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ON THE EXISTENCE OF CODIMENSION-ONE MINIMAL SPHERES IN COMPACT SYMMETRIC SPACES OF RANK 2. II

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Introduction

This is the second one of a series of joint papers devoted to the construction of basic examples of codimension-one closed minimal submanifolds in spheres and compact symmetric spaces. In a previous paper [3], we constructed some new examples of codimension-one closed minimal submanifolds in $S^{3p+2}(1)$, and the following four symmetric spaces of rank 2, namely,

$$SU(3)/SO(3), \quad SU(3), \quad SU(6)/Sp(3), \quad E_6/F_4,$$

which are respectively of the diffeomorphic types of

$$S^1 \times S^p \times S^p \times S^p, \quad S^1 \times \frac{SO(3)}{\mathbf{Z}_2^2}, \quad S^1 \times \frac{SU(3)}{T^2},$$
$$S^1 \times \frac{Sp(3)}{Sp(1)^3}, \quad S^1 \times \frac{F_4}{\mathrm{Spin}(8)}.$$

In [3] we also ask the following problem of "generalized equator", namely,

Problem 1. Let M^n be a given compact symmetric space. Among all codimension-one submanifolds of M^n, which one is the "simplest" and the "best" that one may consider to be the generalized equator of M^n?

Of course, the above problem is as yet not precise because the "simplicity" as well as "value judgment" are purely a matter of taste and viewpoint. From differential topological viewpoint, spheres are the simplest and the most basic closed manifolds. Therefore it is rather natural to ask the following related problem of codimension-one minimal sphere.

Problem 2. Let M^n be a compact irreducible simply connected symmetric space. Are there codimension-one minimal imbeddings of the differentiable $(n-1)$-sphere into M^n?

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If the answer of above problem happens to be positive in general, then such minimal spheres are nice, simple candidates of generalized equators of symmetric spaces. The main result of this paper is the following existence theorem of such minimal spheres for some symmetric spaces of rank 2.

Theorem. *There exists codimension-one K-invariant minimal spheres in the following four symmetric spaces*:

$$G/K \simeq \frac{SU(3)}{SO(3)}, \quad SU(3), \quad \frac{SU(6)}{Sp(3)}, \quad \frac{E_6}{F_4}.$$

Similar to [3], the proof of the above theorem is again based on the method of equivariant differential geometry. Following the framework of [5], [3], one may equip the orbit space $K \backslash G/K$ with a specific Riemannian metric defined in terms of the orbital geometry of the K-action on G/K. Then the proof of the existence of an K-invariant minimal sphere is reduced to showing the existence of an imbedded geodesic arc both started and ended at the boundary. Due to the fact that the above metric is degenerate along the boundary, the geodesic equation becomes singular at boundary points. Therefore one of the major analytical in-put needed for the existence proof is the understanding of the behavior of solutions originated at such singularities.

In §2 we shall use the method of power series substitution and majorization to prove the existence and uniqueness of analytic solutions of certain type of nonlinear ordinary differential equations in the neighborhood of singularity. This technical result will also be useful in the proof of the existence of constant mean curvature imbeddings of exotic spheres and knotted spheres into the standard Riemannian spheres [cf. 4].

1. The basic ideas of construction and the orbital geometry of K-action on G/K

Let G/K be one of the following symmetric spaces, namely,

$$\frac{SU(3)}{SO(3)}, \quad SU(3), \quad \frac{SU(6)}{Sp(3)}, \quad \frac{E_6}{F_4}.$$

Then one has the following orbital geometry of the K-action on G/K (Cf. [3] for the proofs of (i) and (ii).)

(i) The orbit space $K \backslash G/K$ equipped with the orbital distance metric (which measures the distance between orbits) can be identified with a flat regular triangle Δ, namely, $ds^2 = dx^2 + dy^2$.

