^2})$, the second claim follows from (1.6).

3.3. Theorem. *Let M be a simply connected, compact Riemannian manifold which is not a $\mathbf{Z}_2$ homology sphere. Let k be the first dimension in which a nontrivial homology class appears.*

(i) *If $1/4p^2 < \delta \leqslant K \leqslant 1$, p some integer $\geqslant 2$, and $i(M) \geqslant \pi$, then there exist at least $]k/(p-1)[$ closed geodesics with lengths in $[2\pi, \pi/\sqrt{\delta}]$. These closed geodesics do not have self-intersections if $\delta > 1/16$.*

(ii) *If $1/16 < \delta \leqslant K \leqslant 1$, $i(M) \geqslant \pi$, and $3(k-1) < n-1$, then there exist at least $g(k)$ closed geodesics without self-intersections and with lengths in $[2\pi, \pi/\sqrt{\delta}] \subset [2\pi, 4\pi)$.*

Remark. The theorem applies to the projective spaces P^mC, P^mH, P^2Ca with $k = 2, 4, 8$ respectively. Here $i(M) \geqslant \pi$ follows from the curvature assumption since dim M is even. The condition $3(k-1) < n-1$ is satisfied for P^mC and P^mH if $m \geqslant 3$.

Proof. It follows from the Poincaré duality that $k \leqslant n/2$. By the $\mathbf{Z}_2$ Hurewicz theorem [18, p. 305], there is a map $f: S^k \to M$ which is $\mathbf{Z}_2$ homologically nontrivial. We can assume that f is smooth and that it is injective on an open subset of S^k, which after a reparametrization contains a closed hemisphere. This prevents a biangle from being mapped into $\Lambda_{O(2)}$ and hence $B(f)(B_e) \subset P$.

There exists an $\varepsilon > 0$ such that $B(f)(B_e^\varepsilon) \subset V$, and hence we get an induced map $\bar{B}(f): (B_e, B_e^\varepsilon)/\Gamma \to (\bar{P}, \bar{V} \cap \bar{P})$ which maps subordinate classes into subordinate ones by (2.4). Hence $(\bar{P}, \bar{V} \cap \bar{P})$ contains $g(k)$ subordinate classes with dimensions in $[k-1, 3(k-1)]$.

We want to show that k of these classes have a representative under the $\pi^2/2\delta$ energy level in the proof of part (i) of the theorem and $g(k)$ in the proof of part (ii).

It follows from [8, (1.8)] that any closed geodesic c with $L(c) > \pi/\sqrt{\delta}$ or $E(c) > \pi^2/2\delta$ has index $\geqslant n-1$. Hence critical point theory implies that

$H_i(\Lambda, \Lambda^{\kappa-}) = 0$ for $i < n-1$, where $\kappa > \pi^2/2\delta$ is arbitrary. E.g., apply [24, p. 121] to a finite dimensional approximation of Λ. Hence by (2.2), $H_i(P, P^{\kappa-}) = 0$ for $i < n-1$. The Gysin sequences of the covering $(P, P^{\kappa-}) \to (P, P^{\kappa-})/\theta$ and of the S^1 bundle $(P, P^{\kappa-})/\theta \to (\bar{P}, \bar{P}^{\kappa-})$ imply $H_i(\bar{P}, \bar{P}^{\kappa-}) = 0$ for $i < n-1$. The homology sequence of the triple $(\bar{P}, \bar{P}^{\kappa-}, \bar{V} \cap \bar{P}^{\kappa-})$ then implies that $H_i(\bar{P}^{\kappa-}, \bar{V} \cap \bar{P}^{\kappa-}) \to H_i(\bar{P}, \bar{V} \cap \bar{P}^{\kappa-})$ is an isomorphism for $i < n-2$ and surjective for $i = n-2$. Since $\bar{V} \cap \bar{P}^{\kappa-}$ is a deformation retract of $\bar{P} \cap \bar{V}$, it follows that any homology class in $H_i(\bar{P}, \bar{V} \cap \bar{P})$, $i < n-1$, has a critical level $< \kappa$, and since $\kappa > \pi^2/2\delta$ was arbitrary, a critical level $\leqslant \pi^2/2\delta$.

We first prove (ii). In this case all $g(k)$ homology classes have critical levels $\leqslant \pi^2/2\delta$ since $3(k-1) < n-1$. Since $\delta > 1/16$ we can apply (1.5)(ii).

In part (i) we can choose k subordinate homology classes in dimension $k-1, \cdots, 2(k-1)$, which are subordinate with respect to the first Stiefel-Whitney class ω_1 of $P/\theta \to \bar{P}$. Indeed, the $g(k)$ subordinate homology class in $(B_e, B_e^{\varepsilon})/\Gamma$ are obtained by applying the one- and two-dimensional cocycles $(0,1)_B$ and $(1,1)_B$ to $\{k-1, k-1\}_B$. Since these cocylces can be applied in an arbitrary order, and since there exist at least k one-dimensional ones among them (see §2), the claim follows. Since $k \leqslant n/2$, we have $2(k-1) < n-1$. Hence these k classes have critical level $\leqslant \pi^2/2\delta$. It follows from (2.5) that (1.5)(ii) applies. This proves (i).

Remark. In (3.1) we had to assume that f is injective, since changing f as in the present proof could violate $f^*(g) \leqslant \alpha^2 g_0$.

If M is simply connected and a $\mathbf{Z}_2$ homology sphere, then the $\mathbf{Z}_2$ Hurewicz theorem implies that there is a map $f\colon S^n \to M$ which is $\mathbf{Z}_2$ homologically nontrivial. With arguments as in the proof of (3.3) we get

3.4. Theorem. *Suppose M is a simply connected compact $\mathbf{Z}_2$ homology sphere. Assume that $1/p^2 < \delta \leqslant K \leqslant 1$, p an integer $\geqslant 2$, and $i(M) \geqslant \pi$. Then M has at least $](n-1)/(p-1)[$ geometrically different closed geodesics with lengths in $[2\pi, 2\pi/\sqrt{\delta}]$. In particular, there exist two such closed geodesics if $\delta > 1/(n-1)^2$ and $i(M) \geqslant \pi$.*

Remarks. (a) Any compact Riemannian manifold has a closed geodesic of length $\leqslant 2\pi/\sqrt{\delta}$ if $K \geqslant \delta > 0$.

(b) (3.4) implies that any metric with $((k-1)/(n-1))^2 < \delta \leqslant K \leqslant 1$ for some integer $k \leqslant n-1$ and $i(M) \geqslant \pi$ has at least k geometrically different closed geodesics with lengths in $[2\pi, 2\pi/\sqrt{\delta}]$.

(c) If also follows from (3.4) that any metric with $1/4 < K \leqslant 1$ has at least $n-1$ closed geodesics. Arguments as in the proof of (3.3)(ii) show that any metric with $9/16 < K \leqslant 1$ has at least $g(n) - 1$ closed geodesics. Using a result of Tsukamoto, see [28], it follows that the proofs also work if $1/4 \leqslant K \leqslant 1$ and $9/16 \leqslant K \leqslant 1$ respectively. Notice that these proofs are, in contrast to the proof of the better result in (4.1), much more elementary.

4. Closed geodesics on 1/4-pinched manifolds

4.1. Theorem. *Suppose M is homeomorphic to S^n and $1/4 \leqslant \delta \leqslant K \leqslant 1$.*

(i) *M has at least $g(n)$ closed geodesics without self-intersections and with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,] \subset [2\pi, 4\pi]$. If all closed geodesics of length $\leqslant 2\pi/\sqrt{\delta}$ are nondegenerate, then there exist at least $n(n+1)/2$ such closed geodesics.*

(ii) *If all closed geodesics with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,]$ have the same length l, then all geodesics are closed of length l. If the closed geodesics with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,]$ have only two different length values, then there exists a family of closed geodesics of equal length in $[2\pi, 2\pi/\sqrt{\delta}\,]$ such that every point of M lies in the image of some geodesic in the family.*

Remark. It is easy to find surfaces of revolution which satisfy the condition in the second part of (ii) with two different length levels. There also exist surfaces of revolution such that all geodesics are closed of the same length l, but K is not constant, see [9]. In particular, there exist perturbations of the standard metric with this property.

