J. DIFFERENTIAL GEOMETRY
17 (1982) 55–138

COMPLEX-ANALYTICITY OF HARMONIC MAPS, VANISHING AND LEFSCHETZ THEOREMS

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Dedicated to Professor S. S. Chern on his 70th birthday

Abstract

In this paper we discuss the relationship between the various techniques of proving vanishing theorems and the method of obtaining the complex-analyticity of harmonic maps between Kähler manifolds. We then obtain sharp results on the complex-analyticity of harmonic maps with the curvature conditions on the target manifold expressed in natural and familiar terms and also results concerning curvature characterizations of compact symmetric Kähler manifolds, Barth-Lefschetz type theorems, the generalization of the strong Lefschetz theorem, and vanishing theorems.

Introduction

This paper is an outgrowth of an attempt to apply the method of proving the strong rigidity of compact Kähler manifolds to obtain vanishing theorems for holomorphic vector bundles. To prove the strong rigidity of negatively curved compact Kähler manifolds, one tried to use harmonic maps $f\colon M \to N$ between compact Kähler manifolds (for definition and background of harmonic maps see [17], [18], or [53]) and the technique of considering $\Delta\,|\,\bar\partial f\,|^2$. (The technique of considering the Laplacian of the square norm was first introduced by Bochner [10] in the case of harmonic tensors on Riemannian manifolds and later applied by Kodaira [33] to $(0, q)$-forms on a Kähler manifold with values in a Hermitian holomorphic line bundle.) With this method of proving strong rigidity one encountered the difficulty of two curvature terms of opposite signs, one involving the Ricci curvature of M and the other involving the full curvature tensor of N. In [53] the author overcame this difficulty by the following variation of the Bochner-Kodaira technique which for convenience's sake we refer to in this paper as the $\partial\bar\partial$ Bochner-Kodaira technique. One considers the integral of $\partial\bar\partial(\Sigma_{\alpha,\beta}\, g_{\alpha\bar\beta}\bar\partial f^\alpha \wedge \partial f^{\bar\beta}) \wedge \omega_M^{n-2}$ over M instead of $\Delta\,|\,\bar\partial f\,|^2$, where $g_{\alpha,\bar\beta}$ is the Kähler metric of N, ω_M is the Kähler form of M, and

Received October 10, 1981. Research partially supported by an NSF grant.

n is complex dimension of M. This $\partial\bar{\partial}$ Bochner-Kodaira technique enables one to get rid of the Ricci tensor of M and thereby to conclude the complex-analyticity (or conjugate complex-analyticity) of the harmonic map f if $\operatorname{rank}_{\mathbf{R}} df \geqslant 4$ and the curvature tensor of M is strongly negative in the sense of [53].

It is natural to attempt to apply this $\partial\bar{\partial}$ Bochner-Kodaira technique to obtain vanishing theorems for negative bundles. Suppose E is a holomorphic vector bundle over a compact Kähler manifold M with Hermitian metric $h_{\alpha\bar{\beta}}$ along its fibers, and suppose φ is a harmonic E-valued $(0, q)$-form. One considers the integral of $\partial\bar{\partial}(\Sigma_{\alpha,\beta} h_{\alpha\bar{\beta}}\varphi^{\alpha} \wedge \overline{\varphi^{\beta}}) \wedge \omega_M^{n-q-1}$ over M. In this way one obtains the following vanishing theorem. Let

$$\Theta_{\alpha\bar{\beta}} = \sqrt{-1} \sum_{\gamma,\nu} h_{\gamma\bar{\beta}}\bar{\partial}\left(h^{\bar{\nu}\gamma}\partial h_{\alpha\bar{\nu}}\right)$$

be the curvature form of E. Let $0 \leqslant q < n$. If for every nonzero E-valued $(0, q)$-form (ξ^{α}) at any point of M

$$(*)_q \qquad \begin{gathered}(-1)^{q(q+1)/2}\left(\sqrt{-1}\right)^q \sum_{\alpha,\beta} \Theta_{\alpha\bar{\beta}} \wedge \xi^{\alpha} \wedge \overline{\xi^{\beta}} \wedge \omega_M^{n-q-1} \\ \text{is a negative multiple of } \omega_M^n,\end{gathered}$$

then $H^q(M, E) = 0$. In particular, if the curvature tensor of an n-dimensional compact Kähler manifold M is very strongly negative in the sense of [53] (as, for example, in the case of a compact quotient of the open n-ball), then for $0 \leqslant q < n$ the tangent bundle T_M of M satisfies $(*)_q$ and $H^q(M, E) = 0$.

At first these vanishing theorems obtained by the $\partial\bar{\partial}$ Bochner-Kodaira technique seemed to the author to be new theorems until he became puzzled by the following situation. Since the curvature tensor of $\mathbf{P}_n$ is the same as that of the n-ball with an opposite sign and since the tangent bundle of the n-ball satisfies $(*)_q$ for $0 \leqslant q < n$, it is natural to expect that the cotangent bundle $\Omega^1_{\mathbf{P}_n}$ of $\mathbf{P}_n$ should also satisfy $(*)_q$ for $0 \leqslant q < n$. However, this would lead to the vanishing of $H^1(\mathbf{P}_n, \Omega^1_{\mathbf{P}_n})$ in the case $n > 1$ which is a contradiction, because the Kähler form of $\mathbf{P}_n$ is a nonzero class in that cohomology group.

To resolve this puzzle, I examined closely the curvature form of the dual bundle of a bundle satisfying $(*)_q$. I discovered that a bundle satisfies $(*)_q$ if and only if its dual bundle satisfies the positivity condition for $(0, q)$-forms in the sense of Nakano [41]. In this paper we call this positivity condition Nakano q-positivity (see §4.1). For example, the curvature tensor of a Kähler manifold is very strongly negative in the sense of [53] if and only if its cotangent bundle is positive in the sense of Nakano [41] (or Nakano 1-positive in the terminology of this paper). The vanishing theorems obtained by the $\partial\bar{\partial}$ Bochner-Kodaira technique turn out to be no other than the usual vanishing theorems for

Nakano q-positive bundles expressed in dual form via Serre duality. The puzzle resulted from the fact that the dual bundle of a Nakano q-positive bundle is in general not Nakano q-negative (see §4.2).

The original Bochner technique is equivalent to applying integration by parts to the global square norm of the gradient of a harmonic tensor. In the case of a Hermitian holomorphic vector bundle E over a compact Kähler manifold, the gradient of a harmonic E-valued $(0, q)$-form can be decomposed into two parts. One part is the $(0, 1)$-gradient and the other is the $(1, 0)$-gradient. Integration by parts applied to the global square norm of the $(0, 1)$-gradient yields the vanishing theorem for positive bundles. Integration by parts applied to the global square norm of the $(1, 0)$-gradient yields the vanishing theorem for negative bundles. For convenience's sake in this paper we refer to these two techniques respectively as the $\overline{\nabla}$ Bochner-Kodaira technique and the ∇ Bochner-Kodaira technique. The $\overline{\nabla}$ and ∇ Bochner-Kodaira techniques are transformed to each other by the Hodge star operator composed with conjugation. The Akizuki-Nakano vanishing theorem [1] for E-valued harmonic (p, q)-forms when E is a line bundle is the consequence of comparing the $\overline{\nabla}$ and ∇ Bochner-Kodaira techniques.

Further investigation shows that the $\partial\bar{\partial}$ Bochner-Kodaira technique is equivalent to the ∇ Bochner-Kodaira technique in the sense that each term in the equation obtained by the $\partial\bar{\partial}$ Bochner-Kodaira technique can be transformed by using identities in multilinear algebra to the corresponding term obtained by the ∇ Bochner-Kodaira technique. The identity in multilinear algebra used in transforming the curvature term is not transparent and is an interesting identity by itself (see §3.6). This identity plays a very useful role in this paper involving the complex-analyticity of harmonic maps. Moreover, this identity, together with the relationship between the curvatures of a Hermitian bundle and its dual, leads us to realize that the underlying reason why the vanishing theorem of Calabi-Vensentini [12] holds is the Nakano q-positivity of the cotangent bundle of a bounded symmetric domain for an appropriate q (see §§6.4 and 6.5).

Since the $\partial\bar{\partial}$ Bochner-Kodaira technique is equivalent to the ∇ Bochner-Kodaira technique which in turn can be transformed to the $\overline{\nabla}$ Bochner-Kodaira technique by the Hodge star operator composed with conjugation, from the method [53] of proving the complex-analyticity of harmonic maps by the $\partial\bar{\partial}$ Bochner-Kodaira technique we can derive two other methods of proving the complex-analyticity of harmonic maps. One uses the ∇ Bochner-Kodaira technique and the other uses the $\overline{\nabla}$ Bochner-Kodaira technique. The one using the ∇ Bochner-Kodaira technique was presented in [55] without explaining how it is related to the method of [53]. In this paper we present the one using the $\overline{\nabla}$ Bochner-Kodaira technique. This is the most natural one, because it is parallel to the proof of the usual Kodaira vanishing theorem and, more importantly, the Morrey trick of handling the boundary term in the case with boundary works most directly in the $\overline{\nabla}$ Bochner-Kodaira technique. The method of proof by means of the $\overline{\nabla}$ Bochner-Kodaira technique explains why one of the curvature conditions on the target manifold is the Nakano q-positivity of the cotangent bundle.

By using the $\overline{\nabla}$ Bochner-Kodaira technique, we get in this paper a proof of the conjecture [55, §6] concerning the complex-analyticity of harmonic maps of appropriately high ranks from compact Kähler manifolds into quotients of irreducible bounded symmetric domains. This proof makes use of the identity in multilinear algebra mentioned above, the eigenvalues of the curvature operator computed by Calabi-Vesentini [12] and Borel [11], and the computation of what we call the degree of the strong nondegeneracy of the bisectional curvature (see §5.8) which is a measure of the dimensions of the null spaces of the bisectional curvature. We have to rely on the computation by Zhong [66] of the degree of the strong nondegeneracy of the bisectional curvature in the case of the two exceptional domains. This conjecture yields as a corollary the strong rigidity of the compact quotients of irreducible bounded symmetric domains of complex dimension $\geqslant 2$. The proof of strong rigidity via this conjecture is the most natural and elegant and by far the simplest proof.

As a corollary of the confirmation of this conjecture we show by using the method of Kalka [32] that, for a complex submanifold of appropriately high dimension in a compact quotient of an irreducible bounded symmetric domain, the deformation as a submanifold agrees with the deformation as an abstract manifold.

Besides the quotient of bounded symmetric domains, these results more generally hold for Kähler manifolds whose cotangent bundle is Nakano 1-semipositive and Nakano p-positive and whose bisectional curvature is strongly p-nondegenerate, when the rank over $\mathbf{R}$ of the map is at least $2p + 1$ or the complex dimension of the submanifold is at least $p + 1$.

Since the Morrey trick is directly applicable to the $\overline{\nabla}$ Bochner-Kodaira technique, we obtain also, in the case where the domain manifold has boundary, results concerning the complex-analyticity of harmonic maps satisfying the tangential Cauchy-Riemann equations and concerning the extension of maps satisfying tangential Cauchy-Riemann equations from the boundary of the domain manifold to holomorphic maps defined on the whole domain manifold.

Though the complex-analyticity of harmonic maps was discussed in [53], [55] and extended to the case with boundary in [42], [65] (see Remarks 5.19), yet there was no real understanding of the geometric meaning of the curvature conditions on the target manifold introduced in [53] and used in the other papers. Our discussion of the various Bochner-Kodaira techniques not only makes sharper the results on the complex-analyticity of harmonic maps but also reduces the curvature conditions on the tangent manifold to the more natural and familiar notions of Nakano 1-semipositivity and p-positivity of the cotangent bundle and the strong p-nondegeneracy of the bisectional curvature.

In [56], [54] the curvature characterizations of the complex projective space and the complex hyperquadric were obtained by proving the complex-analyticity of energy-minimizing harmonic maps by the second variation formula. In this paper we obtain the following partial result on the curvature characterization of general compact symmetric Kähler manifolds. If the cotangent bundle of a compact Kähler manifold is Nakano 1-seminegative and if at some point the bisectional curvature is irreducible, then either the Kähler manifold is an irreducible Hermitian symmetric manifold with respect to the given Kähler metric, or its cohomology righ with coefficients in $\mathbf{R}$ is isomorphic to that of the complex projective space. As a consequence, on an irreducible compact Hermitian symmetric space of rank > 1 any other Kähler metric which makes the cotangent bundle Nakano 1-seminegative must be a constant multiple of the standard invariant Kähler metric. Here the irreducibility of the bisectional curvature at a point means that it is not possible to decompose the holomorphic tangent space into two orthogonal direct summands so that the bisectional curvature in the direction of two tangent vectors, one from each summand, is always zero. For the proof of this result we do not use energy-minimizing harmonic maps. Instead, we use multilinear algebra (cf. [8], [21], [37], [43]) to transform the curvature term in the $\overline{\nabla}$ Bochner-Kodaira technique to show that harmonic (p, q)-forms are parallel. Then we use Simon's result [51] on the transitivity of holonomy systems and Weyl's theory [64] of the invariants of the unitary group to obtain our result.

In Schneider's scheme [49] of using the Grauert-Riemanschneider vanishing theorem [22] to prove Barth-Lefschetz type theorems for compact symmetric Kähler manifolds, he had trouble with the curvature term when the rank of the symmetric manifold is > 1. The multilinear algebra used in proving the parallelism of harmonic (p, q)-forms in the curvature characterization of compact symmetric Kähler manifolds can be used to complete Schneider's scheme. However, Schneider's proof of the strong hyper-q-convexity of the complement of a complex submanifold in a compact symmetric Kähler manifold seems to be invalid. If one indeed has the hyper-q-convexity as Schneider claimed, the Barth-Lefschetz theorems at the homotopy level can easily be proved by using Morse theory, which we do in this paper instead of completing Schneider's scheme. More precisely, we prove the following. Let M be a compact Kähler manifold of complex dimension n with nonnegative bisectional curvature. Let V be a complex submanifold of M admitting a tubular neighborhood U with smooth boundary such that $M - U$ has strongly hyper-q-convex boundary. Then $\pi_\nu(M, V)$ vanishes for $\nu \leqslant n - q$. The proof involves using the second variation formula for arc length and what we call q-plurisubharmonic functions which has the property of being subharmonic on local complex submanifolds of complex dimension q.

We also prove the surjectivity portion of a Barth-Lefschetz type theorem at the homology level for compact Kähler manifolds whose bisectional curvature is nonnegative and appropriately nondegenerate. This is done by proving a generalized strong Lefschetz theorem which asserts that cupping with the top Chern class of a Hermitian vector bundle is surjective (respectively injective) for cohomology groups of dimensions greater than (respectively smaller than) a certain number when the curvature of the bundle is semipositive in the sense of Griffiths and is appropriately nondegenerate.

Finally the close look we have at the various Bochner-Kodaira techniques leads us to two, though very minor, results on vanishing theorems. One is a generalization of the Akizuki-Nakano theorem to the case of a semi-negative line bundle over a compact Kähler manifold and a corresponding statement for vector bundles (see §§4.7 and 4.8). Another is a vanishing theorem for semipositive line bundles over a non-Kähler compact complex manifold which is motivated by the Grauert-Riemanschneider conjecture (see §10).

In this paper we will use the summation convention of summing over any index which appears once as a subscript and once as a superscript. The usual process of raising and lower indices by using metric tensors will be performed without explicit mention. Standard notations in Kähler and Riemannian geometries which carry obvious meanings will not be explained. For example, when z^i are the local holomorphic coordinates, ∂_i means $\partial/\partial z^i$ and $\partial_{\bar{i}}$ means $\partial/\partial \bar{z}^i$; the components of a (p, q)-form φ with values in a vector bundle are $\varphi^\alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}$. For a complex manifold M we denote the holomorphic tangent bundle by T_M, and the bundle of holomorphic q-forms by Ω^q_M. We denote the holomorphic tangent space of M at P by $T_{M,P}$. The space $T_{M,P}$ as a vector space over $\mathbf{R}$ is isomorphic to the real tangent space of M at P (when M is regarded as a real manifold) under the isomorphism defined by taking the real part of a tangent vector with complex coefficients. This isomorphism is actually an isomorphism over $\mathbf{C}$, when the real tangent space of M at P is made into a $\mathbf{C}$-vector space by the almost complex structure operator of M. Because of this isomorphism, we denote the real tangent space of M at P also by $T_{M,P}$.

We will explicitly mention which of the two meanings $T_{M,P}$ takes on when it is used.

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1. The $\overline{\nabla}$ and ∇ Bochner-Kodaira techniques

1.1. The original Bochner technique [10] is to integrate the Laplacian of the pointwise square norm of a harmonic form over a compact Riemannian manifold, yielding thereby two terms. One is the global square norm of the gradient (i.e., the covariant derivative) of the harmonic form. The other involves the curvature tensor. If the curvature tensor satisfies some suitable positivity condition, then it follows that the harmonic form must be zero or parallel. Equivalently, one can regard this procedure as transforming the global square norm of the gradient of the harmonic tensor by integration by parts to a term involving the curvature tensor. In the process of integration by parts the principlal step of computation is to compute the Laplacian of the harmonic tensor.

1.2. In the case of a Hermitian holomorphic vector bundle E over a compact Kähler manifold, the gradient of an E-valued (p, q)-form can be decomposed into two parts. One part is the $(0, 1)$-gradient, and the other is the $(1, 0)$-gradient. Integration by parts applied to the global square norm of the $(0, 1)$-gradient in the case $p = 0$ yields the vanishing theorem for positive line bundles. This is due to Kodaira [33]. We call this technique *the* $\overline{\nabla}$ *Bochner-Kodaira technique*.

One can also apply integration by parts to the global square norm of the $(1, 0)$-gradient in the case $p = 0$ and get the vanishing theorem for negative line bundles. We call this technique the ∇ *Bochner-Kodaira technique*. Usually the vanishing theorem for the negative bundle is obtained from the vanishing theorem for the positive bundle and Serre duality or from the Akizuki-Nakano vanishing theorem [1]. It is not proved by using the ∇ Bochner-Kodaira technique.

One can obtain the Akizuki-Nakano vanishing theorem [1] by comparing the integration by parts for the (0, 1)-gradient of φ with that for the (1, 0)-gradient of $\bar{\varphi}$ in the case of a harmonic (p, q)-form φ. This is not the usual proof which uses the identity $[\bar{\partial}, \Lambda] = -\sqrt{-1}\,\partial^*$ where Λ is the transpose of the operator defined by multiplication by the Kähler form.

The principal step in the $\bar{\nabla}$ and ∇ Bochner-Kodaira techniques is the computation of the Laplacian of an E-valued (p, q)-form. The formulas for such computations are well-known. We collect them below and fix notations.

1.3. Let M be a Kähler manifold with Kähler metric $g_{i\bar{j}}$. Its curvature tensor is

$$R_{i\bar{j}k\bar{l}} = \partial_i\partial_{\bar{j}}g_{k\bar{l}} - g^{\bar{s}t}\partial_i g_{k\bar{s}}\partial_{\bar{j}}g_{t\bar{l}},$$

and its Ricci curvature tensor is

$$R_{i\bar{j}} = g^{k\bar{l}}R_{i\bar{j}k\bar{l}}.$$

Note that in this convention $R_{i\bar{j}}$ is negative definite as a Hermitian matrix when the sectional curvature of M is positive.

Let E be a Hermitian holomorphic vector bundle over M with Hermitian metric $h_{\alpha\bar{\beta}}$ along its fibers. The curvature form

$$\Theta_{\alpha\bar{\beta}} = -\sqrt{-1}\sum_{i,j}\Omega_{\alpha\bar{\beta}i\bar{j}}dz^i \wedge dz^{\bar{j}}$$

of E is given by

$$\Omega_{\alpha\bar{\beta}i\bar{j}} = \partial_i\partial_{\bar{j}}h_{\alpha\bar{\beta}} - h^{\lambda\bar{\mu}}\partial_i h_{\alpha\bar{\mu}}\partial_{\bar{j}}h_{\lambda\bar{\beta}}.$$

This convention is chosen so that $\Theta_{\alpha\bar{\beta}}$ is a positive (1, 1)-form when E is a positive line bundle, and $\Omega_{\alpha\bar{\beta}i\bar{j}}$ agrees with the curvature tensor when E is the holomorphic tangent bundle of M. Let

$$\Omega_{\alpha\bar{\beta}} = g^{i\bar{j}}\Omega_{\alpha\bar{\beta}i\bar{j}}.$$

Let ∇_i, $\nabla_{\bar{i}}$ denote the covariant differential operators. Let $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and $\bar{\Box} = \partial\partial^* + \partial^*\partial$. Let

$$\varphi^\alpha = \frac{1}{p!q!}\sum \varphi^\alpha_{I_p\bar{J}_q}dz^{I_p} \wedge dz^{\bar{J}_q}$$

be an E-valued (p, q)-form on M, where $I_p = (i_1,\cdots,i_p)$, $\bar{J}_q = (\bar{j}_1,\cdots,\bar{j}_q)$, $dz^{I_p} = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, and $dz^{\bar{J}_q} = dz^{\bar{j}_1} \wedge \cdots \wedge dz^{\bar{j}_q}$. From

$$(1.3.1) \qquad (\bar{\partial}\varphi)^\alpha_{I_p\bar{j}_0\cdots\bar{j}_q} = (-1)^p\sum_{\nu=0}^{q}(-1)^\nu\nabla_{\bar{j}_\nu}\varphi^\alpha_{I_p\bar{j}_0\cdots\hat{\bar{j}}_\nu\cdots\bar{j}_p},$$

where $\hat{\bar{j}}_\nu$ means that the index $\bar{j}_\nu$ is removed, it follows that

$$
\begin{aligned}
(\bar{\partial}^*\varphi)^\alpha_{I_p\bar{j}_1\cdots\bar{j}_{q-1}} &= (-1)^{p+1} g^{\bar{j}i}\nabla_i\varphi^\alpha_{I_p\bar{j}\bar{j}_1\cdots\bar{j}_{q-1}}, \\
(\Box\varphi)^\alpha_{I_p\bar{J}_q} &= -g^{i\bar{j}}\nabla_{\bar{j}}\nabla_i\varphi^\alpha_{I_p\bar{J}_q} - \sum_{\nu=1}^{q} \Omega^{\alpha\bar{l}}_{\beta\,\bar{j}_\nu}\varphi^\beta_{I_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q} \\
&\quad - \sum_{\nu=1}^{q} R^{\bar{l}}{}_{\bar{j}_\nu}\varphi^\alpha_{I_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q} \\
&\quad + \sum_{\mu=1}^{p}\sum_{\nu=1}^{q} R_{i_\mu}{}^{k\bar{l}}{}_{\bar{j}_\nu}\varphi^\alpha_{i_1\cdots(k)_\mu\cdots i_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q},
\end{aligned}
\tag{1.3.2}
$$

where $(k)_\mu$ means that the index in the μth place is replaced by k.

When M is compact, by contracting formula (1.3.2) with $\bar{\varphi}/(p!q!)$ and integrating over M one obtains

$$
\begin{aligned}
\|\bar{\partial}\varphi\|^2_M + \|\bar{\partial}^*\varphi\|^2_M &= \|\overline{\nabla}\varphi\|^2_M - \frac{1}{p!(q-1)!}\int_M \Omega_{\alpha\bar{\beta}}{}^{\bar{s}}{}_{\bar{t}}\varphi^\alpha_{I_p\bar{s}\bar{J}_{q-1}}\overline{\varphi^{\beta\bar{I}_p tJ_{q-1}}} \\
&\quad - \frac{1}{p!(q-1)!}\int_M R^{\bar{s}}{}_{\bar{t}}\varphi^\alpha{}_{I_p\bar{s}\bar{J}_{q-1}}\overline{\varphi_{\bar{\alpha}}^{\bar{I}_p tJ_{q-1}}} \\
&\quad + \frac{1}{(p-1)!(q-1)!}\int_M R_l{}^{k\bar{s}}{}_{\bar{t}}\varphi^\alpha{}_{kI_{p-1}\bar{s}\bar{J}_{q-1}}\overline{\varphi_{\bar{\alpha}}{}^{\bar{l}\bar{I}_{p-1}tJ_{q-1}}},
\end{aligned}
\tag{1.3.3}
$$

where $\|\cdot\|_M$ denotes the global L^2 norm over M, and $\overline{\nabla}\varphi$ denotes the E-valued tensor with components $\nabla_{\bar{j}}\varphi^\alpha_{I_p\bar{J}_q}$. Formula (1.3.3) is the $\overline{\nabla}$ Bochner-Kodaira technique which yields the vanishing theorem for positive line bundles.

By applying the commutation formula for $[\nabla_i, \nabla_{\bar{j}}]\varphi$ to formula (1.3.2), we obtain

$$
\begin{aligned}
(\Box\varphi)^\alpha_{I_p\bar{J}_q} &= -g^{i\bar{j}}\nabla_{\bar{j}}\nabla_i\varphi^\alpha_{I_p\bar{J}_q} - \sum_{\nu=1}^{q} \Omega_\beta{}^{\alpha\bar{l}}{}_{\bar{j}_\nu}\varphi^\beta{}_{I_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q} + \Omega_\beta{}^\alpha\varphi^\beta_{I_p\bar{J}_q} \\
&\quad - \sum_{\mu=1}^{p} R_{i_\mu}{}^k\varphi^\alpha_{i_1\cdots(k)_\mu\cdots i_p\bar{J}_q} \\
&\quad + \sum_{\mu=1}^{p}\sum_{\nu=1}^{q} R_{i_\mu}{}^{k\bar{l}}{}_{\bar{j}_\nu}\varphi^\alpha_{i_1\cdots(k)_\mu\cdots i_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q}.
\end{aligned}
\tag{1.3.4}
$$

When M is compact, by contracting formula (1.3.4) with $\bar{\varphi}/(p!q!)$ and integrating over M one obtains

$$
\begin{aligned}
\|\bar{\partial}\varphi\|_M^2 + \|\bar{\partial}^*\varphi\|_M^2 &= \|\nabla\varphi\|_M^2 - \frac{1}{p!\,(q-1)!}\int_M \Omega_{\alpha\bar{\beta}}{}^{\bar{s}}{}_{\bar{t}}\varphi^{\alpha}_{I_p\bar{s}\bar{J}_{q-1}}\overline{\varphi^{\beta\bar{I}_p t J_{q-1}}} \\
&\quad + \frac{1}{p!q!}\int_M \Omega_{\alpha\bar{\beta}}\varphi^{\alpha}_{I_p\bar{J}_q}\overline{\varphi^{\beta\bar{I}_p J_q}} - \frac{1}{(p-1)!q!}\int_M R_s{}^t\varphi^{\alpha}_{tI_{p-1}\bar{J}_q}\overline{\varphi_{\bar{\alpha}}^{\bar{s}\bar{I}_{p-1}J_q}} \\
&\quad + \frac{1}{(p-1)!\,(q-1)!}\int_M R_l{}^{k\bar{s}}{}_{\bar{t}}\varphi^{\alpha}{}_{kI_{p-1}\bar{s}\bar{J}_{q-1}}\overline{\varphi_{\bar{\alpha}}^{\bar{l}\bar{I}_{p-1}tJ_{q-1}}},
\end{aligned}
\tag{1.3.5}
$$

where $\nabla\varphi$ denotes the E-valued tensor with components $\nabla_i\varphi^{\alpha}_{I_p\bar{J}_q}$. This formula is the ∇ Bochner-Kodaira technique which yields the vanishing theorem for negative line bundles. In the same way as deriving (1.3.4), by using $\overline{\Box}$ instead of $\Box$, we obtain

$$
\begin{aligned}
(\overline{\Box}\varphi)^{\alpha}_{I_p\bar{J}_q} &\doteq -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\varphi^{\alpha}_{I_p\bar{J}_q} \\
&\quad + \sum_{\mu=1}^{p}\Omega_{\beta}{}^{\alpha}{}_{i_\mu}{}^{k}\varphi^{\beta}_{i_1\cdots(k)_\mu\cdots i_p\bar{J}_q} - \Omega_{\beta}{}^{\alpha}\varphi^{\beta}_{I_p\bar{J}_q} \\
&\quad - \sum_{\nu=1}^{q} R^{\bar{l}}_{\bar{j}_\nu}\varphi^{\alpha}_{I_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q} \\
&\quad + \sum_{\mu=1}^{p}\sum_{\nu=1}^{q} R_{i_\mu}{}^{k\bar{l}}{}_{\bar{j}_\nu}\varphi^{\alpha}_{i_1\cdots(k)_\mu\cdots i_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q}.
\end{aligned}
\tag{1.3.6}
$$

Another way to derive this formula is to apply formula (1.3.4) to $\bar{\varphi}$ and taking complex conjugates of both sides. In this derivation one has to be careful about the interpretation of $\bar{\varphi}$. One has to lower the index α of $\overline{\varphi^{\alpha}}$, and regard $\psi_\beta = h_{\beta\bar{\alpha}}\overline{\varphi^{\alpha}}$ as a (q,p)-form with coefficients in the dual bundle E^* of E. The curvature form of E^* is the negative of that of E (cf. Lemma 4.3). This accounts for the fact that the terms of (1.3.6) which involve the curvature form of E differ in sign from those obtained by formally applying (1.3.4) to $\bar{\varphi}$ and taking complex conjugates.

Subtracting (1.3.6) from (1.3.2) we obtain

$$
\begin{aligned}
((\Box - \overline{\Box})\varphi)^{\alpha}_{I_p\bar{J}_q} &= -\sum_{\mu=1}^{p}\Omega_{\beta}{}^{\alpha}{}_{i_\mu}{}^{k}\varphi^{\beta}_{i_1\cdots(k)_\mu\cdots i_p\bar{J}_q} \\
&\quad - \sum_{\nu=1}^{q}\Omega_{\beta}{}^{\alpha\bar{l}}{}_{\bar{j}_\nu}\varphi^{\beta}_{I_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q} + \Omega_{\beta}{}^{\alpha}\varphi^{\beta}_{I_p\bar{J}_q}.
\end{aligned}
\tag{1.3.7}
$$

When E is the trivial line bundle, formula (1.3.7) gives an alternative proof of the well-known formula $\square = \overline{\square}$ for (p, q)-forms on a Kähler manifold, with coefficients in the trivial line bundle.

When M is compact, by contracting formula (1.3.7) with $\overline{\varphi}/(p!q!)$ and integrating over M one obtains

$$\begin{aligned}
&\|\bar\partial\varphi\|^2_M + \|\bar\partial^*\varphi\|^2_M - \|\partial\varphi\|^2_M - \|\partial^*\varphi\|^2_M \\
&\quad = \frac{-1}{(p-1)!q!}\int_M \Omega_{\alpha\bar\beta s}{}^{t}\varphi^{\alpha}_{tI_{p-1}\bar J_q}\overline{\varphi^{\beta\bar s I_{p-1}J_q}} \\
&\qquad - \frac{1}{p!(q-1)!}\int_M \Omega_{\alpha\bar\beta}{}^{\bar s}{}_{\bar t}\varphi^{\alpha}_{I_p\bar s\bar J_{q-1}}\overline{\varphi^{\beta\bar I_p t J_{q-1}}} \\
&\qquad + \frac{1}{p!q!}\int_M \Omega_{\alpha\bar\beta}\varphi^{\alpha}_{I_p\bar J_q}\overline{\varphi^{\beta\bar I_p J_q}}\,.
\end{aligned} \tag{1.3.8}$$

This formula yields the Akizuki-Nakano vanishing theorem [1] in the case of a negative line bundle. This derivation is more transparent than the usual proof using the identity $[\bar\partial, \Lambda] = \sqrt{-1}\,\partial^*$. It shows that the Akizuki-Nakano vanishing theorem holds because of the failure of $\square - \overline{\square}$ due to the curvature of the bundle.

1.4. The Hodge star operator $*$ composed with complex conjugation can be extended to an operator $\bar *$ mapping E-valued (p, q)-forms to E^*-valued $(n-p, n-q)$-forms, where n is the complex dimension of M. It is straightforward to verify that the $\overline{\nabla}$ Bochner-Kodaira technique applied to an E-valued (p, q)-form φ is equivalent to the ∇ Bochner-Kodaira technique applied to the E^*-valued $(n-p, n-q)$-form $\bar *\varphi$.

2. The Morrey trick for the boundary term

2.1. The Morrey trick was introduced by Morrey [40, p. 176, Th. 6.1] to handle the boundary terms when the $\overline{\nabla}$ Bochner-Kodaira technique is applied to a domain with boundary. He did the case of (0, 1)-forms, and Kohn [35, p. 113, Th. 5.6] extended it to the case of (p, q)-forms.

We use the notations of §1. Let G be a relatively compact subdomain of M with smooth boundary given by $G = \{\rho < 0\}$, where ρ is a smooth function on M so that the pointwise norm of $d\rho$ is identically one on ∂G. Let $\mathrm{Dom}_G\,\bar\partial^*$ denote the domain of the adjoint operator of the operator $\bar\partial$ defined for smooth E-valued $(0, q-1)$-forms. Then a smooth E-valued (p, q)-form φ on $\overline{G}$ belongs to $\mathrm{Dom}_G\,\bar\partial^*$ if and only if

$$g^{s\bar t}\rho_s\varphi^{\alpha}_{I_p\bar t\bar J_{q-1}} = 0 \tag{2.1.1}$$

on ∂G, where $\rho_s = \partial_s\rho$.