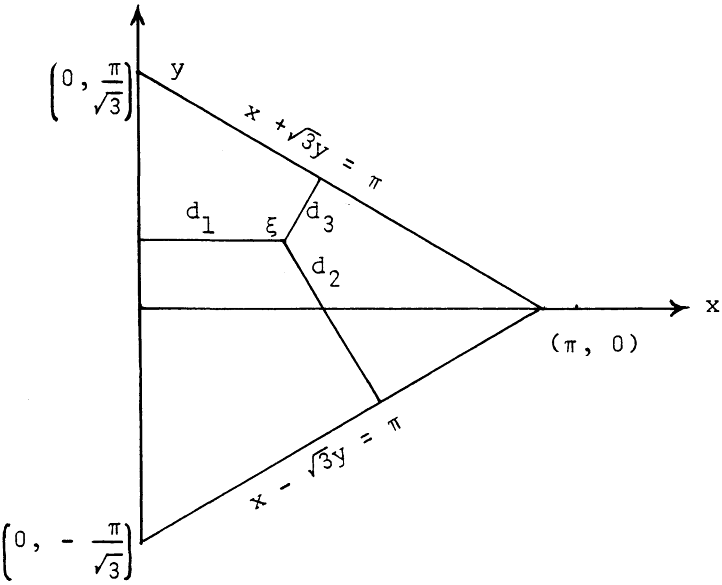

FIG. 1

(ii) The volume function $v(\xi)$ defined on the above orbit space which records the volume of principal orbits (its value on lower dimensional orbits is defined to be zero) is given by

$$v(\xi) = c\left[\sin(d_1(\xi)) \cdot \sin(d_2(\xi)) \cdot \sin(d_3(\xi))\right]^k,$$

where c is a constant, $d_i(\xi)$, $i = 1, 2, 3$, are the distance of ξ to the three sides of Δ, and $k = 1, 2, 4, 8$ according to the above four cases of G/K.

Combining the above two geometric structures of K-orbits, one may define a new metric on $\Delta = K \setminus G/K$, namely, $d\bar{s}^2 = v^2\, ds^2 = v^2(dx^2 + dy^2)$. Then it is not difficult to verify the following relationships between codimension-one K-invariant submanifolds Γ in G/K and their images γ in the orbit space Δ:

(1) The codimension-one volume of Γ equals the $\bar{s}$-length of γ. Γ is a minimal submanifold if and only if γ is an $\bar{s}$-geodesic.

(2) Let $p \in \Gamma$, and $\xi = K(p) \in \gamma \subset \Delta$ be a principal orbit. Let H be the mean curvature of Γ at p, and k', $\bar{k}$ be the geodesic curvatures of γ at ξ with respect to ds, $d\bar{s}$ respectively. Then

$$H = \frac{\bar{k}}{v(\xi)} = k' - \frac{d}{d\vec{n}} \ln v,$$

where $d/d\vec{n}$ is the directional derivative with respect to the unit normal $\vec{n}$.

(3) Let γ be a curve in Δ given by $y = y(x)$. Then γ is an $\bar{s}$-geodesic if and only if $y(x)$ is a solution of the following equation:

$$(*)\qquad \begin{aligned}\sin x\left(\cos x + \cos\sqrt{3}\,y\right)y'' = -k\left(1 + y'^2\right)\Big\{\sqrt{3}\,\sin x \sin\sqrt{3}\,y\\ +y'\left(\cos 2x + \cos x\cos\sqrt{3}\,y\right)\Big\}.\end{aligned}$$

(4) Geometrically, if γ is an imbedded curve in Δ which starts at one side and ends at another side perpendicularly (as indicated in the following), then the inverse image of γ, $\Gamma = \pi^{-1}(\gamma)$, is automatically an imbedded sphere.

Therefore our basic idea of construction of such a minimal sphere is to prove the existence of an $\bar{s}$-geodesic $\bar{\gamma}$ which starts at one side and ends at another side of Δ perpendicularly. There are the following major steps in the construction of such an $\bar{s}$-geodesic $\bar{\gamma}$.