The sphere theorem states that M is homeomorphic to S^n if $1/4 < \delta \leqslant K \leqslant 1$ and $\pi_1(M) = 0$. This is proved by showing that one can cover M with two balls: for each point p and a point q furthest away from p, two balls of radius ρ with $\pi/2\sqrt{\delta} < \rho < \pi$ around p and q cover M. One can then cover M with geodesic quadrangles with corners at p, q and on the equator consisting of equidistant points from p and q. This enabled Klingenberg to represent the classes $\{0,0\}, \cdots, \{0, n-1\}$ by curves with energy $\leqslant 2\pi^2/\delta$ if $\delta > 1/4$; see [19].

In [27] Thorbergsson used the fact that one can cover M by balls of radius π around the endpoints of a geodesic segment of length π if $4/9 < K \leqslant 1$. Choosing a fixed ball of radius $\pi/2$ one can then perform the above equator construction for each pair of antipodal points on the boundary of this ball. Thorbergsson attempted to use this construction to push the classes $\{i, n-1\}$, $0 \leqslant i \leqslant n-1$, under the energy level $2\pi^2/\delta$. However, the deformations involved were only Γ equivariant and not $O(2)$ invariant as claimed in [27]. Klingenberg observed that this is still sufficient to bring all the classes $\{a, b\}$ below the energy level $2\pi^2/\delta$ if $\delta > 4/9$; see [21].

To extend Thorbergsson's idea to $1/4 \leqslant \delta \leqslant K \leqslant 1$ we need the following.

4.2. Lemma. *Let* $K \geqslant \delta > 0$ *and* $\gamma : [0, \pi/\sqrt{\delta}] \to M$ *a normal geodesic. Set* $p = \gamma(0)$, $q = \gamma(\pi/\sqrt{\delta})$.

(i) *Any* $r \in M$ *satisfies* $\min\{d(p, r), d(q, r)\} \leqslant \frac{1}{2}\pi/\sqrt{\delta}$.

(ii) *Suppose* $1/4 \leqslant \delta \leqslant K \leqslant 1$ *and* $p \neq q$. *If* $\tilde{\gamma}$ *is a normal geodesic emanating at* p, *then there is precisely one* $t \in (0, 2\pi^2/\delta]$ *such that* $d(p, \tilde{\gamma}(t)) = d(q, \tilde{\gamma}(t))$.

4.3. Lemma. *Suppose* $1/4 \leqslant K \leqslant 1$ *and assume that K is not constant. Given* $\alpha > 0$ *there exist a* $\beta > 0$ *and a constant* a, $0 < a < 1$, *such that for every* p, q *with* $d(p, q) \leqslant \pi + \beta$ *there exists a curve* $c_{pq}(t)$, $0 \leqslant t \leqslant 1$, *from p to q such that* $(p, q, t) \to c_{pq}(t)$ *is continuous, smooth on* $\{(p, q) \mid d(p, q) \leqslant \pi + \beta\} \times [0, a]$ *and on* $\{(p, q) \mid d(p, q) \leqslant \pi + \beta\} \times [a, 1]$, *and such that* $E(c_{pq}) < \pi^2/2 + \alpha$.

Remarks. (a) If $1/4 < \delta \leqslant K \leqslant 1$ and $\pi_1(M) = 0$, then $i(M) \geqslant \pi > \pi/2\sqrt{\delta}$. Then (4.2) implies that M can be covered by two balls around p and q with radius ρ, where $2\pi^2/\delta < \rho < \pi$, which proves that M is homeomorphic to S^n. This proof differs from the usual proofs of the sphere theorem.

(b) In the proof of (4.3) we associate, as in [15] and [16], vector fields to distance functions, such that the associated flow shortens distances. Our presentation follows [15]. The essential observation is that the distance function does not have any critical points (in the sense of [15]) at distance π if $1/4 \leqslant K \leqslant 1$. The curves c_{pq} in (4.3) will be used to replace the minimal geodesics connecting p and q, which are not uniquely defined if $d(p, q) \geqslant \pi$.

Proof of (4.2). (i) Let c_1 be a minimal geodesic joining q and r, and c_2 a minimal geodesic joining r and p. If $\min\{d(q, r), d(r, p)\} > \pi/2\sqrt{\delta}$, then (c_1, c_2, γ) is a generalized δ-triangle of parameter $> 2\pi/\sqrt{\delta}$. This is a contradiction by the Toponogov comparison theorem, see [11].

(ii) This follows from (i) as the corresponding statement follows in the proof of (6.4) in [11].

Proof of (4.3). As in [15] we say that q is a critical point of the function $x \to d(p, x)$ if for every unit vector v at q there exists a minimal geodesic γ from q to p such that $\measuredangle(\dot{\gamma}(0), v) \leqslant \pi/2$. It follows from Klingenberg's injectivity radius estimate that there is no critical point q with $0 < d(p, q) < \pi$. We now prove that there is also none with $d(p, q) = \pi$:

Suppose q is a critical point with $d(p, q) = \pi$. Let C be the set of unit tangent vectors at q tangent to minimal geodesics from q to p. It follows from Lemma 6.9 in [11] that C is convex in the sense that the shortest great circle arc joining $v, w \in S^{n-1}$ is contained in C if $v \neq -w$. Since we assume that K is not constant, there is no closed geodesic of length 2π on M by a result of Tsukamoto; see [28]. Since $i(M) \geqslant \pi$ there also does not exist a geodesic loop of length 2π. Hence $-v \notin C$ if $v \in C$. But a closed convex subset of S^{n-1} containing no antipodal points is easily seen to be contained in an open hemisphere. This is a contradiction since q was assumed to be a critical point. Therefore there exists no critical point q with $d(p, q) = \pi$.

We conclude that there is a $\beta_1 > 0$ such that there is no critical point q with $0 < d(p, q) \leqslant \pi + \beta_1$. If q is not a critical point, then there exists a C^∞ vector field X_p in a neighborhood of q such that $\| X_p \| \leqslant 1$ and such that $\sphericalangle (X_p, v) < \pi/2$ for all v tangent to a minimal geodesic from p to q, and, hence for all v tangent to a minimal geodesic from p to a point sufficiently close to q. Using a partition of unity we get a smooth vector field $X_p(q)$ on $\{(p, q) \mid d(p, q) \leqslant \pi + \beta_1\}$ such that $\| X_p(q) \| \leqslant 1$ and such that $d(p, \phi_{pq}(s))$ is strictly monotonically decreasing for $p \neq q$, where $\phi_{pq}(s)$ is the integral curve of X_p starting at q.

Choose $0 < a < 1$ such that $\pi^2/a + 1 - a < \pi^2 + 2\alpha$. There exists a $\beta > 0$ such that $d(\phi_{pq}(1-a), p) < \pi$ for all p, q with $d(p, q) \leqslant \pi + \beta$. Define

$$c_{pq}(t) = \begin{cases} \gamma(t) & \text{for } 0 \leqslant t \leqslant a, \\ \phi_{pq}(1-t) & \text{for } a \leqslant t \leqslant 1, \end{cases}$$

where $\gamma : [0, a] \to M$ is the unique minimal geodesic from p to $\phi_{pq}(1-a)$. Then $E(c_{pq}) < \pi^2/2 + \alpha$, and the lemma is proved.

Proof of (4.1). (i) We first prove this part of the theorem under the assumption $\delta > 1/4$, since the main ideas of the proof seem to be clearer without the additional arguments needed for the case $\delta = 1/4$. We then explain the changes which have to be carried out in the case $\delta = 1/4$.

Choose a point $p_0 \in M$ and set $R = \{r \in M \mid d(p_0, r) = \pi/2\sqrt{\delta}\}$. R is a smooth submanifold of M diffeomorphic to S^{n-1} since $\pi/2\sqrt{\delta} < \pi \leqslant i(M)$. For any $r \in R$ there exists a unique normal geodesic γ_r such that $\gamma_r(0) = r$ and $\gamma_r(\pi/2\sqrt{\delta}) = p_0$. Set $I(r) = \gamma_r(\pi/\sqrt{\delta}) \in R$. I defines an involution of R.