When one tries to derive (1.3.3) from (1.3.2) with M replaced by G, one has the following two additional terms on the right-hand side:

$$\frac{1}{q!}\int_{\partial G}\rho_{\bar s}\varphi^{\alpha}_{I_p\bar J_q}\overline{(\bar\partial\varphi)_{\bar\alpha}{}^{\bar I_p s J_q}} - \frac{1}{q!}\int_{\partial G}\rho_{\bar s}\varphi^{\alpha}_{I_p\bar J_q}\overline{\nabla^{s}\varphi_{\bar\alpha}{}^{\bar I_p J_q}},$$

coming respectively from $\|\bar\partial\varphi\|^2_G$ and $\|\overline{\nabla}\varphi\|^2_G$. (The boundary term form $\|\bar\partial^*\varphi\|\|^2_G$ vanishes because φ belongs to $\operatorname{Dom}_G\bar\partial^*$.) By using (1.3.1) we combine these two boundary terms together to get

$$\text{(2.1.2)}\qquad \frac{-1}{q!}\sum_{\nu=1}^{q}\int_{\partial G}\rho_{\bar s}\varphi^{\alpha}_{I_p\bar J_q}\overline{\nabla^{j_\nu}\varphi_{\bar\alpha}{}^{\bar I_p j_1\cdots(s)_\nu\cdots j_q}}.$$

From (2.1.1) it follows that for $1\leqslant\nu\leqslant q$

$$\text{(2.1.3)}\qquad \rho_{\bar s}\overline{\varphi_{\bar\alpha}{}^{\bar I_p j_1\cdots(s)_\nu\cdots j_q}} = \rho\psi_{\alpha}{}^{I_p\bar j_1\cdots\hat{\bar j}_\nu\cdots\bar j_q}$$

for some smooth $\psi_\alpha{}^{I_p\bar j_1\cdots\hat{\bar j}_\nu\cdots\bar j_q}$ on $\overline{G}$. Applying $\sum_{\nu=1}^{q}\varphi^{\alpha}_{I_p\bar J_q}\nabla^{\bar j_\nu}$ to (2.1.1) we obtain

$$\begin{aligned}\sum_{\nu=1}^{q}\varphi^{\alpha}_{I_p\bar J_q}\left(\nabla^{\bar j_\nu}\rho_{\bar s}\right)\overline{\varphi_{\bar\alpha}{}^{\bar I_p j_1\cdots(s)_\nu\cdots j_q}} + \sum_{\nu=1}^{q}\rho_{\bar s}\varphi^{\alpha}_{I_p\bar J_q}\overline{\nabla^{j_\nu}\varphi_{\bar\alpha}{}^{\bar I_p j_1\cdots(s)_\nu\cdots j_q}}&\\ = \sum_{\nu=1}^{q}\varphi^{\alpha}_{I_p\bar J_q}\left(\nabla^{\bar j_\nu}\rho\right)\psi_{\alpha}{}^{I_p\bar j_1\cdots\hat{\bar j}_\nu\cdots\bar j_q}&\\ =0&\end{aligned}$$

on ∂G, because $\varphi^{\alpha}_{I_p\bar J_q}(\nabla^{\bar j_\nu}\rho)=0$ on ∂G due to (2.1.1). Hence the boundary term in (2.1.2) becomes

$$\frac{1}{(q-1)!}\int_{\partial G}(\nabla_s\rho_{\bar t})\varphi^{\alpha\,s}_{I_p\,\bar J_{q-1}}\overline{\varphi_{\bar\alpha}{}^{\bar I_p t J_{q-1}}}.$$

We thus have the following formula:

$$\begin{aligned}\|\bar\partial\varphi\|^2_G+\|\bar\partial^*\varphi\|^2_G=\|\overline{\nabla}\varphi\|^2_G&+\frac{1}{(q-1)!}\int_{\partial G}(\partial_s\partial_{\bar t}\rho)\varphi^{\alpha\,s}_{I_p\,\bar J_{q-1}}\overline{\varphi_{\bar\alpha}{}^{\bar I_p t J_{q-1}}}\\ &+\frac{1}{p!(q-1)!}\int_G\Omega_{\alpha\bar\beta}{}^{\bar s}{}_{t}\varphi^{\alpha}_{I_p\bar s\bar J_{q-1}}\overline{\varphi^{\beta\bar I_p t J_{q-1}}}\\ \text{(2.1.4)}\qquad &+\frac{1}{p!(q-1)!}\int_G R^{\bar s}{}_{t}\varphi^{\alpha}_{I_p\bar s\bar J_{q-1}}\overline{\varphi_{\bar\alpha}{}^{\bar I_p t J_{q-1}}}\\ &+\frac{1}{(p-1)!(q-1)!}\int_G R_l{}^{k\bar s}{}_{t}\varphi^{\alpha}{}_{kI_{p-1}\bar s\bar J_{q-1}}\overline{\varphi_{\bar\alpha}{}^{\bar l\bar I_{p-1} t J_{q-1}}}\end{aligned}$$

for smooth E-valued (p,q)-form on $\overline{G}$ belonging to $\operatorname{Dom}_G\bar\partial^*$.

2.2. It is natural to ask whether there is a similar Morrey trick to take care of the boundary terms of the ∇ Bochner-Kodaira technique in the case of a domain with boundary. Hopefully this trick can yield vanishing theorems in the case of pseudoconcave boundary so that it is in some sense dual to the original Morrey trick used to take care of pseudoconvex boundaries. Unfortunately so far such a dual Morrey trick has not been found. However, as observed in §1.4 the ∇ and $\overline{\nabla}$ Bochner-Kodaira techniques can be transformed to each other by the generalized Hodge star operator $\bar{*}$. When the ∇ Bochner-Kodaira technique is applied to a smooth E-valued (p, q)-form φ on $\overline{G}$, the boundary terms can be handled by expressing the integrand in terms of $\psi = \bar{*}\varphi$. A computation strictly analogous to that carried out above yields

$$
\begin{aligned}
&\|\bar\partial\varphi\|_G^2 + \|\bar\partial^*\varphi\|_G^2 \\
&\quad= \|\nabla\varphi\|_G^2 \frac{1}{(n-q-1)!}\int_{\partial G}(\partial_s\partial_{\bar t}\rho)\psi_{\alpha I_{n-p}}{}^{s}{}_{\bar J_{n-q-1}}\overline{\psi^{\bar\alpha \bar I_{n-p} t J_{n-q-1}}} \\
&\qquad+\frac{1}{p!(q-1)!}\int_G \Omega_{\alpha\bar\beta}{}^{\bar s}{}_{t}\varphi^{\alpha}{}_{I_p \bar s \bar J_{q-1}}\overline{\varphi^{\beta \bar I_p t J_{q-1}}} \\
&\qquad-\frac{1}{p!q!}\int_G \Omega_{\alpha\bar\beta}\varphi^{\alpha}_{I_p\bar J_q}\overline{\varphi^{\beta\bar I_p J_q}} - \frac{1}{(p-1)!q!}\int_G R_s{}^{t}\varphi^{\alpha}_{tI_{p-1}\bar J_q}\overline{\varphi_{\bar\alpha}{}^{\bar s\bar I_{p-1}J_q}} \\
&\qquad+\frac{1}{(p-1)!(q-1)!}\int_G R_l{}^{k\bar s}{}_{\bar t}\varphi^{\alpha}_{kI_{p-1}\bar s\bar J_{q-1}}\overline{\varphi_{\bar\alpha}{}^{\bar l\bar I_{p-1}tJ_{q-1}}},
\end{aligned}
\tag{2.2.1}
$$

when $\psi = \bar{*}\varphi$ belongs to $\mathrm{Dom}_G\,\bar\partial^*$. We need the condition $\psi \in \mathrm{Dom}_G\,\bar\partial^*$ instead of $\varphi \in \mathrm{Dom}_G\,\bar\partial^*$ because the boundary terms are handled by the Morrey trick for the $\overline{\nabla}$ Bochner-Kodaira technique after transformation by the generalized Hodge star operator $\bar{*}$. The condition $\psi \in \mathrm{Dom}_G\,\bar\partial^*$ is easily seen to be equivalent to the condition

$$\bar\partial\rho \wedge \varphi = 0 \quad \text{at every point of } \partial G. \tag{2.2.2}$$

Because of a multilinear algebra lemma proved in §3.6 formula (2.2.1) for φ is identical to formula (2.1.4) for $\bar{*}\varphi$.

3. The $\partial\bar\partial$ Bochner-Kodaira technique

3.1. In [53] the complex-analyticity of a harmonic map $f\colon M \to N$ between compact Kähler manifolds is proved under suitable negative curvature and rank conditions by considering the integral of $\partial\bar\partial(h_{\alpha\bar\beta}\bar\partial f^\alpha \wedge \overline{\partial f^\beta}) \wedge \omega^{n-2}$ over M, where $h_{\alpha\bar\beta}$ is the Kähler metric of N, ω is the Kähler form of M, and n is the complex dimension of M. This leads one to using this kind of integral to get vanishing theorems for holomorphic vector bundles over compact Kähler manifolds. More precisely, let E be a Hermitian holomorphic vector bundle with Hermitian metric $h_{\alpha\bar\beta}$ over an n-dimensional compact Kähler manifold M with Kähler form ω. For an E-valued $(0, q)$-form φ on M the integral of $\partial\bar\partial(h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{\varphi^\beta}) \wedge \omega^{n-q-1}$ over M vanishes by Stokes' theorem. By expanding the integrand one obtains a vanishing theorem. We call this technique the $\partial\bar\partial$ Bochner-Kodaira technique. The vanishing theorem obtained this way looks at first sight different from the vanishing theorems obtained from the $\overline{\nabla}$ and ∇ Bochner-Kodaira techniques. However, upon closer observation this $\partial\bar\partial$ Bochner-Kodaira technique is equivalent to the ∇ Bochner-Kodaira technique. This equivalence can easily be obtained by using the exterior algebra of Hermitian vector spaces [62, Chap. I]. This is done in this section. The knowledge of this equivalence will be used in later sections of this paper to get new results on the complex-analyticity of harmonic maps.

3.2. Direct computation (by using normal coordinates of M and normal fiber coordinates of E) yields

$$\begin{aligned}\partial\bar\partial\left(h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1}\right) &= \sqrt{-1}\,\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1} \\ &+ h_{\alpha\bar\beta} D\bar\partial\varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1} + (-1)^{q+1} h_{\alpha\bar\beta}\bar\partial\varphi^\alpha \wedge \overline{\bar\partial\varphi^\beta} \wedge \omega^{n-q-1} \\ &+ (-1)^q h_{\alpha\bar\beta} D\varphi^\alpha \wedge \overline{D\varphi^\beta} \wedge \omega^{n-q-1} - h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{D\bar\partial\varphi^\beta} \wedge \omega^{n-q-1},\end{aligned} \tag{3.2.1}$$

where $\Theta_{\alpha\bar\beta}$ is the curvature form of E (see §1.3), and $D\varphi^\alpha$ (respectively $D\bar\partial\varphi^\alpha$) is the E-valued $(1, q)$-form (respectively $(1, q + 1)$-form) obtained from φ (respectively $\bar\partial\varphi$) by covariant differentiation. Integrating it over M yields

$$\begin{aligned}&\sqrt{-1}\int_M \Theta_{\alpha\bar\beta} \wedge \varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1} + \int_M h_{\alpha\bar\beta} D\bar\partial\varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1} \\ &+ (-1)^{q+1}\int_M h_{\alpha\bar\beta}\bar\partial\varphi^\alpha \wedge \overline{\bar\partial\varphi^\beta} \wedge \omega^{n-q-1} + (-1)^q\int_M h_{\alpha\bar\beta} D\varphi^\alpha \wedge \overline{D\varphi^\beta} \wedge \omega^{n-q-1} \\ &- \int_M h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{D\bar\partial\varphi^\beta} \wedge \omega^{n-q-1} = 0.\end{aligned}$$

We now apply integration by parts to the second term and the last term. From

$$\begin{aligned}&d\left(h_{\alpha\bar\beta}\,\bar\partial\varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1}\right) \\ &\qquad = h_{\alpha\bar\beta} D\bar\partial\varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1} + (-1)^{q+1} h_{\alpha\bar\beta}\,\bar\partial\varphi^\alpha \wedge \overline{\bar\partial\varphi^\beta} \wedge \omega^{n-q-1},\end{aligned}$$

it follows that

$$\int_M h_{\alpha\bar\beta} D\bar\partial\varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1} = (-1)^q\int_M h_{\alpha\bar\beta}\,\bar\partial\varphi^\alpha \wedge \overline{\bar\partial\varphi^\beta} \wedge \omega^{n-q-1}.$$

Likewise

$$\int_M h_{\alpha\bar\beta}\,\varphi^\alpha \wedge \overline{D\bar\partial\varphi^\beta} \wedge \omega^{n-q-1} = (-1)^{q+1}\int_M h_{\alpha\bar\beta}\,\bar\partial\xi^\alpha \wedge \overline{\bar\partial\xi^\beta} \wedge \omega^{n-q-1}.$$

Hence

$$(3.2.2)\qquad \begin{aligned}(-1)^q\int_M h_{\alpha\bar\beta}\,\bar\partial\varphi^\alpha\wedge\overline{\bar\partial\varphi^\beta}\wedge\omega^{n-q-1}+(-1)^q\int_M h_{\alpha\bar\beta}D\varphi^\alpha\wedge\overline{D\varphi^\beta}\wedge\omega^{n-q-1}\\ +\sqrt{-1}\int_M\Theta_{\alpha\bar\beta}\wedge\varphi^\alpha\wedge\overline{\varphi^\beta}\wedge\omega^{n-q-1}=0.\end{aligned}$$

We are going to transform, by using the exterior algebra of Hermitian vector spaces, each term in (3.2.2) to a corresponding term obtained from the ∇ Bochner-Kodaira technique.

3.3. We collect together the formulas we need concerning exterior algebras of Hermitian vector spaces. We use the following standard notations. Let L be the operator of taking wedge product with the Kähler form ω. Let $\langle\cdot\,,\cdot\rangle$ be the pointwise inner product. Let Λ be the adjoint operator of L with respect to $\langle\cdot\,,\cdot\rangle$. The Hodge star operator $*$ is with respect to $\langle\cdot\,,\cdot\rangle$. A k-form ψ is called *primitive* if $\Lambda\psi=0$.

For any primitive k-form ψ and $s\leqslant r$

$$(3.3.1)\qquad \Lambda^sL^r\psi=\left(\prod_{i=0}^{s-1}(r-i)\right)\left(\prod_{j=1}^{s}(n-k-r+j)\right)L^{r-s}\psi.$$

Let $\varepsilon_{p,q}=(-1)^{\frac12(p+q)(p+q+1)}(\sqrt{-1})^{p-q}$. For any primitive (p,q)-form ψ with $p+q=k$

$$(3.3.2)\qquad *L^l\psi=\varepsilon_{p,q}\frac{l!}{(n-k-l)!}L^{n-k-l}\psi$$

for $0\leqslant l\leqslant n-k$. One has $*L^l\psi=0$ if $l>n-k$.

Every k-form v can be uniquely written as

$$(3.3.3)\qquad v=\sum_r L^rv_r,$$

where each v_r is primitive, and r runs from $\max(0,k-n)$ to the largest integer $[k/2]$ not exceeding $k/2$.

For proofs of these three formulas see [62, pp. 21–28].

3.4. Lemma. *For any* $(1,q)$*-form* η

$$\overline{\varepsilon_{1,q}}\,\eta\wedge\bar\eta\wedge\frac{\omega^{n-q-1}}{(n-q-1)!}=(\langle\eta,\eta\rangle-\langle\Lambda\eta,\Lambda\eta\rangle)\frac{\omega^n}{n!}.$$

Proof. By (3.3.3) we can write uniquely $\eta=\eta_0+L\eta_1$, where η_0,η_1 are both primitive. Then by (3.3.1)

$$(3.4.1)\qquad \Lambda\eta=\Lambda L\eta_1=(n-q+1)\eta_1.$$

One has

$$\begin{aligned}\langle \eta, \eta\rangle &= \langle \eta_0, \eta_0\rangle + \langle \eta_0, L\eta_1\rangle + \langle L\eta_1, \eta_0\rangle + \langle L\eta_1, L\eta_1\rangle \\ &= \langle \eta_0, \eta_0\rangle + \langle \Lambda\eta_0, \eta_1\rangle + \langle \eta_1, \Lambda\eta_0\rangle + \langle \eta_1, \Lambda L\eta_1\rangle.\end{aligned}$$

Hence

$$\langle \eta, \eta\rangle = \langle \eta_0, \eta_0\rangle + (n-q+1)\langle \eta_1, \eta_1\rangle. \tag{3.4.2}$$

Since $\varepsilon_{1,q} = -\varepsilon_{0,q-1}$ by direct computation, it follows from (3.3.2) that

$$\begin{aligned}\overline{\varepsilon_{1,q}}\eta \wedge \bar\eta \wedge \omega^{n-q-1} &= \bar\varepsilon_{1,q}(\eta_0 + L\eta_1) \wedge (\bar\eta_0 + L\bar\eta_1) \wedge \omega^{n-q-1} \\ &= \overline{\varepsilon_{1,q}}\eta_0 \wedge L^{n-q-1}\bar\eta_0 - \overline{\varepsilon_{0,q-1}}\eta_0 \wedge L^{n-q}\bar\eta_1 \\ &\quad + \overline{\varepsilon_{1,q}}L\eta_1 \wedge L^{n-q-1}\bar\eta_0 - \overline{\varepsilon_{0,q-1}}\eta_1 \wedge L^{n-q-1}\bar\eta_1 \\ &= \big[(n-q-1)!\langle \eta_0, \eta_0\rangle - (n-q)!\langle \eta_0, L\eta_1\rangle \\ &\quad + (n-q+1)!\langle L\eta_1, \eta_0\rangle - (n-q+1)!\langle \eta_1, \eta_1\rangle\big]\frac{\omega^n}{n!} \\ &= \big[(n-q-1)!\langle \eta_0, \eta_0\rangle - (n-q+1)!\langle \eta_1, \eta_1\rangle\big]\frac{\omega^n}{n!},\end{aligned}$$

because $\langle \eta_0, L\eta_1\rangle = \langle \Lambda\eta_0, \eta_1\rangle = 0$. By (3.4.2) we have

$$\begin{aligned}&\overline{\varepsilon_{1,q}}\eta \wedge \bar\eta \wedge \frac{\omega^{n-q-1}}{(n-q-1)!} \\ &= \big[\langle \eta_0, \eta_0\rangle - (n-q)(n-q+1)\langle \eta_1, \eta_1\rangle\big]\frac{\omega^n}{n!} \\ &= \big[\langle \eta, \eta\rangle - (n-q+1)\langle \eta_1, \eta_1\rangle - (n-q)(n-q+1)\langle \eta_1, \eta_1\rangle\big]\frac{\omega^n}{n!} \\ &= \big[\langle \eta, \eta\rangle - (n-q+1)^2\langle \eta_1, \eta_1\rangle\big]\frac{\omega^n}{n!} \\ &= \big[\langle \eta, \eta\rangle - \langle \Lambda\eta, \Lambda\eta\rangle\big]\frac{\omega^n}{n!}.\end{aligned}$$

3.5. Lemma. (a)

$$\begin{aligned}&-\overline{\varepsilon_{0,q}}\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \frac{\omega^{n-q-1}}{(n-q-1)!} \\ &= \left(\frac{1}{q!}\Omega_{\alpha\bar\beta}\,\varphi^\alpha_{\bar J_q}\overline{\varphi^{\beta J_q}} - \frac{1}{(q-1)!}\Omega_{\alpha\bar\beta s\bar t}\varphi^{\alpha s}{}_{\bar J_{q-1}}\overline{\varphi^{\beta t J_{q-1}}}\right)\frac{\omega^n}{n!},\end{aligned}$$

where as before $\Theta_{\alpha\bar\beta} = -\sqrt{-1}\,\Omega_{\alpha\bar\beta s\bar t}dz^s \wedge d\bar z^{t}$ *and* $\Omega_{\alpha\bar\beta} = \Omega_{\alpha\bar\beta s\bar t}g^{s\bar t}$ *with* $\omega = \sqrt{-1}\,g_{s\bar t}dz^s \wedge d\bar z^{t}$.

(b) *Let* $\zeta = \bar{*}\varphi$. *Then*

$$-\overline{\varepsilon_{0,q}}\,\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \frac{\omega^{n-q-1}}{(n-q-1)!}$$

$$= \frac{1}{n!\,(n-q)!}\Omega_{\alpha\bar\beta s\bar t}\zeta^{\bar\beta}{}_{I_n}{}^{s}{}_{\bar J_{n-q-1}}\overline{\zeta^{\bar\alpha \bar I_n t J_{n-q-1}}}\,\frac{\omega^n}{n!}.$$

Proof. (a) Since φ^β is a $(0, q)$-form, it is primitive. By (3.3.2),

$$-\bar\varepsilon_{0,q}\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \frac{\omega^{n-q-1}}{(n-q-1)!} = -\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha \wedge *L\,\overline{\varphi^\beta}$$

$$= \langle -\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha, L\varphi^\beta\rangle \frac{\omega^n}{n!}$$

$$= \langle \Lambda(-\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha), \varphi^\beta\rangle \frac{\omega^n}{n!}.$$

Using local coordinates we have

$$-\left(\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha\right)_{s\bar t \bar J_q} = \sqrt{-1}\left(\Omega_{\alpha\bar\beta s\bar t}\varphi^\alpha_{\bar J_q} - \sum_{\nu=1}^{q} \Omega_{\alpha\bar\beta s\bar j_\nu}\varphi^\alpha_{\bar j_1\cdots(\bar t)_\nu\cdots \bar j_q}\right).$$

Contracting both sides with $g^{s\bar t}$ we obtain

$$-\left(\Lambda\left(\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha\right)\right)_{\bar J_q} = \Omega_{\alpha\bar\beta}\varphi^\alpha_{\bar J_q} - \sum_{\nu=1}^{q} \Omega_{\alpha\bar\beta}{}^{\bar t}{}_{\bar j_\nu}\varphi^\alpha_{\bar j_1\cdots(\bar t)_\nu\cdots\bar j_q},$$

from which the desired equation follows upon taking the inner product with φ^β.

(b) We choose local coordinates and fiber coordinates such that both Hermitian matrices $h_{\alpha\bar\beta}$ and $g_{i\bar j}$ are identity matrices. Let Σ' denote summation over distinct indices. Then

$$-\overline{\varepsilon_{0,q}}\,\Theta_{\alpha\bar\beta} \wedge \varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1}$$

$$= \frac{1}{(q+1)!}\sum_{j_1,\cdots,j_n}{}' \sum_{\mu,\nu=1}^{q+1} (-1)^{\mu+\nu}\Omega_{\alpha\bar\beta j_\nu \bar j_\mu}\varphi^\alpha_{\bar j_1\cdots\hat{\bar j}_\mu\cdots\bar j_{q+1}}\overline{\varphi^\beta_{\bar j_1\cdots\hat{\bar j}_\nu\cdots\bar j_{q+1}}}\,\frac{\omega^n}{n!}$$

$$= \frac{1}{(q+1)!}\sum_{\alpha,\beta}\sum_{j_1,\cdots,j_n}{}' \sum_{u,\nu=1}^{q+1} \Omega_{\alpha\bar\beta j_\nu \bar j_\mu}\overline{\zeta_{\alpha 1\cdots n \bar j_\mu \bar j_{q+2}\cdots\bar j_n}}\,\zeta_{\beta 1\cdots n\bar j_\nu\bar j_{q+2}\cdots\bar j_n}\frac{\omega^n}{n!}$$

$$= \sum_{\alpha,\beta}\sum_{j_{q+2},\cdots,j_n}{}' \sum_{s,t}\Omega_{\alpha\bar\beta s\bar t}\overline{\zeta_{\alpha 1\cdots n\bar t\bar j_{q+2}\cdots\bar j_n}}\,\zeta_{\beta 1\cdots n\bar s\bar j_{q+2}\cdots\bar j_n}\frac{\omega^n}{n!},$$

from which the desired equation follows. q.e.d.

In the proof of Lemma 3.5 computation is carried out pointwise, and only multilinear algebra is used. By using local coordinates so that both Hermitian matrices $h_{\alpha\bar\beta}$ and $g_{i\bar j}$ are identity matrices, we can combine Part (a) and Part (b) of Lemma 3.5 together to obtain the following lemma in multilinear algebra which will be used later in this paper.

3.6. Lemma. *Let α, β run from 1 to r, and $j_1,\cdots,j_n$, s, t from 1 to n. Fix a positive integer $q < n$. Let $\xi^\alpha_{\bar j_1\cdots\bar j_q}$, $\Xi_{\alpha\bar\beta s\bar t}$ be complex numbers. Assume $\xi^\alpha_{\bar j_1\cdots\bar j_q}$ is skew-symmetric in $j_1,\cdots,j_q$. Define*

$$\theta_{\alpha\bar j_{q+1}\cdots\bar j_n} = \sum_{j_1,\cdots,j_q} \operatorname{sgn}\begin{pmatrix} j_1 & \cdots & j_n \\ 1 & \cdots & n \end{pmatrix} \overline{\xi^\alpha_{\bar j_1\cdots\bar j_q}}.$$

Then

$$\frac{1}{(n-q-1)!}\sum_{\alpha,\beta,s,t,j_{q+2},\cdots,j_n} \Xi_{\alpha\bar\beta s\bar t}\theta_{\beta\bar s\bar j_{q+2}\cdots\bar j_n}\overline{\theta_{\alpha\bar t\bar j_{q+2}\cdots\bar j_n}}$$
$$= \frac{1}{q!}\sum_{\alpha,\beta,j_1,\cdots,j_q}\Xi_{\alpha\bar\beta}\xi^\alpha_{\bar j_1\cdots\bar j_q}\overline{\xi^\beta_{\bar j_1\cdots\bar j_q}}$$
$$- \frac{1}{(q-1)!}\sum_{\alpha,\beta,s,t,j_1,\cdots,j_{q-1}}\Xi_{\alpha\bar\beta s\bar t}\xi^\alpha_{\bar s\bar j_1\cdots\bar j_{q-1}}\overline{\xi^\beta_{\bar t\bar j_1\cdots\bar j_{q-1}}},$$

where $\Xi_{\alpha\bar\beta} = \sum_{s=1}^n \Xi_{\alpha\bar\beta s\bar s}$.

3.7. We are now ready to transform the terms in (3.2.2) to terms obtained from the ∇ Bochner-Kodaira technique. The procedure is done pointwise. Fix a point of M, and choose local coordinates of M and fiber coordinates of E so that at that point both Hermitian matrices $g_{i\bar j}$ and $h_{\alpha\bar\beta}$ are equal to identity matrices.

By applying (3.3.2) to the case $\psi = \bar\partial\varphi$ and $l = 0$, we obtain

$$\overline{\varepsilon_{0,q+1}}\, h_{\alpha\bar\beta}\,\bar\partial\varphi^\alpha \wedge \overline{\bar\partial\varphi^\beta} \wedge \frac{\omega^{n-q-1}}{(n-q-1)!} = \sum_{\alpha=1}^r \bar\partial\varphi^\alpha \wedge *\,\overline{\bar\partial\varphi^\alpha} = \langle\bar\partial\varphi, \bar\partial\varphi\rangle\frac{\omega^n}{n!}.$$

By applying Lemma 3.4 to the case $\eta = D\varphi^\alpha$, we obtain

$$\overline{\varepsilon_{1,q}}\, h_{\alpha\bar\beta} D\varphi^\alpha \wedge \overline{D\varphi^\beta} \wedge \frac{\omega^{n-q-1}}{(n-q-1)!} = \left(\langle\nabla\varphi, \nabla\varphi\rangle - \langle\bar\partial^*\varphi, \bar\partial^*\varphi\rangle\right)\frac{\omega^n}{n!},$$

because $\nabla\varphi = D\varphi$ and $\bar\partial^*\varphi^\alpha = \Lambda D\varphi^\alpha$.

Using Lemma 3.5(a) and $\varepsilon_{1,q} = -\varepsilon_{0,q+1} = \overline{(-1)^{q+1}\sqrt{-1}}\,\varepsilon_{0,q}$, we conclude that, after we multiply the equation (3.2.2) by $(-1)^q\varepsilon_{1,q}/(n-q-1)!$, we obtain

$$-\|\bar\partial\varphi\|_M^2 + \|\nabla\varphi\|_M^2 - \|\bar\partial^*\varphi\|_M^2$$
$$+\int_M\left(\frac{1}{q!}\Omega_{\alpha\bar\beta}\varphi^\alpha_{\bar J_q}\overline{\varphi^{\beta \bar J_q}} - \frac{1}{(q-1)!}\Omega_{\alpha\bar\beta s\bar t}\varphi^{\alpha s}{}_{\bar J_{q-1}}\overline{\varphi^{\beta t J_{q-1}}}\right) = 0,$$

which is the same as the equation obtained by the ∇ Bochner-Kodaira technique.

3.8. Even in the case of Kähler manifolds with boundaries the $\partial\bar{\partial}$ Bochner-Kodaira technique is equivalent to the ∇ Bochner-Kodaira technique. We drop the condition that M is compact. Let G be a relatively compact subdomain of M with smooth boundary. Let φ be a smooth E-valued $(0, q)$-form on $\bar{G}$. When we apply the $\partial\bar{\partial}$ Bochner-Kodaira technique to G instead of M, we get the following three additional boundary terms on the left-hand side of (3.2.2) (besides the three integrals over G).

$$-\int_{\partial G} \bar{\partial}\left(h_{\alpha\bar{\beta}}\varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1}\right) + \int_{\partial G} h_{\alpha\bar{\beta}}\bar{\partial}\varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1}$$

$$+ (-1)^{q+1}\int_{\partial G} h_{\alpha\bar{\beta}}\varphi^\alpha \wedge \overline{\bar{\partial}\varphi^\beta} \wedge \omega^{n-q-1},$$

which is equal to

$$(-1)^{q+1}\int_{\partial G} h_{\alpha\bar{\beta}}\varphi^\alpha \wedge \overline{D\varphi^\beta} \wedge \omega^{n-q-1} + (-1)^{q+1}\int_{\partial G} h_{\alpha\bar{\beta}}\varphi^\alpha \wedge \overline{\bar{\partial}\varphi^\beta} \wedge \omega^{n-q-1}.$$

Let $G = \{\rho < 0\}$, where ρ is a smooth function on M so that the pointwise norm of $d\rho$ is identically one on ∂D. We use the following formula to convert the above two boundary integrals. For any $(2n-1)$-form η on ∂G

$$\int_{\partial G} \eta = \int_{\partial G}\left\langle d\rho \wedge \eta, \frac{\omega^n}{n!}\right\rangle,$$

where on the right-hand side the integral is with respect to the volume form of ∂G which is omitted.

As in the ∇ Bochner-Kodaira technique in the case with boundary, we assume that $\bar{*}\varphi$ belongs to $\mathrm{Dom}_G\,\bar{\partial}^*$. According to (2.2.2) this condition is equivalent to $\bar{\partial}\rho \wedge \varphi = 0$ at every point of ∂G. Hence

$$\int_{\partial G} h_{\alpha\bar{\beta}}\varphi^\alpha \wedge \overline{\bar{\partial}\varphi^\beta} \wedge \omega^{n-q-1} = \int_{\partial G}\left\langle \bar{\partial}\rho \wedge h_{\alpha\bar{\beta}}\varphi^\alpha \wedge \overline{\bar{\partial}\varphi^\beta} \wedge \omega^{n-q-1}, \frac{\omega^n}{n!}\right\rangle$$

vanishes. For the other boundary term we have

$$(-1)^{q+1}\int_{\partial G} h_{\alpha\bar{\beta}}\varphi^\alpha \wedge \overline{D\varphi^\beta} \wedge \omega^{n-q-1}$$

$$= (-1)^{q+1}\int_{\partial G}\left\langle \partial\rho \wedge h_{\alpha\bar{\beta}}\varphi^\alpha \wedge \overline{D\varphi^\beta} \wedge \omega^{n-q-1}, \frac{\omega^n}{n!}\right\rangle.$$

From the vanishing of $\bar{\partial}\rho \wedge \varphi$ at every point of ∂G it follows that $\bar{\partial}\rho \wedge \varphi = \rho\psi$ for some E-valued $(0, q+1)$-form ψ on $\bar{G}$. Applying D to both sides, we obtain

$$\partial\bar{\partial}\rho \wedge \varphi - \bar{\partial}\rho \wedge D\varphi = \partial\rho \wedge \psi$$

at every point of ∂G. Hence

$$\begin{aligned}\partial\rho \wedge h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{D\varphi^\beta} &= (-1)^q h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{\bar\partial\rho \wedge D\varphi^\beta}\\ &= (-1)^q h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{\partial\bar\partial\rho \wedge \varphi^\beta} + (-1)^{q+1} h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{\bar\partial\rho \wedge \psi^\beta}\\ &= (-1)^q \partial\bar\partial\rho \wedge h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{\varphi^\beta}\end{aligned}$$

at every point of ∂G, because $\bar\partial\rho \wedge \varphi = 0$ at every point of ∂G. Thus

$$\begin{aligned}&(-1)^{q+1}\int_{\partial G} h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{D\varphi^\beta} \wedge \omega^{n-q-1}\\ &\qquad = -\int_{\partial G}\left\langle \partial\bar\partial\rho \wedge h_{\alpha\bar\beta}\varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \omega^{n-q-1}, \frac{\omega^n}{n!}\right\rangle.\end{aligned}$$

The integrand corresponds to the value of the Levi form at $\bar{*}\varphi$.