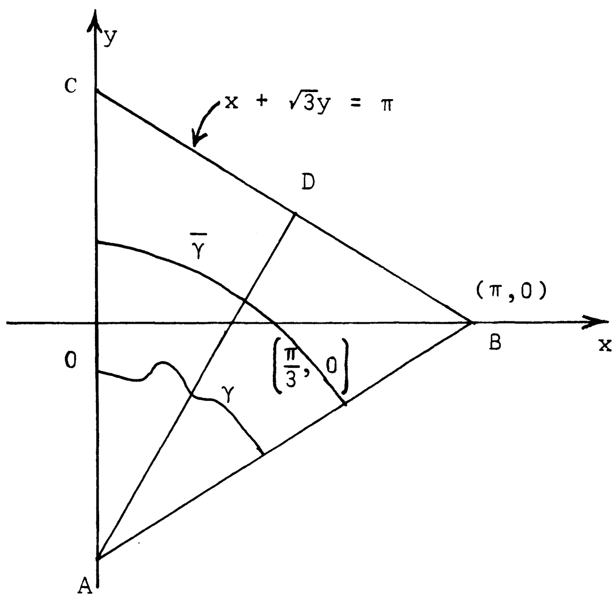

FIG. 2

Step 1. In §2 we shall use the method of power series substitution and majorization to establish the existence of an analytical family of $\bar{s}$-geodesics locally expressible in terms of analytic functions, namely,

$$\gamma_t\colon y = y(t, x) \quad \text{with}\, y(t,0) = t,\ |t| < \pi/\sqrt{3}\,.$$

Step 2. Study some global behavior of the above family $\{\gamma_t\}$ of $\bar{s}$-geodesics, especially the intersection angle $\theta(t)$ of γ_t and the bisector $\overline{AD}$.

Step 3. Use Gauss-Bonnet formula to show that there exist t_1, t_2 such that $\theta(t_1) < \frac{1}{2}\pi < \theta(t_2)$. Therefore it follows from the continuity of $\theta(t)$ with respect to t that there exists $\bar{\gamma} = \gamma_{t_0}$ with $\theta(t_0) = \frac{1}{2}\pi$. Then by reflection principle and uniqueness of geodesic one may simply continue $\bar{\gamma}$ by reflecting it with respect to $\overline{AD}$ and thus obtaining an $\bar{s}$-geodesic perpendicular to both sides. The inverse image $\Gamma = \pi^{-1}(\bar{\gamma})$ of such an $\bar{s}$-geodesic is then a minimal imbedding of sphere into M^n.

2. On the existence and uniqueness of analytic solutions for a type of ordinary differential equations with singularity

Analytically, the differential equation $(*)$ is singular at $x = 0$ and hence requires a special treatment. However, from a geometric viewpoint, those $\bar{s}$-geodesic are simply images of nonsingular minimal submanifolds in M^n. Therefore it follows from the interior regularity theorem (for codimension-one nonsingular minimal submanifolds) that such solutions, if exist, are automatically analytic. In this section, we shall prove the existence and uniqueness of analytic solutions for a type of ordinary differential equations with singularity which arises naturally in the study of equivariant differential geometric problems around singular orbits.

Proposition 1. *There exists a unique analytic solution $y = y(t, x)$ for the following system $(**)$ which is a convergent power series of (t, x) in a neighborhood of $(0, 0)$ and $y(t, 0) = 0$, $dy/dx(t, 0) = p(t, 0) = 0$:*

$$(**)\qquad \begin{aligned} \frac{dy}{dx} &= p, \\ x\frac{dp}{dx} &= \lambda p + a_{0,1,0,0}x + \psi(t, x, y, p), \end{aligned}$$

where λ is not a positive integer, t is a parameter, and

$$\psi = \sum_{\substack{l+m+n+\nu \geq 2 \\ m+n+\nu \geq 1}} a_{l,m,n,\nu} t^l \cdot x^m y^n p^\nu.$$