We now construct a homeomorphism $h : S^n \to M$ which maps the equator of S^n onto R. Let $N = (0, \cdots, 0, 1)$ be the north pole and $S = (0, \cdots, 0, -1)$ be the south pole of S^n, and let $A : T_N S^n \to T_{p_0} M$ be an isometry. Let $q_0 \in M$ be a point at maximal distance from p_0. Then by (6.3) in [11], $\min\{d(p, p_0), d(p, q_0)\} \leqslant \pi/2\sqrt{\delta} < \pi$ for any $p \in M$. Hence we can define $h : S^n \to M$ by

$$h(x) = \begin{cases} \exp_{p_0}\left(\left(A \circ \exp_N^{-1}(x) \right) / \sqrt{\delta} \right), & \text{if } x \in B_{\pi/2}(N), \\ \exp_{q_0}\left(\dfrac{d(x, S)}{\pi/2} \exp_{q_0}^{-1} \circ \exp_{p_0}\left(\dfrac{\pi/2}{\sqrt{\delta}} \dfrac{A \circ \exp_N^{-1}(x)}{\| A \circ \exp_N^{-1}(x) \|} \right) \right), & \\ \qquad \text{if } x \in B_{\pi/2}(S) - S, & \\ q_0, \quad \text{if } x = S, & \end{cases}$$

where $B_t(y)$ denotes the closed ball of radius t around y in the corresponding manifold. By (6.10) in [11] the complement of the ball $B_{\pi/2\sqrt{\delta}}(p_0)$ is $\pi/\sqrt{\delta}$-convex. Hence any geodesic of length $\leqslant \pi/2\sqrt{\delta}$ starting at q_0 meets the boundary of the ball at most once. Therefore h is injective and hence a homeomorphism. However, h is not necessarily differentiable. We make h differentiable by reparametrizing h close to the equator and around q_0 along the geodesics emanating from p_0 and q_0. Denote this new map by f. $f \circ c$ is in $P(M)$ for $c \in B_e$ since f is a homeomorphism.

f induces a continuous and Γ equivariant map $B(f)\colon B_e \to P(M)$. There is an $\varepsilon > 0$ such that $B(f)(B_e^\varepsilon) \subset V$. Note that f is $\mathbf{Z}_2$ homologically nontrivial since f is a homeomorphism. By (1.5)(ii), (1.6), and (2.4) we will be done if we can show that the classes $\overline{B}(f)_*\{a, b\}_B$ have critical levels $\leqslant 2\pi^2/\delta < 8\pi^2$. It suffices to show that the classes $(B(f)/\Gamma)_*\{a, b\}_B$ have critical levels $\leqslant 2\pi^2/\delta$ in $(P, V \cap P)/\Gamma$. To do this we define a deformation of these cycles which possibly brings a curve into a θ invariant curve. By excision we then bring these cycles back into $(P, V \cap P)/\Gamma$.

We first give some definitions. For $r \in R$ set

$$E(r) = \{p \in M \mid d(r, p) = d(Ir, p)\}.$$

Since r and Ir are connected by a geodesic of length $\pi/\sqrt{\delta}$, it follows from (4.2) that $E(r)$ divides M into two cells $B_+(r) = \{p \in M \mid d(r, p) \leqslant d(Ir, p)\}$ and $B_-(r) = \{p \in M \mid d(r, p) \leqslant d(Ir, p)\}$. $E(r)$ is diffeomorphic to S^{n-1} and depends smoothly on r: $\bigcup_{r\in R} E(r)$ is the set of zeros of the function $g\colon R \times M \to \mathbf{R}$, $g(r, p) = d(r, p) - d(Ir, p)$. g is C^∞ in a neighborhood of $g^{-1}(0)$ and it follows from the Gauss lemma that g has 0 as regular value. Let $D_{r,s}$, $r \in R$, $0 \leqslant s \leqslant 1$, be the radial projection of $M - \{r, Ir\}$ into $E(r)$, which is well defined by (4.2)(ii). The s parameter is chosen such that $D_{r,s}(p)$ is smooth in r, s, and p and such that $D_{Ir,s}(p) = D_{r,s}(p)$.

We will deform $f \circ c$ into the geodesic quadrangle with corners $f \circ c(0)$, $D_{f \circ c(0), 1}(f \circ c(1/4))$, $f \circ c(1/2)$, and $D_{f \circ c(0), 1}(f \circ c(3/4))$. We choose the parametrization of the quadrangle such that the corners occur at $i/4$, $i = 0, 1, 2, 3$. Notice that this construction is Γ equivariant and that the geodesic quadrangle has energy $\leqslant 2\pi^2/\delta$ since each side has length $\leqslant \pi/2\sqrt{\delta}$. Since it can happen that $D_{f \circ c(0), 1}(f \circ c(1/4)) = D_{f \circ c(0), 1}(f \circ c(3/4))$, this geodesic quadrangle can be an element of $\Lambda_\theta = \{c \in \Lambda \mid \theta c = c\}$. Λ_θ is invariant under Γ.

We now define the desired homotopy of $B(f)/\Gamma$. Choose $\eta > 0$ such that $2\eta < 1/4$ and $d(f \circ c(0), f \circ c(t)) < d(f \circ c(1/2), f \circ c(t))$ for all $c \in B_e$ and all $t \in [-2\eta, 2\eta]$. By the Γ equivariance of $B(f)$ this also implies $d(f \circ c(1/2), f \circ c(t)) < d(f \circ c(0), f \circ c(t))$ for $t \in [1/2 - 2\eta, 1/2 + 2\eta]$.

Define $H_1 : B_e \times I \to P \cup \Lambda_\theta$ by

$$H_1(c, s)(t) = \begin{cases} f \circ c(t), & \text{if } t \in [0, \eta], \\ D_{f \circ c(0), s(t-\eta)/\eta}(f \circ c(t)), & \text{if } t \in [\eta, 2\eta], \\ D_{f \circ c(0), s}(f \circ c(t)), & \text{if } t \in [2\eta, 1/4]. \end{cases}$$

Extend this definition to $t \in [0, 1]$ such that H_1 is Γ equivariant. $\tilde{c} = H_1(c, 1)$ then satisfies $d(\tilde{c}(0), \tilde{c}(t)) \leqslant \pi/2\sqrt{\delta} < \pi$ for $t \in [-1/4, 1/4]$, $d(\tilde{c}(1/2), \tilde{c}(t)) \leqslant \pi/2\sqrt{\delta} < \pi$ for $t \in [1/4, 3/4]$, and $\tilde{c}(0) = f \circ c(0)$, $\tilde{c}(1/2) = f \circ c(1/2)$, $\tilde{c}(1/4) = D_{f \circ c(0), 1}(f \circ c(1/4))$, $\tilde{c}(3/4) = D_{f \circ c(0), 1}(f \circ c(3/4))$. Thus we can define a further Γ equivariant homotopy of $H_1(\ , 1)$, $H_2 : B_e \times [0, 1/4] \to P \cup \Lambda_\theta$, by

$$H_2(c, s)(t) = \begin{cases} \gamma_{\tilde{c}(0), \tilde{c}(s)}, & \text{if } t \in [0, s], \\ \tilde{c}(t), & \text{if } t \in [s, 1/4], \end{cases}$$

where $\gamma_{p,q}$ is the unique shortest geodesic connecting p and q and $\tilde{c} = H_1(c, 1)$. H_2 is extended to $t \in [0, 1]$ such that it becomes Γ equivariant. $H_2(c, 1/4)$ is now the desired geodesic quadrangle. H_1 and H_2 are continuous since $B(f)$ is continuous, D is differentiable, and $\gamma_{p,q}$ and its derivatives depend continuously on p and q if $d(p, q) < \pi$. Throughout the homotopy, 0 is the only parameter value mapped into $f \circ c(0)$. Therefore the multiplicity of all curves in the deformation is equal to one. By compactness we can choose $\varepsilon > 0$ such that $H_i(c, s) \in V$ for all $c \in B_e^\varepsilon$, all s, and $i = 1, 2$. Hence we have a homotopy between $B(f)/\Gamma$ and $H_2(\ , 1/4)/\Gamma$ as maps from $(B_e, B_e^\varepsilon)/\Gamma \to (P \cup \Lambda_\theta, (V \cap P) \cup \Lambda_\theta)/\Gamma$. But $(P, V \cap P)/\Gamma \to (P \cup \Lambda_\theta, (V \cap P) \cup \Lambda_\theta)/\Gamma$ is an excision map which induces an isomorphism in homology since Λ_θ/Γ is closed and $((V \cap P) \cup \Lambda_\theta)/\Gamma$ is open with respect to the topology of $(P \cup \Lambda_\theta)/\Gamma$: Λ_θ/Γ is closed since it is closed in Λ/Γ. $((V \cap P) \cup \Lambda_\theta)/\Gamma$ is open because V/Γ is open in Λ/Γ and $V/\Gamma \cap (P \cup \Lambda_\theta)/\Gamma = ((V \cap P) \cup \Lambda_\theta)/\Gamma$. Similarly we have an excision map $(P^\kappa, V \cap P^\kappa)/\Gamma \to (P^\kappa \cup \Lambda_\theta^\kappa, (V \cap P^\kappa) \cup \Lambda_\theta^\kappa)/\Gamma, \kappa = 2\pi^2/\delta$. We can thus first map a cycle $(B(f)/\Gamma)_*\{a, b\}_B$ into $(P \cup \Lambda_\theta, (V \cap P) \cup \Lambda_\theta)/\Gamma$ and apply the above deformation to bring it into $(P^\kappa \cup \Lambda_\theta^\kappa, (V \cap P^\kappa) \cup \Lambda_\theta^\kappa)/\Gamma$. Using the inverse of the second excision map we get the cycle into $(P^\kappa, V \cap P^\kappa)/\Gamma$. This cycle is homologous to $(B(f)/\Gamma)_*\{a, b\}_B$. Hence the critical level of $(B(f)/\Gamma)_*\{a, b\}_B$ and therefore also the critical level of $\bar{B}(f)_*\{a, b\}_B$ is $\leqslant 2\pi^2/\delta$.