4. Various notions of positivity

4.1. In this section we give the various notions of positivity which will be needed for the discussion of the complex-analyticity of harmonic maps and other results of this paper.

Definition. Let M be a Kähler manifold with Kähler metric $g_{i\bar j}$. A Hermitian holomorphic vector bundle E over M with curvature form $\Theta_{\alpha\bar\beta} = -\sqrt{-1}\sum_{i,j}\Omega_{\alpha\bar\beta i\bar j}dz^i \wedge d\bar z^j$ is said to be *Nakano q-positive* (respectively *semipositive, negative, seminegative*) if at any point of M with $g_{i\bar j} = \delta_{ij}$ (the Kronecker delta), for any nonzero set of complex numbers $\zeta^\alpha_{I_q}$ which is skew-symmetric in the q-tuple I_p of indices, the expression

$$-\sum_{\alpha,\beta,k,l,I_{q-1}} \Omega_{\alpha\bar\beta k\bar l}\zeta^\alpha_{kI_{q-1}}\overline{\zeta^\beta_{lI_{q-1}}}$$

is positive (respectively nonnegative, negative, nonpositive).

Remarks. 1. Any subbundle of a Nakano q-(semi)negative vector bundle is also Nakano q-(semi)negative (cf. the computations of [26, p. 197, (2.14)]).

2. In the case when E is a line bundle, Nakano q-positivity (respectively semipositivity, negativity, seminegativity) means that at every point the sum of any set of q eigenvalues of the curvature form is positive (respectively nonnegative, negative, nonpositive) when the eigenvalues are computed with respect to the Kähler metric of M.

Definition. E is said to be *q-positive* (respectively *semipositive, negative, seminegative*) *in the dual Nakano sense* if at every point of M with $g_{i\bar j} = \delta_{ij}$, for any nonzero set of complex numbers $\zeta^{\alpha}_{I_q}$ which is skew-symmetric in the q-tuple I_q of indices, the expression

$$-\sum_{\alpha,\beta,k,l,I_{q-1}} \Omega_{\alpha\bar{\beta}k\bar{l}}\zeta^{\alpha}_{lI_{q-1}}\overline{\zeta^{\beta}_{kI_{q-1}}}$$

is positive (respectively nonnegative, negative, nonpositive).

Remarks. 1. When E is a line bundle, Nakano k-positivity is equivalent to k-positivity in the dual Nakano sense. The same holds for semipositivity, negativity, and seminegativity.

2. When the exterior product $\bigwedge^q E$ of q copies of E is given the Hermitian metric induced from E, E is Nakano q-positive (respectively q-positive in the dual Nakano sense) if and only if $\bigwedge^q E$ is Nakano 1-positive (respectively 1-positive and the dual Nakano sense). The same holds for semipositivity, negativity, and seminegativity.

3. The condition that the curvature tensor of a Kähler manifold is very strongly (semi)negative as defined in [53] is equivalent to its tangent bundle being 1-(semi)negative in the dual Nakano sense.

4.2. For vector bundles of rank > 1 Nakano q-positivity is in general different from q-positivity in the dual Nakano sense. As an illustration we give below the example which is responsible for motivating part of the discussion which leads to the results of this paper. We take the holomorphic tangent bundle $T_{\mathbf{P}_2}$ of $\mathbf{P}_2$. Using the Fubini-Study metric and a suitable coordinate system, we have $g_{i\bar{j}} = \delta_{ij}$ and

$$\Omega_{1\bar{1}1\bar{1}} = \Omega_{2\bar{2}2\bar{2}} = -2,$$
$$\Omega_{1\bar{1}2\bar{2}} = \Omega_{1\bar{2}2\bar{1}} = \Omega_{2\bar{1}1\bar{2}} = \Omega_{2\bar{2}1\bar{1}} = -1,$$

with all the other components $\Omega_{\alpha\bar{\beta}i\bar{j}}$ being zero. The bundle $T_{\mathbf{P}_2}$ is 1-positive in the dual Nakano sense, because the Hermitian matrix

$$-\begin{pmatrix} \Omega_{1\bar{1}1\bar{1}} & \Omega_{1\bar{1}2\bar{1}} & \Omega_{1\bar{2}1\bar{1}} & \Omega_{1\bar{2}2\bar{1}} \\ \Omega_{1\bar{1}1\bar{2}} & \Omega_{1\bar{1}2\bar{2}} & \Omega_{1\bar{2}1\bar{2}} & \Omega_{1\bar{2}2\bar{2}} \\ \Omega_{2\bar{1}1\bar{1}} & \Omega_{2\bar{1}2\bar{1}} & \Omega_{2\bar{2}1\bar{1}} & \Omega_{2\bar{2}2\bar{1}} \\ \Omega_{2\bar{1}1\bar{2}} & \Omega_{2\bar{1}2\bar{2}} & \Omega_{2\bar{2}1\bar{2}} & \Omega_{2\bar{2}2\bar{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

is positive definite. However, the bundle $T_{\mathbf{P}_2}$ is only Nakano 1-semipositive and not Nakano 1-positive, because the Hermitian matrix

$$-\begin{pmatrix} \Omega_{1\bar{1}1\bar{1}} & \Omega_{1\bar{1}1\bar{2}} & \Omega_{1\bar{2}1\bar{1}} & \Omega_{1\bar{2}1\bar{2}} \\ \Omega_{1\bar{1}2\bar{1}} & \Omega_{1\bar{1}2\bar{2}} & \Omega_{1\bar{2}2\bar{1}} & \Omega_{1\bar{2}2\bar{2}} \\ \Omega_{2\bar{1}1\bar{1}} & \Omega_{2\bar{1}1\bar{2}} & \Omega_{2\bar{1}1\bar{2}} & \Omega_{2\bar{2}1\bar{2}} \\ \Omega_{2\bar{1}2\bar{1}} & \Omega_{2\bar{1}2\bar{2}} & \Omega_{2\bar{2}2\bar{1}} & \Omega_{2\bar{2}2\bar{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

is only positive semidefinite and not positive definite (the second and third rows being equal).

4.3. Lemma. *A Hermitian holomorphic vector bundle E over a Kähler manifold is q-positive (respectively semipositive, negative, seminegative) in the dual Nakano sense if and only if its dual bundle E^* is Nakano q-negative (respectively seminegative, positive, semipositive).*

Proof. Let $h_{\alpha\bar\beta}$ be the Hermitian metric of E. Then $h^{\alpha\bar\beta}$ is the Hermitian metric of E^*. Fix a point and choose local trivializations of E and E^* so that at that point $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$ and $dh_{\alpha\bar\beta} = 0$. The curvature form $\Theta_{\alpha\bar\beta} = -\sqrt{-1}\,\Sigma_{i,j}\,\Omega_{\alpha\bar\beta i\bar j}dz^i \wedge \overline{dz^j}$ of E is given by $\Omega_{\alpha\bar\beta i\bar j} = \partial_i\partial_{\bar j}h_{\alpha\bar\beta}$. The curvature form $\Theta^*_{\alpha\bar\beta} = -\sqrt{-1}\,\Sigma_{i,j}\,\Omega^*_{\alpha\bar\beta i\bar j}dz^i \wedge \overline{dz^j}$ of E^* is given by $\Omega^*_{\alpha\bar\beta i\bar j} = \partial_i\partial_{\bar j}h^{\alpha\bar\beta}$. Since

$$0 = \partial\bar\partial\left(h^{\alpha\bar\gamma}h_{\beta\bar\gamma}\right) = (\partial\bar\partial h^{\alpha\bar\gamma})h_{\beta\bar\gamma} + h^{\alpha\bar\gamma}\partial\bar\partial h_{\beta\bar\gamma} = \partial\bar\partial h^{\alpha\bar\beta} + \partial\bar\partial h_{\beta\bar\alpha},$$

it follows that $\Omega^*_{\alpha\bar\beta i\bar j} = -\Omega_{\beta\bar\alpha i\bar j}$, from which the assertions of the lemma are clear. q.e.d.

As a corollary of Lemma 4.3, any quotient bundle of a q-(semi)positive bundle in the dual Nakano sense is also q-(semi)positive in the dual Nakano sense.

4.4. Definition. Let E be a Hermitian holomorphic vector bundle over a complex manifold M of complex dimension n. Let $\Theta_{\alpha\bar\beta} = -\sqrt{-1}\,\Sigma_{i,j}\,\Omega_{\alpha\bar\beta i\bar j}dz^i \wedge \overline{dz^j}$ be the curvature form of E. The bundle E is said to be *Griffiths q-positive* (respectively *semipositive*, *negative*, *seminegative*) if at any point of M and for any nonzero set of complex number ξ^α the $(1,1)$-form $\Sigma_{\alpha,\beta}\,\Theta_{\alpha\bar\beta}\xi^\alpha\overline{\xi^\beta}$ has at least $n - q + 1$ positive (respectively nonnegative, negative, nonpositive) eigenvalues.

Remarks. 1. Clearly E is Griffiths q-positive (respectively semipositive, negative, seminegative) if and only if its dual bundle E^* is Griffiths q-negative (respectively seminegative, positive, semipositive). This property is different from the case of Nakano positivity and negativity.

2. E is Griffiths q-positive if E is Nakano q-positive of if E is q-positive in the dual Nakano sense. The same holds for semipositivity, negativity, and seminegativity. On the other hand, Demailly and Skoda [14], [15] proved that if E is Griffiths 1-(semi)positive over M, then $E \otimes \det E$ and $E^* \otimes (\det E)^k$ are both Nakano 1-(semi)positive and are also 1-(semi)positive in the dual Nakano sense, where k is the minimum of the rank of E and the complex dimension of M.

3. When E is a line bundle, Griffiths q-positivity (respectively semipositivity, negativity, seminegativity) means that the curvature form has at least $n - q + 1$ positive (respectively semipositive, negative, seminegative) eigenvalues at every point.

4. Any subbundle of a Griffiths q-(semi)negative vector bundle is also Griffiths q-(semi)negative when given the induced Hermitian metric (see [26, p. 197, (2.14)]). Any quotient bundle of a Griffiths q-(semi)positive vector bundle is also Griffiths q-(semi)positive when given the induced Hermitian metric.

The following lemma follows from the standard projectivization argument (see [26, pp. 201–203]).

4.5. Lemma. *Let E be a Hermitian holomorphic vector bundle of rank r over a complex manifold M. Let $\pi\colon \mathbf{P}(E) \to M$ be the projective bundle over M associated to E, and let L be the tautological line bundle over $\mathbf{P}(E)$ with the Hermitian metric induced from that of E. Then the following two statements hold*:

(a) *E is Griffiths q-(semi)negative if and only if L is Griffiths q-(semi)negative.*

(b) *E is Griffiths q-(semi)positive if and only if L is Griffiths $(q + r - 1)$-(semi)positive.*

To conclude this section, we give a generalization of the Akizuki-Nakano theorem to the case of vector bundles which are 1-seminegative and k-negative in the sense of Griffiths. This generalization will not be used in this paper, but the idea of its proof will be used later in §9.

4.6. Lemma. *Let M be a compact Kähler manifold of complex dimension n with Kähler form ω. Let $1 \leqslant k \leqslant n$ and let u be a smooth $(1,1)$-form on M, which is positive semidefinite and has at least k positive eigenvalues at every point of M. For $\varepsilon > 0$ let $\omega_\varepsilon = u + \varepsilon\omega$. Then for ε sufficiently small the following condition is satisfied at every point x of M: For $p + q < k$ and for any subset A of p distinct elements and any subset B of q distinct elements in $\{1, \cdots, n\}$,*

$$\sum_{\gamma=1}^{n} \lambda_\gamma(\varepsilon, x) - \sum_{\alpha \in A} \lambda_\alpha(\varepsilon, x) - \sum_{\beta \in B} \lambda_\beta(\varepsilon, x)$$

is positive, where $\lambda_1(\varepsilon, x) \geqslant \lambda_2(\varepsilon, x) \geqslant \cdots \geqslant \lambda_n(\varepsilon, x)$ are the eigenvalues of u with respect to the Kähler metric whose Kähler form is ω_ε.

Proof. For $x \in M$ let $\lambda_1(x) \geqslant \lambda_2(x) \geqslant \cdots \geqslant \lambda_n(x)$ be the eigenvalues of u at x with respect to the Kähler metric whose Kähler form is ω. The functions $\lambda_\alpha(x)$, $1 \leqslant \alpha \leqslant n$, are continuous functions of x. Clearly

$$\lambda_\alpha(\varepsilon, x) = \frac{\lambda_\alpha(x)}{\lambda_\alpha(x) + \varepsilon}, \quad 1 \leqslant \alpha \leqslant n.$$

Let $f(\varepsilon, x)$ be the minimum of

$$\sum_{\gamma=1}^{n} \lambda_\gamma(\varepsilon, x) - \sum_{\alpha \in A} \lambda_\alpha(\varepsilon, x) - \sum_{\beta \in B} \lambda_\beta(\varepsilon, x)$$

over all A and B, where A is a subset of p distinct elements, and B is a subset of q distinct elements in $\{1,\cdots,n\}$. For $p+q$ even, let $l_1 = l_2 = \frac{1}{2}(p+q)$. For $p+q$ odd, let $l_1 = \frac{1}{2}(p+q-1)$ and $l_2 = \frac{1}{2}(p+q-1)$. Clearly the minimum $f(\varepsilon, x)$ is achieved when $A = \{1,\cdots,l_1\}$ and $B = \{1,\cdots,l_2\}$. Thus

$$\begin{aligned} f(\varepsilon, x) &= \sum_{\alpha=l_1+1}^{k} \lambda_\alpha(\varepsilon, x) - \sum_{\beta=1}^{l_2} \lambda_\alpha(\varepsilon, x) \\ &= \sum_{\alpha=l_1+1}^{k} \frac{\lambda_\alpha(x)}{\lambda_\alpha(x)+\varepsilon} - \sum_{\beta=1}^{l_2} \frac{\lambda_\alpha(x)}{\lambda_\alpha(x)+\varepsilon}, \end{aligned}$$

which approaches $k-(p+q)$ uniformly in x as ε approaches zero, because each $\lambda_\alpha(x)$ is a positive continuous function on M for $1 \leqslant \alpha \leqslant k$. Hence for ε sufficiently small, $f(\varepsilon, x)$ is positive for every $x \in M$.

4.7. Proposition. *Let M be a compact Kähler manifold of complex dimension n. Let $1 \leqslant s \leqslant n$. Let L be a Hermitian holomorphic line bundle over M, which is Griffiths 1-seminegative and Griffiths s-negative. Then $H^p(M, \Omega_M^q \otimes L)$ vanishes for $p+q \leqslant n-s$.*

Proof. Let ω be the Kähler form of M, and v the curvature form of L. Let $u=-v$ and $k=n-s+1$. We use the notations of Lemma 4.6. Then the assumptions of Lemma 4.6 are satisfied, and we obtain a sufficiently small $\varepsilon > 0$. We give M the new Kähler form ω_ε. Fix p and q with $p+q<k$. Let φ be an L-valued (p,q)-form on M, which is harmonic with respect to the new Kähler form ω_ε. On the manifold M with the new Kähler metric ω_ε (and with $E=L$), when we take the global inner product of both sides of (1.3.7) with φ, we obtain $-\|\partial\varphi\|_M^2 - \|\bar\partial^*\varphi\|_M^2$ from the left-hand side, and obtain from the right-hand side an expression which is $\geqslant \eta\|\varphi\|_M^2$, where η is the minimum over $x \in M$ of the function $f(\varepsilon, x)$ defined in the proof of Lemma 4.6. Since η is positive, it follows that φ is identically zero.

4.8. Theorem. *Let M be a compact Kähler manifold of complex dimension n, and let E be a Hermitian holomorphic vector bundle of rank r over M, which is Griffiths 1-seminegative and Griffiths k-negative for some $1 \leqslant k \leqslant n$. Then $H^p(M, \Omega_M^q \otimes E)$ vanishes for $p+q \leqslant n-k-r+1$.*

Proof. This theorem follows from Proposition 4.7 and the argument of Schneider [48].

5. Complex-analyticity of harmonic maps

5.1. In [53] the complex-analyticity of a harmonic map between compact Kähler manifolds when the target manifold is strongly negatively curved in the sense of [53] was proved by using the $\partial\bar\partial$ Bochner-Kodaira technique. The difference between Bochner-Kodaira techniques for bundle-valued forms and Bochner-Kodaira techniques for harmonic maps lies only in the curvature terms in the formula. Since the $\partial\bar{\partial}$ Bochner-Kodaira technique is equivalent to the ∇ Bochner-Kodaira technique, the complex-analyticity of harmonic maps can also be proved by using the ∇ Bochner-Kodaira technique. Such a proof was given in [55] without explaining how it is related to the old proof of [53].

Here we investigate the complex-analyticity of harmonic maps when the target manifold is strongly seminegatively curved in the sense of [53]. A class of examples of such target manifolds are the compact quotients of bounded symmetric domains of rank higher than one. We obtain the complex-analyticity of the harmonic map, when the curvature tensor of the target manifold is very strongly seminegative in the sense of [53] and is sufficiently nondegenerate and when the harmonic map has sufficiently high rank. This result for the case of compact quotients of irreducible bounded symmetric domains of rank higher than one was conjectured in [55, §8]. In the previous proofs [53], [55] of the complex-analyticity of a harmonic map f, either the $\partial\bar{\partial}$ or the ∇ Bochner-Kodaira technique was applied to $\bar{\partial}f$. This corresponds to the Bochner-Kodaira techniques for bundle-valued (0, 1)-forms. The vanishing of $\bar{\partial}f$ (or ∂f) follows when one has the strongest kind of negativity for the curvature tensor of the target manifold but only the weakest condition on the rank of f. Here we apply the Bochner-Kodaira techniques to $\bar{\partial}f \wedge \cdots \wedge \bar{\partial}f$ (p times). This corresponds to applying the Bochner-Kodaira techniques to bundle-valued $(0, p)$-forms. The negativity required of the curvature tensor of the target manifold is weaker. On the other hand, one has to assume that f has a higher rank. Since the $\partial\bar{\partial}$ and the ∇ Bochner-Kodaira techniques are equivalent and since the ∇ and the $\overline{\nabla}$ Bochner-Kodaira techniques can be transformed to each other by the generalized Hodge star operator $\bar{*}$, to prove the complex-analyticity of a harmonic map f one can apply any one of the three Bochner-Kodaira techniques to $\bar{\partial}f \wedge \cdots \wedge \bar{\partial}f$. The $\partial\bar{\partial}$ Bochner-Kodaira technique was used in the proof given in [53], and the ∇ Bochner-Kodaira technique in the proof given in [55]. Here we choose the $\overline{\nabla}$ Bochner-Kodaira technique to show how it is applied to prove the complex-analyticity of a harmonic map. There is another reason for this choice. In this paper we will consider also the complex-analyticity of harmonic maps when the domain manifold has a boundary. To take care of the boundary term one needs the Morrey trick which works most directly in the $\overline{\nabla}$ Bochner-Kodaira technique. The Morrey tricks for the other two kinds of Bochner-Kodaira techniques are obtained only after transformation back to the case of the $\overline{\nabla}$ Bochner-Kodaira technique.

5.2. Let M, N be Kähler manifolds with Kähler metrics $\sum_{\alpha,\beta=1}^{m} h_{\alpha\bar{\beta}} dz^{\alpha}\overline{dz^{\beta}}$, $\sum_{i,j=1}^{n} g_{i\bar{j}} dw^{i}\overline{dw^{j}}$ respectively. We use the lower-case Greek letters $\alpha, \beta, \gamma, \cdots$

to denote the coordinate indices for M, and the lower-case Latin letters $i, j, k, \cdots$ to denote the coordinate indices for N. Let G be a relatively compact subdomain of N with smooth boundary. Let ρ be a smooth function on N, such that $G = \{\rho < 0\}$ and $d\rho$ is of unit length at every point of the boundary ∂G of G. Let $f\colon G \to M$ be a map smooth up to ∂G. (The results in this section hold also when $\overline{G}$ is a compact Kähler manifold with boundary instead of being the closure of a relatively compact subdomain of a Kähler manifold N.)

Denote by $\wedge^p(\bar\partial f)$ the exterior product $\bar\partial f \wedge \cdots \wedge \bar\partial f$ (p times) which is a $(0, p)$-form on G with values in the bundle $f^* \wedge^p T^{1,0}M$. We apply the generalized Hodge star operator $\bar{*}$ of N to $\wedge^p(\bar\partial f)$, and obtain an $(n, n-p)$-form ζ on G with values in $f^*\Omega_M^p$. That is, $\zeta = (\zeta_{\alpha_1 \cdots \alpha_p})$ and

$$\zeta_{\alpha_1\cdots\alpha_p} = \frac{1}{(n-p)!}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n} dz^1 \wedge \cdots \wedge dz^n \wedge d\overline{z^{i_{p+1}}} \wedge \cdots \wedge d\overline{z^{i_n}},$$

with

$$\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n} = \frac{\left(\sqrt{-1}\right)^n(-1)^{\frac{1}{2}n(n-1)}}{p!} h_{\alpha_1\bar\beta_1}\cdots h_{\alpha_p\bar\beta_p}$$
$$\cdot g_{1\cdots n\bar{i}_1\cdots\bar{i}_n} g^{\bar{i}_1 j_1}\cdots g^{\bar{i}_p j_p}\left(\partial_{j_1}\overline{f^{\beta_1}}\right)\cdots\left(\partial_{j_p}\overline{f^{\beta_p}}\right),$$

where $g_{1\cdots n\bar{i}_1\cdots\bar{i}_n} = \det(g_{k\bar{i}_l})_{1\le k,\, l\le n}$. We denote also ζ by $\bar{*}(\wedge^q(\bar\partial f))$.

We now apply to ζ the $\overline{\nabla}$ Bochner-Kodaira technique. The vector bundle $f^*\Omega_M^p$ over N is in general not holomorphic. We can still use the $\overline{\nabla}$ Bochner-Kodaira technique, but the curvature terms are more complicated. We denote by ∇_i (respectively $\nabla_{\bar{i}}$) the covariant differential operator with respect to $\partial/\partial w^i$ (respectively $\partial/\partial\overline{w^i}$) for $f^*\Omega_M^p$-valued forms. Denote by $\overline{D}$ the exterior differential operator defined by covariant differentiation which sends $f^*\Omega_M^p$-valued (n, q)-forms to $f^*\Omega_M^p$-valued $(n, q+1)$-forms. Let $\overline{D}^*$ be the adjoint operator of $\overline{D}$ and let $\square = \overline{D}^*\overline{D} + \overline{D}\overline{D}^*$. We now compute $\square\zeta$.

From

$$\left(\overline{D}^*\overline{D}\zeta\right)_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n} = (-1)^{n+1} g^{\bar{j}i}\nabla_i\left(\overline{D}\zeta\right)_{\alpha_1\cdots\alpha_p\bar{j}\bar{i}_{p+1}\cdots\bar{i}_n}$$
$$= -g^{\bar{j}i}\nabla_i\Big(\nabla_{\bar{j}}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}$$
$$+ \sum_{\nu=p+1}^{n}(-1)^{\nu-p}\nabla_{\bar{i}_\nu}\zeta_{\alpha_1\cdots\alpha_p\bar{j}\bar{i}_{p+1}\cdots\hat{\bar{i}}_\nu\cdots\bar{i}_n},$$

$$
\begin{aligned}
&(\bar{D}\bar{D}^*\zeta)_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}\\
&\quad=(-1)^n\sum_{\nu=p+1}^{n}(-1)^{\nu-p+1}\nabla_{\bar{i}_\nu}(\bar{D}^*\zeta)_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\hat{\bar{i}}_\nu\cdots\bar{i}_n}\\
&\quad=-\sum_{\nu=p+1}^{n}(-1)^{\nu-p+1}\nabla_{\bar{i}_\nu}\left(g^{j\bar{i}}\nabla_i\zeta_{\alpha_1\cdots\alpha_p\bar{j}\bar{i}_{p+1}\cdots\hat{\bar{i}}_\nu\cdots\bar{i}_n}\right),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
(\square\zeta)_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}&=-g^{j\bar{i}}\nabla_i\nabla_{\bar{j}}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}\\
&\quad+\sum_{\nu=p+1}^{n}(-1)^{\nu-p+1}g^{j\bar{i}}\left[\nabla_i,\nabla_{\bar{i}_\nu}\right]\zeta_{\alpha_1\cdots\alpha_p\bar{j}\bar{i}_{p+1}\cdots\hat{\bar{i}}_\nu\cdots\bar{i}_n}.
\end{aligned}
$$

The above computation runs exactly in the same way as the case of the $\bar{\nabla}$ Bochner-Kodaira technique applied to a holomorphic vector bundle; the difference lies in the computation of the commutation $[\nabla_i, \nabla_{\bar{i}_\nu}]$ which we are going to carry out.

5.3. For the computation of $[\nabla_i, \nabla_{\bar{i}_\nu}]$ we choose normal coordinates of M and N at the points under consideration. Let $f_i^\alpha = \partial_i f^\alpha$, $f_{\bar{i}}^\alpha = \partial_{\bar{i}} f^\alpha$, $f_i^{\bar{\alpha}} = \partial_i\overline{f^\alpha}$, and $f_{\bar{i}}^{\bar{\alpha}} = \partial_{\bar{i}}\overline{f^\alpha}$. Clearly $f_{\bar{i}}^{\bar{\alpha}} = \overline{f_i^\alpha}$ and $f_i^{\bar{\alpha}} = \overline{f_{\bar{i}}^\alpha}$. Denote the Christoffel symbols and the curvature tensors of M and N respectively by $\Gamma^\alpha_{\beta\gamma}$, Γ^i_{jk} and $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$, $R_{i\bar{j}k\bar{l}}$. Though Γ and R are used for both manifolds, confusion is avoided by using Greek letters for coordinate indices for M and Latin letters for coordinate indices for N. We denote the Ricci tensor of N by $R_{i\bar{j}}$. From

$$
\begin{aligned}
&\nabla_k\nabla_{\bar{l}}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}\\
&\quad=\nabla_k\Bigg(\partial_{\bar{l}}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}-\sum_{\mu=1}^{n}\Gamma^\gamma_{\alpha_\mu\beta}\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}f_{\bar{l}}^\beta\\
&\qquad\qquad-\sum_{\nu=p+1}^{n}\overline{\Gamma^s_{i_\nu l}}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots(\bar{s})_\nu\cdots\bar{i}_{n-p}}\Bigg)\\
&\quad=\partial_k\partial_{\bar{l}}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}-\sum_{\mu=1}^{n}\partial_{\bar{\delta}}\Gamma^\gamma_{\alpha_\mu\beta}\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}f_k^{\bar{\delta}}f_{\bar{l}}^\beta\\
&\qquad-\sum_{\nu=p+1}^{n}\partial_{\bar{k}}\overline{\Gamma^s_{i_\nu l}}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots(\bar{s})_\nu\cdots\bar{i}_{n-p}},
\end{aligned}
$$

$$\nabla_{\bar l}\nabla_k \zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n}$$

$$= \nabla_{\bar l}\Big(\partial_k \zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n} - \sum_{\mu=1}^{p} \Gamma^{\gamma}_{\alpha_\mu\beta}\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n} f^{\beta}_k$$

$$-\Gamma^{j}_{jk}\zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n}\Big)$$

$$= \partial_{\bar l}\partial_k \zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n} - \sum_{\mu=1}^{p} \partial_{\bar\delta}\Gamma^{\gamma}_{\alpha_\mu\beta}\zeta_{\alpha_1\cdots(\gamma)\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n} f^{\bar\delta}_{\bar l} f^{\beta}_k$$

$$-\partial_{\bar l}\Gamma^{j}_{jk}\zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n}$$

it follows that

$$[\nabla_k, \nabla_{\bar l}]\zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n} = \sum_{\mu=1}^{p} R_{\alpha_\mu}{}^{\gamma}{}_{\beta\bar\delta}\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n}\Big(f^{\beta}_k f^{\bar\delta}_{\bar l} - f^{\beta}_{\bar l} f^{\bar\delta}_k\Big)$$

$$- \sum_{\nu=p+1}^{n} R^{\bar s}{}_{\bar i_\nu k\bar l}\zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots(\bar s)_\nu\cdots \bar i_{n-p}}$$

$$+R_{k\bar l}\zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n}.$$

Hence

$$(\Box\zeta)_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n} = -g^{\bar j i}\nabla_{\bar i}\nabla_{\bar j}\zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots \bar i_n}$$

$$+ \sum_{\nu=p+1}^{n}(-1)^{\nu-p+1} g^{\bar j i}\sum_{\mu=1}^{p} R_{\alpha_\mu}{}^{\gamma}{}_{\beta\bar\delta}\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p \bar j \bar i_{p+1}\cdots\hat{\bar i}_\nu\cdots \bar i_n}\Big(f^{\beta}_{\bar i} f^{\bar\delta}_{\bar i_\nu} - f^{\beta}_{\bar i_\nu} f^{\bar\delta}_{i}\Big)$$

$$- \sum_{\nu=p+1}^{n}(-1)^{\nu-p+1} g^{\bar j i}\Big(R^{\bar s}{}_{\bar j i\bar i_\nu}\zeta_{\alpha_1\cdots\alpha_p \bar s \bar i_{p+1}\cdots\hat{\bar i}_\nu\cdots \bar i_n}$$

$$+ \sum_{\substack{p<\mu\le n\\ \mu\ne\nu}} R^{\bar s}{}_{\bar i_\mu i\bar i_\nu}\zeta_{\alpha_1\cdots\alpha_p \bar j \bar i_{p+1}\cdots(\bar s)_\mu\cdots\hat{\bar i}_\nu\cdots \bar i_n}\Big)$$

$$+ \sum_{\nu=p+1}^{n}(-1)^{\nu-p+1} g^{\bar j i} R_{i\bar i_\nu}\zeta_{\alpha_1\cdots\alpha_p \bar j \bar i_{p+1}\cdots\hat{\bar i}_\nu\cdots \bar i_n}.$$

Since

$$(-1)^{\nu-p+1} g^{\bar j i} R^{\bar s}{}_{\bar i_\mu i\bar i_\nu}\zeta_{\alpha_1\cdots\alpha_p \bar j \bar i_{p+1}\cdots(\bar s)_\mu\cdots\hat{\bar i}_\nu\cdots \bar i_n} = R^{\bar s}{}_{\bar i_\mu}{}^{\bar j}{}_{\bar i_\nu}\zeta_{\alpha_1\cdots\alpha_p \bar i_{p+1}\cdots(\bar s)_\mu\cdots(\bar j)_\nu\cdots \bar i_n}$$

vanishes by the symmetry of $R^{\bar{s}\,\bar{j}}{}_{\bar{i}_\mu \bar{i}_\nu}$ in s, j and the skew-symmetry of $\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots(\bar{s})_\mu\cdots(\bar{j})_\nu\cdots\bar{i}_n}$ in s, j, it follows that

$$\begin{aligned}(\Box\zeta)_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n} &= -g^{\bar{j}i}\nabla_i\nabla_{\bar{j}}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n} \\ &+\sum_{\mu=1}^{p}\sum_{\nu=p+1}^{n} g^{\bar{j}i}R_{\alpha_\mu}{}^{\gamma}{}_{\beta\bar{\delta}}\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p\bar{i}_{p+1}\cdots(\bar{j})_\nu\cdots\bar{i}_n}\left(f_i^\beta f_{\bar{i}}^{\bar{\delta}} - f_{\bar{i}_\nu}^\beta f_i^{\bar{\delta}}\right).\end{aligned} \tag{5.3.1}$$

5.4. We now consider the curvature term obtained from (5.3.1) by taking the pointwise inner product of $\Box\zeta$ and ζ, namely, the term

$$\frac{1}{p!\,(n-p)!}\sum_{\mu=1}^{p}\sum_{\nu=p+1}^{n} g^{\bar{j}i}R_{\alpha_\mu}{}^{\gamma}{}_{\beta\bar{\delta}}\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p\bar{i}_{p+1}\cdots(\bar{j})_\nu\cdots\bar{i}_n} \cdot\left(f_i^\beta f_{\bar{i}_\nu}^{\bar{\delta}} - f_{\bar{i}_\nu}^\beta f_i^{\bar{\delta}}\right)\overline{\zeta^{\bar{\alpha}_1\cdots\bar{\alpha}_p i_{p+1}\cdots i_n}}.$$

We denote this term by C and we want to simplify it.