Proof. Let us first try to solve the above system of equations by a formal power series substitution, namely,

$$y = \sum_{\substack{i \geq 0 \\ j \geq 1}} b_{i,j+1} t^i x^{j+1}, \quad p = \sum_{\substack{i \geq 0 \\ j \geq 1}} b'_{i,j} t^i x^j.$$

Substituting the above formal expression of y and p into $(**)$, one gets

$$b_{i,j+1} = \frac{1}{j+1} b'_{i,j}, \quad i \geqslant 0, j \geqslant 1,$$

$$\sum_{\substack{i \geqslant 0 \\ j \geqslant 1}} (j-\lambda) b'_{ij} t^i x^j = a_{0100} x + \psi(t, x, y, p).$$

By comparing the coefficients of both sides of the above formal identity, one obtains

$$(j-\lambda) b'_{ij} = Q_{ij}\big(a_{lmn\nu};\, b_{\alpha\beta};\, b'_{\gamma\delta}\big), \quad (j-\lambda) \neq 0.$$

where Q_{ij} is a polynomial with nonnegative integral coefficients in the variables $a_{lmn\nu}$, $b_{\alpha\beta}$ and $b'_{\gamma\delta}$ with $\alpha, \gamma \leqslant i$, $\beta, \delta \leqslant j$ and $\alpha+\beta$, $\gamma+\delta < i+j$. Therefore one may inductively determine all the coefficients $b_{i,j+1}$ and $b'_{i,j}$ uniquely from the above algebraic equations. This proves the uniqueness of such an analytic solution of $(**)$.

Next we shall use the method of majorization to prove the convergence of the above formal power series of y and p. Since λ is *not* a positive integer, there exists $N > 0$ such that $|j - \lambda| \geqslant 1/N$ for all $j \geqslant 1$. For the purpose of comparison, let us consider the following system of equations, namely,

$$(***) \qquad \begin{aligned} &Y = x \cdot P, \\ &\frac{1}{N} P = A_{1000} t + A_{0100} x + \Psi(t, x, Y, P), \\ &Y|_{t=0,x=0} = 0, \quad P|_{t=0,x=0} = 0, \end{aligned}$$

where $\Psi(t, x, Y, P) = \sum_{l+m+n+\nu \geqslant 2} A_{lmn\nu} t^l x^m Y^n P^\nu$ is analytic in a neighborhood of origin, and $A_{lmn\nu} \geqslant |a_{lmn\nu}|$. It follows from the implicit function theorem that Y, P are analytic functions of t, x in the neighborhood of $(0,0)$. Let

$$Y = \sum_{i+j \geqslant 1} B_{i,j+1} t^i x^{j+1}, \quad P = \sum_{i+j \geqslant 1} B'_{ij} t^i x^j.$$

Then it is not difficult to see that $B_{i,j+1} = B'_{i,j}$ and

$$\frac{1}{N} B'_{ij} = Q'_{ij}\big(A_{lmn\nu};\, B_{\alpha\beta};\, B'_{\gamma\delta}\big),$$

where Q'_{ij} is the same polynomial as that of the formal solution of $(**)$. Therefore it follows from $A_{lmn\nu} \geqslant |a_{lmn\nu}|$ that

$$B_{i,j+1} \geqslant |b_{i,j+1}|, \quad B'_{i,j} \geqslant |b'_{i,j}|,$$

and hence $Y(t, x) \gg y(t, x)$. Explicitly, one may choose the right hand side of the second equation of (***) as follows:

$$F(t, x, Y, P) = \frac{M}{\left(1 - \frac{t+X}{\rho_1}\right)\left(1 - \frac{Y+P}{\rho_2}\right)} - M\left(1 + \frac{Y+P}{\rho_2}\right)$$

for suitable M, ρ_1 and ρ_2. Then we may further majorize the two auxiliary functions Y and P by a single function $Z = Z(t, x)$ satisfying the following quadratic equation:

$$Z = \frac{NM}{\left(1 - \frac{t+x}{\rho_1}\right)\left(1 - \frac{2Z}{\rho_2}\right)} - NM\left(1 + \frac{2Z}{\rho_2}\right), \quad Z|_{t=0,x=0} = 0,$$

Therefore it is easy to show the convergence of Z and hence the convergence of Y, P and y, p. This completes the proof of existence of such an analytic solution of (**).