We now explain which changes have to be made in the case $\delta = 1/4$.

We can assume that K is not constant. By a result of Tsukamoto, see [28], there does not exist a geodesic loop of length 2π. Choose $p_0, q_0 \in M$ such that $d(p_0, q_0) = d(M)$ and set $R = \{p \in M \mid d(p, p_0) = \pi\}$. Note that R is not necessarily a submanifold of M in contrast to the case $\delta > 1/4$ above. By (6.5) and (6.6) in [11] we have $\pi < d(M) < 2\pi$, and the proof of (6.6)(1) in [11] then implies that $d(r, q_0) < \pi$ for any $r \in R$. Let S^n be the sphere of constant curvature 1 with north pole N, south pole S, and equator R_0. Let $A : T_N S^n \to T_{p_0} M$ be an isometry. Define $h : S^n \to M$ by

$$h(x) = \begin{cases} \exp_{p_0}\left(2A \circ \exp_N^{-1}(x)\right), & \text{if } x \in B_{\pi/2}(N), \\ \exp_{q_0}\left(\dfrac{d(x,S)}{\pi/2} \exp_{q_0}^{-1} \circ \exp_{p_0}\left(\pi \dfrac{A \circ \exp_N^{-1}(x)}{\|A \circ \exp_N^{-1}(x)\|}\right)\right), & \text{if } x \in B_{\pi/2}(S), \\ & \text{and } x \neq S, \\ q_0, \quad \text{if } x = S. \end{cases}$$

Since $d(r, q_0) < \pi$ for any $r \in R$, h is well defined but not necessarily injective. As above we make h differentiable by reparametrizing h close to the equator and around q_0 along the geodesics emanating from p_0 and q_0. Denote this map by f. f clearly has degree 1 and hence is $\mathbf{Z}_2$ homologically nontrivial. f can possibly map a biangle in B_e into Λ_θ. To overcome this difficulty we consider a continuous family $F_t : S^n \to S^n$ of diffeomorphisms where $t \in [0,1]$, $F_0 = \mathrm{id}$, and F_t maps the equator into the open northern hemisphere for $t \neq 0$. $f \circ F_t$ maps B_e into P for $t \neq 0$ since $f \circ F_t$ is a diffeomorphism on an open set containing the equator and the northern hemisphere. Set $\tilde{f} = f \circ F_1$. Then (2.4) applies to $\tilde{f}$. To prove the theorem for $\delta = 1/4$ it suffices to show that the images of the homology classes $\{a, b\}_B$ under $\tilde{f}$ in $(P, V \cap P)/\Gamma$ can be deformed under the energy level $8\pi^2$. As in the proof of the case $\delta > 1/4$ it follows by an excision argument, that it suffices to deform them in $(P \cup \Lambda_\theta, (V \cap P) \cup \Lambda_\theta)/\Gamma$ under the energy level $8\pi^2$. Now $f \circ F_t$ defines a deformation of the images of the homology classes $\{a, b\}_B$ under $\tilde{f}$ to those under f in $(P \cup \Lambda_\theta, (V \cap P) \cup \Lambda_\theta)/\Gamma$. Hence it suffices to deform the images of the homology classes $\{a, b\}_B$ under f under the energy level $8\pi^2$.

Since K is not constant, there do not exist closed geodesics of length 4π by a result of Tsukamoto-Sugimoto [28], [26], and therefore none of energy $8\pi^2$. Hence there exists some $\alpha > 0$ such that there is no closed geodesic with energy in $[8\pi^2 - 16\alpha, 8\pi^2 + 16\alpha]$. Therefore it suffices, using the gradient flow of E, to show that the critical level of any $B(f)_*\{a, b\}$ is $\leqslant 8\pi^2 + 16\alpha$. We will do this by representing these classes by curves consisting of four of the curves c_{pq} in (4.3).

Notice that R does not necessarily have an involution as in the case $\delta > 1/4$. But we have a corresponding involution I on R_0 which will suffice for our purposes. As before we can use (4.2) to define the equators $E(r) = \{p \in M \mid d(f(r), p) = d(f(Ir), p)\}$ for $r \in R_0$ which divide M into two regions $B_+(r)$

and $B_-(r)$. Note that $f(r) \neq f(Ir)$ for every $r \in R_0$ since these points are connected by a geodesic of length 2π by the definition of f. Thus $E(r)$ is never the whole manifold M. $E(r)$ is not necessarily a submanifold of M since there may exist points p such that $d(f(r), p) = d(f(Ir), p) = \pi$. Similarly the radial deformation retractions $D_{r,s}$, $r \in R_0$, $s \in [0,1]$, of $M - \{f(r), f(Ir)\}$ into $E(r)$ are well defined by (4.2)(ii) and also continuous, but may not be differentiable. To make these retractions differentiable we proceed as follows: Let $N = \{(r, v) \mid r \in R_0, v \in T_{f(r)}M, \|v\| = 1\}$, and let $g(r, v) \in (0, \pi]$ be the unique parameter such that $\gamma_v(g(r, v)) \in E(r)$. Choose a smooth map

$$h : N \times (0, \pi) \times [0,1] \to (0, \pi)$$

such that $h(r, v, t, 0) = t$, $h(r, v, t, 1) \geqslant g(r, v) - \beta$, and $h(r, v, t, s) = t$ if $t \geqslant g(r, v) - \beta/2$, where $\beta > 0$ is chosen with respect to α as in (4.3). Define

$$D'_{r,s}(p) = \begin{cases} \gamma_v(h(r, v, t, s)), & \text{if } p = \gamma_v(t), v \in T_{f(r)}M, \|v\| = 1 \\ & \quad \text{and } d(p, f(r)) < g(r, v), \\ \gamma_v(h(Ir, v, t, s)), & \text{if } p = \gamma_v(t), v \in T_{f(Ir)}M, \|v\| = 1 \\ & \quad \text{and } d(p, f(Ir)) < g(Ir, v), \\ p, & \text{otherwise.} \end{cases}$$

It is clear that $D'_{r,s}(p)$ is differentiable in r, s and p, and that $D'_{r,s}(p)$ defines deformations of $M - \{f(r), f(Ir)\}$ such that $D'_{r,s} = D'_{Ir,s}$, $d(f(r), D'_{r,s}(p)) \leqslant \pi + \beta$ if $d(f(r), p) \leqslant d(f(Ir), p)$, and $d(f(r), D'_{r,1}(p)) \leqslant \pi + \beta$, $d(f(Ir), D'_{r,1}(p)) \leqslant \pi + \beta$. We replace D by D' in the definition of H_1 above. $\tilde{c} = H_1(c, 1)$ then satisfies $d(\tilde{c}(0), \tilde{c}(t)) \leqslant \pi + \beta$ for $t \in [-1/4, 1/4]$, $d(\tilde{c}(1/2), \tilde{c}(t)) \leqslant \pi + \beta$ for $t \in [1/4, 3/4]$, and $\tilde{c}(0) = f \circ c(0)$, $\tilde{c}(1/2) = f \circ c(1/2)$. Furthermore, throughout the homotopy, 0 is the only parameter value mapped into $f \circ c(0)$. Therefore the multiplicity of all curves in the deformation is equal to one. In the definition of H_2 we replace the segments $\gamma_{\tilde{c}(0),\tilde{c}(s)}$ by the curves $c_{\tilde{c}(0),\tilde{c}(s)}$ in Lemma 4.3. It is clear that H_1 and H_2 are continuous. The energy of $H_2(c, 1/4)$ is less than $8\pi^2 + 16\alpha$ and hence the critical level of $B(f)_*\{a, b\}$ is $< 8\pi^2$. The rest of the proof is as above.