Let

$$\begin{aligned}&\partial\overline{f^{\alpha_1}}\wedge\cdots\wedge\partial\overline{f^{\alpha_s}}\wedge\partial f^{\beta_1}\wedge\cdots\wedge\partial f^{\beta_t} \\ &= \frac{1}{(s+t)!}f^{\bar{\alpha}_1\cdots\bar{\alpha}_s\beta_1\cdots\beta_t}_{i_1\cdots i_{s+t}}\,dw^{i_1}\wedge\cdots\wedge dw^{i_{s+t}}.\end{aligned}$$

We will use this only for the cases $(s,t)=(p,0)$, $(p+1,0)$. Clearly $f^{\bar{\alpha}_1\cdots\bar{\alpha}_s\beta_1\cdots\beta_t}_{i_1\cdots i_{s+t}}$ is the skew-symmetrization of $f_{i_1}^{\bar{\alpha}_1}\cdots f_{i_s}^{\bar{\alpha}_s}f_{i_{s+1}}^{\beta_1}\cdots f_{i_{s+t}}^{\beta_t}$ with respect to its subscripts $i_1,\cdots,i_{s+t}$. We will need the following obvious identity

$$f^{\bar{\alpha}_1\cdots\bar{\alpha}_p\beta}_{i_1\cdots i_p j} = \frac{1}{p+1}\left(f^{\bar{\alpha}_1\cdots\bar{\alpha}_p}_{i_1\cdots i_p}f_j^\beta - \sum_{\lambda=1}^{p} f^{\bar{\alpha}_1\cdots\bar{\alpha}_p}_{i_1\cdots(j)_\lambda\cdots i_p}f^\beta_{i_\lambda}\right). \tag{5.4.1}$$

This identity also holds when β is replaced by $\bar{\beta}$.

We use normal coordinates at both points under consideration. Then

$$\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n} = \left(\sqrt{-1}\right)^n(-1)^{\frac{1}{2}n(n-1)}\frac{1}{p!}\sum_{i_1,\cdots,i_p}\operatorname{sgn}\begin{pmatrix}1 & \cdots & n\\ i_1 & \cdots & i_n\end{pmatrix}f^{\bar{\alpha}_1\cdots\bar{\alpha}_p}_{i_1\cdots i_p}.$$

In the following computation of C, in addition to the usual summation convention, we use the summation convention that an index is being summed if it occurs either twice as a subscript or twice as a superscript, once with a bar and once without a bar. An index with (respectively without) a bar in an expression carrying a bar is considered without (respectively with) a bar. As before, we use Σ' to denote summation over distinct indices. Since we have normal coordinates at both points under consideration, we have

$$C=\frac{1}{p!\,(n-p)!}\sum_{\mu=1}^{p}\sum_{\nu=p+1}^{n}R_{\alpha_\mu\bar\gamma\beta\bar\delta}\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p\bar i_{p+1}\cdots(\bar j)_\nu\cdots\bar i_n}\cdot\left(f_j^\beta f_{\bar i_\nu}^{\bar\delta}-f_{\bar i_\nu}^\beta f_j^{\bar\delta}\right)\overline{\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}}\,.$$

We separate the summation in j into two parts, the first part with $j=i_\nu$ and the second part with $j\neq i_\nu$.

$$\begin{aligned}
C &= \frac{1}{p!\,(n-p)!}\sum_{\mu=1}^{p}\sum_{\nu=p+1}^{n}R_{\alpha_\mu\bar\gamma\beta\bar\delta}\Big(\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}\\
&\qquad\cdot\left(f_{i_\nu}^\beta f_{\bar i_\nu}^{\bar\delta}-f_{\bar i_\nu}^\beta f_{i_\nu}^{\bar\delta}\right)\overline{\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}}\\
&\quad+\sum_{j\neq i_\nu}\zeta_{\alpha_1\cdots(\gamma)_\mu\cdots\alpha_p\bar i_{p+1}\cdots(\bar j)_\nu\cdots\bar i_n}\left(f_j^\beta f_{\bar i_\nu}^{\bar\delta}-f_{\bar i_\nu}^\beta f_j^{\bar\delta}\right)\overline{\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}}\Big)\\
&= \frac{1}{(p-1)!\,(n-p)!\,p!}\sum_{i_1,\cdots,i_n}{}'\sum_{\nu=p+1}^{n}R_{\sigma\bar\gamma\beta\bar\delta}\\
&\quad\cdot\Big(f_{i_1\cdots i_p}^{\gamma\bar\alpha_2\cdots\bar\alpha_p}\left(f_{i_\nu}^\beta f_{\bar i_\nu}^{\bar\delta}-f_{\bar i_\nu}^\beta f_{i_\nu}^{\bar\delta}\right)\overline{f_{i_1\cdots i_p}^{\sigma\bar\alpha_2\cdots\bar\alpha_p}}\\
&\quad-\sum_{\lambda=1}^{p}f_{i_1\cdots(i_\nu)_\lambda\cdots i_p}^{\gamma\bar\alpha_2\cdots\bar\alpha_p}\left(f_{i_\nu}^\beta f_{\bar i_\nu}^{\bar\delta}-f_{\bar i_\nu}^\beta f_{i_\nu}^{\bar\delta}\right)\overline{f_{i_1\cdots i_p}^{\sigma\bar\alpha_2\cdots\bar\alpha_p}}\Big)\\
&= \frac{1}{(p-1)!\,p!}\sum_{i_1,\cdots,i_{p+1}}{}'R_{\sigma\bar\gamma\beta\bar\delta}\Big(f_{i_1\cdots i_p}^{\gamma\bar\alpha_2\cdots\bar\alpha_p}\left(f_{i_{p+1}}^\beta f_{\bar i_{p+1}}^{\bar\delta}-f_{\bar i_{p+1}}^\beta f_{i_{p+1}}^{\bar\delta}\right)\overline{f_{i_1\cdots i_p}^{\sigma\bar\alpha_2\cdots\bar\alpha_p}}\\
&\quad-\sum_{\lambda=1}^{p}f_{i_1\cdots(i_{p+1})_\lambda\cdots i_p}^{\gamma\bar\alpha_2\cdots\bar\alpha_p}\left(f_{i_\lambda}^\beta f_{\bar i_{p+1}}^{\bar\delta}-f_{\bar i_{p+1}}^\beta f_{i_\lambda}^{\bar\delta}\right)\overline{f_{i_1\cdots i_p}^{\sigma\bar\alpha_2\cdots\bar\alpha_p}}\Big)\\
&= \frac{1}{(p-1)!\,p!}\sum_{i_1,\cdots,i_{p+1}}{}'R_{\sigma\bar\gamma\beta\bar\delta}\\
&\quad\cdot\Big(f_{i_1\cdots i_p}^{\gamma\bar\alpha_2\cdots\bar\alpha_p}f_{i_{p+1}}^\beta\overline{f_{i_1\cdots i_p}^{\sigma\bar\alpha_2\cdots\bar\alpha_p}f_{i_{p+1}}^\delta}-f_{i_1\cdots i_p}^{\gamma\bar\alpha_2\cdots\bar\alpha_p}f_{i_{p+1}}^{\bar\delta}\overline{f_{i_1\cdots i_p}^{\sigma\bar\alpha_2\cdots\bar\alpha_p}f_{i_{p+1}}^{\bar\beta}}\\
&\quad-\sum_{\lambda=1}^{p}f_{i_1\cdots(i_{p+1})_\lambda\cdots i_p}^{\gamma\bar\alpha_2\cdots\bar\alpha_p}f_{i_{p+1}}^\beta\overline{f_{i_1\cdots i_p}^{\sigma\bar\alpha_2\cdots\bar\alpha_p}f_{i_{p+1}}^\delta}\\
&\quad+\sum_{\lambda=1}^{p}f_{i_1\cdots(i_{p+1})_\lambda\cdots i_p}^{\gamma\bar\alpha_2\cdots\bar\alpha_p}f_{i_\lambda}^{\bar\delta}\overline{f_{i_1\cdots i_p}^{\sigma\bar\alpha_2\cdots\bar\alpha_p}f_{i_{p+1}}^{\bar\beta}}\Big).
\end{aligned}$$

By (5.4.1) and the identity obtained from (5.4.1) by replacing the β there by $\bar{\beta}$, we have

$$C = \frac{p+1}{(p-1)!p!} R_{\sigma\bar{\gamma}\beta\bar{\delta}} \left(f^{\bar{\gamma}\bar{\alpha}_2\cdots\bar{\alpha}_p\beta}_{i_1\cdots i_{p+1}} \overline{f^{\bar{\sigma}\bar{\alpha}_2\cdots\bar{\alpha}_p}_{i_1\cdots i_p} f^{\delta}_{i_{p+1}}} - f^{\bar{\gamma}\bar{\alpha}_2\cdots\bar{\alpha}_p\bar{\delta}}_{i_1\cdots i_{p+1}} \overline{f^{\bar{\sigma}\bar{\alpha}_2\cdots\bar{\alpha}_p}_{i_1\cdots i_p} f^{\bar{\beta}}_{i_{p+1}}} \right).$$

Since $f^{\bar{\gamma}\bar{\alpha}_2\cdots\bar{\alpha}_p\bar{\delta}}_{i_1\cdots i_{p+1}}$ is skew-symmetric in γ, δ, and $R_{\sigma\bar{\gamma}\beta\bar{\delta}}$ is symmetric in γ, δ, it follows that

$$R_{\sigma\bar{\gamma}\beta\bar{\delta}} f^{\bar{\gamma}\bar{\alpha}_2\cdots\bar{\alpha}_p\bar{\delta}}_{i_1\cdots i_{p+1}} = 0.$$

Hence

$$C = \frac{p+1}{(p-1)!p!} R_{\sigma\bar{\gamma}\beta\bar{\delta}} f^{\bar{\gamma}\bar{\alpha}_2\cdots\bar{\alpha}_p\beta}_{i_1\cdots i_{p+1}} \overline{f^{\bar{\sigma}\bar{\alpha}_2\cdots\bar{\alpha}_p}_{i_1\cdots i_p} f^{\delta}_{i_{p+1}}}.$$

Since the inner product of a skew-symmetric tensor with another tensor remains unchanged when the second tensor is replaced by its skew-symmetrization, we have

$$C = \frac{p+1}{(p-1)!p!} R_{\sigma\bar{\gamma}\beta\bar{\delta}} f^{\bar{\gamma}\bar{\alpha}_2\cdots\bar{\alpha}_p\beta}_{i_1\cdots i_{p+1}} \overline{f^{\bar{\sigma}\bar{\alpha}_2\cdots\bar{\alpha}_p\delta}_{i_1\cdots i_{p+1}}}. \tag{5.4.2}$$

As before, we use $\langle\cdot,\cdot\rangle$ to denote the pointwise inner product. By combining together (5.3.1) and (5.4.2), we obtain the following.

5.5. Lemma.

$$\langle \Box\zeta, \zeta\rangle = -\langle g^{i\bar{j}} \nabla_{\bar{j}}\nabla_i\zeta, \zeta\rangle + \frac{p+1}{p!(p-1)!} R_{\alpha_1\bar{\beta}_1\gamma\bar{\delta}} f^{\bar{\beta}_1\cdots\bar{\beta}_p\gamma}_{i_1\cdots i_{p+1}} \cdot \overline{f^{\bar{\alpha}_1\cdots\bar{\alpha}_p\delta}_{j_1\cdots j_{p+1}}} h_{\alpha_2\bar{\beta}_2}\cdots h_{\alpha_p\bar{\beta}_p} g^{i_1\bar{j}_1}\cdots g^{i_{p+1}\bar{j}_{p+1}}.$$

5.6. Lemma. (a) $\bar{D}^*\zeta$ *vanishes identically on* G *for any smooth map* f.

(b) *If* $p=1$ *and* f *is harmonic on* G, *then* $\bar{D}\zeta$ *vanishes identically on* G.

(c) *For general* p, *if* f *is pluriharmonic on* G, *i.e.*, $D\bar{\partial}f \equiv 0$ *on* G, *then* $\bar{D}\zeta$ *vanishes identically on* G.

(d) *If* $\nabla_{\bar{j}}\zeta \equiv 0$ *on* G *for all* j *when* $p=1$, *then* f *is pluriharmonic on* G.

Proof. For the proof we use normal coordinates at both points under consideration.

(a) From the definition of ζ it follows that

$$
\begin{aligned}
&(\bar{D}^*\zeta)_{\alpha_1\cdots\alpha_p\bar{i}_{p+2}\cdots\bar{i}_n}\\
&\quad= (-1)^{n+1}\sum_{i_{p+1}=1}^{n}\nabla_{i_{p+1}}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_{p+1}\cdots\bar{i}_n}\\
&\quad= \left(\sqrt{-1}\right)^n(-1)^{\frac{1}{2}n(n-1)+n+1}\sum_{i_1,\cdots,i_{p+1}}\operatorname{sgn}\begin{pmatrix} i_1 & \cdots & i_n\\ 1 & \cdots & n\end{pmatrix}\\
&\qquad\qquad\cdot\partial_{i_{p+1}}\left(\left(\partial_{i_1}\overline{f^{\alpha_1}}\right)\cdots\left(\partial_{i_p}\overline{f^{\alpha_p}}\right)\right)\\
&\quad= \left(\sqrt{-1}\right)^n(-1)^{\frac{1}{2}n(n-1)+n+1}\sum_{\nu=1}^{p}\sum_{i_1,\cdots,i_{p+1}}\operatorname{sgn}\begin{pmatrix} i_1 & \cdots & i_n\\ 1 & \cdots & n\end{pmatrix}\\
&\qquad\cdot\left(\partial_{i_1}\overline{f^{\alpha_1}}\right)\cdots\left(\partial_{i_{\nu-1}}\overline{f^{\alpha_{\nu-1}}}\right)\left(\partial_{i_{p+1}}\partial_{i_\nu}\overline{f^{\alpha_\nu}}\right)\left(\partial_{i_{\nu+1}}\overline{f^{\alpha_{\nu+1}}}\right)\cdots\left(\partial_{i_p}\overline{f^{\alpha_p}}\right),
\end{aligned}
$$

which vanishes because $\operatorname{sgn}\binom{i_1\ \cdots\ i_n}{1\ \cdots\ n}$ is skew-symmetric i_{p+1} and i_ν, whereas $\partial_{i_{p+1}}\partial_{i_\nu}\overline{f^{\alpha_\nu}}$ is symmetric in i_{p+1} and i_ν.

(b) Assume $p = 1$. Then

$$
\begin{aligned}
&(\bar{D}\zeta)_{\alpha_1\bar{i}_1\cdots\bar{i}_n}\\
&\quad= \sum_{\nu=1}^{n}(-1)^{\nu-1}\nabla_{\bar{i}_\nu}\zeta_{\alpha_i\bar{i}_1\cdots\hat{\bar{i}}_\nu\cdots\bar{i}_n}\\
&\quad= \left(\sqrt{-1}\right)^n(-1)^{\frac{1}{2}n(n-1)}\sum_{\nu=1}^{n}(-1)^{\nu-1}\sum_{j=1}^{n}\operatorname{sgn}\begin{pmatrix} ji_1 & \cdots & \hat{i}_\nu & \cdots & i_n\\ & 1 & \cdots & n & \end{pmatrix}\partial_{\bar{i}_\nu}\partial_j\overline{f^{\alpha_1}}\\
&\quad= \left(\sqrt{-1}\right)^n(-1)^{\frac{1}{2}n(n-1)}\sum_{j=1}^{n}\partial_{\bar{j}}\partial_j\overline{f^{\alpha_1}},
\end{aligned}
$$

which vanishes because f is harmonic.

(c) Assume that f is pluriharmonic. Then $\partial_{\bar{i}}\partial_j\overline{f^\alpha} = 0$. Hence

$$
\begin{aligned}
(\bar{D}\zeta)_{\alpha_1\cdots\alpha_p\bar{i}_p\cdots\bar{i}_n} &= \sum_{\nu=p}^{n}(-1)^{\nu-p}\nabla_{\bar{i}_\nu}\zeta_{\alpha_1\cdots\alpha_p\bar{i}_p\cdots\hat{\bar{i}}_\nu\cdots\bar{i}_n}\\
&= \left(\sqrt{-1}\right)^n(-1)^{\frac{1}{2}n(n-1)}\sum_{\nu=p}^{n}(-1)^{\nu-p}\\
&\quad\cdot\sum_{j_1,\cdots,j_p}\operatorname{sgn}\begin{pmatrix} j_1 & \cdots & j_pi_p & \cdots & \hat{i}_\nu & \cdots & i_n\\ & & 1 & \cdots & n & & \end{pmatrix}\partial_{\bar{i}_\nu}\left(\left(\partial_{j_1}\overline{f^{\alpha_1}}\right)\cdots\left(\partial_{j_p}\overline{f^{\alpha_p}}\right)\right) = 0.
\end{aligned}
$$

(d) Assume $p = 1$ and $\nabla_{\bar{j}}\zeta = 0$ for all j. Since

$$\nabla_{\bar{j}}\zeta_{\alpha_1 \bar{i}_2 \cdots \bar{i}_n} = \left(\sqrt{-1}\right)^n (-1)^{\frac{1}{2}n(n-1)} \sum_{i_1} \operatorname{sgn}\begin{pmatrix} i_1 & \cdots & i_n \\ 1 & \cdots & n \end{pmatrix} \partial_{\bar{j}}\partial_{i_1}\overline{f^{\alpha_1}},$$

where $(i_1, \cdots, i_n)$ is a permutation of $(1, \cdots, n)$, it follows that $\partial_{\bar{j}}\partial_{i_1}\overline{f^{\alpha_1}} = 0$ for all j and i_1.

5.7. Proposition. *Suppose f satisfies the tangential Cauchy-Riemann equations of the boundary ∂G of G (in notation $\bar{\partial}_b f = 0$ on ∂G). Then*

$$\begin{aligned} &\|\bar{D}\zeta\|_G^2 + \|\bar{D}^*\zeta\|_G^2 \\ &= \frac{1}{(n-p-1)!}\int_{\partial G} (\partial_k \partial_{\bar{l}} \rho) \zeta_{\alpha_1 \cdots \alpha_p}{}^{k}{}_{\bar{i}_{p+2} \cdots \bar{i}_n} \overline{\zeta^{\bar{\alpha}_1 \cdots \bar{\alpha}_p l i_{p+2} \cdots i_n}} + \|\overline{\nabla}\zeta\|_G^2 \\ &\quad + \frac{p+1}{p!\ (p-1)!}\int_G R_{\alpha_1 \bar{\beta}_1 \gamma \bar{\delta}} f_{i_1 \cdots i_{p+1}}^{\beta_1 \cdots \beta_p \gamma} \overline{f_{j_1 \cdots j_{p+1}}^{\bar{\alpha}_1 \cdots \bar{\alpha}_p \delta}} h_{\alpha_2 \bar{\beta}_2} \\ &\qquad \cdots h_{\alpha_p \bar{\beta}_p} g^{i_1 \bar{j}_1} \cdots \overline{g^{i_{p+1} j_{p+1}}}, \end{aligned}$$

where $\|\cdot\|_G$ means the global L^2 norm over G, and $\overline{\nabla}\zeta$ means the $f^*\Omega_M^p$-valued tensor of rank $n - p + 1$ on G whose components are $\nabla_{\bar{j}}\zeta_{\alpha_1 \cdots \alpha_p \bar{i}_{p+1} \cdots \bar{i}_n}$.

Proof. This proof is a straightforward adaptation of the Morrey trick. First we verify that the vanishing of $\bar{\partial}_b f$ on ∂G implies that

$$g^{l\bar{k}}\rho_l \zeta_{\alpha_1 \cdots \alpha_p \bar{k} \bar{i}_{p+1} \cdots \bar{i}_n} = 0 \quad \text{on } \partial G. \tag{5.7.1}$$

To verify (5.7.1), we fix a point P on ∂G and choose local coordinates, so that $dw^1, \cdots, dw^n$ are orthonormal at P and the tangent space of ∂G at P is defined by $d(Re\, w^n) = 0$. It follows from the vanishing of $\bar{\partial}_b f$ at P that $\partial_{\bar{k}} f^\alpha = 0$ at P for $1 \leqslant k \leqslant n - 1$. At P

$$\begin{aligned} \zeta_{\alpha_1 \cdots \alpha_p \bar{n} \bar{i}_{p+2} \cdots \bar{i}_n} &= \left(\sqrt{-1}\right)^n (-1)^{\frac{1}{2}n(n-1)} \\ &\cdot \sum_{k_1, \cdots, k_p} \operatorname{sgn}\begin{pmatrix} k_1 & \cdots & k_p & n & i_{p+2} & \cdots & i_n \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & n \end{pmatrix} \left(\partial_{k_1}\overline{f^{\alpha_1}}\right) \cdots \left(\partial_{k_p}\overline{f^{\alpha_p}}\right) \end{aligned}$$

vanishes, because the only possible nonzero terms in the sum are those with $k_1, \cdots, k_p$ not equal to n. (5.7.1) follows from $g_{i\bar{j}} = \delta_{ij}$ and $\rho_l = 0$ for $l \neq n$ at P.

By integrating the equation in Lemma 5.5 over G and performing three integration by parts, we obtain an equation which is the desired one except that the boundary term is replaced by the sum of the following three boundary terms:

$$B_1 = \frac{1}{(n-p)!}\int_{\partial G}\rho_{\bar s}\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}\overline{(\bar D\zeta)^{\bar\alpha_1\cdots\bar\alpha_p s i_{p+1}\cdots i_n}},$$

$$B_2 = \frac{-1}{(n-p-1)!}\int_{\partial G} g^{j\overline{i_{p+1}}}\rho_j\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}\overline{(\bar D^*\zeta)^{\bar\alpha_1\cdots\bar\alpha_p i_{p+2}\cdots i_n}},$$

$$B_3 = \frac{1}{(n-p)!}\int_{\partial G}\rho_{\bar s}\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}\overline{\nabla^s\zeta^{\bar\alpha_1\cdots\bar\alpha_p i_{p+1}\cdots i_n}},$$

where $\rho_{\bar s} = \partial_{\bar s}\rho$ and $\rho_j = \partial_j\rho$.

It follows from (5.7.1) that B_2 vanishes. By

$$(\bar D\zeta)_{\alpha_1\cdots\alpha_p\bar s\bar i_{p+1}\cdots\bar i_n} = \nabla_{\bar s}\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n} - \sum_{\nu=p+1}^{n}\nabla_{\bar i_\nu}\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots(\bar s)_\nu\cdots\bar i_n}$$

we have

$$B_1 + B_3 = \frac{-1}{(n-p)!}\sum_{\nu=p+1}^{n}\int_{\partial G}\rho_{\bar s}\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}\overline{\nabla^{i_\nu}\zeta^{\bar\alpha_1\cdots\bar\alpha_p i_{p+1}\cdots(s)_\nu\cdots i_n}}.$$

From (5.7.1) it follows that

$$\rho_{\bar s}\overline{\zeta^{\bar\alpha_1\cdots\bar\alpha_p i_{p+1}\cdots(s)_\nu\cdots i_n}} = \rho\eta^{\alpha_1\cdots\alpha_p\overline{i_{p+1}}\cdots\hat{\bar i}_\nu\cdots\bar i_n} \tag{5.7.2}$$

for some smooth $\eta^{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\hat{\bar i}\cdots\bar i_n}$ on $\bar G$. Applying $\Sigma_{\nu=p+1}^{n}\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}\nabla^{\bar i_\nu}$ to (5.7.2), we obtain

$$\sum_{\nu=p+1}^{n}\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}\left(\nabla^{\bar i_\nu}\rho_{\bar s}\right)\overline{\zeta^{\bar\alpha_1\cdots\bar\alpha_p i_{p+1}\cdots(s)_\nu\cdots i_n}}$$
$$+\sum_{\nu=p+1}^{n}\rho_{\bar s}\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}\overline{\nabla^{i_\nu}\zeta^{\bar\alpha_1\cdots\bar\alpha_p i_{p+1}\cdots(s)_\nu\cdots i_n}}$$
$$=\sum_{\nu=p+1}^{n}\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}\left(\nabla^{\bar i_\nu}\rho\right)\eta^{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\hat{\bar i}_\nu\cdots\bar i_p} = 0$$

on ∂G, because $\zeta_{\alpha_1\cdots\alpha_p\bar i_{p+1}\cdots\bar i_n}(\nabla^{\bar i_\nu}\rho) = 0$ on ∂G due to (5.7.1). Hence $B_1 + B_3$ equals the boundary term of the desired equation.

5.8. Before we introduce our main result on the complex-analyticity of harmonic maps, we need some definitions.

Definition. Let $s \geqslant 1$. The bisectional curvature of a Kähler manifold M is said to be *strongly s-nondegenerate* at a point P of M when the following holds. If k and l are positive integers, and $\xi_{(1)},\cdots,\xi_{(k)}$ (respectively $\eta_{(1)},\cdots,\eta_{(l)}$) are

$\mathbf{C}$-linearly independent tangent vectors of M of type $(1,0)$ at P such that

$$(5.8.1) \qquad R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^{\alpha}_{(\mu)}\overline{\xi^{\beta}_{(\mu)}}\eta^{\gamma}_{(\nu)}\overline{\eta^{\beta}_{(\nu)}} = 0$$

at P for all $1 \leq \mu \leq k$ and $1 \leq \nu \leq l$, then $k + l \leq s$, where $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is the curvature tensor of M.

When the above condition is satisfied only for the special case $k = 1$, we say that the bisectional curvature of M is *s-nondegenerate* at P.

The smallest s so that the bisectional curvature of M is (strongly) s-nondegenerate at P is called the *degree of the* (*strong*) *nondegeneracy* of the bisectional curvature of M at P.

Remarks. 1. If the holomorphic tangent bundle T_M of M is s-negative (or s-positive) in the dual Nakano sense, then the bisectional curvature of M is s-nondegenerate. For we can apply the inequality in the definition of negativity of positivity in the dual Nakano sense to the set of complex numbers

$$\zeta^{\delta}_{\alpha_1\cdots\alpha_l} = \overline{\xi^{\delta}_{(1)}}\det\left(\eta^{\alpha_\lambda}_{(\nu)}\right)_{1\leq\lambda,\,\nu\leq l}.$$

The same statement holds when T_M is Nakano s-negative (or Nakano s-positive) in which case we use

$$\zeta^{\delta}_{\alpha_1\cdots\alpha_l} = \xi^{\delta}_{(1)}\deg\left(\eta^{\alpha_\lambda}_{(\nu)}\right)_{1\leq\lambda,\,\nu\leq l}.$$

2. If the bisectional curvature of M is s-nondegenerate, then the bisectional curvature of M is strongly t-nondegenerate, where $t = \max(1, 2s - 2)$. For we have $1 + l \leq s$ by considering $\xi_{(1)}, \eta_{(1)}, \cdots, \eta_{(l)}$ and we have $k + 1 \leq s$ by changing the roles of $\xi_{(\mu)}$ and $\eta_{(\nu)}$. Hence $k + l \leq 2s - 2$.

Definition. Let G be a relatively compact subdomain of a Kähler manifold N given by $G = \{\rho < 0\}$ for some smooth function ρ on N whose gradient is of unit length at every point of the boundary ∂G of G. The boundary ∂G of G is said to be *hyper-q-convex* (respectively *strongly hyper-q-convex*) at $P \in \partial G$ if the sum of any q eigenvalues of the Hermitian matrix $\partial_i\partial_{\bar{j}}\rho$ computed with respect to the given Kähler metric is nonnegative (respectively positive). When ∂G is hyper-q-convex at every point of ∂G, we simply say that ∂G is hyper-q-convex.

We now continue to use the notations M, N, G, f, ζ etc. introduced in §§5.1 through 5.7.

5.9. Lemma. *Let $1 \leq p \leq n - 1$ and $\zeta = \bar{*}(\bigwedge^p(\bar{\partial}f))$. Suppose $\overline{D}\zeta$ vanishes identically on G. Assume that ∂G is hyper-$(n - p)$-convex, and the holomorphic tangent bundle T_M of M is p-seminegative in the dual Nakano sense. Then the following statements hold*:

(a) $\overline{\nabla}\zeta$ *and the following two expressions vanish identically on* G:

$$(5.9.1)\qquad (\partial_k\partial_{\bar l}\rho)\zeta_{\alpha_1\cdots\alpha_p}{}^{k}{}_{\bar i_{p+2}\cdots\bar i_n}\overline{\zeta^{\bar\alpha_1\cdots\bar\alpha_p l\bar i_{p+2}\cdots\bar i_n}},$$

$$(5.9.2)\qquad R_{\alpha_1\bar\beta_1\gamma\bar\delta}f^{\beta_1\cdots\beta_p\gamma}_{i_1\cdots i_{p+1}}\overline{f^{\bar\alpha_1\cdots\bar\alpha_p\delta}_{j_1\cdots j_{p+1}}}h_{\alpha_2\bar\beta_2}\cdots h_{\alpha_p\bar\beta_p}g^{i_1\bar j_1}\cdots g^{i_{p+1}\bar j_{p+1}}.$$

(b) *If* ∂G *is strongly hyper-p-convex at some* $Q\in\partial G$, *then* $\bigwedge^p(\bar\partial f)$ *vanishes at* Q.

(c) *If* T_M *is p-negative in the dual Nakano sense at some* $Q\in M$, *then* $(\bigwedge^p(\bar\partial f))\wedge\overline{\partial f}$ *vanishes at* Q.

Proof. The expression (5.9.1) is nonnegative, because ∂G is hyper-$(n-1)$-convex. The expression (5.9.2) is nonnegative, because T_M is p-seminegative in the dual Nakano sense. By Lemma 5.6 (a), $\overline{D}^*\zeta\equiv 0$ on G. Statement (a) now follows from Proposition 5.7. The other two statements (b) and (c) are clear, because ζ vanishes at a point if and only if $\bigwedge^p(\bar\partial f)$ vanishes.

5.10. Lemma. *Suppose* ∂G *is hyper-*$(n-1)$*-convex, and* T_M *is* 1*-seminegative in the dual Nakano sense. Let* $1\leqslant p\leqslant n-1$ *and* $\zeta=\bar*(\bigwedge^p(\bar\partial f))$. *If* f *is harmonic on* G, *then* f *is pluriharmonic on* G *and, consequently,* $\overline{D}\zeta$ *vanishes identically on* G.

Proof. Let $\eta=\bar*(\bar\partial f)$. By Lemma 5.6 (b), $\overline{D}\eta\equiv 0$ on G. By Lemma 5.9 (a) for the case $p=1$, we have $\overline{\nabla}\eta\equiv 0$ on G which, according to Lemma 5.6 (d), implies that f is pluriharmonic. Hence by Lemma 5.6 (c), $\overline{D}\zeta\equiv 0$ on G.

5.11. Lemma. *Suppose* f *is pluriharmonic on* G. *If* $\operatorname{rank}_{\mathbf{C}}\bar\partial f<q$ *at every point of some nonempty open subset* H *of* ∂G, *then* $\operatorname{rank}_{\mathbf{C}}\bar\partial f<q$ *at every point of* $\overline{G}$.

Proof. Take a connected open subset U of N such that $U\cap\partial G$ is nonempty and is contained in H. Let $\varphi=\bigwedge^q(\bar\partial f)$. Since $\operatorname{rank}_{\mathbf{C}}\bar\partial f<q$ at every point of H, φ vanishes at every point of $U\cap\partial G$. From the pluriharmonicity of f it follows that $\nabla\varphi\equiv 0$ on G. Extend φ to $\tilde\varphi$ on $G\cup U$ by setting $\tilde\varphi\equiv 0$ on $U-\overline{G}$. Since $\varphi\equiv 0$ on $U\cup\partial G$, and the differential operator ∇ is of first order, it follows that $\nabla\tilde\varphi\equiv 0$ on $G\cup U$ in the sense of distributions. Hence

$$(5.11.1)\qquad g^{k\bar j}\nabla_{\bar k}\nabla_j\tilde\varphi\equiv 0\quad\text{on } G\cup U$$

in the sense of distributions. From the ellipticity of the differential operator $g^{k\bar j}\nabla_{\bar k}\nabla_j$ we conclude that $\tilde\varphi$ is smooth on $G\cup U$ and (5.11.1) holds in the usual sense. Since $\tilde\varphi\equiv 0$ on the nonempty open subset $U-\overline{G}$ of $G\cup U$, from the identity theorem for solutions of second-order elliptic equations [2] it follows that $\tilde\varphi\equiv 0$ on $G\cup U$. Hence $\operatorname{rank}_{\mathbf{C}}\bar\partial f<q$ at every point of G.

5.12. Lemma. *Let* $Q\in\overline{G}$ *and* $1\leqslant p\leqslant n-1$.

(a) *If* $\operatorname{rank}_{\mathbf{R}}df\geqslant 4p-3$ *at* Q, *then either* $\operatorname{rank}_{\mathbf{C}}\partial f\geqslant p$ *or* $\operatorname{rank}_{\mathbf{C}}\bar\partial f\geqslant p$ *at* Q.

(b) *If* $(\wedge^p(\bar\partial f)) \wedge \overline{\partial f} = 0$ *at* Q *and* $\operatorname{rank}_{\mathbf{C}} \bar\partial f \geqslant p+1$ *at* Q, *then* $\partial f = 0$. *The statement remains true when* ∂ *and* $\bar\partial$ *are interchanged.*

(c) *Suppose* $\operatorname{rank}_{\mathbf{R}} df \geqslant \max(4p-3, 2p+1)$ *at* Q. *If* $(\wedge^p(\bar\partial f)) \wedge \overline{\partial f}$ *and* $(\wedge^p(\partial f)) \wedge \partial \bar f$ *both vanish at* Q, *then either* $\partial f = 0$ *at* Q *or* $\bar\partial f = 0$ *at* Q.