Corollary 1. *There exists an analytical family of $\bar{s}$-geodesics $\{\gamma_t\}$ which are locally given by $y = y(t, x)$, $y(t, 0) = t$, $|t| < \pi/\sqrt{3}$.*

Proof. Set $y = t + \tilde{y}$, and $\tilde{y} = \tilde{y}(t, x)$ be the unique analytic solution of the following system of equations of (**)-type:

$$\frac{d\tilde{y}}{dx} = p,$$

$$x\frac{dp}{dx} = \frac{-k(1+p^2)}{\cos x + \cos\sqrt{3}(t+\tilde{y})}\Big\{\sqrt{3}\,x \sin\sqrt{3}(t+\tilde{y}) + \frac{px}{\sin x}\left[\cos 2x + \cos x \cos\sqrt{3}(t+\tilde{y})\right]\Big\},$$

$$\tilde{y}|_{x=0} = 0, \quad p|_{x=0} = 0.$$

3. Some global behavior of the family of $\bar{s}$-geodesics $\{\gamma_t; 0 \leqslant t < \pi/\sqrt{3}\}$

In §2 we prove the existence and uniqueness of an analytical family of $\bar{s}$-geodesics $\{\gamma_t\}$ *locally* given by $y = y(t, x)$, $y(t, 0) = t$. In this section we shall proceed to study some of their global behavior, especially their intersection property with the bisector $\overline{AD}$. From the local analytical dependence of $y(t, x)$ on the parameter t and the Lipschitz condition on any compact subset of Δ it follows that both the position and the direction of γ_t vary continuously with respect to the parameter t. It is easy to see that γ_0 is simply given by $y(0, x) \equiv 0$ which intersects with $\overline{AD}$ at an angle $\frac{1}{3}\pi$. Therefore, in case t is close to 0, γ_t will intersect with $\overline{AD}$ and the segment of γ_t between $\overline{AC}$ and $\overline{AD}$ is expressible as the graph of $y = y(t, x)$. For a given $0 < t < \pi/\sqrt{3}$, let $I_t = (0, \alpha_t)$ be the maximal interval on the x-axis such that the above $\bar{s}$-geodesic γ_t is expressible as the graph of a differentiable function $y = y(t, x)$ and $y(t, x) > 0$ for all $x \in I_t$.

Lemma 1. *If* $|y'(t, x)| < 1$ *for all* $0 < x < \frac{1}{3}\pi$, *then* $\alpha_t \geqslant \frac{1}{3}\pi$, *and* $y'(t, x)$, $y''(t, x)$ *are both negative for all* $x \in I_t$.

Proof. (i) Since $y''(t, 0) < 0$ and $y'(t, 0) = 0$, one has $y'(t, x) < 0$ for sufficiently small x. We claim that $y'(t, x) < 0$ for all $x \in I_t$. Suppose the contrary. Then there exists $x_0 \in I_t$ such that $y'(t, x_0) = 0$ but $y'(t, x) < 0$ for all $0 < x < x_0$. However, it follows from the differential equation $(*)$ that

$$\sin x_0 \left(\cos x_0 + \cos\sqrt{3}\, y_0\right) y_0'' = -k\sqrt{3}\sin x_0 \sin\sqrt{3}\, y_0 < 0,$$

and hence $y_0'' = y''(t, x_0) < 0$ which contradicts the assumption that $y_0 = 0$ and $y'(t, x) < 0$ for $0 < x < x_0$.