(ii) The images h_1 and h_3 of the homology classes $\{0, 0\}_B$ and $\{n-1, n-1\}_B$ are subordinate classes in $(\bar{P}, \bar{V} \cap \bar{P})$, and it follows from the proof of (i) that $\kappa(h_1) \leqslant \kappa(h_3) \leqslant 2\pi^2/\delta \leqslant 8\pi^2$ and $< 8\pi^2$ if K is not constant.

Let $q : P/\theta \to \bar{P}$ be the projection. In the proof of (2.3) it was shown that there exists an $\tilde{h}_1 \in H_{n-1}((P, V \cap P)/\theta)$ such that $q_*(\tilde{h}_1) = h_1$. We first show that $H_{n-1}((P, V \cap P)/\theta) = \mathbf{Z}_2$ since we will need that $\tilde{h}_1$ is uniquely determined. Since M is homeomorphic to S^n, we have $H_{n-1}(P, V \cap P) \cong H_{n-1}(P) \cong H_{n-1}(\Lambda)$. Because $\pi_i(\Lambda) = 0$ for $i < n-1$ and $\pi_{n-1}(\Lambda) \cong \pi_n(M) \cong \mathbf{Z}$, it follows from the Hurewicz theorem that $H_{n-1}(\Lambda) = \mathbf{Z}_2$. Now the

Gysin sequence of $(P, V \cap P) \to (P, V \cap P)/\theta$ implies $H_{n-1}((P, V \cap P)/\theta) = \mathbf{Z}_2$.

Clearly $\kappa(\tilde{h}_1) \geqslant \kappa(h_1)$. Since q is an S^1 bundle, we have the transfer map $\tau: H_*(\bar{P}, \bar{V} \cap \bar{P}) \to H_{*+1}(P, V \cap P)/\theta)$ of the associated Gysin sequence. Define $\tilde{h}_3 = \tau h_3$. It follows from the naturality of the transfer that $\kappa(\tilde{h}_3) \leqslant \kappa(h_3)$. We will now show that $\tilde{h}_1$ and $\tilde{h}_3$ are subordinate, which implies $\kappa(\tilde{h}_3) \geqslant \kappa(\tilde{h}_1)$. Hence $\kappa(h_1) \leqslant \kappa(\tilde{h}_1) \leqslant \kappa(\tilde{h}_3) \leqslant \kappa(h_3)$.

We first compute the cohomology of $G/\theta = T_1S^n/\theta$, where $\theta: T_1S^n \to T_1S^n$ is defined by $\theta(v) = -v$. $H^i(T_1S^n) = \mathbf{Z}_2$ for $i = 0,\ n-1,\ n, 2n-1$ and vanishes otherwise. The generator of $H^{n-1}(T_1S^n)$ is dual to the fundamental class F of the fibre S^{n-1} of $T_1S^n \to S^n$. Under $T_1S^n \to T_1S^n/\theta$, F is mapped into twice the fundamental class of the fibre $P^{n-1}\mathbf{R}$ of $T_1S^n/\theta \to S^n$. Hence $H^{n-1}(T_1S^n/\theta) \to H^{n-1}(T_1S^n)$ is the zero map. The generator of $H^n(T_1S^n)$ is the pullback of the fundamental coclass $[S^n]$ of S^n. Since $T_1S^n \to S^n$ factors through T_1S^n/θ the pullback ω_n of $[S^n]$ to T_1S^n/θ is nonzero. From the Gysin sequence of $T_1S^n \to T_1S^n/\theta$ we conclude that $H^i(T_1S^n/\theta) = \mathbf{Z}_2$ for $0 \leqslant i \leqslant 2n-1$ with generator ω_1^i for $0 \leqslant i \leqslant n-1$ and $\omega_n \cup \omega_1^{i-n}$ for $n \leqslant i \leqslant 2n-1$, where ω_1 is the Stiefel-Whitney class of $T_1S^n \to T_1S^n/\theta$. In particular $\omega_n \cup \omega_1^{n-1} = [T_1S^n/\theta]$.

ω_1 and ω_n both have preimage under $(G(\tilde{f})/\theta)^*$, where $G(\tilde{f})/\theta: T_1S^n/\theta = G/\theta \to P/\theta$ is the map induced by $\tilde{f}$ ($\tilde{f}$ as in the proof of (i)): Let $\tilde{\omega}_1$ be the Stiefel-Whitney class of the covering $P \to P/\theta$. By naturally $(G(\tilde{f})/\theta)^*\tilde{\omega}_1 = \omega_1$. The evaluation $e(c) = c(0)$ defines a map $e: P/\theta \to M$ such that $e \circ G(\tilde{f})/\theta: G/\theta \to M$ coincides with $G/\theta \to S^n \xrightarrow{f} M$. Hence $\tilde{\omega}_n = e^*[M]$ satisfies $(G(\tilde{f})/\theta)^*\tilde{\omega}_n = \omega_n$.

Set $\tilde{\xi} = \tilde{\omega}_n \cup \tilde{\omega}_1^{n-1}$. Then $(G(\tilde{f})/\theta)^*\tilde{\xi} = [G/\theta]$. Let $\eta: H^*(G/\theta) \to H^{*-1}(\bar{G})$ and $\tilde{\eta}: H^*(P/\theta) \to H^{*-1}(\bar{P})$ be the transfers in cohomology of the corresponding Gysin sequences. Then $\eta[G/\theta] = [\bar{G}]$, and hence $\xi = \tilde{\eta}(\tilde{\xi}) \in H^{2n-2}(\bar{P})$ satisfies $\bar{G}(\tilde{f})^*\xi = [\bar{G}]$. As at the end of the proof of (2.4) it follows that $\xi \cap h_3 = h_1$. By the definition of τ and $\tilde{\eta}$ and (12.14) in [13, p. 240], it follows that $q_*(\tilde{\xi} \cap \tau h_3) = \tilde{\eta}(\tilde{\xi}) \cap h_3$, and hence

$$q_*(\tilde{\xi} \cap \tilde{h}_3) = q_*(\tilde{\xi} \cap \tau h_3) = \xi \cap h_3 = h_1.$$

Since $H_{n-1}((P, V \cap P)/\theta) = \mathbf{Z}_2$ it follows that $\tilde{\xi} \cap \tilde{h}_3 = \tilde{h}_1$. Set $\tilde{h}_2 = \tilde{\omega}_n \cap \tilde{h}_3 \in H_{2(n-1)}((P, V \cap P)/\theta)$. Then $\tilde{\omega}_1^{n-1} \cap \tilde{h}_2 = \tilde{h}_1$ and hence $\tilde{h}_1$ is subordinate to $\tilde{h}_2$ and $\tilde{h}_2$ is subordinate to $\tilde{h}_3$.

Assume first that all closed geodesics whose lengths lie in $[2\pi, 2\pi/\sqrt{\delta}\,]$ have the same length l. Then $\kappa(\tilde{h}_3) = \kappa(\tilde{h}_1) = l^2/2$. From (1.3), applied to $X = \Lambda/\theta$, $Y = P/\theta$ and $Z = (V \cap P)/\theta$, it follows that $\tilde{\xi} \,|\, U \neq 0$ for any neighborhood U of $K = \{c \in P/\theta \,|\, c \text{ closed geodesic with } E(c) = l^2/2\}$ in P/θ. Since

P/θ is a locally contractible metric space, it follows that $H^*(U)$ is naturally isomorphic to $\bar{H}^*(U)$ for any open subset U of P/θ, where $\bar{H}$ denotes Alexander cohomolgy, see [25, (6.9.5)]. Since K is closed we have $\bar{H}^*(K) = \varinjlim \bar{H}^*(U)$, where the direct limit is over all open neighborhoods U of K, ordered by $U < V$ if $V \subset U$; see [25, (6.6.2)]. Hence $\tilde{\xi}$ defines a nonzero element in $\bar{H}^{2n-1}(K)$ and therefore $\bar{H}^{2n-1}(K) \neq 0$. On the other hand, K can be embedded into T_1M/θ since a closed geodesic is determined by its initial vector, and the topology induced on K by T_1M/θ coincides with the topology induced by P/θ. We have again that $\bar{H}^*(K) = \varinjlim \bar{H}^*(U)$, where the limit is over all open neighborhoods U of K in T_1M/θ. If U is not all of T_1M/θ, then U is an open $(2n-1)$-dimensional manifold which implies $\bar{H}^{2n-1}(U) = 0$. Hence $\bar{H}^{2n-1}(K) = 0$ if K is not all of T_1M/θ. Therefore $K = T_1M/\theta$ and all geodesics are closed of length l.