Proof. (a) Using $\operatorname{Re} f^\alpha = \frac{1}{2}(f^\alpha + \overline{f^\alpha})$ and $\operatorname{Im} f^\alpha = (1/2\sqrt{-1})(f^\alpha - \overline{f^\alpha})$ to compute $\operatorname{rank}_{\mathbf{R}} df$, we conclude from $\operatorname{rank}_{\mathbf{R}} df \geqslant 4p-3$ at Q that

$$\left(\bigwedge_{\mu=1}^{l-1} d\left(f^{\alpha_\mu} + \overline{f^{\alpha_\mu}}\right)\right) \wedge \left(\bigwedge_{\nu=l}^{4p-3} d\left(f^{\alpha_\nu} - \overline{f^{\alpha_\nu}}\right)\right) \tag{5.12.1}$$

is nonzero at Q for some (not necessarily distinct) indices $\alpha_1 \cdots \alpha_{4p-3}$ and some $1 \leqslant l \leqslant 4p-2$. We use $df^\alpha = \partial f^\alpha + \bar\partial f^\alpha$ and $d\overline{f^\alpha} = \overline{\partial f^\alpha} + \overline{\bar\partial f^\alpha}$ to write the expression (5.12.1) as a linear combination of terms of the form

$$\left(\bigwedge_{\kappa=1}^{k} \partial f^{\beta_\kappa}\right) \wedge \left(\bigwedge_{\rho=k+1}^{r} \bar\partial f^{\beta_\rho}\right) \wedge \left(\bigwedge_{\sigma=r+1}^{s} \partial \overline{f^{\beta_\sigma}}\right) \wedge \left(\bigwedge_{\tau=s+1}^{4p-3} \overline{\partial f^{\beta_\tau}}\right). \tag{5.12.2}$$

At least one expression of the form (5.12.2) is nonzero at Q. Since each of the four factors in that expression is nonzero at Q, we have

$$\operatorname{rank}_{\mathbf{C}} \partial f + \operatorname{rank}_{\mathbf{C}} \bar\partial f + \operatorname{rank}_{\mathbf{C}} \partial \bar f + \operatorname{rank}_{\mathbf{C}} \overline{\partial f} \geqslant 4p-3$$

at Q. From $\operatorname{rank}_{\mathbf{C}} \partial f = \operatorname{rank}_{\mathbf{C}} \overline{\partial f}$ and $\operatorname{rank}_{\mathbf{C}} \bar\partial f = \operatorname{rank}_{\mathbf{C}} \partial \bar f$ it follows that either $\operatorname{rank}_{\mathbf{C}} \partial f \geqslant p$ at Q or $\operatorname{rank}_{\mathbf{C}} \bar\partial f \geqslant p$ at Q.

(b) Let $r = \operatorname{rank}_{\mathbf{C}} \bar\partial f$. We can choose local coordinates at Q and at $f(Q)$ such that $\bar\partial f^\alpha = d\overline{w^\alpha}$ for $1 \leqslant \alpha \leqslant r$, and $\bar\partial f^\beta = 0$ for $\beta > r$. Take γ and i arbitrarily, and we want to show that $\partial_i f^\gamma = 0$ at Q. Since $r \geqslant p+1$, we can choose $1 \leqslant \alpha_1 \cdots \alpha_p \leqslant r$ so that they are all distinct and all different from i. Then

$$\bar\partial f^{\alpha_1} \wedge \cdots \wedge \bar\partial f^{\alpha_p} \wedge \overline{\partial f^\gamma} = \left(\overline{\partial_j f^\gamma}\right) d\overline{w^{\alpha_1}} \wedge \cdots \wedge d\overline{w^{\alpha_p}} \wedge d\overline{w^j}.$$

From the vanishing of $(\wedge^p(\bar\partial f)) \wedge \overline{\partial f}$ at Q it follows that $\partial_j f^\gamma = 0$ at Q for $j \neq \alpha_1 \cdots \alpha_p$. In particular, $\partial_i f^\gamma = 0$ at Q. The statement with ∂ and $\bar\partial$ interchanged is proved analogously.

(c) Assume that neither ∂f nor $\bar\partial f$ vanishes at Q and we want to derive a contradiction. By (a), either $\operatorname{rank}_{\mathbf{C}} \partial f \geqslant p$ or $\operatorname{rank}_{\mathbf{C}} \bar\partial f \geqslant p$ at Q. We consider only the case $\operatorname{rank}_{\mathbf{C}} \bar\partial f \geqslant p$ at Q, because the other case is completely analogous. By (b), $\operatorname{rank}_{\mathbf{C}} \bar\partial f = p$. We can choose local coordinates at Q and at $f(Q)$ so that $\bar\partial f^\alpha = d\overline{w^\alpha}$ for $1 \leqslant \alpha \leqslant p$, and $\bar\partial f^\beta = 0$ for $\beta > p$. As in the proof of (b), it follows from the vanishing of $\bar\partial f^1 \wedge \cdots \wedge \bar\partial f^p \wedge \overline{\partial f^\gamma}$ at Q that $\partial_j f^\gamma = 0$ for all γ and all $p < j \leqslant n$. Hence both $\partial_j f$ and $\partial_{\bar j} f$ vanish for $p < j \leqslant n$. In the computation of $\operatorname{rank}_{\mathbf{R}} df$ we can ignore the variables $w^{p+1}, \cdots, w^n$. Thus $\operatorname{rank}_{\mathbf{R}} df \leqslant 2p$ and we have a contradiction.

5.13. Lemma. *Let $1 \leqslant p \leqslant n-1$ and $Q \in \bar{G}$. Assume that at Q*

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} f_{ik}^{\bar{\beta}\gamma} \overline{f_{jl}^{\bar{\alpha}\delta}} g^{i\bar{j}} g^{k\bar{l}} = 0. \tag{5.13.1}$$

Suppose T_M is 1-seminegative in the dual Nakano sense at $f(Q)$, and the bisectional curvature of M is strongly p-nondegenerate at $f(Q)$. Suppose $\operatorname{rank}_{\mathbf{R}} df \geqslant 2p+1$ *at Q. If either $(\bigwedge^p(\bar{\partial} f)) \wedge \overline{\partial f}$ or $(\bigwedge^p(\partial f)) \wedge \partial \bar{f}$ vanishes at Q, then either $\bar{\partial} f$ or ∂f vanishes at Q.*

Proof. We consider only the case $(\bigwedge^p(\partial f)) \wedge \partial \bar{f} = 0$ at Q, because the other case is completely analogous. Let $r = \operatorname{rank}_{\mathbf{C}} \partial f$ at Q. We assume that $r > 0$, and we are going to prove that $\bar{\partial} f = 0$ at Q. By Lemma 5.12(b) it suffices to consider the case $r \leqslant p$.

$\operatorname{Ker} df$ is a real subspace of real dimension $\leqslant 2n-2p-1$ in the real tangent space $T_{N,Q}$ of N at Q (when N is reparded as a real manifold). Let Gr be the Grassmannian of all complex linear subspaces of complex dimension $p+1$ in $T_{N,Q}$ (when $T_{N,Q}$ is given the complex structure from N). Take a real linear subspace F of real codimension 1 in $\operatorname{Ker} df$. Since $\dim_{\mathbf{R}} F \leqslant 2n-2p-2$, the set of all $L \in Gr$ with $L \cap F = 0$ is dense in Gr (see e.g. [53, p. 85, Lemma 1]). Thus the set of all $L \in Gr$ with $\dim_{\mathbf{R}} L \cap \operatorname{Ker} df \leqslant 1$ is dense in Gr. Hence we can choose normal coordinates $w^1, \cdots, w^n$ of N at Q such that if we denote by L the complex linear subspace of $T_{N,Q}$ spanned over $\mathbf{R}$ by $\operatorname{Re} \frac{\partial}{\partial w^i}$, $\operatorname{Im} \frac{\partial}{\partial w^i}$ $(1 \leqslant i \leqslant p)$, then $\dim_{\mathbf{R}} L \cap \operatorname{Ker} df \leqslant 1$ and $\partial f \mid L \neq 0$. Let s be the rank of $\partial f \mid L$ over $\mathbf{C}$. Then $1 \leqslant s \leqslant r \leqslant p$. Choose local coordinates z^α of M at $f(Q)$ such that $\partial f^\alpha \mid L = dw^\alpha$ for $1 \leqslant \alpha \leqslant s$ and $\partial f^\beta \mid L = 0$ for $\beta > s$.

Fix $1 \leqslant i \leqslant s < j \leqslant p+1$. Then $f_j^\alpha = 0$ for all α, and $f_i^\beta = \delta_{\beta i}$ (the Kronecker delta) for all β. Since $w^1, \cdots, w^n$ are normal coordinates at Q, we have $g_{k\bar{l}} = \delta_{kl}$ at Q. Since T_M is 1-seminegative in the dual Nakano sense at $f(Q)$, it follows from (5.13.1) that at Q

$$R_{\alpha\bar{\beta}\gamma\bar{\sigma}}\left(f_i^{\bar{\beta}} f_j^\alpha - f_j^{\bar{\beta}} f_i^\alpha\right)\left(f_{\bar{i}}^\gamma f_{\bar{j}}^{\bar{\sigma}} - f_{\bar{j}}^\gamma f_{\bar{i}}^{\bar{\sigma}}\right) = 0.$$

Hence

$$0 = \sum_{\alpha, \beta, \gamma, \sigma} R_{\alpha\bar{\beta}\gamma\bar{\sigma}}\left(-f_j^{\bar{\beta}} \delta_{\alpha i}\right)\left(-f_{\bar{j}}^\gamma \delta_{\sigma i}\right) = \sum_{\beta, \gamma} R_{i\bar{\beta}\gamma\bar{i}} f_j^{\bar{\beta}} f_{\bar{j}}^\gamma.$$

Let $\xi_{(\mu)} = \frac{\partial}{\partial z^\mu}$ $(1 \leqslant \mu \leqslant s)$ and $\eta_{(\nu)} = \sum_\alpha f_{\bar{\nu}}^\alpha \frac{\partial}{\partial z^\alpha}$ $(s < \nu \leqslant p+1)$ at Q. Then

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi_{(\mu)}^\alpha \overline{\xi_{(\mu)}^\delta} \eta_{(\nu)}^\gamma \overline{\eta_{(\nu)}^\delta} = 0 \qquad (1 \leqslant \mu \leqslant s < \nu \leqslant p+1).$$

It follows from the strong p-nondegeneracy of the bisectional curvature of M at $f(Q)$ that $\eta_{(s+1)}, \cdots, \eta_{(p+1)}$ cannot be $\mathbf{C}$-linearly independent. There exist complex numbers $a_\nu (s < \nu \leqslant p+1)$ not all zero such that $\sum_{\nu=s+1}^{p+1} a_\nu \eta_{(\nu)} = 0$. Let $X = \sum_{\nu=s+1}^{p+1} a_\nu \frac{\partial}{\partial w^\nu}$, $Y_1 = \operatorname{Re} X$, and $Y_2 = \operatorname{Im} X$ at Q. Then Y_1, Y_2 are

R-linearly independent elements of L. Since $X(f) = 0$ and $\overline{X}(f) = 0$, it follows that $(df)(Y_\mu) = 0$ $(\mu = 1, 2)$, contradicting $\dim_{\mathbf{R}} L \cap \operatorname{Ker} df \leqslant 1$.

5.14. Theorem. *Let M be a Kähler manifold whose holomorphic tangent bundle T_M is 1-seminegative in the dual Nakano sense. Let G be a relatively compact subdomain with smooth boundary in an n-dimensional Kähler manifold N such that ∂G is hyper-$(n-1)$-convex. Let $f\colon G \to M$ be a harmonic map smooth up to ∂G such that $\bar\partial_b f \equiv 0$ on ∂G. Assume that one of the following three conditions* (a), (b), (c) *is satisfied*:

(a) *∂G is strongly hyper-$(n-1)$-convex at some point of ∂G.*

(b) *There exists $1 \leqslant p \leqslant n-1$ such that* (i) *∂G is hyper-$(n-p)$-convex,* (ii) *T_M is p-negative in the dual Nakano sense, and* (iii) $\operatorname{rank}_{\mathbf{R}} df \geqslant \max(4p-3, 2p+1)$ *at some point Q of $\overline{G}$.*

(c) *There exists $1 \leqslant p \leqslant n-1$ such that* (i) *∂G is hyper-$(n-p)$-convex,* (ii) *T_M is p-negative in the dual Nakano sense,* (iii) *the bisectional curvature of M is strongly p-nondegenerate, and* (iv) $\operatorname{rank}_{\mathbf{R}} df \geqslant 2p+1$ *at some point Q of $\overline{G}$. Then f is holomorphic when ∂G is nonempty, and f is either holomorphic or antiholomorphic when ∂G is empty.*

Proof. We continue to use the notations we have been using in this section. Let p be the positive integer given in condition (b) or condition (c). When condition (a) is satisfied, we set $p = 1$. The conclusion for condition (a) follows from Lemmas 5.10, 5.9(b) both for the case $p = 1$, and Lemma 5.11 for the case $q = 1$.

We now assume the common subconditions (i) and (ii) of conditions (b) and (c). By Lemmas 5.10 and 5.0(a), we have the vanishing of the expression (5.9.2), which, by the p-negativity of T_M in the dual Nakano sense, implies $(\wedge^p(\bar\partial f)) \wedge \overline{\partial f} \equiv 0$ on G. By applying the same argument to the Kähler manifold which is the complex conjugate of N instead of to N, we conclude that $(\wedge^p(\partial f)) \wedge \overline{\bar\partial f} \equiv 0$ on G.

When condition (b) is satisfied, it follows from Lemma 5.12(c) that either ∂f or $\bar\partial f$ vanishes identically on some open neighborhood of Q in $\overline{G}$ and hence on all of $\overline{G}$ because of the harmonicity of f (cf. [53, p. 88, Prop. 4]).

Since ∂G is hyper-$(n-1)$-convex and T_M is 1-seminegative in the dual Nakano sense, it follows from Lemma 5.9(a) that (5.13.1) is satisfied at every point of $\overline{G}$. When condition (c) is satisfied, it follows from Lemma 5.13 that either ∂f or $\bar\partial f$ vanishes identically on some open neighborhood of Q in $\overline{G}$ and hence on all of $\overline{G}$.

What remains to be proved is that when ∂G is nonempty and condition (b) or (c) is satisfied, ∂f cannot vanish identically on $\overline{G}$. Suppose the contrary. Since $\bar\partial_b f \equiv 0$ on ∂G, it follows that $\operatorname{rank}_{\mathbf{C}} \bar\partial f \leqslant 1$ at every point of ∂G. By

Lemma 5.11, $\operatorname{rank}_{\mathbf{C}} \bar{\partial} f \leqslant 1$ at every point of $\bar{G}$. Thus $\operatorname{rank}_{\mathbf{R}} df \leqslant 2$ at every point of $\bar{G}$, contradicting $\operatorname{rank}_{\mathbf{R}} df \geqslant 2p+1 \geqslant 3$ at Q.

5.15 Remarks. 1. In Theorem 5.14, condition (c) implies condition (b) because of Remark 2 of §5.8. We give a direct proof of the case of condition (b) here because its proof is much easier than the proof of the case of condition (c).

2. Instead of using the $\bar{\nabla}$ Bochner-Kodaira technique as is done here, we can also use the $\partial\bar{\partial}$ Bochner-Kodaira technique to prove Theorem 5.14. We give the key step of such an approach here. We first show that $\partial\bar{\partial} f \equiv 0$ by considering the integral of $\partial\bar{\partial}(h_{\alpha\bar{\beta}}\, \bar{\partial} f^{\alpha} \wedge \overline{\partial f^{\beta}} \wedge \omega^{n-2})$ over G, where ω is the Kähler form of N. The boundary term has to be taken care of as in §3.8. Then we consider

$$\partial\bar{\partial}\Big(h_{\alpha_1\bar{\beta}_1} \cdots h_{\alpha_p\bar{\beta}_p} \bar{\partial} f^{\alpha_1} \wedge \cdots \wedge \bar{\partial} f^{\alpha_p} \partial f^{\overline{\beta_1}} \wedge \cdots \wedge \partial f^{\overline{\beta_p}} \wedge \omega^{n-p-1} \Big)$$

$$= \sum_{\nu=1}^{p} h_{\alpha_1\bar{\beta}_1} \cdots h_{\alpha_{\nu-1}\bar{\beta}_{\nu-1}} R_{\lambda\bar{\mu}\alpha_\nu\bar{\beta}_\nu} h_{\alpha_{\nu+1}\bar{\beta}_{\nu+1}} \cdots h_{\alpha_p\bar{\beta}_p} \big(\partial f^{\lambda} \wedge \overline{\partial f^{\mu}} + \overline{\partial f^{\mu}} \wedge \bar{\partial} f^{\lambda}\big)$$

$$\wedge \bar{\partial} f^{\alpha_1} \wedge \cdots \wedge \bar{\partial} f^{\alpha_p} \wedge \partial\, \overline{f^{\beta_1}} \wedge \cdots \wedge \partial\, \overline{f^{\beta_p}} \wedge \omega^{n-p-1}$$

$$= \sum_{\nu=1}^{p} h_{\alpha_1\bar{\beta}_1} \cdots h_{\alpha_{\nu-1}\bar{\beta}_{\nu-1}} R_{\lambda\bar{\mu}\alpha_\nu\bar{\beta}_\nu} h_{\alpha_{\nu+1}\bar{\beta}_{\nu+1}} \cdots h_{\alpha_p\bar{\beta}_p} \partial f^{\lambda} \wedge \overline{\partial f^{\mu}}$$

$$\wedge \bar{\partial} f^{\alpha_1} \wedge \cdots \wedge \bar{\partial} f^{\alpha_p} \wedge \partial\, \overline{f^{\beta_1}} \wedge \cdots \wedge \partial\, \overline{f^{\beta_p}} \wedge \omega^{n-p-1}$$

(because $R_{\lambda\bar{\mu}\alpha_\nu\bar{\beta}_\nu}$ is symmetric in λ, α_ν, and $\bar{\partial} f^{\lambda} \wedge \bar{\partial} f^{\alpha_\nu}$ is skew-symmetric in λ, α_ν)

$$= (-1)^{p+1} p((n-p-1)!) R_{\lambda\bar{\mu}\alpha_1\bar{\beta}_1} f^{\bar{\beta}_1 \cdots \bar{\beta}_p \lambda}_{i_1 \cdots i_{p+1}} \overline{f^{\bar{\alpha}_1 \cdots \bar{\alpha}_p \mu}_{j_1 \cdots j_{p+1}} \cdots j_{p+1}}$$

$$\cdot h_{\alpha_2\bar{\beta}_2} \cdots h_{\alpha_p\bar{\beta}_p} g^{i_1\overline{j_1}} \cdots g^{i_{p+1}\overline{j_{p+1}}},$$

and integrate it over G. Again we have to take care of the boundary term as in §3.8. The computation of the curvature term by using the $\partial\bar{\partial}$ Bochner-Kodaira technique is easier than by using the $\bar{\nabla}$ Bochner-Kodaira technique, because skew-symmetrization is a built-in process in the exterior algebra of forms. However, when one uses the $\bar{\nabla}$ Bochner-Kodaira technique, it is slightly easier to deal with the boundary term and is by far much easier to get the $\bar{D}^*$ and $\bar{D}$ terms for a general smooth map satisfying the tangential Cauchy-Riemann equations at the boundary. We choose the $\bar{\nabla}$ Bochner-Kodaira technique here, because we want to have the formula in §5.7 for a general smooth map satisfying the tangential Cauchy-Riemann equations at the boundary.

5.16. Theorem. *Let M be a compact Kähler manifold of complex dimension $m \geqslant 2$ whose holomorphic tangent bundle T_M is 1-seminegative in the dual Nakano sense. Then M is strongly rigid (in the sense that any compact Kähler manifold which is a homotopic to M must be either biholomorphic or antibiholomorphic to it) if one of the following two conditions is satisfied:*

(a) *T_M is p-negative in the dual Nakano sense for some $p \leqslant \min(m-1, \frac{m+1}{2})$.*

(b) *T_M is $(m-1)$-negative in the dual Nakano sense, and the bisectional curvature of M is strongly $(m-1)$-nondegenerate.*

Proof. Since T_M is 1-seminegative in the dual Nakano sense, the sectional curvature of M is nonpositive. By the theorem of Eells-Sampson [18], if N is a compact Kähler manifold homotopic to M, then we can find a harmonic map $f\colon N \to M$ which is a homotopic equivalence. Since $\operatorname{rank}_{\mathbf{R}} df \geqslant 2m$ at some point of N, it follows from Theorem 5.14 that f is holomorphic or antiholomorphic. For every $P \in M$, $f^{-1}(P)$ is a subvariety which must be 0-dimensional otherwise the homology class represented by $f^{-1}(P)$ is mapped to 0 by f, contradicting that f is a homotopy equivalence. Since the degree of f must be one, f is a homeomorphism and is therefore either a biholomorphism or an antibiholomorphism.

5.17. Theorem. *Let M be a Kähler manifold whose holomorphic tangent bundle T_M is 1-seminegative and p-negative in the dual Nakano sense. Let N be a compact complex submanifold of M. Then the deformation of N as a complex submanifold of M agrees with the deformation of N as an abstract complex manifold if one of the following two conditions is satisfied:*

(a) *The complex dimension of N is $\geqslant \max(2p-1, p+1)$.*

(b) *The bisectional curvature of M is strongly p-nondegenerate, and the complex dimension of N is $\geqslant p+1$.*

Proof. This follows from Theorem 5.14 and the method of Kalka [32]. The only thing we have to show is that every holomorphic cross section $s = s^\alpha \partial/\partial z^\alpha$ of $T_M \mid N$ over N must be identically zero. Let P be the point of N where the maximum of the pointwise square norm $|s|^2$ of s on N is achieved. Let n be the complex dimension of N, and let $X_\nu = \xi^\alpha_{(\nu)}\partial/\partial z^\alpha$, $1 \leqslant \nu \leqslant n$, be holomorphic tangent vector fields of N defined on an open neighborhood of P so that they are $\mathbf{C}$-linearly independent at P. Then for $1 \leqslant \nu \leqslant n$ we have at P

$$0 \geqslant X_\nu \overline{X_\nu} \, |s|^2 = |\nabla_{X_\nu} s|^2 + R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^\alpha_{(\nu)} \overline{\xi^\beta_{(\nu)}} s^\gamma \overline{s^\delta},$$

where $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is the curvature tensor of M. Since $n \geqslant p+1$ and the bisectional curvature of M is p-nondegenerate (see Remark 1 of §5.8), it follows that s vanishes at P. Hence s is identically zero on N.

5.18. Theorem. *Let M be a complete Kähler manifold whose holomorphic tangent bundle T_M is* 1*-seminegative in the dual Nakano sense. Let N be a Kähler manifold of complex dimension n, and G be a relatively compact subdomain with smooth nonempty boundary in N such that ∂G is hyper-$(n-1)$-convex. Let φ: $\partial G \to M$ be a smooth map satisfying the tangential Cauchy-Riemann equation of ∂G. Suppose φ can be extended to a continuous map Φ from $\overline{G}$ to M. Then φ can be extended to a smooth map from $\overline{G}$ to M, which is holomorphic on G if one of the following three conditions is satisfied*:

(a) *∂G is strongly hyper-$(n-1)$-convex at some point of ∂G.*

(b) *There exists $1 \leqslant p \leqslant n-1$ such that* (i) *∂G is hyper-$(n-p)$-convex,* (ii) *T_M is p-negative in the dual Nakano sense, and* (iii) *either* $\operatorname{rank}_{\mathbf{R}} d\varphi \geqslant \max(4p-3, 2p+1)$ *at some point of ∂G or Φ_*: $H_q(\overline{G}, \partial G, \mathbf{R}) \to H_q(M, \varphi(\partial G), \mathbf{R})$ is nonzero for some* $q \geqslant \max(4p-3, 2p+1)$.

(c) *There exists $1 \leqslant p \leqslant n-1$ such that* (i) *∂G is hyper-$(n-p)$-convex,* (ii) *T_M is p-negative in the dual Nakano sense,* (iii) *the bisectional curvature of M is strongly p-nondegenerate, and* (iv) *either* $\operatorname{rank}_{\mathbf{R}} d\varphi \geqslant 2p+1$ *at some point of ∂G or Φ_*: $H_q(\overline{G}, \partial G, \mathbf{R}) \to H_q(M, \varphi(\partial G), \mathbf{R})$ is nonzero for some $q \geqslant 2p+1$.*

Proof. Since M is complete and has nonpositive sectional curvature, by Schoen's result [50, p. 115] there exists a harmonic map f: $G \to M$ smooth up to ∂G such that f agrees with φ on ∂G and f is homotopic to Φ relative to ∂G. Now the desired result follows from Theorem 5.14. The condition of Φ_* being nonzero is used to conclude that $\operatorname{rank}_{\mathbf{R}} df \geqslant q$ at some point of G.

5.19. Remarks. 1. In Theorem 5.18, instead of assuming that M is complete, we can assume that M is compact with convex boundary. In that case we use Hamilton's result [28] instead of Schoen's result to get the harmonic map f.

2. Wood [65] gave an extension theorem proved by extending to the case with boundary the method given in [53] of showing the complex-analyticity of harmonic maps. Wood does not assume any hyperconvexity condition on the boundary of the domain space and claims that the boundary term which occurs in the proof automatically vanishes because the given map satisfies the tangential Cauchy-Riemann equations. His claim and his final results are both incorrect. His argument can be made to work only when it is possible to choose, in his notations, a local coordinate system $z^1, \cdots, z^m$ at p such that ∂X is defined by $\operatorname{Im} z^m \equiv$ constant in a neighborhood of p. The existence of such a local coordinate system implies that ∂X is Levi flat at p.

3. After the author wrote up this paper, he received a preprint from S. Nishikawa and K. Shiga [42] in which they applied to the case with boundary his $\partial\bar{\partial}$ Bochner-Kodaira method [53] and proved the following. Let M, N be complete Kähler manifolds of complex dimension $n \geqslant 2$. Let $D_1 \subset M$ and $D_2 \subset N$ be relatively compact subdomains in M and N with smooth boundaries

∂D_1 and ∂D_2. Suppose N has adequately negative curvature in the sense of [53, p. 84], and ∂D_1 is pseudoconvex (or more generally hyper-$(n-1)$-convex). If $f\colon \partial D_1 \to \partial D_2$ is a smooth map which satisfies the tangential Cauchy-Riemann equations and extends to a homotopy equivalence of $\overline{D_1}$ and $\overline{D_2}$, then f extends to a biholomorphic map from D_1 to D_2 diffeomorphic up to the boundary.

6. Negativity of Einstein bundles

6.1. Let M be a Kähler manifold of complex dimension n with Kähler metric $g_{i\bar{j}}dz^i\overline{dz^j}$. Let E be a holomorphic vector bundle with Hermitian metric $h_{\alpha\bar{\beta}}$ along its fibers. Let $\Theta_{\alpha\bar{\beta}} = -\sqrt{-1}\,\Omega_{\alpha\bar{\beta}i\bar{j}}dz^i\overline{dz^j}$ be the curvature form of E. Let $\Omega_{\alpha\bar{\beta}} = g^{i\bar{j}}\Omega_{\alpha\bar{\beta}i\bar{j}}$.

Definition. E is said to be *Einstein* if there exists a real-valued function κ_E on M such that $\Omega_{\alpha\bar{\beta}} = \kappa_E h_{\alpha\bar{\beta}}$ at every point of M.

For every point P of M let $\chi_E(P)$ be the largest eigenvalue of the Hermitian form

$$(\zeta^{\alpha i}) \mapsto \Omega_{\alpha\bar{\beta}i\bar{j}}\zeta^{\alpha i}\overline{\zeta^{\beta j}}.$$

That is, χ_E equals the supremum of $\Omega_{\alpha\bar{\beta}i\bar{j}}\zeta^{\alpha i}\overline{\zeta^{\beta j}}$ for all $(\zeta^{\alpha i})$ satisfying $h_{\alpha\bar{\beta}}g_{i\bar{j}}\zeta^{\alpha i}\overline{\zeta^{\beta j}} = 1$.

6.2. Lemma. *Suppose E is Einstein, and $0 \leqslant q < n$ is an integer. If $\kappa_E > q\chi_E$ (respectively $\kappa_E \geqslant q\chi_E$) at some point P of M, then E is $(n-q)$-negative (respectively $(n-q)$-seminegative) in the dual Nakano sense at P. Hence if $\kappa_E \geqslant q\chi_E$ on M with strict inequality at some point, then $H^\nu(M, E) = 0$ for $\nu \leqslant q$.*

Proof. Choose local coordinates of M at P and fiber coordinates of E at P such that $g_{i\bar{j}} = \delta_{ij}$ and $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$. Let $\zeta^\alpha_{I_{n-q}}$ be a nonzero set of complex numbers which is skew-symmetric in the $(n-q)$-tuple I_{n-q} of indices. Let

$$\xi_{\alpha J_q} = \sum_{I_{n-q}} \operatorname{sgn}\begin{pmatrix} J_q & I_{n-q} \\ 1 & \cdots n \end{pmatrix} \overline{\zeta^\alpha_{I_{n-q}}}.$$

By Lemma 3.6, at P

$$\begin{aligned}
&\frac{1}{(n-q-1)!}\sum_{\alpha,\beta,s,t,I_{n-q-1}} \Omega_{\alpha\bar{\beta}s\bar{t}}\zeta^\beta_{sI_{n-q-1}}\overline{\zeta^\alpha_{tI_{n-q-1}}} \\
&= \frac{1}{q!}\sum_{\alpha,\beta,J_q} \Omega_{\alpha\bar{\beta}}\xi_{\alpha J_q}\overline{\xi_{\beta J_q}} - \frac{1}{(q-1)!}\sum_{\alpha,\beta,s,t,J_{q-1}} \Omega_{\alpha\bar{\beta}s\bar{t}}\xi_{\alpha sJ_{q-1}}\overline{\zeta_{\beta tJ_{q-1}}} \\
&\geqslant \frac{\kappa_E}{q!}\sum_{\alpha,J_q} |\xi_{\alpha J_q}|^2 - \frac{\chi_E}{(q-1)!}\sum_{\alpha,s,J_{q-1}} |\zeta_{\alpha sJ_{q-1}}|^2 \\
&= \frac{\kappa_E - q\chi_E}{q!}\sum_{\alpha,J_q} |\xi_{\alpha J_q}|^2.
\end{aligned}$$

The last statement follows from the ∇ Bochner-Kodaira technique (1.3.5).

q.e.d.

For the special case where E is the holomophic tangent bundle T_M of M, there is a trick of Calabi-Vesentini [12] which can be used to improve the result of Lemma 6.2.