(ii) Next let us prove that $y''(t, x) < 0$ for all $x \in I_t$. If $\cos 2x + \cos x \cos\sqrt{3}\, y \leqslant 0$, then it follows directly from $(*)$ and $y' < 0$ that $y'' < 0$. Hence we need only to show that for the remaining case where

$$\cos 2x + \cos x \cos\sqrt{3}\, y > 0,$$

we have $y''(t, x) < 0$, which clearly implies that $x < \frac{1}{3}\pi$. Suppose the contrary. Then there exists $0 < x_1 < \frac{1}{3}\pi$ such that $y''(t, x_1) = 0$ but $y''(t, x) < 0$ for all $0 < x < x_1$. Differentiate $(*)$ at $x = x_1$, one obtains

$$\begin{aligned} y_1''' &= y'''(t, x_1) \\ &= \frac{-k(1 + y_1')^2}{\sin x_1 \cdot \left(\cos x_1 + \cos\sqrt{3}\, y_1\right)} \\ &\quad \cdot \left\{\left(1 - y_1'^2\right)\sqrt{3}\cos x_1 \sin\sqrt{3}\, y_1 + 2\left(\sin x_1 \cos\sqrt{3}\, y_1 - \sin 2x_1\right) y_1'\right\}, \end{aligned}$$

and hence from the assumption $|y_1'| < 1$ and $y_1' < 0$ it follows that $y_1''' < 0$, which is clear a contradiction to the above choice of x_1. Therefore $y''(t, x) < 0$ for all $x \in I_t$.

(iii) Finally, we shall prove that $\alpha_t \geqslant \frac{1}{3}\pi$. From $y'(t, x) < 0$ for all $0 < x < \alpha_t$ it follows that $y'(t, \alpha_t) \leqslant 0$. On the other hand, the uniqueness of geodesic gives that $y'(t, \alpha_t) \neq 0$, hence $y'(t, \alpha_t) < 0$. Therefore

$$y''(t, \alpha_t) = \lim_{x \to \alpha_t} y''(t, x) \leqslant 0$$

and $(*)$ implies that

$$(\cos 2\alpha_t + \cos \alpha_t) \leqslant 0, \text{ or } \alpha_t \geqslant \tfrac{1}{3}\pi.$$

This completes the proof of Lemma 1.

4. The existence proof

As it was explained in §1, the existence of K-invariant minimal spheres in the four specific symmetric spaces, namely, $SU(3)/SO(3)$, $SU(3)$, $SU(6)/Sp(3)$ and E_6/F_4, can be reduced to the existence of a specific $\bar{s}$-geodesic curve $\bar{\gamma} = \gamma_{t_0}$ which starts at the boundary $\overline{AC}$ and perpendicular to the bisector $\overline{AD}$. Let $\mathcal{F} = \{\gamma_t;\ t \in \mathcal{S} \subseteq (0, \pi/\sqrt{3})\}$ be the subfamily of those $\bar{s}$-geodesics γ_t satisfying the following conditions:

(i) γ_t intersects $\overline{AD}$,

(ii) the part of γ_t between $\overline{AD}$ and $\overline{AD}$ is expressible as the graph of a differentiable function $y = y(t, x)$ with $y''(t, x) < 0$ and

$$-\frac{1}{\sqrt{3}} < y'(t, x) \leqslant 0.$$

From Lemma 1 and the continuous dependence of γ_t on the parameter t it follows that $\mathcal{S}$ is an open subset of $(0, \pi/\sqrt{3})$ containing an open interval $(0, t_0)$, $t_0 > 0$.

Lemma 2. $t_0 < \pi/\sqrt{3}$.