If the closed geodesics with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,]$ have only two different length values, then $\kappa(\tilde{h}_1) = \kappa(\tilde{h}_2)$ or $\kappa(\tilde{h}_2) = \kappa(\tilde{h}_3)$. Assume first $\kappa = \kappa(\tilde{h}_2) = \kappa(\tilde{h}_3)$. From (1.3) it follows that $\tilde{\omega}_n \,|\, U \neq 0$ for any neighborhood U of $K = \{c \in P/\theta \mid c \text{ closed geodesic with } E(c) = \kappa\}$. Suppose there is a point $x \in M$ such that no $c \in K$ meets x, i.e., given any representative $\tilde{c} \in P$ of any $c \in K$, then $c(t) \neq x$ for all $t \in S^1$. Then there is also a neighborhood U_0 of K such that no $c \in U_0$ meets x. U_0 contains a cycle z such that $\tilde{\omega}_n(z) \neq 0$. Since $\tilde{\omega}_n$ is the pullback $e^*[M]$ of the fundamental class $[M]$ of M, $e_*(z)$ is a fundamental cycle of M. Hence the carrier of $e_*(z)$ contains x, a contradiction to the choice of U_0.

To discuss the case $\kappa(\tilde{h}_1) = \kappa(\tilde{h}_2) = \kappa$, we first define a map $H_*(P/\theta) \to H_{*+1}(M)$: Let σ be a singular k-simplex in P/θ. σ has a lift $\tilde{\sigma}$ to P since $P \to P/\theta$ is a covering. Then $e \circ (S^1 \cdot \tilde{\sigma})$ is a $(k+1)$-chain of M, where $S^1 \cdot \tilde{\sigma}$ is defined as in the proof of (2.3). σ has exactly two lifts, $\tilde{\sigma}$ and $\theta\tilde{\sigma}$, to P. Since we use $\mathbf{Z}_2$ coefficients we have $e \circ (S^1 \cdot \tilde{\sigma}) = e \circ (S^1 \cdot \theta\tilde{\sigma})$. Hence $\sigma \to e \circ (S^1 \cdot \tilde{\sigma})$ induces a map $\tilde{e}$ of the singular chain complexes. $\tilde{e}$ commutes with differentials since $\partial(S^1 \cdot \tilde{\sigma}) = S^1 \cdot \partial\tilde{\sigma}$. We denote by $\tilde{e}$ also the induced map of the homology groups.

Assume that there is some $x \in M$ such that no geodesic in the set K, defined as above, meets x. Let U_0 be a neighborhood of K in P/θ such that no curve in U_0 meets x. Then there is a cycle z in U_0 such that $\tilde{\omega}_1^{n-1}(z) \neq 0$ by (1.3). It follows that z represents the unique nonzero element in $H_{n-1}(P/\theta) = \mathbf{Z}_2$. On the other hand, it is also clear that $\tilde{e}$ maps the nonzero element in $H_{n-1}(P/\theta)$ onto the fundamental class of M. Therefore the carrier of $\tilde{e}(z)$ is equal to M, contradicting the choice of U_0.

Remarks. (a) In all three cases in the proof of (ii) it is essential to work in $(P, V \cap P)/\theta$. For example, in the first case we use the embedding of K into T_1M/θ. Note that there is no such natural embedding of $\bar{K}$.

(b) As claimed in Theorem B, one can prove (ii) also under the assumption $g \leqslant \alpha^2 g_0$, $1 \leqslant \alpha < 2$, and $l(M) \geqslant 2\pi$, where the lengths of all closed geodesics considered are in the interval $[2\pi, 2\pi\alpha]$. This follows as in (ii) since the assumptions imply $\kappa(\tilde{h}_1) = \kappa(\tilde{h}_3)$ and $\kappa(\tilde{h}_1) = \kappa(\tilde{h}_2)$ or $\kappa(\tilde{h}_2) = \kappa(\tilde{h}_3)$ respectively.

(c) The method of the proof of (ii) can also be used to prove the following result claimed by Lusternik and Schnirelmann; see [22, p. 82]: If the closed geodesics without self-intersections on S^2 with respect to some Riemannian metric g all have the same length α, then all geodesics on (S^2, g) are closed of length α and have no self-intersections. If the lengths of the closed geodesics without self-intersections only take on two values, then there is a family of closed geodesics without self-intersections and of constant length which meets every point of S^2.

(d) It is interesting to note that one can easily prove the following perturbation version of (4.1)(i):

Let g_t, $0 \leqslant t \leqslant 1$, be a C^2 family of metrics on S^n such that g_0 has constant curvature 1 and such that the sectional curvature K_t of g_t satisfies $1/4 \leqslant K_t \leqslant 1$. Then g_t has $g(n)$ closed geodesics without self-intersections and with lengths in $[2\pi, 4\pi]$.

To see this let C be the set of metrics g such that $1/4 \leqslant K_g \leqslant 1$, endowed with the C^2 topology. Let $g \in G$. If g has a closed geodesic of length 4π, then $K_g \equiv 1$ or $1/4$ by a result of Tsukamoto-Sugimoto; see [28] and [26]. Hence there exists an $\alpha > 0$ and a neighborhood U of g such that no $g' \in U$ has a closed geodesic with length in $(8\pi^2 - \alpha, 8\pi^2)$. It follows that $(\bar{P}_g^{8\pi^2-}, P_g^{\varepsilon})$ has exactly as many subordinate homology classes as $(\bar{P}_{g'}^{8\pi^2-}, \bar{P}_{g'}^{\varepsilon})$. Therefore the set A of metrics $g \in G$ such that $(\bar{P}_g^{8\pi^2-}, \bar{P}_g^{\varepsilon})$ contains $g(n)$ subordinate homology classes is open and contains g_0. The same arguments imply that $G - A$ is open, and hence A contains the connected component of g_0 in G.

Similarly it follows that the number of subordinate homology classes of $(\bar{P}_g^{8\pi^2-}, \bar{P}_g^{\varepsilon})$ depends only on the connected component of g in G.

5. Stability properties

In this section we discuss some properties of the linearized Poincaré map of the closed geodesics found in the previous sections. For definitions we refer to [8].

5.1. Theorem. *Let $\delta \leqslant K \leqslant 1$. Suppose that c is a closed geodesic with $L(c) \geqslant 2\pi$ and $\operatorname{ind}(c) \leqslant k$, where k is some integer with $n-1 \leqslant k < 2(n-1)$. Then the following hold.*

(i) *$\delta \geqslant (k/2(n-1))^2$ implies that c is nonhyperbolic.*

(ii) *$\delta \geqslant ((2s+3)/(2s+4))^2$, $s = k-(n-1)$, implies that the linearized Poincaré map of c has at least $2(n-1)-k$ Jordan blocks $J_R(e^{i\phi}, m, \sigma)$ with $0 \leqslant \phi < \pi$ and $\sigma \geqslant 0$. If $\phi \neq 0$, then m is odd and $\sigma = +1$.*

(iii) *$\delta \geqslant 9/16$ and $k < 3(n-1)/2$ implies that there exist at least $3(n-1)-2k$ Jordan blocks with $0 \leqslant \phi < \pi$ and $\sigma \geqslant 0$.*

5.2. Theorem. *Let $\delta \leqslant K \leqslant 1$. Suppose that c is a closed geodesic with $L(c) \leqslant 2\pi/\sqrt{\delta}$ and $\operatorname{ind}_0(c) \geqslant k$, where k is some integer with $2(n-1) < k \leqslant 3(n-1)$. Then the following hold.*

(i) *$\delta \geqslant (2(n-1)/k)^2$ implies that c is nonhyperbolic.*

(ii) *$\delta \geqslant ((2s+4)/(2s+5))^2$, $s = 3(n-1)-k$, implies that the linearized Poincaré map of c has at least $k-2(n-1)$ Jordan blocks $J_R(e^{i\phi}, m, \sigma)$ with $0 \leqslant \phi < \pi$ and $\sigma \leqslant 0$. If $\phi \neq 0$, then m is odd and $\sigma = -1$.*

(iii) *$\delta \geqslant 16/25$ and $k > 5(n-1)/2$ imply that there exist at least $2k-5(n-1)$ Jordan blocks with $0 \leqslant \phi < \pi$ and $\sigma \leqslant 0$.*

Proof. Notice that for all pinching assumptions $\delta \geqslant 1/4$. Hence by results of Tsukamoto and Sugimoto we can assume $L(c) > 2\pi$ (see [28]) and $L(c) < 2\pi/\sqrt{\delta}$ (see [28] and [26]).