6.4. Lemma. *For a Kähler-Einstein manifold M of complex dimension n, the holomorphic tangent bundle T_M of M is $(n-q)$-negative (respectively $(n-q)$-seminegative) in the dual Nakano sense at P if $\kappa_{T_M} > \frac{1}{2}(q+1)\chi_{T_M}$ (respectively $\kappa_{T_M} \geqslant \frac{1}{2}(q+1)\chi_{T_M}$) at P.*

Proof. We choose local coordinates of M at P such that $g_{i\bar{j}} = \delta_{ij}$ at P. To use the same notations as in §6.3, instead of $R_{i\bar{j}k\bar{l}}$ we use $\Omega_{\alpha\bar{\beta}s\bar{t}}$ to denote the curvature tensor of T_M. Since $\Omega_{\alpha\bar{\beta}s\bar{t}}$ is symmetric in α, s and in β, t, it follows that the Hermitian form

$$(\theta^{\alpha s}) \mapsto \Omega_{\alpha\bar{\beta}s\bar{t}}\theta^{\alpha s}\overline{\theta^{\beta t}}$$

factors through the orthogonal projection $\pi: T_M^{\otimes 2} \to S^2T_M$, where S^2T_M is the bundle of symmetric contravariant 2-tensors. At P the Hermitian form

$$(\theta^{\alpha s}) \mapsto \frac{1}{2}(\delta_{\alpha\beta}\delta_{st} + \delta_{\alpha t}\delta_{\beta s})\theta^{\alpha s}\overline{\theta^{\beta t}}$$

sends $\theta \in T_M^{\otimes 2}$ to the square norm of $\pi(\theta)$. Hence at P

$$\Omega_{\alpha\bar{\beta}s\bar{t}}\theta^{\alpha s}\overline{\theta^{\beta t}} \leqslant \chi_{T_M}\frac{1}{2}(\delta_{\alpha\beta}\delta_{st} + \delta_{\alpha t}\delta_{\beta s})\theta^{\alpha s}\overline{\theta^{\beta t}}. \tag{6.4.1}$$

Let $\zeta^{\alpha}_{I_{n-q}}$ and $\xi_{\alpha J_q}$ be as in §6.3, when E is replace by T_M. Let

$$\varphi_{\beta J_q} = \frac{1}{q+1}\sum_{\mu=1}^{q}\left(\xi_{\beta J_q} + \xi_{j_\mu j_1\cdots(\beta)_\mu\cdots j_q}\right),$$

$$\psi_{\beta J_q} = \frac{1}{q+1}\left(\xi_{\beta J_q} - \sum_{\mu=1}^{q}\xi_{j_\mu j_1\cdots(\beta)_\mu\cdots j_q}\right),$$

where $J_q = (j_1,\cdots,j_q)$. Then

$$\xi_{\beta J_q} = \varphi_{\beta J_q} + \psi_{\beta J_q}. \tag{6.4.2}$$

Since $\varphi_{\beta J_q}$ is a sum of $q+1$ tensors each of which is symmetric in two indices and since $\psi_{\beta J_q}$ is skew-symmetric in all of its $q+1$ indices, it follows that

$$\sum_{\beta, J_q}\varphi_{\beta J_q}\psi_{\beta J_q} = 0. \tag{6.4.3}$$

By (6.4.1) we have at P

$$\sum_{\alpha,\beta,s,t,J_{q-1}} \Omega_{\alpha\bar\beta s\bar t}\xi_{\alpha s J_{q-1}}\overline{\xi_{\beta J_{q-1}}}$$
$$= \frac{1}{q}\sum_{\alpha,\beta,s,J_q}\sum_{\mu=1}^{q}\Omega_{\alpha\bar\beta s\bar j_\mu}\xi_{\alpha j_1\cdots(s)_\mu\cdots j_q}\overline{\xi_{\beta J_q}}$$
$$\leqslant \frac{1}{q}\chi_{T_M}\sum_{\alpha,\beta,s,J_q}\sum_{\mu=1}^{q}\frac{1}{2}\left(\delta_{\alpha\beta}\delta_{sj_\mu}+\delta_{\alpha j_\mu}\delta_{\beta s}\right)\xi_{\alpha j_1\cdots(s)_\mu\cdots j_q}\overline{\xi_{\beta J_q}}$$
$$= \frac{1}{2q}\chi_{T_M}\sum_{\beta,J_q}\sum_{\mu=1}^{q}\left(\xi_{\beta J_q}+\xi_{j_\mu j_1\cdots(\beta)_\mu\cdots j_q}\right)\overline{\xi_{\beta J_q}}$$
$$= \frac{q+1}{2q}\chi_{T_M}\sum_{\beta,J_q}\varphi_{\beta J_q}\overline{\xi_{\beta J_q}}$$
$$= \frac{q+1}{2q}\chi_{T_M}\sum_{\beta,J_q}|\xi_{\beta J_q}|^2$$

by (6.4.2) and (6.4.3). We now use Lemma 3.6 to conclude, as in the proof of Lemma 6.3, that

$$\frac{1}{(r-q-1)!}\sum_{\alpha,\beta,s,t,I_{n-q-1}}\Omega_{\alpha\bar\beta s\bar t}\zeta^{\beta}{}_{sI_{n-q-1}}\overline{\zeta^{\alpha}{}_{tI_{n-q-1}}}$$
$$= \frac{1}{q!}\left(\kappa_{T_M}-\frac{q+1}{2}\chi_{T_M}\right)\sum_{\beta,I_q}|\xi_{\beta J_q}|^2.$$

6.5. Remarks. 1. By Remark 2 of §4.1 and Lemma 4.3, the conclusion of Lemma 6.4 is equivalent to the 1-negativity of $\Lambda^{n-q}T_M$ in the dual Nakano sense which is equivalent to the Nakano 1-positivity of Ω_M^{n-q}.

2. From the ∇ Bochner-Kodaira technique (1.3.5) and Lemma 6.4 it follows that $H^\mu(M, T_M) = 0$ for $\mu \leqslant q$ and $H^\nu(M, \Lambda^{n-q}T_M) = 0$ for $\nu < n$ if M is a compact Kähler-Einstein manifold with $\kappa_{T_M} \geqslant \frac{1}{2}(q+1)\chi_{T_M}$ on M and strict inequality at some point of M.

3. The k-negativity of T_M in the dual Nakano sense for an appropriate k is the underlying reason why the vanishing theorems proved in [12] by Calabi-Vesentini hold.

4. It is unknown whether $\Lambda^k T_M$ is $(n-q+1-k)$-negative in the dual Nakano sense if $\kappa_{T_M} \geqslant \frac{1}{2}(q+1)\chi_{T_M}$.

6.6. We apply the above considerations to the case of bounded symmetric domains, and compute the negativity of the tangent bundle in the dual Nakano sense and the strong nondegeneracy of the bisectional curvature. For the negativity of the tangent bundle in the dual Nakano sense we use Lemma 6.4 and the computation given in [11], [12] of the eigenvalues of the curvature operator. For the strong nondegeneracy of the bisectional curvature we use the computation concerning the classical domains given in [55] and the exceptional domains given in [66].

Recall that for an n-dimensional Kähler manifold M the Hermitian form

$$(\zeta^{ik}) \mapsto R_{i\bar{j}k\bar{l}}\zeta^{ik}\overline{\zeta^{jl}} \tag{6.6.1}$$

factors through the orthogonal projection $T_M^{\otimes 2} \to S^2T_M$. Hence the sum of its eigenvalues for the subspace of symmetric 2-tensors equals its trace $g^{i\bar{j}}g^{k\bar{l}}R_{i\bar{j}k\bar{l}}$ on $T_M^{\otimes 2}$ which, when M is Kähler-Einstein and of complex dimension n, equals $n\kappa_{T_M}$, where $g_{i\bar{j}}$ is the Kähler metric of M. In the following we will denote κ_{T_M} and χ_{T_M} simply by κ and χ.

A. Let D_{mn}^I be the set of all complex $m \times N$ matrices Z such that $I_n - {}^t Z\overline{Z}$ is positive definite, where I_n is the $n \times n$ identity matrix, ${}^tZ$ is the transpose of Z, and $\overline{Z}$ is the complex conjugate of Z. By [12], for the case $M = D_{mn}^I$ the Hermitian form (6.6.1) has the following eigenvalues: 2 with multiplicity $\frac{1}{4}mn(m+1)(n+1)$, and -2 with multiplicity $\frac{1}{4}mn(m-1)(n-1)$. (Note that our sign convention for $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is opposite to that of [11], [12].) Hence $\chi = 2$. It follows from $\dim_{\mathbf{C}} D_{mn}^I = mn$ that

$$\kappa = \frac{1}{mn}\left(2 \times \tfrac{1}{4}mn(m+1)(n+1) - 2 \times \tfrac{1}{4}mn(m-1)(n-1)\right) = m+n.$$

Thus $\kappa > \frac{1}{2}(q+1)\chi$ for $q < m-n-1$. *The tangent bundle of* D_{mn}^I *is* $((m-1)(n-1)+1)$*-negative in the dual Nakano sense. By* [55], *the bisectional curvature of* D_{mn}^I *is strongly* $((m-1)(n-1)+1)$*-nondegenerate.*

B. Let D_n^{II} be the set of all complex skew-symmetric $n \times n$ matrices Z such that $I_n - {}^tZ\overline{Z}$ is positive definite. By [12], for the case $M = D_n^{II}$ the Hermitian form (6.6.1) has the following eigenvalues: 2 with multiplicity $\frac{1}{4}n^2(n^2-1)$, and -4 with multiplicity $\binom{n}{4}$. Hence $\chi = 2$. It follows from $\dim_{\mathbf{C}} D_n^{II} = \frac{1}{2}n(n-1)$ that

$$\kappa = \frac{1}{\frac{1}{2}n(n-1)}\left(2 \times \tfrac{1}{4}n^2(n^2-1) - 4 \times \binom{n}{4}\right) = 2(n-1).$$

Thus $\kappa > \frac{1}{2}(q+1)\chi$ for $q < 2n-3$. *The tangent bundle of* D_n^{II} *is* $(\frac{1}{2}(n-2)(n-3)+1)$*-negative in the dual Nakano sense.* By [55], *the bisectional curvature of* D_n^{II} *is strongly* $(\frac{1}{2}(n-2)(n-3)+1)$*-nondegenerate.*

C. Let D_n^{III} be the set of all complex symmetric $n \times n$ matrices Z such that $I_n - {}^tZ\overline{Z}$ is positive definite. By [12], for the case $M = D_n^{III}$ the Hermitian form (6.6.1) has the following eigenvalues: 4 with multiplicity $\binom{n+3}{4}$, and -2 with multiplicity $\frac{1}{12}n^2(n^2-1)$. Hence $\chi = 4$. It follows from $\dim_{\mathbf{C}} D_n^{III} = \frac{1}{2}n(n+1)$ that

$$\kappa = \frac{1}{\frac{1}{2}n(n+1)}\left(4 \times \binom{n+3}{4} - 2 \times \tfrac{1}{2}n^2(n^2-1)\right) = 2(n+1).$$

Thus $\kappa > \frac{1}{2}(q+1)\chi$ for $q < n$. *The tangent bundle of D_n^{III} is $(\frac{1}{2}n(n-1)+1)$-negative in the dual Nakano sense. By* [55] *the bisectional curvature of D_n^{III} is strongly $(\frac{1}{2}n(n-1)+1)$-nondegenerate.*

D. Let D_n^{IV} be the set of all complex column n-vector z with ${}^t z\bar{z} < 1$ and $2{}^t z\bar{z} < 1 + |{}^t zz|^2$. By [12], for $M = D_n^{IV}$ the Hermitian form (6.6.1) has the following eigenvalues: 2 with multiplicity $\frac{1}{2}(n-1)(n+2)$ and $2-n$ with multiplicity 1. Hence $\chi = 2$. It follows from $\dim_{\mathbf{C}} D_n^{IV} = n$ that

$$\kappa = \frac{1}{n}\left(2 \times \tfrac{1}{2}(n-1)(n+2) + 2 - n\right) = n.$$

Thus $\kappa > \frac{1}{2}(q+1)\chi$ for $q < n-1$. *The tangent bundle of D_n^{IV} is 2-negative in the dual Nakano sense. By* [55] *the bisectional curvature of D_n^{IV} is strongly 2-nondegenerate.*

E. Let D^V be the exceptional bounded symmetric domain $E_6/\mathrm{Spin}(10) \times T^1$. By [11], for the case $M = D^V$ the Hermitian form (6.6.1) has the following eigenvalues: 1 with multiplicity 126, and –3 with mltiplicity 10. Hence $\chi = 1$. It follows from $\dim_{\mathbf{C}} D^V = 16$ that $\kappa = \frac{1}{16}(126 - 3 \times 10) = 6$. Thus $\kappa \geqslant \frac{1}{2}(q+1)\chi$ for $q < 11$. *The tangent bundle of D^V is 6-negative in the dual Nakano sense.* By [66] *the bisectional curvature of D^V is strongly 6-nondegenerate.*

F. Let D^{VI} be the exceptional bounded symmetric domain $E^7/E^6 \times T^1$. By [11], for the case $M = D^{VI}$ the Hermitian form (6.6.1) has the following eigenvalues: 1 with multiplicity 351, and –4 with multiplicity 27. Hence $\chi = 1$. It follows from $\dim_{\mathbf{C}} D^{VI} = 27$ that $\kappa = \frac{1}{27}(351 - 4 \times 27) = 9$. Thus $\kappa \geqslant \frac{1}{2}(q+1)\chi$ for $q < 17$. *The tangent bundle of D^{VI} is 11-negative in the dual Nakano sense.* By [66] *the bisectional curvature of D^{VI} is strongly 11-nondegenerate.*

6.7. Theorem. *Let M be a compact quotient of an irreducible bounded symmetric domain D, and N a compact Kähler manifold of complex dimension n. Let $f\colon N \to M$ be a harmonic map, and r the maximum of the rank of df over* $\mathbf{R}$. *Then f is either holomorphic or antiholomorphic if $r \geqslant 2p(D)+1$, where $p(D_n^I) = (m-1)(n-1)+1$, $p(D_n^{II}) = \frac{1}{2}(n-2)(n-3)+1$, $p(D_n^{III}) = \frac{1}{2}n(n-1)+1$, $p(D_n^{IV}) = 2$, $p(D^V) = 6$, and $p(D^{VI}) = 11$. In particular, any compact quotient of an irreducible bounded symmetric domain of complex dimension $\geqslant 2$ is strongly rigid.*

This follows from Theorems 5.14 and 5.16, and confirms the conjecture of [55, §6].

6.8. Remark. Since $2 \times 16 > \max(4 \times 6 - 3,\ 2 \times 6 - 1)$ and $2 \times 27 > \max(4 \times 11,\ 2 \times 11 + 1)$, to conclude from Theorem 5.16 the strong rigidity of the compact quotients of the two exceptional domains D^V and D^{VI} one needs only the k-negativity of the tangent bundle in the dual Nakano sense for the appropriate k and does not need to know the strong k-nondegeneracy of the bisectional curvature. However, the computation given in [11] of the eigenvalues of the Hermitian form (6.6.1) of the curvature operator for the two exceptional cases is by no means simple. The simplest way to get the strong rigidity of the compact quotients of the bounded symmetric domains is the one given in [55].

The following theorems are obtained by applying Theorems 5.14, 5.17, and 5.18 to the case of a compact quotient of a bounded symmetric domain. The number $p(D)$ carries the same meaning as in Theorem 6.7.

6.9. Theorem. *Let M be a quotient of an irreducible bounded symmetric domain D, and N a compact complex submanifold of complex dimension $> p(D)$. Then the deformation of N as a complex submanifold of M agrees with the deformation of N as an abstract complex manifold.*

6.10. Theorem. *Let M be a quotient of an irreducible bounded symmetric domain D. Let N be a Kähler manifold of complex dimension n, and G a relatively compact subdomain with smooth nonempty boundary in N such that ∂G is hyper-$(n-1)$-convex. Let $f\colon G \to M$ be a harmonic map smooth up to ∂G such that $\bar\partial_b f \equiv 0$ on ∂G. Then f is holomorphic if either*

(i) *∂G is strongly hyper-$(n-1)$-convex at some point of ∂G, or*

(ii) *∂G is hyper-$(n - p(D))$-convex and* $\operatorname{rank}_{\mathbf{R}} df \geqslant 2p(D) + 1$ *at some point of $\bar G$.*

In particular, a smooth map $\varphi\colon \partial G \to M$ with $\bar\partial_b\varphi \equiv 0$ on ∂G can be extended to a smooth map from $\bar G$ to M which is holomorphic on G if φ can be extended to a continuous map $\Phi\colon \bar G \to M$ and if one of the following two conditions holds:

(i) *∂G is strongly hyper-$(n-1)$-convex at some point of ∂G.*

(ii) *∂G is hyper-$(n - p(D))$-convex and either* $\operatorname{rank}_{\mathbf{R}} d\varphi \geqslant 2p(D) + 1$ *at some point of ∂G or $\Phi_*\colon H_q(\bar G, \partial G, \mathbf{R}) \to H_q(M, \varphi(\partial G), \mathbf{R})$ is nonzero for some $q \geqslant 2p(D) + 1$.*

6.11. Besides using the eigenvalues and the dimensions of the eigenspaces of the Hermitian curvature operator, at least in the case of classical bounded symmetric domains one can also straightforwardly use the explicit form of the curvature tensor and direct computation to get the numbers q so that the tangent bundles are q-negative in the dual Nakano sense. To illustrate this method we do the cases D^I_{mn} and D^{IV}_n.

The Case of D^I_{mn}. For every double index α, we denote its first component by α', and its second component by α'' so that $\alpha = (\alpha', \alpha'')$. The double index α with first component α' is said to be on the α'th row. At $Z = 0$ the curvature tensor of the invariant metric with potential $\log\det(I_n - {}^tZ\bar{Z})^{-1}$ satisfies

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}\zeta^{\alpha\bar{\beta}}\overline{\zeta^{\delta\bar{\gamma}}} = \sum_{\alpha'', \beta''}\left|\sum_{\alpha'}\zeta^{(\alpha', \alpha'')\overline{(\alpha'', \beta'')}}\right|^2 + \sum_{\alpha', \beta'}\left|\sum_{\alpha''}\zeta^{(\alpha', \alpha'')\overline{(\beta', \alpha'')}}\right|^2.$$

It follows that

$$\sum_{\alpha, \beta, \gamma, \delta, \lambda_1, \cdots, \lambda_q} R_{\alpha\bar{\beta}\gamma\bar{\delta}}\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}\overline{\zeta_{\delta\gamma\lambda_1\cdots\lambda_q}}$$
$$= \sum_{\alpha, \beta, \lambda_1, \cdots, \lambda_q}\left|\sum_{\alpha'}\zeta_{(\alpha', \alpha'')(\alpha', \beta'')\lambda_1\cdots\lambda_q}\right|^2$$
$$+ \sum_{\alpha', \beta', \lambda_1, \cdots, \lambda_q}\left|\sum_{\alpha''}\zeta_{(\alpha', \alpha'')(\beta', \alpha'')\lambda_1\cdots\lambda_q}\right|^2.$$

To prove that the tangent bundle of D^I_{mn} is $((m-1)(n-1)+1)$-negative in the dual Nakano sense, it suffices to show that for $q \geqslant mn - m - n + 1$ the equations

$$\text{(6.11.1)} \qquad \begin{aligned} &\sum_{\alpha'}\zeta_{(\alpha', \alpha'')(\alpha', \beta'')\lambda_1\cdots\lambda_q} = 0, \\ &\sum_{\alpha''}\zeta_{(\alpha', \alpha'')(\beta', \alpha'')\lambda_1\cdots\lambda_q} = 0 \end{aligned}$$

imply the vanishing $\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}$ when it is skew-symmetric in $\beta, \lambda_1, \cdots, \lambda_q$. We are going to prove this by induction on $m + n \geqslant 3$.

We observe that, if either $\tau(\alpha', \alpha'') = (\alpha'', \alpha')$ for all double indices $\alpha = (\alpha', \alpha'')$ or $\tau(\alpha', \alpha'') = (\sigma(\alpha'), \alpha'')$ for all double indices $\alpha = (\alpha', \alpha'')$, where σ is a permutation of $\{1, \cdots, m\}$, then the transformation

$$\tau_*: \zeta_{\alpha\beta\lambda_1\cdots\lambda_q} \mapsto \zeta_{\tau(\alpha)\tau(\beta)\tau(\lambda_1)\cdots\tau(\lambda_q)}$$

leaves the set of equations (6.11.1) invariant. We further observe that the component $\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}$ vanishes if every double index on the same row as the index β is one of $\lambda_1, \cdots, \lambda_q$, as one can easily see from the second equation of (6.11.1) and the skew-symmetry of $\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}$ in $\beta, \lambda_1, \cdots, \lambda_q$. That is, the component $\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}$ vanishes if there is some row such that all indices on that row belong to the set $\{\beta, \lambda_1, \cdots, \lambda_q\}$.

Consider the initial step of the induction where $m + n = 3$. We can assume without loss of generality that $m = 2$ and $n = 1$. Every component $\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}$ vanishes for $q \geqslant mn - m - n + 1 = 0$, because β is the only index on its row.

Assume now $m+n>3$. Without loss of generality we can assume that $m \geqslant n$. It follows from $q \geqslant mn-m-n+1 \geqslant mn-2m+1$ that, for any given set of indices $\lambda_0, \cdots, \lambda_q$, either

(i) there exists a row such that every index on that row belongs to $\{\lambda_0, \lambda_1, \cdots, \lambda_q\}$, or

(ii) there exist $1 \leqslant \beta_1' < \beta_2' \leqslant m$ such that, for $\nu = 1, 2$, exactly one index on the β_ν'th row does not belong to $\{\lambda_0, \lambda_1, \cdots, \lambda_q\}$.

According to our earlier observation, $\zeta_{\gamma\lambda_0\cdots\lambda_q}$ vanishes in Case (i). Because of the transformation τ_*, for the proof of the vanishing of $\zeta_{\gamma\lambda_0\cdots\lambda_q}$, it suffices to consider Case (ii) with the additional assumption that $\beta_2' = m$, γ is not on the mth row, and $\lambda_i = (m, i+1)$ for $0 \leqslant i \leqslant n-2$. Define

$$\varphi_{\alpha\mu_{n-1}\cdots\mu_q} = \zeta_{\alpha\lambda_0\cdots\lambda_{n-2}\mu_{n-1}\cdots\mu_q}$$

for indices $\alpha, \mu_{n-1}, \cdots, \mu_q$ not on the mth row. We claim that $\varphi_{\alpha\mu_{n-1}\cdots\mu_q}$ satisfies the following set of equations corresponding to (6.11.1):

$$(6.11.2)\quad \begin{aligned} &\sum_{\alpha'=1}^{m-1} \varphi_{(\alpha',\alpha'')(\alpha',\beta'')\mu_n\cdots\mu_q} = 0 \quad \text{for all fixed } 1 \leqslant \alpha'', \beta'' \leqslant n, \\ &\sum_{\alpha''=1}^{n} \varphi_{(\alpha',\alpha'')(\beta',\alpha'')\mu_n\cdots\mu_q} = 0 \quad \text{for all fixed } 1 \leqslant \alpha', \beta' < m. \end{aligned}$$

The second equation is clearly satisfied. To verify the first equation, it suffices to show the vanishing of

$$\varphi_{(m,\alpha'')(m,\beta'')\mu_n\cdots\mu_q} = \zeta_{(m,\alpha'')\lambda_0\cdots\lambda_{n-2}(m,\beta'')\mu_n\cdots\mu_q}.$$

This is clear, because by skew-symmetry it suffices to consider the case $(m, \beta'') = (m, n)$ and in this case every index on the mth row belongs to the set $\{\lambda_0, \cdots, \lambda_{n-2}, (m, \beta''), \mu_n \cdots \mu_q\}$. Since the equations in (6.11.2) are satisfied, by induction hypothesis $\varphi_{\alpha\mu_{n-1}\cdots\mu_q}$ vanishes. Hence $\zeta_{\gamma\lambda_0\cdots\lambda_q}$ vanishes.

The Case of D_n^{IV}. At $z=0$ the curvature tensor of the invariant metric with potential $-\log(1 - 2\Sigma_\alpha |z_\alpha|^2 + |\Sigma_\alpha z_\alpha^2|^2)$ satisfies

$$R_{\alpha\bar\beta\gamma\bar\delta}\zeta^{\alpha\bar\beta}\,\overline{\zeta^{\delta\bar\gamma}} = 4\left|\sum_\alpha \zeta^{\alpha\bar\alpha}\right|^2 + 2\sum_{\alpha\neq\beta} |\zeta^{\alpha\bar\beta} - \zeta^{\beta\bar\alpha}|^2.$$

It follows that

$$\sum_{\alpha,\beta,\gamma,\delta,\lambda_1,\cdots,\lambda_q} R_{\alpha\bar\beta\gamma\bar\delta}\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}\overline{\zeta_{\delta\gamma\lambda_1\cdots\lambda_q}}$$

$$= 4\sum_{\lambda_1,\cdots,\lambda_q}\left|\sum_\alpha \zeta_{\alpha\alpha i_1\cdots i_q}\right|^2 + 2\sum_{\substack{\lambda_1,\cdots,\lambda_q \\ \alpha\neq\beta}} \left|\zeta_{\alpha\beta\lambda_1\cdots\lambda_q} - \zeta_{\beta\alpha\lambda_1\cdots\lambda_q}\right|^2.$$

To prove that the tangent bundle of D_n^{IV} is 2-negative in the dual Nakano sense, it suffices to show that, for $q \geqslant 1$, the equations

$$\zeta_{\alpha\alpha\lambda_1\cdots\lambda_q} = 0,$$
$$\zeta_{\alpha\beta\lambda_1\cdots\lambda_q} = \zeta_{\beta\alpha\lambda_1\cdots\lambda_q} \quad \text{for } \alpha \neq \beta$$

imply the vanishing of $\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}$ when it is skew-symmetric in $\beta, \lambda_1, \cdots, \lambda_q$. This follows from the fact that $\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}$ is skew-symmetric in β, λ_1 and symmetric in α, β. For

$$\begin{aligned}\zeta_{\alpha\beta\lambda_1\cdots\lambda_q} &= -\zeta_{\alpha\lambda_1\beta\lambda_2\cdots\lambda_q} = -\zeta_{\lambda_1\alpha\beta\lambda_2\cdots\lambda_q} \\ &= \zeta_{\lambda_1\beta\alpha\lambda_2\cdots\lambda_q} = \zeta_{\beta\lambda_1\alpha\lambda_2\cdots\lambda_q} = -\zeta_{\beta\alpha\lambda_1\cdots\lambda_q} = -\zeta_{\alpha\beta\lambda_1\cdots\lambda_q}.\end{aligned}$$

7. Curvature characterization of compact symmetric Kähler manifolds

7.1. Besides the strong rigidity of suitably negatively curved compact Kähler manifolds, another major application of the complex-analyticity of harmonic maps is the curvature characterizations of the complex projective space and the complex hyperquadric [56], [19], [54]. (For the projective space Mori [39] obtained the stronger result of characterization by the ampleness of the tangent bundle by methods of algebraic geometry.) The strong rigidity is a result of the Bochner-Kodaira technique for vanishing theorems applied to the $\bar\partial$ differential of a harmonic map. For this the target manifold has to be suitably negatively curved. We know that strong rigidity holds for a compact quotient of any irreducible bounded symmetric domain of complex dimension $\geqslant 2$, [53], [55]. On the other hand, the curvature characterizations of the complex projective space and the complex hyperquadric [56], [19], [54] are proved by using the second variation formula for energy-mininizing harmonic maps. For this the target manifold has to be suitably positively curved. So far the curvature characterization of general compact Kähler manifolds has not been found. In this section we will deal with this question of curvature characterization. We will not use the method of energy-minimizing harmonic maps. Instead, we will use the Bochner-Kodaira technique and holonomy groups. The result we get is only a partial answer to the question of curvature characterization. First we give a definition.

Definition. The bisectional curvature of a Kähler manifold M is said to be *irreducible* at a point P of M if it is not possible to decompose the holomorphic tangent space $T_{M,P}$ of M at P into two nonzero orthogonal direct summands Ξ and H such that $R_{i\bar{j}k\bar{l}}\xi^i\overline{\xi^j}\eta^k\overline{\eta^l} = 0$ for all $\xi \in \Xi$ and all $\eta \in H$.

Clearly if the complex dimension of M is n, and the bisectional curvature of M is strongly $(n-1)$-nondegenerate at P, then the bisectional curvature of M is irreducible at P.

7.2. Theorem. *Let M be a compact Kähler manifold whose holomorphic tangent bundle T_M is* 1*-semipositive in the dual Nakano sense. Suppose the bisectional curvature of M is irreducible at some point P of M. Then either M is an irreducible Hermitian symmetric manifold with respect to the given Kähler metric or the cohomology ring of M with coefficients in* **R** *is isomorphic to that of the complex projective space of the same dimension.*

7.3. Corollary. *Let M be an irreducible compact Hermitian symmetric space of* rank > 1. *Then any Kähler metric on M with respect to which the holomorphic tangent bundle of M is* 1*-semipositive in the dual Nakano sense must be a constant multiple of the standard invariant Kähler metric on M.*

The rest of this section is devoted to the proof of Theorem 7.2 and Corollary 7.3.

7.4. Proposition. *Let M be a compact Kähler manifold whose holomorphic tangent bundle is* 1*-semipositive in the dual Nakano sense. Then every harmonic (p,q)-form is parallel (i.e., has zero covariant derivative).*

Proof. Let $g_{i\bar j}$ be the Kähler metric of M, and let $\varphi_{I_p\bar J_q}$ be a harmonic (p,q) form on M. Let

$$\eta_{k\bar l I_p\bar J_q}=\sum_{\mu=1}^{p}\varphi_{i_1\cdots(k)_\mu\cdots i_p\bar J_q}g_{i_\mu\bar l}-\sum_{\nu=1}^{q}\varphi_{I_p\bar j_1\cdots(\bar l)_\nu\cdots\bar j_q}g_{k\bar j_\nu},$$

where $I_p=(i_1,\cdots,i_p)$ and $J_q=(j_1,\ldots,j_q)$. Let

$$F=R^{\bar lk}{}_{r\bar s}\eta_{k\bar l I_p\bar J_q}\overline{\eta^{\bar r s\bar I_p J_q}}.$$

Then

$$F=\sum_{\mu=1}^{p}\sum_{\sigma=1}^{p}I_{\mu\sigma}-\sum_{\mu=1}^{p}\sum_{\tau=1}^{q}II_{\mu\tau}-\sum_{\nu=1}^{q}\sum_{\sigma-1}^{p}III_{\nu\sigma}+\sum_{\nu=1}^{q}\sum_{\tau=1}^{q}IV_{\nu\tau},$$

where

$$I_{\mu\sigma}=R^{\bar lk}{}_{r\bar s}\varphi_{i_1\cdots(k)_\mu\cdots i_p\bar J_q}g_{i_\mu\bar l}\,\overline{\varphi^{\bar i_1\cdots(\bar r)_\sigma\cdots\bar i_pJ_q}g^{\bar i_\sigma s}},$$

$$II_{\mu\tau}=R^{\bar lk}{}_{r\bar s}\varphi_{i_1\cdots(k)_\mu\cdots i_p\bar J_q}g_{i_\mu\bar l}\,\overline{\varphi^{\bar I_pj_1\cdots(s)_\tau\cdots j_q}g^{\bar rj_\tau}},$$

$$III_{\nu\sigma}=R^{\bar lk}{}_{r\bar s}\varphi_{I_p\bar j_1\cdots(\bar l)_\nu\cdots\bar j_q}g_{k\bar j_\nu}\,\overline{\varphi^{\bar i_1\cdots(\bar r)_\sigma\cdots\bar i_pJ_q}g^{\bar i_\sigma s}},$$

$$IV_{\nu\tau}=R^{\bar lk}{}_{r\bar s}\varphi_{I_p\bar j_1\cdots(\bar l)_\nu\cdots\bar j_q}g_{k\bar j_\nu}\,\overline{\varphi^{\bar I_pj_1\cdots(s)_\tau\cdots j_q}g^{\bar rj_\tau}}.$$

To calculate $I_{\mu\sigma}$ we first consider the case $\mu = \sigma$.

$$I_{\mu\mu} = R^k{}_r \varphi_{i_1 \cdots (k)_\mu \cdots i_p \bar{J}_q} \overline{\varphi^{\bar{i}_1 \cdots (\bar{r})_\mu \cdots \bar{i}_p J_q}} .$$

For $\mu \neq \sigma$,

$$I_{\mu\sigma} = R_l{}^k{}_r{}^s \varphi_{i_1 \cdots (k)_\mu \cdots (s)_\sigma \cdots i_p \bar{J}_q} \overline{\varphi^{\bar{i}_1 \cdots (\bar{l})_\mu \cdots (\bar{r})_\sigma \cdots \bar{i}_p J_q}}$$

vanishes, because $R_l{}^k{}_r{}^s$ is symmetric in k, s whereas $\varphi_{i_1 \cdots (k)_\mu \cdots (s)_\sigma \cdots i_p \bar{J}_q}$ is skew-symmetric in k, s.

$$II_{\mu\tau} = R_l{}^{k\bar{r}}{}_{\bar{s}} \varphi_{i_1 \cdots (k)_\mu \cdots i_p \bar{j}_1 \cdots (\bar{r})_\tau \cdots \bar{j}_q} \overline{\varphi^{\bar{i}_1 \cdots (\bar{l})_\mu \cdots \bar{i}_p j_1 \cdots (s)_\tau \cdots j_q}} ,$$

$$\begin{aligned} III_{\nu\sigma} &= R^{\bar{l}}{}_{\bar{k}r}{}^s \varphi_{i_1 \cdots (s)_\sigma \cdots i_p \bar{j}_1 \cdots (\bar{l})_\nu \cdots \bar{j}_q} \overline{\varphi^{\bar{i}_1 \cdots (\bar{r})_\sigma \cdots \bar{i}_p j_1 \cdots (k)_\nu \cdots j_q}} \\ &= II_{\sigma\nu} . \end{aligned}$$

To calculate $IV_{\nu\tau}$ we first consider the case $\nu = \tau$.

$$IV_{\nu\nu} = R^{\bar{l}}{}_{\bar{s}} \varphi_{I_p \bar{j}_1 \cdots (\bar{l})_\nu \cdots \bar{j}_q} \overline{\varphi^{\bar{I}_p j_1 \cdots (s)_\tau \cdots j_q}} .$$

For $\nu \neq \tau$

$$IV_{\nu\tau} = R^{\bar{l}}{}_{\bar{k}}{}^{\bar{r}}{}_{\bar{s}} \varphi_{I_p \bar{j}_1 \cdots (\bar{l})_\nu \cdots (\bar{r})_\tau \cdots \bar{j}_q} \overline{\varphi^{\bar{I}_p j_1 \cdots (k)_\nu \cdots (s)_\tau \cdots j_q}}$$

vanishes, because $R^{\bar{l}}{}_{\bar{k}}{}^{\bar{r}}{}_{\bar{s}}$ is symmetric in l, r, whereas $\varphi_{I_p \bar{j}_1 \cdots (\bar{l})_\nu \cdots (\bar{r})_\tau \cdots \bar{j}_q}$ is skew-symmetric in l, r.