Proof. Suppose the contrary that $t_0 = \pi/\sqrt{3}$, so that $\gamma_t \in \mathcal{F}$ for $t = \pi/\sqrt{3} - \delta$, where δ is an arbitrary small positive number. Then, by assumption, the $\bar{s}$-geodesic γ_t lies between the following two parallel lines:

$$l_0 : y = \frac{-1}{\sqrt{3}} x + \frac{\pi}{\sqrt{3}},$$

$$l_\delta : y = \frac{-1}{\sqrt{3}} x + \frac{\pi}{\sqrt{3}} - \delta.$$

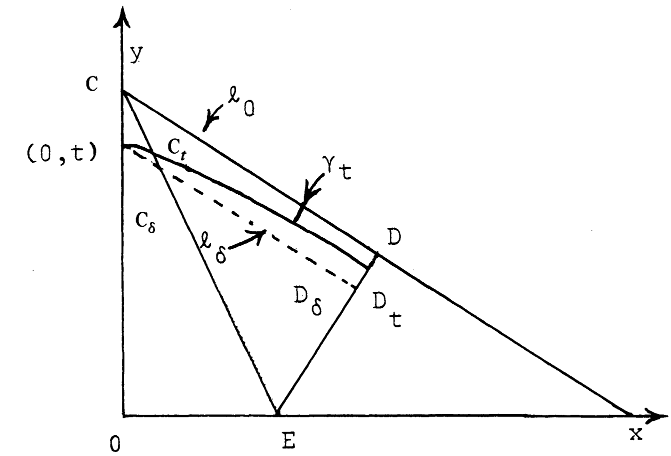

FIG. 3

Let C_t, D_t be the intersection points of γ_t with $\overline{EC}$ and $\overline{ED}$ respectively, and C_δ, D_δ be the intersection points of l_δ with $\overline{EC}$ and $\overline{ED}$ respectively. (Cf. Fig. 3.) Then it follows from condition (ii) of γ_t that

$$\angle C_t < \frac{\pi}{3}, \quad \angle D_t < \frac{\pi}{2}, \quad \angle E = \frac{\pi}{3}$$

for the three interior angles of the $\bar{s}$-geodesic triangle $\bar{\Delta} ED_tC_t$. Therefore one has the following inequality from the positivity of the Gaussian curvature $\bar{K}$ and the Gauss-Bonnet formula:

$$\int_{\bar{\Delta} ED_\delta C_\delta} \bar{K} d\bar{S} < \int_{\bar{\Delta} ED_t C_t} \bar{K} d\bar{S} = \angle C_t + \angle D_t + \angle E - \pi < \frac{\pi}{6},$$

where $\bar{K} = (\sin d_1 \cdot \sin d_2 \cdot \sin d_3)^{-k}(csc^2 d_1 + csc^2 d_2 + csc^2 d_3)$, and $d\bar{S} = (\sin d_1 \cdot \sin d_2 \cdot \sin d_3)^k \, dx\, dy$. However, the above inequality is clearly impossible to hold for small δ, and this contradiction proves that $t_0 < \pi/\sqrt{3}$.

Existence theorem. $\bar{\gamma} = \gamma_{t_0}$ intersects perpendicularly with $\overline{AD}$. Hence $\bar{\gamma}$ is an $\bar{s}$-geodesic starting at $\overline{AC}$ and ending at $\overline{AB}$, and $\Gamma = \pi^{-1}(\bar{\gamma})$ is a K-invariant minimal sphere of the symmetric space G/K.

Proof. By the definition of t_0, $\gamma_t \in \mathfrak{F}$ for $0 < t < t_0$ but $\gamma_{t_0} \notin \mathfrak{F}$. Hence from the continuity of γ_t with respect to the parameter t it follows that γ_{t_0} still intersects with $\overline{AD}$, and the part of γ_{t_0} between $\overline{AC}$ and $\overline{AD}$ is also expressible as the graph of a differentiable function $y = y(t_0, x)$ with $-1/\sqrt{3} \leqslant y'(t_0, x) \leqslant 0$. Therefore it follows from Lemma 1 that $y''(t_0, x) < 0$ and that $y'(t_0, x)$ is a monotonic decreasing function of x. Let $(x_0, y(x_0))$ be the intersection point of γ_{t_0} and $\overline{AD}$. Then one must have $y'(t_0, x_0) = -1/\sqrt{3}$, and hence $\gamma_{t_0} \perp \overline{AD}$, for otherwise, $\gamma_{t_0} \in \mathfrak{F}$ which contradicts to the definition of t_0. This completes the proof of the existence of $\bar{\gamma}$ and Γ.