(5.1)(i) then follows as at the end of the proof of [8, (3.9)].

Assume now that $\sqrt{\delta} > p/2q$. Then $L(c^q) > 2\pi q \geqslant p\pi/\sqrt{\delta}$, and hence $\operatorname{ind}(c^q) \geqslant p(n-1)$ by [8, (1.8)]. As in the proof of [8, (3.3)] it follows that there exists a z with $z^q = 1$, $z \neq 1$, such that $I(z) - I(1) \geqslant (p(n-1)-kq)/(q-1)$. To prove (5.1)(ii) choose $p = 2s+3$ and $q = s+2$. Then $I(z) - I(1) > 2(n-1) - k - (s/(s+1))$, and hence (5.1)(ii) follows from [8, (3.2)]. Similarly, for (5.1)(iii) choose $p = 3$ and $q = 2$. This implies $I(z) - I(1) \geqslant 3(n-1) - 2k$.

To prove (5.2)(i) choose a rational number p/q such that $L(c) < p\pi/q < 2\pi/\sqrt{\delta}$. Then $L(c^q) < p\pi$, and hence $\operatorname{ind}_0(c^q) \leqslant p(n-1) < 2q(n-1)/\sqrt{\delta} \leqslant qk \leqslant q\operatorname{ind}_0(c)$ since $\sqrt{\delta} \geqslant 2(n-1)/k$. Therefore c is nonhyperbolic by [8, (2.3)].

Assume now that $\sqrt{\delta} \geqslant 2q/p$. Then $L(c^q) < 2\pi q/\sqrt{\delta} \leqslant p\pi$, so that [8, (1.9)] implies $\operatorname{ind}_0(c^q) \leqslant p(n-1)$. Hence there exists a z with $z^q = 1$, $z \neq 1$, such that $I_0(z) - I_0(1) \leqslant -(kq - p(n-1))/(q-1)$. To prove (ii) let $q = s+2$ and $p = 2s+5$. This implies $I_0(z) - I_0(1) \leqslant -(k-2(n-1)) + s/(s+1)$, and (ii) now follows from [8, (3.2)]. To prove (iii) let $q = 2$ and $p = 5$. Then $I_0(z) - I_0(1) \leqslant -(2k - 5(n-1))$.

Remark. As in [8, (3.3)] and [8, (3.4)] one could estimate the number of Jordan blocks with eigenvalues on the unit circle and their angles as δ increases.

In Remark (b) following [8, (4.2)] we gave a local example with $K = 1$, $\mathrm{ind}(c) = k$, $n - 1 < k \leqslant 2(n-1)$, such that all eigenvalues of the linearized Poincaré map are on the unit circle, but $k - (n-1)$ of them could be chosen to be -1. Notice that Jordan blocks with eigenvalues -1 cannot be detected with our methods. This shows that in (5.1) one cannot obtain more than $2(n-1) - k$ Jordan blocks on the unit circle with our methods. A similar remark applies to (5.2).

In Chapter 4 we showed that there exist $g(n)$ closed geodesics with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,]$ if $\pi_1(M) = 0$ and $1/4 \leqslant \delta \leqslant K \leqslant 1$. In order to apply (5.1) and (5.2) we have to find estimates for their index.

If all closed geodesics of length $\leqslant 4\pi$ are nondegnerate, we can apply (1.6): For each $a, b, 0 \leqslant a \leqslant b \leqslant n-1$, there exists a closed geodesic c with length in $[2\pi, 2\pi/\sqrt{\delta}\,]$ and $\mathrm{ind}(c) = \mathrm{ind}_0(c) = a + b + n - 1$. Hence $]n^2/2[$ of these closed geodesics have $\mathrm{ind}(c) = \mathrm{ind}_0(c) \neq 2(n-1)$. Using (5.1)(i) and (5.2)(i) they are nonhyperbolic if δ is sufficiently large. The strongest pinching which we need occurs if $\mathrm{ind}_0(c) = 2(n-1) + 1$, and hence we obtain $]n^2/2[$ nonhyperbolic closed geodesics if $\delta \geqslant ((2n-2)/(2n-1))^2$.

To get an estimate for the number of nonhyperbolic closed geodesics in the general case we need the following two lemmas.

5.3. Lemma. *Suppose $i(M) \geqslant \pi$ and $\kappa < 8\pi^2$. If $h \in H_k(\bar{P}^{\kappa-}, \bar{V} \cap \bar{P}^{\kappa-})$ is nonzero, then there exists a closed geodesic of energy $\kappa(h)$ satisfying* $\mathrm{ind}(c) \leqslant k \leqslant \mathrm{ind}_0(c)$.

Proof. Let $\kappa(h) = \alpha$. If (5.3) is not true, then every closed geodesic of energy α satisfies $\mathrm{ind}(c) > k$ or $\mathrm{ind}_0(c) < k$. Given $\varepsilon > 0$ we can approximate the metric g by a metric g_* in the C^2 topology such that g_* has only nondegenerate closed geodesics and such that $(1-\varepsilon)g \leqslant g_* \leqslant (1+\varepsilon)g$, see [1]. If we set $\bar{P}_*^{\beta} = E_*^{-1}[0, \beta]$, where E_* is the energy of g_*, we have

$$\bar{P}_*^{\alpha(1-\varepsilon)^2} \subset \bar{P}^{\alpha(1-\varepsilon)} \subset \bar{P}^{\alpha} \subset \bar{P}^{\alpha(1+\varepsilon)} \subset \bar{P}_*^{\alpha(1+\varepsilon)^2}.$$

We can furthermore assume that all closed geodesics of g_* in $\bar{P}_*^{\alpha(1+\varepsilon)^2} - \bar{P}_*^{\alpha(1-\varepsilon)^2-}$ satisfy $\mathrm{ind}(c) \neq k$. Otherwise we can take a sequence $\varepsilon_i \to 0$ and metrics $g_i \to g$ such that all closed geodesics of g_i are nondegenerate, and such that g_i has a closed geodesic c_i with $\mathrm{ind}(c_i) = k$ and $E_i(c_i) \in [\alpha(1-\varepsilon_i)^2, \alpha(1+\varepsilon_i)^2]$. But then a subsequence of the c_i converges to a closed geodesic c of g with energy α, and by continuity of the eigenvalues of the Hessian, $\mathrm{ind}(c) \leqslant k \leqslant \mathrm{ind}_0(c)$.

As in the proof of (1.6) it follows that $H_k(\bar{P}_*^{\alpha(1+\varepsilon)^2}, \bar{P}_*^{\alpha(1-\varepsilon)^2}) = 0$ since there exist no closed geodesics of g_* of index k in $\bar{P}_*^{\alpha(1+\varepsilon)^2} - \bar{P}_*^{\alpha(1-\varepsilon)^2-}$. But by the definition of α, there exists a representative of h in $(\bar{P}^{\alpha(1+\varepsilon)}, \bar{V} \cap \bar{P}^{\alpha(1+\varepsilon)})$, and since $\bar{V}$ contains no closed geodesics, also one in $(\bar{P}^{\alpha(1+\varepsilon)}, \bar{V} \cap \bar{P}^{\varepsilon})$. But since $(\bar{P}^{\alpha(1+\varepsilon)}, \bar{V} \cap \bar{P}^{\varepsilon}) \subset (\bar{P}_*^{\alpha(1+\varepsilon)^2}, \bar{P}_*^{\alpha(1-\varepsilon)^2})$, the above implies that h has a representative in $(\bar{P}_*^{\alpha(1-\varepsilon)^2}, \bar{V} \cap \bar{P}^{\varepsilon}) \subset (\bar{P}^{\alpha(1-\varepsilon)}, \bar{V} \cap \bar{P}^{\varepsilon})$ which contradicts the definition of α.