Combining these calculations together, we obtain

$$\begin{aligned} F = {} & \sum_{\mu=1}^{p} R^k{}_r \varphi_{i_1 \cdots (k)_\mu \cdots i_p \bar{J}_q} \overline{\varphi^{\bar{i}_1 \cdots (\bar{r})_\mu \cdots \bar{i}_p J_q}} \\ & + \sum_{\nu=1}^{q} R^{\bar{l}}{}_{\bar{s}} \varphi_{I_p \bar{j}_1 \cdots (\bar{l})_\nu \cdots \bar{j}_q} \overline{\varphi^{\bar{I}_p j_1 \cdots (s)_\tau \cdots j_q}} \\ & - 2 \sum_{\mu=1}^{p} \sum_{\nu=1}^{q} R_l{}^{k\bar{r}}{}_{\bar{s}} \varphi_{i_1 \cdots (k)_\mu \cdots i_p \bar{j}_1 \cdots (\bar{r})_\nu \cdots \bar{j}_q} \overline{\varphi^{\bar{i}_1 \cdots (\bar{l})_\mu \cdots \bar{i}_p j_1 \cdots (s)_\nu \cdots j_q}} \\ = {} & p\, R^k{}_r \varphi_{k I_{p-1} \bar{J}_q} \overline{\varphi^{\bar{r} \bar{I}_{p-1} J_q}} + q R^{\bar{l}}{}_{\bar{s}} \varphi_{I_p \bar{l} \bar{J}_{q-1}} \overline{\varphi^{\bar{I}_p s J_{q-1}}} \\ & - 2pq R_l{}^{k\bar{r}}{}_{\bar{s}} \varphi_{k I_{p-1} \bar{r} \bar{J}_{q-1}} \overline{\varphi^{\bar{l} \bar{I}_{p-1} s J_{q-1}}} . \end{aligned}$$

By (1.3.3) for the case of the trivial line bundle, we have

$$
\begin{aligned}
\|\bar\partial\varphi\|_M^2 + \|\bar\partial^*\varphi\|_M^2 = \|\overline{\nabla}\varphi\|_M^2 &- \frac{F}{2p!q!} \\
&+ \frac{1}{2(p-1)!q!}\int_M R^k{}_r \varphi_{kI_{p-1}\bar J_q}\overline{\varphi^{\bar r \bar I_{p-1} J_q}} \\
&- \frac{1}{2p!(q-1)!}\int_M R^{\bar l}{}_{\bar s}\varphi_{I_p \bar l \bar J_{q-1}}\overline{\varphi^{\bar I_p s J_{q-1}}}.
\end{aligned}
\tag{7.4.1}
$$

Since the holomorphic tangent bundle of M is 1-semipositive in the dual Nakano sense, it follows that $F \leq 0$. Since $\bar\partial\varphi$ and $\bar\partial^*\varphi$ both vanish identically, we have

$$
\begin{aligned}
\|\overline{\nabla}\varphi\|_M^2 \leq \frac{1}{2p!(q-1)!}&\int_M R^{\bar l}{}_{\bar s}\varphi_{I_p \bar l \bar J_{q-1}}\overline{\varphi^{\bar I_p s J_{q-1}}} \\
&- \frac{1}{2(p-1)!q!}\int_M R^k{}_r \varphi_{kI_{p-1}\bar J_q}\overline{\varphi^{\bar r \bar I_{p-1} J_q}}.
\end{aligned}
$$

Applying the same argument to $\bar\varphi$ instead of φ we obtain

$$
\begin{aligned}
\|\nabla\varphi\|_M^2 \leq \frac{-1}{2p!(q-1)!}&\int_M R^{\bar l}{}_{\bar s}\varphi_{I_p \bar l \bar J_{q-1}}\overline{\varphi^{\bar I_p s J_{q-1}}} \\
&+ \frac{1}{2(p-1)!q!}\int_M R^k{}_r \varphi_{kI_{p-1}\bar J_q}\overline{\varphi^{\bar r \bar I_{p-1} J_q}}.
\end{aligned}
$$

Adding the two inequalities together, we obtain $\|\overline{\nabla}\varphi\|_M^2 + \|\nabla\varphi\|_M^2 \leq 0$. Thus φ is parallel. q.e.d.

The proof of Proposition 7.4 is a generalization of the method of Bishop-Goldberg [8] and Goldberg-Kobayashi [21], because when $p = q = 1$ and $\varphi_{i\bar j} = \lambda_i\delta_{ij}$ and $g_{i\bar j} = \delta_{ij}$, we have $F = \Sigma_{k,l} R_{k\bar l l\bar k}(\lambda_k - \lambda_l)^2$. This method of generalization was introduced by Meyer [37] who used it in the case of real Riemannian manifolds. Later Ogiue-Tachibana [43] applied Meyer's method [37] to the case of effective harmonic forms on Kähler manifolds. (In conjunction with the paper of Ogiue-Tachibana [43] we would like to point out that there exists no compact Kähler manifold M of positive pure curvature operator in the sense of [43], because in such a case T_M is Nakano 1-positive and $H^1(M, \Omega_M^1)$ vanishes by the Nakano vanishing theorem and Serre duality, contradicting the existence of a Kähler form.) Proposition 7.4 can also be proved by considering only the effective harmonic forms. The reason for using the present method of proof is that this method of proof of Proposition 7.4 can readily yield the following proposition which can be used to complete Schneider's scheme [49] of proving Barth-Lefschetz type theorems for compact symmetric Kähler manifolds, though we will not pursue it further in this paper because of the stronger results obtained by Morse theory (see §8).

7.5. Proposition. *Let M be a Kähler manifold whose holomorphic tangent bundle is* 1*-semipositive in the dual Nakano sense. Let λ be a positive number $\geqslant$ the ratio of any two eigenvalues of the Ricci tensor of M at any point. Let G be a relatively compact subdomain of M with smooth boundary ∂G such that ∂G is hyper-k-convex. Let E be a Hermitian holomorphic vector bundle over M which is Nakano* 1*-semipositive. Then $H^q(G, E \otimes \Omega^p_M)$ vanishes if one of the following conditions is satisfied*:

(a) *$q \geqslant k$, and $q > \lambda p$.*

(b) *$q \geqslant k$, $q \geqslant \lambda p$, and ∂G is strongly hyper-q-convex at some point of ∂G.*

(c) *$q \geqslant k$, $q \geqslant \lambda p$, and E is Nakano* 1*-positive at some point of G.*

The proof of Proposition 7.5 differs from that of Proposition 7.4 only in the following. From the E-valued harmonic (p, q)-form $\varphi^{\alpha}_{I_p\bar{J}_q}$ we form $\eta^{\alpha}_{k\bar{l}I_p\bar{J}_q}$ in the same way, but in defining F we have to contract the index α of $\eta^{\alpha}_{k\bar{l}I_p\bar{J}_q}$ with the corresponding index from $\bar{\eta}$. On the right-hand side of the equation corresponding to (7.4.1) we have an additional boundary term and another additional term involving the curvature form of E. The conclusions of Proposition 7.5 result from this equation. The conditioin $q > \lambda p$ or $q \geqslant \lambda p$ is used to compare the two terms involving the Ricci tensor. The strong hyper-q-convexity of ∂G at some point of ∂G is used in the same way as in the proof of Lemma 5.11.

We now continue with the proof of Theorem 7.2. For the rest of this section M denotes the Kähler manifold of Theorem 7.2.

7.6. The 1-semipositivity of T_M in the dual Nakano sense implies that the bisectional curvature of M is nonnegative. Since the bisectional curvature of M is irreducible at P, it follows from the method of proof of [54, p. 647, Theorem 3] that the second Betti number $b_2(M)$ of M is 1.

For the rest of this section, for $Q \in M$, $T_{M,Q}$ means the real tangent space of M when M is regarded as a real manifold. It is given the complex structure J from the complex manifold M so that it is a **C**-vector space. Let H_Q be the holonomy group of M at Q. Clearly H_Q is a subgroup of the unitary group of the **C**-vector space $T_{M,Q}$.

The Ricci tensor of M is positive definite at P, because the bisectional curvature of M is irreducible at P, otherwise some decomposition of the complex vector space $T_{M,P}$ into a 1-dimensional complex linear subspace and its orthogonal complement leads to a contradiction. According to [31], this together with $b_2(M) = 1$ implies that M is simply connected and irreducible in the sense of the de Rham decomposition theorem [16]. Hence for $Q \in M$, H_Q is connected and acts irreducibly on $T_{M,Q}$.

7.7. Suppose M is not Hermitian symmetric (and therefore not Riemannian symmetric) with respect to the given Kähler metric. We want to prove that the cohomology ring of M over $\mathbf{R}$ is isomorphic to that of the complex projective space of the same dimension. The case $\dim_{\mathbf{C}} M = 1$ is clear. So we assume $\dim_{\mathbf{C}} M \geqslant 2$.

For $Q \in M$ let R^Q denote the Riemannian curvature tensor of M at Q, i.e., let $R^Q_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ for $X, Y \in T_{M,Q}$, where ∇_X denotes the covariant differentiation in the direction X with respect to the Levi-Civita connection. Since M is not Riemannian symmetric, it follows from [51, p. 233, Theorem 8] that for some $Q \in M$ the holonomy system $\{T_{M,Q}, R^Q, H_Q\}$ is not symmetric in the sense of [51, p. 215]. By [51, p. 221, Theorem 4], the action of H_Q on the unit sphere of $T_{M,Q}$ must be transitive. Hence the action of H_P on the unit sphere of $T_{M,P}$ is transitive.

7.8. Define R' by

$$R' = \int_{h \in H_P} h(R^P),$$

where $h(R^P)$ is the tensor obtained by the natural action of h on R^P. Let $\mathfrak{h}$ be the Lie algebra of H_P. Define as follows a Lie algebra $\mathfrak{g}$ whose underlying vector space is $\mathfrak{h} \oplus T_{M,P}$:

$$[A, B] = [A, B] \text{ in } \mathfrak{h} \text{ for } A, B \in \mathfrak{h}, \tag{7.8.1}$$

$$[X, Y] = R'_{X,Y} \text{ for } X, Y \in T_{M,P}, \tag{7.8.2}$$

$$[A, X] = A(X) \text{ for } A \in \mathfrak{h} \text{ and } X \in T_{M,P}. \tag{7.8.3}$$

Because of [51, p. 213, Theorem 1], $(h(R^P))_{X,Y}$ belongs to $\mathfrak{h}$ for $X, Y \in T_{M,P}$ and $h \in H_P$. Hence $R'_{X,Y}$ belongs to $\mathfrak{h}$ for $X, Y \in T_{M,P}$. The Jacobi identity for $\mathfrak{g}$ is satisfied because

$$[A, R'_{X,Y}] = R'_{X,A(Y)} - R'_{Y,A(X)} \tag{7.8.4}$$

for $A \in \mathfrak{h}$, and $X, Y \in T_{M,P}$ due to the invariance of R' under the action of H_P. So $\mathfrak{g}$ is well-defined and is indeed a Lie algebra.

In the same way as one forms the Ricci tensor from a curvature tensor, one forms the symmetric quadratic form $r(X, Y)$ on $T_{M,P}$ from R'. That is, $r(X, Y)$ is the trace of the endomorphism

$$Z \mapsto R'_{X,Z} Y.$$

Being equal to the Ricci tensor of M at P averaged over the action of H_P, the quadratic form r is positive definite.

Since H_P is compact, the Killing form $B_{\mathfrak{h}}(\cdot,\cdot)$ of $\mathfrak{h}$ is negative semidefinite [30, p. 122, Prop. 6.6]. The Killing form $B_{\mathfrak{g}}(\cdot,\cdot)$ of $\mathfrak{g}$ is strictly negative definite.

For, when $A \in H_P$, it follows from (7.8.1) and (7.8.3) that

$$B_{\mathfrak{g}}(A, A) = B_{\mathfrak{h}}(A, A) - \sum_{i,j} A_{ij}^2$$

due to the skew-symmetry of $A = (A_{ij})$ with respect to an orthonormal basis over $\mathbf{R}$ of the $\mathbf{R}$-vector space $T_{M,P}$. Moreover, when $X \in T_{M,P}$, we have $R'_{X,A(X)} = 0$ by (7.8.4), and it follows from (7.8.2) and (7.8.3) that

$$B_{\mathfrak{g}}(X, X) = -r(X, X). \tag{7.8.5}$$

Let G be the adjoint group $\mathrm{Int}(\mathfrak{g})$ of $\mathfrak{g}$ [30, p. 116]. Since $B_{\mathfrak{g}}$ is strictly negative definite, G is a compact subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ [30, p. 122, Prop. 6.6]. Let H be the analytic subgroup of G which corresponds to the subalgebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$. The adjoint representation Ad_{H_P} of H_P on $\mathfrak{h}$ together with the action of H_P on $T_{M,P}$ defines a monomorphism from H_P to $\mathrm{Aut}(\mathfrak{g})$ whose image clearly equals H. This monomorphism is compatible with the actions of H and H_P on $T_{M,P}$.

The self-map of $\mathfrak{g}$ which sends X to $-X$ for $T_{M,P}$ and leaves every element of $\mathfrak{h}$ fixed defines an involutive automorphism s of $\mathfrak{g}$. The automorphism s of $\mathfrak{g}$ induces an involutive analytic automorphism σ of G such that the identity component of the set of fixed points of σ is H. Since H is compact, the pair (G, H) of Lie groups is a Riemannian symmetric pair. Since the tangent space of G/H at the point H is naturally isomorphic to $T_{M,P}$, and H acts effectively on $T_{M,P}$, it follows that G acts effectively on G/H.

The restriction of $-B_{\mathfrak{g}}$ to $T_{M,P}$ defines a G-invariant Riemannian structure on G/H, which is clearly invariant under the complex structure operator J of $T_{M,P}$ because of (7.8.5). Hence G/H is a Hermitian symmetric space [30, p. 302, Prop. 4.2]. By [51, p. 213, Theorem 1] and [30, p. 207, Theorem 4.1] the holonomy group of G/H at the point H is equal to H. Since the action of H_P on the unit sphere of $T_{M,P}$ (and therefore the action of H on the unit sphere of the tangent space of G/H at H) is transitive, it follows that the Hermitian symmetric space G/H is irreducible and of rank one. So H is the full unitary group of the tangent space of G/H at H. Hence H_P is the full unitary group of $T_{M,P}$.

7.9. Let φ be a harmonic (p, q)-form on M. By Proposition 7.4, φ is parallel. Since H_P is the full unitary group of $T_{M,P}$ by §7.8, it follows that at P the form φ is invariant under the full unitary group of $T_{M,P}$. We can regard φ as a linear function on the set of all contravariant tensors $\xi = (\xi^{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q})$ of type (p, q) at P. This linear function is invariant under the full unitary group of $T_{M,P}$. By Weyl's theory [64] of invariants of the unitary group (see [3, p. 291, Theorem (3.12)]), such linear invariants are zero when $p \neq q$ and are of the form

$$\sum_{\tau} c_\tau \xi^{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} g_{i_{\tau(1)} \bar{j}_1} \cdots g_{i_{\tau(p)} \bar{j}_p} \tag{7.9.1}$$

when $p = q$, where $c_\tau \in \mathbf{C}$, and the sum is over all permutations τ of $\{1, \cdots, p\}$. Since $\varphi_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_p}$ is skew-symmetric in $i_1, \cdots, i_p$ and in $\bar{j}_1, \cdots, \bar{j}_q$, it follows that at P, φ vanishes when $p \neq q$ and $\varphi_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_p} \xi^{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_p}$ is of the form (7.9.1) with $c_\tau = (\operatorname{sgn} \tau) c$ for some c and for all τ when $p = q$. Hence φ is a constant multiple of ω^p at P when $p = q$, where ω is the Kähler form of M. Since φ is parallel, φ is a constant multiple of ω^p on all of M. Thus the cohomology ring of M over $\mathbf{R}$ is isomorphic to that of the complex projective space of the same dimension. This concludes the proof of Theorem 7.2.

Observe that in the proof of Theorem 7.2 the irreducibility of the bisectional curvature of M at one point is used only to show that $b_2(m) = 1$ and the Ricci tensor is positive definite at some point. For the proof of Corollary 7.3 the assumption of the irreducibility of the bisectional curvature at some point is not needed because the second Betti number is clearly 1, and the Ricci tensor must be positive definite at some point due to the nonnegativity of the bisectional curvature and the simple connectedness of the manifold [31].

7.10. Remarks. 1. In the last step of the proof of Theorem 7.2 one can avoid using Weyl's theory of invariants by using the following observation. The cohomology ring of $\mathbf{P}_n$ is isomorphic to the ring of all exterior forms *at one point* which are invariant under its holonomy group, because any such an exterior form at one point yields a harmonic form by parallel transport.

2. There is a Riemannian analog of Theorem 7.2 which can be proved in the same way. In the Riemannian case the 1-semipositivity of the tangent bundle in the dual Nakano sense is replaced by the nonnegativity of the curvature operator of [37]. The irreducibility of the bisectional curvature at one point is replaced by simple connectedness and irreducibility of the manifold in the sense of de Rham and the positive definiteness of the Ricci curvature at one point. The conclusion is that either the manifold is an irreducible symmetric Riemannian manifold with respect to the given Riemannian metric or its cohomology ring with coefficients in $\mathbf{R}$ is isomorphic to that of one of the compact symmetric Riemannian manifolds of rank 1.

3. It is unknown whether the assumption in Theorem 7.2 of the 1-semipositivity of T_M in the dual Nakano sense can be weakened to the nonnegativity of the bisectional curvature, and whether one can conclude that M is biholomorphic to an irreducible compact Hermitian symmetric manifold in all cases.

8. Barth-Lefschetz type theorems

8.1. Schneider [49] introduced a scheme of using the vanishing theorem of Grauert-Riemenschneider [22] for strongly hyper-q-convex domains to generalize theorems of Barth-Lefschetz type [4], [5], [6], [29], [36] for submanifolds of low codimension in the complex projective space (and submanifolds of codimension 2 in the Grassmannian [7]) to the more general case of compact symmetric Kähler manifolds. He encountered difficulties because the holomorphic tangent bundle of a compact symmetric Kähler manifold is not Nakano semipositive except in the case of the complex projective space. By using the method of proof of Proposition 7.5, it is possible to overcome his difficulty. However, Schneider's proof of the strong hyper-q-convexity of the complement of a complex submanifold in a compact symmetric Kähler manifold seems to be invalid, even for the special case of the complex projective space, except in the obvious case of codimension one. For this and two other reasons given below we do not complete Schneider's scheme here. In the meantime Sommese [58] announced some generalizations of the Barth-Lefschetz type theorems to homogeneous compact complex manifolds with details to be given in a series of papers quite a number of which have already appeared [59], [60], [61]. Moreover, if the complement of a complex submanifold of a compact Kähler manifold is strongly hyper-q-convex, then one can easily get Barth-Lefschetz theorems at the homotopy level by means of Morse theory under the very weak assumption that the bisectional curvature of the compact Kähler manifold is nonnegative. This can be achieved by using the second variation formula for arc-length and by introducing an appropriate class of functions lying between the classes of subharmonic and plurisubharmonic functions. We will devote the rest of this section to this method of getting Barth-Lefschetz type theorems.

First we introduce the new class of functions which we need.

Definition. Let f be a real-valued C^2 function on a Kähler manifold M of complex dimension n. Let $1 \leqslant q \leqslant n$. The function f is said to be (*strongly*) *q-plurisubharmonic* at a point P of M if for every local complex submanifold N of M at P of complex dimension q the restriction of f to N is (strongly) subharmonic at P when N is given the induced Kähler metric. In other words, f is strongly q-plurisubharmonic (respectively q-plurisubharmonic) at P if for any unitary q-frame $\Sigma_{i=1}^n a_\nu^i(\partial/\partial z^i)$ at P $(1 \leqslant \nu \leqslant q)$ the expression

$$\sum_{\nu=1}^{q} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} a_\nu^i \overline{a_\nu^j}$$

is positive (respectively nonnegative) at P.

For $\kappa > 0$ the q-plurisubharmonicity of f is said to be $\geqslant \kappa$ at P if $f - \kappa(d_P)^2$ is q-plurisubharmonic at P where d_P is the distance function measured from P.

If g is a real-valued C^2 function such that $g = f$ at P, and $g \geqslant f$ on some neighborhood of P, then the q-plurisubharmonicity of g at P is $\geqslant$ the q-plurisubharmonicity of f at P.

Clearly (strong) n-plurisubharmonicity coincides with (strong) subharmonicity and (strong) 1-plurisubharmonicity coincides with (strong) plurisubharmonicity. Except for the case $q = 1$, (strong) q-plurisubharmonicity depends on the Kähler metric.

8.2. *Let M be a Kähler manifold with nonnegative holomorphic bisectional curvature, and G a relatively compact subdomain of M with smooth boundary ∂G. Assume that ∂G is strongly hyper-q-convex. Then there exists a smooth exhaustion function on G which is strongly q-plurisubharmonic, where an exhaustion function on G means a function approaching ∞ on ∂G.*

The proof of this theorem and its immediate consequences will occupy the rest of this section. Let d_G be the distance function from ∂G. The exhaustion function will be obtained by smoothing out $\tau \circ (-d_G)$, where τ is a sufficiently convex increasing smooth function. To apply this theorem to obtain Barth-Lefschetz type theorems by means of Morse theory, it suffices to conclude the existence of a smooth strongly q-pseudoconvex exhaustion function on G. (A strongly q-pseudoconvex function means a function whose Levi form at every point has no more than $q - 1$ nonpositive eigenvalues.) However, for this method it is essential to assume that ∂G is strongly hyper-q-convex instead of merely strongly q-pseudoconvex, even if one only wants to conclude the existence of a smooth strongly q-pseudoconvex exhaustion function on G. (The strong q-pseudoconvexity of ∂G means that the Levi form of the function defining ∂G has no more than $q - 1$ nonpositive eigenvalues on the complex tangent space of ∂G at every point.) The difficulty is with the smoothing process. If ∂G is strongly q-pseudoconvex, then the continuous exhaustion function $\tau \circ (-d_G)$ is strongly q-pseudoconvex in the sense that for any point there is a local complex submanifold of complex codimension $q - 1$ so that the restriction of $\tau \circ (-d_G)$ to this submanifold is strongly plurisubharmonic. In general one cannot smooth out such a continuous strongly q-pseudoconvex function to get a smooth strongly q-pseudoconvex function. The problem is that the sum of two continuous strongly q-pseudoconvex functions may fail to be strongly q-pseudoconvex, because at a given point the local complex submanifolds of complex codimension $q - 1$ on which the restrictions of the two functions are strongly plurisubharmonic respectively may have tangent spaces whose intersection is of complex codimension $2(q - 1)$. Moreover, one cannot modify Morse theory to make it applicable to the case of a continuous strongly q-pseudoconvex exhaustion function, as is seen from the following counter-example. For $q > 2$ consider the Segre embedding. Let $M = \mathbf{P}_{2q+1}$ and $G =$ the set of points of M of distance $> \eta$ from $\mathbf{P}_1 \times \mathbf{P}_q$ with η being a sufficiently small positive number. Then ∂G is strongly q-pseudoconvex. By the argument of the second variation of the arc-length given later in this section, one can easily see that $\tau \circ (-d_G)$ is a continuous strongly q-pseudoconvex exhaustion function on G for some smooth sufficiently convex increasing function τ. However, G admits no smooth strongly q-pseudoconvex exhaustion function, otherwise $4q - 1 > (2q + 1) + q$ implies the vanishing of $H^{4q-1}(G, \mathbf{C}) \approx H_c^3(G, \mathbf{C})$ and the surjectivity of $H^2(\mathbf{P}_{2q+1}, \mathbf{C}) \to H^2(\mathbf{P}_1 \times \mathbf{P}_q, \mathbf{C})$, where H_c denotes cohomology with compact support. The nonvanishing of $H^{4q-1}(G, \mathbf{C})$ shows also that one cannot modify Morse theory to make it applicable to the case of a continuous strongly q-pseudoconvex exhaustion function. The introduction of strongly q-plurisubharmonic functions is to overcome the difficulty of smoothing. We will not follow the path of introducing a continuous strongly q-plurisubharmonic function, proving that $\tau \circ (-d_G)$ is such a function, and then smoothing it out. Instead we will construct, for each $P \in G$, a smooth function δ_P defined on a geodesic ball of radius r_P centered at P so that

(i) $\delta_P = d_G$ at P and $\delta_P \geqslant d_G$ on the geodesic ball,

(ii) the q-plurisubharmonicity of $\tau \circ (-\delta_P)$ is $\geqslant \kappa_P > 0$ at every point of the geodesic ball,

(iii) the partial derivatives of δ_P up to the third order are bounded by E_P on the geodesic ball,

(iv) as functions of $P \in G$ both functions r_P and κ_P are locally bounded away from zero, and the function E_P is locally bounded from above.

We will cover G by a locally finite countable family of coordinate charts, and smooth out $\tau \circ (-d_G)$ successively on a sufficiently large compact subset of each chart so that these compact subsets still cover G. The smoothing will be done by using diffeomorphisms of G which fix each point outside the coordinate chart and which on the compact subset are translations with respect to the coordinates of the chart. In each step of the smoothing process we will obtain in a natural way from the family of functions $\tau \circ (-\delta_P)$, $P \in G$, another family of functions which have the same properties and stand in the same relation to the partially smoothed-out $\tau \circ (-d_G)$ as the family $\tau \circ (-\delta_P)$, $P \in G$, to $\tau \circ (-d_G)$. Thus at the end of the smoothing process the smooth function obtained from $\tau \circ (-d_G)$ will be strongly q-plurisubharmonic.

8.3. Let M and G be as in the assumptions of Theorem 8.2. We first fix our notations. Let n be the complex dimension of M. For $P \in M$ and $r > 0$ let $B(P, r)$ be the set of all points of M whose distances from P are $< r$. For a curve γ in M we denote the length of γ by $L(\gamma)$. For the rest of this section, for $P \in M$, $T_{M,P}$ means the real tangent space of M when M is regarded as a real manifold. As before, we denote by $\langle \cdot\,,\cdot \rangle$ the inner product on $T_{M,P}$ defined by the Kähler metric; for $X \in T_{M,P}$ we let ∇_X denote covariant differentiation in the direction X with respect to the Levi-Civita connection; and for $X, Y \in T_{M,P}$ we let $R_{X,Y} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. Let J be the complex structure of M. We use Bisect(X, Y) to denote the holomorphic bisectional curvature

$$\langle R_{X,Y}X, Y\rangle + \langle R_{JX,Y}JX, Y\rangle.$$

For a function f we use the following notation for the real Hessian and the Levi form of f respectively:

$$H(X,Y)f = XYf - (\nabla_X Y)f,$$
$$\mathfrak{L}(X,Y)f = H(X,Y)f + H(JX,JY)f.$$

The strongly hyper-q-convexity of ∂G means that there exists a positive number η such that if $V_1,\cdots,V_q, JV_1,\cdots,JV_q$ are orthonormal vectors tangent to ∂G at a point Q of ∂G and if V_0 is the unit outward normal of ∂G at Q, then

$$\sum_{\nu=1}^{q} \langle [V_\nu, JV_\nu], JV_0\rangle \leqslant -\eta.$$

This one can easily see by using the formula, which defines the exterior derivative of a 1-form in terms of the Lie bracket, and using the relation between V_0 and the gradient of a defining function of ∂G.

Fix $P \in G$. Then the distance $d_G(P)$ of P from ∂G is realized by a geodesic $\gamma(t)$: $[0, l] \to M$ in M parametrized by its arc-length with $\gamma(0) = 0$ and $\gamma(l) \in \partial G$. Let $Q = \gamma(l)$. We observe that, since the geodesic $\gamma(t)$, $0 \leqslant t \leqslant l$, is perpendicular to ∂G at Q, the point P is determined by the point Q of ∂G and the positive number l. We lengthen the geodesic $\gamma(t)$, $0 \leqslant t \leqslant l$, at both ends so that we can assume that $\gamma(t)$ is defined for $-\varepsilon < t < l + \varepsilon$, where ε is some positive number. In the following we denote by γ the lengthened geodesic $\gamma(t)$, $-\varepsilon < t < l + \varepsilon$. Choose $2n$ parallel vector fields $X_1,\cdots,X_{2n}$ along γ such that at every point of γ

(i) $X_1,\cdots,X_{2n}$ are orthonormal,

(ii) X_1 is the unit tangent vector of γ,

(iii) $JX_{2\nu-1} = X_{2\nu}$ for $1 \leqslant \nu \leqslant n$.

Define a coordinate system $y_1,\cdots,y_{2n}$ on an open neighborhood of γ by the map

$$\Phi\colon (y_1,\cdots,y_{2n}) \mapsto \exp_{\gamma(y_1)}\left(\sum_{\nu=2}^{2n} y_\nu X_\nu(\gamma(y_1))\right).$$

The set $\Phi^{-1}(\partial G)$ near $y_1 = l, y_2 = \cdots = y_{2n} = 0$ is defined by

$$y_1 = l + f(y_2, \cdots, y_{2n})$$

with f and its gradient vanishing at $y_2 = \cdots = y_{2n} = 0$.

Let $0 \leqslant \rho(y_1) \leqslant 1$ be a smooth function on $\mathbf{R}$ which is identically zero on some neighborhood of $y_1 = 0$, and identically 1 on some open neighborhood of $y_1 = l$. We introduce another coordinate system $(x_1, \cdots, x_{2n})$ on some open neighborhood U of γ which is related to the coordinate system $(y_1, \cdots, y_{2n})$ by

$$x_1 = y_1 - \rho(y_1) f(y_2, \cdots, y_{2n}),$$
$$x_\nu = y_\nu \qquad (2 \leqslant \nu \leqslant 2n).$$

The coordinate system $(x_1, \cdots, x_{2n})$ satisfies the following conditions.

(i) γ is given by $x_2 = \cdots = x_{2n} = 0$, $-\varepsilon < x_1 < l + \varepsilon$,

(ii) ∂G is defined by $x_1 = l$ near $\gamma(l)$,

(iii) $\frac{\partial}{\partial x_\nu} = X_\nu$ $(1 \leqslant \nu \leqslant 2n)$ at every point of γ,

(iv) $\nabla_{X_\mu} \frac{\partial}{\partial x_\nu} = 0$ at P for $2 \leqslant \mu, \nu \leqslant 2n$.

Choose a positive number a such that the set F defined by $-\frac{1}{2}\varepsilon < x_1 < l + \frac{1}{2}\varepsilon$, $|x_\nu| < a$, $2 \leqslant \nu \leqslant 2n$, is relatively compact in U, and $F \cap \partial G = F \cap \{x_1 = l\}$. For the rest of this section a $2n$-tuple $(b_1, \cdots, b_{2n})$ of real numbers denotes the point whose coordinates are $(b_1, \cdots, b_{2n})$ with respect to the coordinate system $(x_1, \cdots, x_{2n})$.

8.4. For any point $(x_1^0, \cdots, x_{2n}^0)$ in F let $\delta(x_1^0, \cdots, x_{2n}^0)$ be the length of the curve

$$t \mapsto (t, x_2^0, \cdots, x_{2n}^0), \quad x_1^0 \leqslant t \leqslant l$$

with respect to the Kähler metric of M. Between δ and the distance function d_G from ∂G we have the following relation.

$$\delta(P) = d_G(P), \quad \delta \geqslant d_G \text{ on } F. \tag{8.4.1}$$

The derivative of δ along γ is clearly -1. Consider the hypersurface $\{\delta = l\}$. We want to computge the Levi form of δ restricted to the complex tangent space of $\{\delta = l\}$ at P.

First we use the first variation formula of arc-length to verify that $X_\nu(P)$ is tangential to $\{\delta = l\}$ for $2 \leqslant \nu \leqslant 2n$. Fix $2 \leqslant \nu \leqslant 2n$. Let C_s be the curve

$$x_1 = t, \quad 0 \leqslant t \leqslant l,$$
$$x_\nu = s,$$
$$x_\mu = 0, \quad \mu \neq 1, \nu.$$

By the first variation formula of arc-length [13, p. 5, (1.3)], we obtain

$$\frac{d}{ds} L(C_s) \Big|_{s=0} = \langle X_\nu, X_1 \rangle \Big|_{t=0}^{l} - \int_{t=0}^{l} \langle X_\nu, \nabla_{X_1} X_1 \rangle \, dt,$$

which vanishes, where X_ν, X_1 are regarded as functions of t through $\gamma(t)$. Since clearly $(\partial/\partial x_\nu)\delta\,|_P$ equals $(d/ds)L(C_s)\,|_{s=0}$, it follows that $X_\nu(\delta - l) = X_\nu\delta = 0$ at P. Hence X_ν is tangential to $\{\delta = l\}$ at P.