5. Concluding remarks on spheres of constant mean curvature

In concluding this paper, we shall briefly discuss the relationship between results of this paper and the isoparametric problem on symmetric spaces. Let M^n be a given symmetric space. As a straightforward generalization of the classical isoparametric problem on Riemannian manifolds of constant curvature, it is rather natural to formulate the following isoparametric problem on the symmetric space M^n:

A ball-like region Ω_0 is called a solution of the isoparametric problem of total volume C if the $(n-1)$-dimensional volume of its boundary $\partial\Omega_0$ is minimal among all ball-like regions Ω of total volume C in M^n. Such a solution Ω_0 is said to be smooth if Ω_0 is a smooth imbedding of D^n. It is easy to show that the boundary $\partial\Omega_0$ of a smooth solution of the above isoparametric problem is automatically an imbedding of S^{n-1} with constant mean curvature. We conjecture the existence of the following nice solution of isoparametric problem in the case of irreducible symmetric spaces.

Conjecture on isoparametric problem of symmetric spaces. Let $M^n = G/K$ be an irreducible compact simply connected symmetric space, and C be a given positive number less than the total volume of M^n. We conjecture that there exists a unique (up to global isometry of M^n) *smooth K-invariant solution* Ω_0 of the isoparametric problem for all ball-like regions of volume C in M^n. The boundary $\partial\Omega_0$ is a smooth imbedding of S^{n-1} with constant mean curvature $h(C)$, where $h(C)$ is a continuous function of C uniquely determined by the symmetric space M^n and

$$\lim_{C\to 0} h(C) = +\infty, \quad \lim_{C\to \mathrm{vol}(M^n)} h(C) = -\infty.$$

In the case where $M^n = G/K$ is an irreducible simply connected noncompact symmetric space, let C be any positive number. We conjecture that there exists a unique smooth K-invariant solution Ω_0 of the isoparametric problem for ball-like regions of volume C in M^n. the boundary $\partial\Omega_0$ is a smooth imbedding of S^{n-1} of constant mean curvature $h(C)$ which is a continuous function determined by M^n and

$$\lim_{C\to 0} h(C) = +\infty, \quad \lim_{C\to +\infty} h(C) = 0.$$

We conjecture further that the boundary $\partial\Omega_0$ of such a solution Ω_0 should be expressible as a radical graph $r = f(\theta)$ with respect to the "polar coordinate system" with the center of gravity of Ω_0 as its origin.

Therefore the study of isoparametric problem on a symmetric space G/K naturally leads to the construction of a family of spheres of constant mean curvatures which are the natural generalization of concentric spheres in Euclidean space. In the special case of rank-one symmetric spaces, the family of spheres given by the graphs of $r =$ constants is obviously such a family of constant mean curvature spheres. The method of this paper will enable us to construct a family of constant mean curvature spheres in rank-two symmetric spaces.

References

[1] É. Cartan, *Sur certaines formes riemmiennes remarquable des geometries à groupe fondmental simple*, Ann. Sci. École. Norm. Sup. **44** (1927) 345–467.

[2] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

[3] W. T. Hsiang & W. Y. Hsiang, *Examples of codimension-one closed minimal submanifolds in some symmetric spaces*. I, J. Differential Geometry **15** (1980) 543–551.

[4] ______, *On the construction of constant mean curvature imbeddings of exotic and/or knotted spheres into the unit sphere*. III, to appear in Invent. Math.

[5] W. Y. Hsiang & B. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Differential Geometry **5** (1971) 1–38.

[6] H. Weyl, *Theorie der Darstellung Kontinnierlicher halbeinfacher Gruppen durch lineare Transformation*. I, II, III *und Nachtrag*, Math. Z. **23** (1925) 271–309, **24** (1926) 328–376, 377–395, 789–791.

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