5.4. Lemma. *Suppose $i(M) \geqslant \pi$ and $\kappa < 8\pi^2$. Let h_1 and h_2 be two subordinate homology classes in $(\bar{P}^{\kappa-}, \bar{V} \cap \bar{P}^{\kappa-})$ with $\kappa(h_1) = \kappa(h_2) = \alpha$. Then there exist infinitely many degenerate closed geodesics of energy α.*

Proof. If there exist only finitely many degenerate closed geodesics $\gamma_1, \cdots, \gamma_k$ of energy α, let U_i be pairwise disjoint contractible neighborhoods of γ_i in $\bar{P}^{\kappa-}$. Then there exist only finitely many nondegenerate closed geodesics $\sigma_1, \cdots, \sigma_l$ of energy α which are not contained in $U_1, \cdots, U_k$. By choosing the U_i appropriately, we can assume that none of the σ_j's lies in ∂U_i. Choose contractible neighborhoods W_i of σ_i such that the $U_1, \cdots, U_k$, $W_1, \cdots, W_l$ are pairwise disjoint. Then $U_1, \cdots, U_k$, $W_1, \cdots, W_l$ cover the set of closed geodesics of energy α, and one obtains a contradiction to (1.3) since the cohomology of $U_1 \cup \cdots \cup U_k \cup W_1 \cup \cdots \cup W_l$ vanishes in positive dimensions.

Remark. One could expect that the existence of l subordinate homology classes $h_1, \cdots, h_l$ in $(\bar{P}^{\kappa-}, V \cap \bar{P}^{\kappa-})$ of dimension $k_1, \cdots, k_l$ implies the existence of l closed geodesics $c_1, \cdots, c_l$ with $E(c_i) = \kappa(h_i)$ and $\mathrm{ind}(c_i) \leqslant k_i \leqslant \mathrm{ind}_0(c_i)$.

5.5. Theorem. *If M is homeomorphic to S^n and $((2n-2)/(2n-1))^2 \leqslant \delta \leqslant K \leqslant 1$, then there exist at least $g(n) - 1$ nonhyperbolic closed geodesics on M without self-intersections and with lengths in $[2\pi, 2\pi/\sqrt{\delta}]$. If $n \neq 2^k$, then there exist at least $g(n)$ such closed geodesics. If all closed geodesics of length $\leqslant 4\pi$ are nondegenerate, then there exist at least $]n^2/2[$ such closed geodesics.*

Proof. The $g(n)$ homology classes in (4.1) have dimension in $[n-1, 3(n-1)]$. By (5.4) we can assume that the critical levels of these $g(n)$ homology classes are all different, since $\mathrm{null}(c) = \dim\ker(P - id)$ implies that a degenerate closed geodesic is nonhyperbolic. Unless the homology class has dimension $2(n-1)$, (5.3) together with (5.1)(i) and (5.2)(i) implies that the corresponding closed geodesic is nonhyperbolic if the pinching is appropriate. The strongest pinching we need occurs if $k = 2(n-1)+1$, hence $\delta \geqslant ((2n-2)/(2n-1))^2$ suffices. This implies the existence of $g(n) - 1$ nonhyperbolic closed geodesics as claimed.

If $n \neq 2^k$, the sequence of $g(n)$ subordinate homology classes is obtained by applying at least once the cap product with a two-dimensional cohomology class. Since the cap product with cohomology classes can be performed in arbitrary order, we obtain a sequence of $g(n)$ subordinate homology classes none of which has dimension $2(n-1)$.

The case where all closed geodesics of length $\leqslant 4\pi$ are nondegenerate was discussed above.

5.6. Theorem. *Let M be a simply connected compact Riemannian manifold which is not a $\mathbf{Z}_2$ homology sphere. Assume that $1/16 < \delta \leqslant K \leqslant 1$ and $i(M) \geqslant \pi$. Let k be the first dimension in which a nontrivial homology class appears. If $4(k-1) < n-1$, then there exist at least k nonhyperbolic closed geodesics without self-intersections and with lengths in $[2\pi, \pi/\sqrt{\delta}]$. If $6(k-1) < n-1$, then there exist at least $g(k)$ such closed geodesics.*

Proof. The homology classes considered in the proof of (3.3)(i) (resp. (ii)) have dimensions $< (n-1)/2$ if $4(k-1) < n-1$ (resp. $6(k-1) < n-1$). The proof now follows from (5.3), (5.4), and [8, (3.8)].

Remark. This theorem applies to $P^n\mathbf{C}$ and $P^n\mathbf{H}$ if $1/16 < K \leqslant 1$. There exist two (resp. four) nonhyperbolic closed geodesics on $P^n\mathbf{C}$ (resp. $P^n\mathbf{H}$) if $n \geqslant 3$ (resp. $n \geqslant 4$), and there exist three (resp. seven) such closed geodesics if $n \geqslant 4$ (resp. $n \geqslant 5$). Notice that in [8, (3.9)] we obtained the existence of one nonhyperbolic closed geodesic on these spaces under weaker pinching assumptions.

References

[1] R.Abrahams, *Bumpy metrics, global analysis*, Proc. Sympos. Pure Math. Vol. 14, Amer. Math. Soc., 1970, 1-3.

[2] S. I. Alber, *On periodicity problems in the calculus of variations in the large*, Uspehi Mat. Nauk (N.S.) **12** (1957) 57-124 (Russian); Amer. Math. Soc. Transl. (2) **14** (1960) 107-172.

[3] ______, *Topology of function spaces*, Dokl. Akad. Nauk. SSSR (N.S.) **168** (1966) 727-730 (Russian); Soviet Math. **7** (1966) 700-704.

[4] ______, *The topology of functional manifolds and the calculus of variations in the large*, Uspehi Mat. Nauk (N.S.) **25** (1970) 57-122 (Russian); Russian Math. Surveys **25** (1970) 51-117.

[5] D. V. Anosov, *Certain homotopies in the space of closed curves*, Izv. Akad. Nauk SSSR **44** (1980) 1219-1254 (Russian); Math. USSR-Izv. **17** (1981) 423-453.

[6] W. Ballmann, *Der Satz von Lusternik und Schnirelmann*, Bonner Math. Schriften **102** (1978) 1-25.

[7] W. Ballmann, G. Thorbergsson & W. Ziller, *On the existence of short closed geodesics and their stability properties*, to appear in Seminar on Minimal Submanifolds, Ed. E. Bombieri, Annals of Math. Studies No. 103, Princeton University Press, Princeton, NJ.

[8] ______, *Closed geodesics on positively curved manifolds*, Ann. of Math. **116** (1982) 213-247.

[9] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse Math. und ihrer Grenzgebiete, Vol. 93, Springer, Berlin, 1978, 236-242.

[10] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[11] J. Cheeger & D. G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.

[12] S. S. Chern, *On the multiplication in the characteristic ring of a sphere bundle*, Ann. of Math. **49** (1948) 362–372.

[13] A. Dold, *Lectures on algebraic topology*, Grundlehren Math. Wiss. 200, Springer, Berlin, 1972.

[14] H. I. Eliasson, *Morse theory for closed curves*, Sympos. infinite dimensional topology, Louisiana State University; Annals of Math. Studies No. 69, 63–77, Princeton University Press, Princeton, 1972.

[15] M. Gromov, *Curvature, diameter and Betti numbers*, Comment. Math. Helv. **56** (1981) 179–195.

[16] K. Grove & K. Shiohama, *A generalized sphere theorem*, Ann. Math. **106** (1977) 201–211.

[17] N. B. Hingston, *Equivariant Morse theory and closed geodesics*, Thesis, Harvard, 1981.

[18] S. T. Hu, *Homotopy theory*, Academic Press, New York, 1959.

[19] W. Klingenberg, *Simple closed geodesics on pinched spheres*, J. Differential Geometry **2** (1968) 225–232.

[20] ______, *Lectures on closed geodesics*, Grundlehren Math. Wiss. 230, Springer, Berlin, 1978.

[21] ______, *On the existence of closed geodesics on spherical manifolds*, Math. Z. **176** (1981) 319–325.

[22] L. A. Ljusternik, *The topology of function spaces and the calculus of variations in the large*, Trudy Mat. Inst. Steklov **19** (1947) (Russian); *The topology of the calculus of variations in the large*, Transl. Math. Monographs 16, Amer. Math. Soc., 1966.

[23] L. Lusternik & L. Schnirelmann, *Sur le problème de trois géodésiques fermées sur les surfaces de genre* 0, C. R. Acad. Sci. Paris **189** (1929) 269–271.

[24] J. Milnor, *Morse theory*, Annals of Math. Studies No. 51, Princeton University Press, Princeton, 1963.

[25] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[26] M. Sugimoto, *On Riemannian manifolds with a certain closed geodesic*, Tôhoku Math. J. **22** (1970) 56–64.

[27] G. Thorbergsson, *Nonhyperbolic closed geodesics*, Math. Scand. **44** (1979) 135–148.

[28] Y. Tsukamoto, *Closed geodesics on certain Riemannian manifolds of positive curvature*, Tôhoku Math. J. **18** (1966) 138–143.

UNIVERSITY OF BONN
UNIVERSITY OF PENNSYLVANIA