8.5. We now compute the Levi form of δ restricted to the complex tangent space of $\{\delta = l\}$ at P. Let $X_\nu = \partial/\partial x_\nu$ $(1 \leqslant \nu \leqslant 2n)$ at every point of F. Fix real numbers $\lambda_\nu(3 \leqslant \nu \leqslant 2n)$ and let $V = \sum_{\nu=3}^{2n} \lambda_\nu X_\nu$. Let Γ_v be the curve

$$\begin{aligned} &x_1 = t, \quad 0 \leqslant t \leqslant l,\\ &x_2 = 0,\\ &x_\nu = \lambda_\nu v, \quad 3 \leqslant \nu \leqslant 2n. \end{aligned}$$

By the second variation formula of arc-length [13, p. 20, §6], we obtain

$$\frac{\partial^2}{\partial v^2} L(\Gamma_v)\,\Big|_{v=0} = \langle \nabla_V V, X_1\rangle\Big|_{t=0}^{l} - \int_{t=0}^{l} \langle R_{V,X_1}V, X_1\rangle\, dt, \tag{8.5.1}$$

where V, X_1 are regarded as functions of t through $\gamma(t)$. Since clearly $VV\delta\,|_P$ equals $(\partial^2/\partial v^2)L(\Gamma_v)\,|_{v=0}$, it follows from (8.5.1) and $\nabla_V V\,|_P = 0$ that

$$VV\delta\,|_P = \langle \nabla_V V, X_1\rangle|_{t=l} - \int_{t=0}^{l} \langle R_{V,X_1}V, X_1\rangle\, dt. \tag{8.5.2}$$

Since M is Kähler, for any real tangent vector field X one has

$$\begin{aligned} \nabla_X X + \nabla_{JX} JX &= \nabla_X X + J\,\nabla_{JX} X\\ &= \nabla_X X + J([JX, X] + \nabla_X JX)\\ &= J[JX, X]. \end{aligned}$$

It follows from (8.5.2) that

$$\mathfrak{L}(V, V)\delta\,|_P = \langle [V, JV], JX_1\rangle\Big|_{t=l} - \int_{t=0}^{l} \operatorname{Bisect}(V, X_1)\, dt. \tag{8.5.3}$$

Since for any orthonormal vectors $V_1, \cdots, V_q$, $JV_1, \cdots, JV_q$ tangential to ∂G at Q one has

$$\sum_{\nu=1}^{q} \langle [V_\nu, JV_\nu], JX_1\rangle\Big|_{t=l} \leqslant -\eta,$$

and since the holomorphic bisectional curvature of M is nonnegative, it follows from (8.5.3) that

$$\sum_{\nu=1}^{q} \mathfrak{L}(V_\nu, V_\nu)\delta \leqslant -\eta \quad \text{at } P. \tag{8.5.4}$$

8.6. To take care of the Levi form for the normal direction of $\{x_1 = \delta\}$, we have to compose $-\delta$ with a convex increasing function. Take $b > 0$ such that $B(P, b)$ is relatively compact in F. Take a real-valued smooth function defined on some open neighborhood of $[-l-b, -l+b]$ in $\mathbf{R}$ such that both the first and second derivatives τ', τ'' of τ are positive at $-l$.

Take orthonormal vectors $W_1, \cdots, W_q$, $JW_1, \cdots, JW_q$ in $T_{M,P}$. We want to compute $\sum_{\nu=1}^{q} \mathfrak{L}(W_\nu, W_\nu)(\tau \circ (-\delta))$. This expression remains unchanged when the set $W_1, \cdots, W_q$, $JW_1, \cdots, JW_q$ is replaced by another orthonormal set spanning the same $\mathbf{R}$-vector subspace of $T_{M,P}$. Hence without loss of generality we can choose an orthonormal set of vectors $V_0, \cdots, V_q$, $JV_0, \cdots, JV_q$ in $T_{M,P}$ such that

(i) $V_1, \cdots, V_q, JV_1, \cdots, JV_q$ are tangential to ∂G at P,
(ii) V_0 is normal to ∂G,
(iii) $W_\nu = V_\nu$, $1 \leqslant \nu \leqslant q-1$,
(iv) $W_q = \sqrt{1-\alpha^2}\, V_q + \alpha V_0$ for some $0 \leqslant \alpha \leqslant 1$.

Let A be the length of the second-order covariant derivative of δ at P. Simple direct computations yield at P

$$\begin{aligned}\sum_{\nu=1}^{q} \mathfrak{L}(W_\nu, W_\nu)(\tau \circ (-\delta)) &= \alpha^2\tau''(-l) - \tau'(-l) \sum_{\nu=1}^{q} \mathfrak{L}(V_\nu, V_\nu)\delta \\ &\quad + \tau'(-l)\Big(\alpha^2\big(\mathfrak{L}(V_q, V_q)\delta + \mathfrak{L}(V_0, V_0)\delta\big) \\ &\quad - 2\alpha\sqrt{1-\alpha^2}\,\mathfrak{L}(V_q, V_0)\delta\Big) \\ &\geqslant \alpha^2\tau''(-l) + \tau'(-l)(\eta - 8\alpha A),\end{aligned}$$

which is $\geqslant \tau'(-l)(\eta/2)$ if $\alpha \leqslant \eta/(16A)$, and which is $\geqslant 8\tau'(-l)A$ if $\alpha \geqslant \eta/(16A)$ and $\tau''(-l)/\tau'(-l) \geqslant (16A)^3/\eta^2$. Let c be the minimum of $\tau'(-l)(\eta/2)$ and $8\tau'(-l)A$. Then

(8.6.1) the q-plurisubharmonicity of $\tau \circ (-\delta) \geqslant c$ at P if $\tau''/\tau' \geqslant (16A)^3/\eta^2$ at $-l$.

8.7. We now want to deal with the Levi form of the smoothing of $\tau \circ (-d_G)$ for a suitable smooth convex increasing function τ. We let the point vary inside G. Since δ, b, A depend on P, we denote then by δ_P, b_P, A_P. Since the point P is determined by Q and l, letting P vary is the same as letting Q and l vary. By considering the varying of Q and l, we readily see that we can assume without loss of generality that, as functions of $P \in G$, b_P is locally bounded away from zero, and both A_P and b_P are locally bounded from above.

Let l_* be the diameter of G. Choose a smooth convex increasing function $\tau: (-l_*, 0) \to \mathbf{R}$ such that $\tau(\lambda)$ goes to ∞ as λ approaches 0 and $\tau''/\tau' \geqslant (16A_P)^3/\eta^2$ at $-d_G(P)$ for all $P \in G$. Let c_P be the minimum of $\frac{1}{2}\tau'(-d_G(P))\eta$ and $8\tau'(-d_G(P))A_P$. It follows from (8.6.1) that

(8.7.1) the q-plurisubharmonicity of $\tau \circ (-\delta_P) \geqslant c_P$ at P.

Let E_P be the supremum on $B(P, b_P)$ of the lengths of the covariant derivatives of $\tau \circ (-\delta_P)$ up to order 3. By the reasoning given above we can also assume without loss of generality that E_P as a function of $P \in G$ is locally bounded from above. It follows from (8.7.1) and the local boundedness of E_P from above that there exists a function u_P of $P \in G$ with $0 < u_P < b_P$, which is locally bounded away from zero such that for $P \in G$

$$\text{(8.7.2)} \qquad \begin{array}{l} \text{the } q\text{-plurisubharmonicity of } \tau \circ (-\delta_P) \geqslant \tfrac{1}{2} c_P \\ \text{at every point of } B(P, u_p). \end{array}$$

We now describe the smoothing process for $\tau \circ (-d_G)$. Take a coordinate chart U of M, which is relatively compact in G. Let $Y_1, \cdots, Y_{2n}$ be the vector fields on U defined by partial differentiation with respect to the coordinate functions of U. Take a compact subset K of U and a smooth function $0 \leqslant \rho \leqslant 1$ with compact support in U such that $\rho \equiv 1$ on some open neighborhood D of K. For $1 \leqslant i \leqslant 2n$ let $T_i(t)$, $t \in \mathbf{R}$, be the 1-parameter subgroup of the diffeomorphism group of G obtained by integrating the vector field ρY_i. For $s = (s_1, \cdots, s_{2n}) \in \mathbf{R}^{2n}$ let $T(s) = T_1(s_1) \cdots T_{2n}(s_{2n})$. Take $e > 0$ such that $T(s)K \subset D$ when the distance $|s|$ of s from the origin of $\mathbf{R}^{2n}$ is $< e$. Let ζ be a nonnegative smooth function on $\mathbf{R}^{2n}$, whose support is contained in the ball of radius e centered at 0, and whose L^1 norm is 1. For any continuous function f on G define for $P \in G$

$$(Sf)(P) = \int_{s \in \mathbf{R}^{2n}} \zeta(s) f(T(s)P),$$

where integration is with respect to the Euclidean measure of $\mathbf{R}^{2n}$. Then Sf is smooth on K and agrees with f outside the support of ρ. Let h be the supremum of the distance between P and $T(s)P$ for $|s| < e$ and $P \in G$. The supremum h is finite because every $T(s)$ fixes every point of G outside the support of ρ. If for some given point P of G the function f is smooth on $B(P, h)$, then Sf is smooth at P.

Because of (8.7.1) and (8.7.2), for any given $0 < \lambda < 1$ we can choose e sufficiently small so that for $P \in G$

$$\text{(8.7.3)} \qquad \begin{array}{l} \text{the } q\text{-plurisubharmonicity of } \tau \circ (-\delta_P) \circ T(s) \text{ is} \geqslant \tfrac{1}{2}\lambda c_P \\ \text{at every point of } B(P, \lambda u_P) \text{ for } |s| < e. \end{array}$$

By (8.4.1) we have for every s and every $P \in G$

$$\text{(8.7.4)} \qquad \begin{array}{ll} \tau \circ (-d_G) \circ T(s) = \tau \circ (-\delta_{T(s)P}) \circ T(s) & \text{at } P, \\ \tau \circ (-d_G) \circ T(s) \geqslant \tau \circ (-\delta_{T(s)P}) \circ T(s) & \text{on } B(P, b_P - 2h). \end{array}$$

For a family of functions $\mathcal{F} = \{f_P\}$ indexed by $P \in G$, we define another family of functions $S'_P(\mathcal{F})$ also indexed by $P \in G$ as follows (when the definition makes sense):

$$S'_P(\mathcal{F}) = \int_{s \in \mathbf{R}^{2n}} \zeta(s) f_{T(s)P} \circ T(s),$$

where integration is with respect to the Euclidean measure. By an abuse of notation we denote $S'_P(\mathcal{F})$ by $S'(f_P)$. We apply this definition to the family $\tau \circ (-\delta_P)$, $P \in G$, and obtain for every $P \in G$ a smooth function $S'(\tau \circ (-\delta_P))$ on $B(P, b_P - 2h)$. We observe that when the distance of P from U is $> b_P$, the function agrees with $\tau \circ (-\delta_P)$, and we can use in such a case $B(P, b_P)$ as the domain of definition of $S'(\tau \circ (-\delta_P))$. For any given $0 < \lambda < 1$ we can choose e sufficiently small so that for $P \in G$

(8.7.5) the lengths of the covariant derivatives of $S'(\tau \circ (-\delta_P))$ up to order 2 are bounded by E_P/λ on $B(P, \lambda b_P - 2h)$.

It follows from (8.7.3) that

(8.7.6) the q-plurisubharmonicity of $S'(\tau \circ (-\delta_P))$ is $\geqslant \frac{1}{2}\lambda c_P$ at every point of $B(P, \lambda b_P - 2h)$.

Moreover, it follows from (8.7.4) that

$$\begin{aligned} &S(\tau \circ (-d_G)) = S'(\tau \circ (-\delta_P)) \quad \text{at } P, \\ &S(\tau \circ (-d_G)) \geqslant S'(\tau \circ (-\delta_P)) \quad \text{on } B(P, b_P - 2h). \end{aligned} \tag{8.7.7}$$

In the case where the distance of P from U is $> b_P$, $S(\tau \circ (-d_G)) = \tau \circ (-d_G)$ and $S'(\tau \circ (-\delta_P)) = \tau \circ (-\delta_P)$, so that (8.7.5), (8.7.6), (8.7.7) all hold with $B(P, \lambda b_P - 2h)$ or $B(P, b_P - 2h)$ replaced by $B(P, b_P)$.

8.8. Now instead of a single coordinate chart U we take a locally finite family of coordinate charts U_μ, $1 \leqslant \mu < \infty$, with a compact subset K_μ in each U_μ so that the family K_μ, $1 \leqslant \mu < \infty$, covers G. Corresponding to e, h, S, S' we have e_μ, h_μ, S_μ, S'_μ. Since the family U_μ, $1 \leqslant \mu < \infty$, is locally finite, we can successively choose e_μ, $1 \leqslant \mu < \infty$, sufficiently small so that the family K'_μ, $1 \leqslant \mu < \infty$, still covers G, where K'_μ is the set of all points of K_μ whose distances from $G - K_\mu$ are less than the sum of all h_ν with $U_\nu \cap U_\mu \neq \varnothing$. Let φ be the resulting function obtained by applying successively the operators S_μ, $1 \leqslant \mu < \infty$, to $\tau \circ (-d_G)$. Since $S_\mu f \equiv f$ on $G - U_\mu$ and the family U_μ, $1 \leqslant \mu < \infty$, is locally finite, the function φ is well-defined. Moreover, the function φ is smooth, because K'_μ, $1 \leqslant \mu < \infty$, covers G. For $P \in G$, let ψ_P be the resulting function obtained by applying successively the operators S'_μ, $1 \leqslant \mu < \infty$, to the family of functions $\tau \circ (-\delta_P)$, $P \in G$. Since the family U_μ, $1 \leqslant \mu < \infty$, is locally finite, we can successively choose e_μ, $1 \leqslant \mu < \infty$, sufficiently small so that

(i) φ is an exhaustion function of G,

(ii) ψ_P is smooth on $B(P, \frac{1}{2}u_P)$, and the q-plurisubharmonicity of ψ_P is $\geqslant \frac{1}{4}c_P$ at every point of $B(P, \frac{1}{2}u_P)$,

(iii) $\varphi = \psi_P$ at P and $\varphi \geqslant \psi_P$ on $B(P, \frac{1}{2}u_P)$.

It is possible to make the choice so that (ii) and (iii) hold because of the statements corresponding to (8.7.5), (8.7.6), (8.7.7) at each stage of the application of the operators S_μ and S'_μ. It follows that the q-plurisubharmonicity of φ is $\geqslant \frac{1}{4}c_P$ at P. Thus φ is a strongly q-plurisubharmonic exhaustion function on G. This smoothing process is essentially the same as the one given by Richberg [45] for strongly plurisubharmonic functions (cf. [24]).

8.9. Theorem. *Let M be a Kähler manifold of complex dimension n with nonnegative holomorphic bisectional curvature. Let G be a relatively compact subdomain of M with smooth boundary ∂G. Let $1 \leqslant q < n$ and assume that ∂G is strongly hyper-q-convex. Then $\pi_\nu(M, M - G)$ vanishes for $\nu \leqslant n - q$.*

Proof. Choose a relatively compact open neighborhood D of $\overline{G}$ in M such that ∂D is smooth and strongly hyper-q-convex. by Theorem 8.2 there exists a smooth exhaustion function φ on D which is strongly q-plurisubharmonic. Let a be a real number such that $\varphi < a$ on $\overline{G}$. We can approximate φ on $\{\varphi \leqslant a\}$ in the C^2 topology uniformly by Morse functions [38, p. 37, Cor. 6.8], i.e., smooth functions whose critical points are all nondegenerate and hence isolated. Choose a Morse function f on $\{\varphi \leqslant a\}$ which approximates φ so closely in the C^2 topology that f is strongly q-plurisubharmonic on $\{\varphi \leqslant a\}$, and for some $b < a$ one has $\overline{G} \subset \{f < b\}$ and $\{f \leqslant b\} \subset \{\varphi < a\}$.

By the strong q-plurisubharmonicity of f the Levi form of f must have at least $n - q + 1$ positive eigenvalues at every point. So the real Hessian of f must have at least $n - q + 1$ positive eigenvalues at every point. It follows that the index of $-f$, which is the number of negative eigenvalues of the real Hessian of $-f$, is $\geqslant n - q + 1$ at every critical point of $-f$.

Let c be the minimum of f on $\{\varphi \leqslant a\}$. Since f is a Morse function, the set $\{f = c\}$ consists of only a finite number of points. Fix $\nu \leqslant n - q$. Let B be the closed unit ball of real dimension ν. Let σ: $B \to M$ be a continuous map with $\sigma(\partial B) \subset M - G$. Since $\{f = c\}$ is a finite set, we can continuously deform σ without changing $\sigma \mid \partial B$ such that $\sigma(B)$ is disjoint from $\{f = c\}$ and hence from $\{f \leqslant d\}$ for some $d > c$. Since the index of $-f$ is $\geqslant n - q + 1$ at every critical point of $-f$, $M - \{-f > -d\}$ has the same homotopy type as $M - \{-f > -b\}$ with finitely many cells of dimension $\geqslant n - q + 1$ attached [38, p. 19, Remark 3.3]. It follows from $\nu < n - q + 1$ that the continuous map

$\sigma: B \to M$ can be continuously deformed into $M - \{-f > -b\}$ and hence into $M - G$ without changing $\sigma \mid \partial B$. q.e.d.

An immediate consequence of Theorem 8.9 is the following.

8.10. Theorem. *Let M be a compact Kähler manifold of complex dimension n with nonnegative holomorphic bisectional curvature. Let $1 \leqslant q < n$. Let V be a complex submanifold of M admitting a tubular neighborhood U with smooth boundary such that $M - U$ has strongly hyper-q-convex boundary. Then $\pi_\nu(M, V)$ vanishes for $\nu \leqslant n - q$.*

8.11. Remark. For Theorem 8.2 to hold, instead of assuming that ∂G is strongly hyper-q-convex, we can assume that ∂G is hyper-q-convex and the bisectional curvature of M is $(q + 1)$-nondegenerate. The same proof works with some obvious modifications. Theorems 8.9 and 8.10 remain true with a similar change of assumptions.

9. A generalization of the strong Lefschetz theorem

9.1. In the last section we proved a Barth-Lefschetz type theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. Unfortunately it works only for complex submanifolds admitting tubular neighborhoods whose complements have strongly hyper-q-convex boundaries. So far there is no way to verify which complex submanifolds admit such tubular neighborhoods except in the obvious case of codimension one. Out of the desire to get Barth-Lefschetz type theorems with only bisectional curvature conditions, in this section we will prove a generalized strong Lefschetz theorem for Hermitian holomorphic vector bundles over compact Kähler manifolds which are 1-semipositive and k-positive in the sense of Griffiths. This generalized strong Lefschetz theorem will be used to give the surjectivity portion of a Barth-Lefschtz type theorem at the homology level for compact Kähler manifolds whose bisectional curvature is nonnegative and k-nondegenerate.

In [59] Sommese proved a generalization of the strong Lefschetz theorem for a class of bundles over algebraic manifolds called k-ample bundles which he introduced. These bundles are characterized by the property that high powers of the associated line bundles over the projectivizations of their duals have enough sections to give a holomorphic map of fiber dimension $\leqslant k$. His generalized strong Lefschetz theorem follows from the usual strong Lefschetz theorem by slicing by ample divisors to reduce the fiber dimension of the holomorphic map to zero to get positive bundles. He used his generalized strong Lefschetz theorem to prove the surjectivity portion of a Barth-Lefschetz type theorem at the homology level for compact algebraic manifolds with

k-ample tangent bundles and, in particular, for certain compact homogeneous algebraic manifolds. However, the nonnegativity and the k-nondegeneracy of the bisectional curvature are locally given conditions in nature, whereas the k-ampleness of the tangent bundle involves the existence of global sections of certain associated line bundles. For Barth-Lefschetz type theorems for manifolds with bisectional curvature conditions, we need the generalized strong Lefschetz theorem given in this section. This generalization cannot be derived from the usual strong Lefschetz theorem and requires a completely different approach.

9.2. Theorem (*the generalized strong Lefschetz theorem*). *Let M be a compact Kähler manifold of complex dimension n. Let E be a Hermitian holomorphic vector bundle of rank r over M, which is Griffiths 1-semipositive and Griffiths k-positive. Let $c_r(E)$ be the rth Chern class of E. Then the map f_i: $H^i(M,\mathbf{C}) \to H^{i+2r}(M,\mathbf{C})$ defined by cupping with $c_r(E)$ is injective for $i \leqslant n-k+1-r$ and surjective for $i \geqslant n+k-1-r$.*

Most of the rest of this section will be devoted to the proof of this theorem.

9.3. Let M be a compact Kähler manifold of complex dimension n with Kähler form ω and local coordinates $z^\alpha (1 \leqslant \alpha \leqslant n)$. Let $u = \sqrt{-1}\, u_{\alpha\bar\beta} dz^\alpha \wedge \overline{dz^\beta}$ be a closed $(1,1)$-form on M, which is positive semidefinite and has at least $n-k+1$ positive eigenvalues at every point of M. For $\varepsilon > 0$ let $\omega_\varepsilon = u + \varepsilon\omega$. According to Lemma 4.6, after replacing ω by ω_ε for some sufficiently small ε, we can assume without loss of generality that at every point of M the eigenvalues $\lambda_1,\cdots,\lambda_n$ of u with respect to ω satisfies the condition that

$$\sum_{\alpha=1}^{n} \lambda_\alpha - \sum_{\alpha\in A} \lambda_\alpha - \sum_{\beta\in B} \lambda_\beta$$

is positive for any subset A of p elements and any subset B of q elements in $\{1,\cdots,n\}$ with $p+q<n-k+1$.

Fix a point P of M, and choose a local coordinate system $z^1,\cdots,z^n$ at P so that $\omega = \sqrt{-1}\sum_{\alpha=1}^n dz^\alpha \wedge \overline{dz^\alpha}$ and $u = \sqrt{-1}\sum_{\alpha=1}^n \lambda_\alpha dz^\alpha \wedge \overline{dz^\alpha}$ at P. For subsets $A = \{\alpha_1,\cdots,\alpha_a\}$ and $B = \{\beta_1,\cdots,\beta_b\}$ of $\{1,\cdots,n\}$ with $\alpha_1 < \cdots < \alpha_a$ and $\beta_1 < \cdots < \beta_b$ we let $dz^A = dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_a}$ and $d\bar z^B = \overline{dz^{\beta_1}} \wedge \cdots \wedge \overline{dz^{\beta_b}}$. For a subset $C = \{\gamma_1,\cdots,\gamma_c\}$ of $\{1,\cdots,n\}$ let

$$(dz\wedge d\bar z)^C = \left(dz^{\gamma_1}\wedge \overline{dz^{\gamma_1}}\right)\wedge\cdots\wedge\left(dz^{\gamma_c}\wedge\overline{dz^{\gamma_c}}\right).$$

Moreover, let

$$\lambda_{ABC} = \sum_{\alpha=1}^{n} \lambda_\alpha - \sum_{\alpha\in A\cup C} \lambda_\alpha - \sum_{\alpha\in B\cup C} \lambda_\alpha .$$

For (p, q)-forms

$$\varphi = \sum_{A,B,C} \varphi_{ABC} dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^C,$$
$$\psi = \sum_{A,B,C} \psi_{ABC} dz^A \wedge dz^B \wedge (dz \wedge d\bar{z})^C$$

at P with $p + q < n - k + 1$ define

$$\langle \varphi, \psi \rangle' = \sum_{A,B,C} \lambda_{ABC} \varphi_{ABC} \overline{\psi_{ABC}}\,.$$

By the assumption on u, $\langle \cdot\,, \cdot \rangle'$ is a positive definite Hermitian form.

The notations $\langle \cdot\,, \cdot \rangle$, L, and Λ used below carry the same meanings as in §3.3.

9.4. Lemma. *For (p, q)-forms φ, ψ on M with $p + q < n - k + 1$,*

$$\langle \Lambda(u \wedge \varphi), \psi \rangle - \langle u \wedge \Lambda\varphi, \psi \rangle = \langle \varphi, \psi \rangle'.$$

Proof. We prove it at the point P of M with the local coordinates described above. Since both sides are linear in φ, it suffices to prove the special case where

$$\varphi = dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^C.$$

We have

$$\Lambda\varphi = \frac{1}{\sqrt{-1}} \sum_{\sigma \in C} dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^{C - \{\sigma\}},$$
$$u \wedge \Lambda\varphi = \sum_{\sigma \in C} \lambda_\sigma dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^C + \sum_{\substack{\tau \notin A \cup B \cup C \\ \sigma \in C}} \lambda_\tau dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^{(C - \{\sigma\}) \cup \{\tau\}}.$$

On the other hand,

$$u \wedge \varphi = \sqrt{-1} \sum_{\tau \notin A \cup B \cup C} \lambda_\tau dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^{C \cup \{\tau\}},$$
$$\Lambda(u \wedge \varphi) = \sum_{\substack{\tau \notin A \cup B \cup C \\ \sigma \in C}} \lambda_\tau dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^{(C - \{\sigma\}) \cup \{\tau\}} + \sum_{\tau \notin A \cup B \cup C} \lambda_\tau dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^C.$$

It follows that

$$\begin{aligned}\Lambda(u \wedge \varphi) - u \wedge \Lambda\varphi &= \sum_{\tau \notin A \cup B \cup C} \lambda_\tau - \sum_{\tau \in C} \lambda_\tau dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^C \\ &= \lambda_{ABC} dz^A \wedge d\bar{z}^B \wedge (dz \wedge d\bar{z})^C,\end{aligned}$$

which yields the wanted formula upon taking the inner product with ψ. q.e.d.

For Lemma 9.4 the condition $p + q < n - k + 1$ is not needed if we define $\langle \cdot, \cdot \rangle'$ also for other forms. Moreover, the lemma holds for any $(1, 1)$-form without any eigenvalue condition.

Let P^l be the set of all primitive l-forms with respect to the Kähler metric of M.

9.5. Lemma. *For any $v_l \in P^l$, $u \wedge v_l = w_{l+2} + Lw_l + L^2 w_{l-2}$, where $w_\nu \in P^\nu (\nu = l + 2, l, l - 2)$.*

Proof. We have the decomposition

$$u \wedge v_l = \sum_{\nu \geqslant 0} L^\nu w_{l-2\nu+2},$$

where $w_{l-2\nu+2} \in P^{l-2\nu+2}$ for $\nu \geqslant 0$. We know that $L^\mu \colon P^\nu \to L^\mu P^\nu$ is an isomorphism for $\mu \leqslant n - \nu$ and $L^\mu P^\nu = 0$ for $\mu > n - \nu$. It follows that

$$\begin{aligned}0 = u \wedge L^{n-l+1} v_l &= L^{n-l+1}(u \wedge v_l) \\ &= \sum_{\nu \geqslant 0} L^{n+\nu-l+1} w_{l-2\nu+2} \\ &= \sum_{\nu \geqslant 3} L^{n+\nu-l+1} w_{l-2\nu+2}.\end{aligned}$$

Since $n + \nu - l + 1 \leqslant n - (l - 2\nu + 2)$ for $\nu \geqslant 3$, we have $w_{l-2\nu+2} = 0$ for $\nu \geqslant 3$.

Lemma 9.5 holds when u is replace by any 2-form.

9.6. Lemma. *Let v_l, w_l be primitive l-forms. Then for $0 \leqslant \mu \leqslant n - l - 1$ with $\mu \neq \frac{1}{2}(n - l)$,*

$$\langle u \wedge L^\mu v_l, L^{\mu+1} w_l \rangle = \frac{(\mu + 1)(n - l - \mu)}{n - l - 2\mu} \langle L^\mu v_l, L^\mu w_l \rangle'.$$

Proof. By (3.3.1), for $0 \leqslant \nu \leqslant n - l$ we have

$$\Lambda L^\nu v_l = \nu(n - l - \nu + 1) L^{\nu-1} v_l,$$

where $L^{-1}v_l$ is taken to mean zero. By Lemma 9.4 we obtain

$$\begin{aligned}
\langle u \wedge L^{\mu}v_l, L^{\mu+1}w_l\rangle &= \langle \Lambda(u \wedge L^{\mu}v_l), L^{\mu}w_l\rangle \\
&= \langle L^{\mu}v_l, L^{\mu}w_l\rangle' + \langle u \wedge L^{\mu}v_l, L^{\mu}w_l\rangle \\
&= \langle L^{\mu}v_l, L^{\mu}w_l\rangle' + \mu(n-l-\mu+1)\langle u \wedge L^{\mu-1}v_l, L^{\mu}w_l\rangle \\
&= \langle L^{\mu}v_l, L^{\mu}w_l\rangle' + \frac{\mu(n-l-\mu+1)}{(\mu+1)(n-l-\mu)}\langle u \wedge L^{\mu-1}v_l, \Lambda L^{\mu+1}w_l\rangle \\
&= \langle L^{\mu}v_l, L^{\mu}w_l\rangle' + \frac{\mu(n-l-\mu+1)}{(\mu+1)(n-l-\mu)}\langle u \wedge L^{\mu}v_l, L^{\mu+1}w_l\rangle.
\end{aligned}$$

The assertion follows from solving for $\langle u \wedge L^{\mu}v_l, L^{\mu+1}w_l\rangle$ in the above equation. q.e.d.

In Lemma 9.6, when $\mu = \frac{1}{2}(n-l)$, the proof gives $\langle L^{\mu}v_l, L^{\mu}w_l\rangle' = 0$. We will use only the case $\mu < \frac{1}{2}(n-l)$.

9.7. Lemma. *Let v_l, w_{l+2} be respectively a primitive l-form and $(l+2)$-form. Then for $1 \leqslant \mu \leqslant n-l-2$,*

$$\langle u \wedge L^{\mu}v_l, L^{\mu}w_{l+2}\rangle = -\tfrac{1}{2}(n-l-\mu-1)\langle L^{\mu}v_l, L^{\mu-1}w_{l+2}\rangle'.$$

Proof.

$$\begin{aligned}
\langle u \wedge L^{\mu}v_l, L^{\mu}w_{l+2}\rangle &= \langle \Lambda(u \wedge L^{\mu}v_l), L^{\mu-1}w_{l+2}\rangle \\
&= \langle L^{\mu}v_l, L^{\mu-1}w_{l+2}\rangle' + \langle u \wedge \Lambda L^{\mu}v_l, L^{\mu-1}w_{l+2}\rangle \quad \text{(by Lemma 9.4)} \\
&= \langle L^{\mu}v_l, L^{\mu-1}w_{l+2}\rangle' + \mu(n-l-\mu+1)\langle u \wedge L^{\mu-1}v_l, L^{\mu-1}w_{l+2}\rangle \\
&= \langle L^{\mu}v_l, L^{\mu-1}w_{l+2}\rangle' + \frac{n-l-\mu+1}{n-l-\mu-1}\langle u \wedge L^{\mu-1}v_l, \Lambda L^{\mu}w_{l+2}\rangle \\
&= \langle L^{\mu}v_l, L^{\mu-1}w_{l+2}\rangle' + \frac{n-l-\mu+1}{n-l-\mu-1}\langle u \wedge L^{\mu}v_l, L^{\mu}w_{l+2}\rangle.
\end{aligned}$$

The assertion follows from solving form $\langle u \wedge L^{\mu}v_l, L^{\mu}w_{l+2}\rangle$ in the above equation.

9.8. Lemma. *Let v_l, w_{l-2} be respectively a primitive l-form and $(l-2)$-form. Then for $0 \leqslant \mu \leqslant n-l$,*

$$\langle u \wedge L^{\mu}v_l, L^{\mu+2}w_{l-2}\rangle = \tfrac{1}{2}(\mu+2)\langle L^{\mu}v_l, L^{\mu+1}w_{l-2}\rangle'.$$

Proof.

$$\begin{aligned}
\langle u \wedge L^\mu v_l, L^{\mu+2} w_{l-2}\rangle &= \langle \Lambda(u \wedge L^\mu v_l), L^{\mu+1} w_{l-2}\rangle \\
&= \langle L^\mu v_l, L^{\mu+1} w_{l-2}\rangle' + \langle u \wedge \Lambda L^\mu v_l, L^{\mu+1} w_{l-2}\rangle \quad \text{(by Lemma 9.4)} \\
&= \langle L^\mu v_l, L^{\mu+1} w_{l-2}\rangle' + \mu(n-l-\mu+1)\langle u \wedge L^{\mu-1} v_l, L^{\mu+1} w_{l-2}\rangle \\
&= \langle L^\mu v_l, L^{\mu+1} w_{l-2}\rangle' + \frac{\mu}{\mu+2}\langle u \wedge L^{\mu-1} v_l, \Lambda L^{\mu+2} w_{l-2}\rangle \\
&= \langle L^\mu v_l, L^{\mu+1} w_{l-2}\rangle' + \frac{\mu}{\mu+2}\langle u \wedge L^{\mu} v_l, L^{\mu+2} w_{l-2}\rangle.
\end{aligned}$$

The assertion follows from solving for $\langle u \wedge L^\mu v_l, L^{\mu+2} w_{l-2}\rangle$ in the above equation.

9.9. Lemma. *For $p + q < n - k + 1$ the map $H^p(M, \Omega^q) \to H^{p+1}(M, \Omega^{q+1})$ defined by multiplication by u is injective.*

Proof. Fix p and q with $p + q < n - k + 1$. Any element of $H^p(M, \Omega^q)$ is represented by a harmonic (p, q)-form φ on M. Assume that $u \wedge \varphi$ is $\bar\partial$-exact. We have to show that φ is identically zero. Let $l = p + q$. Uniquely φ can be written in the form

$$\varphi = \sum_{0 \leqslant \mu \leqslant \frac{1}{2} l} L^\mu v_{l-2\mu},$$

where $v_{l-2\mu}$ is a primitive harmonic $(l - 2\mu)$-form on M. Let c_μ, $0 \leqslant \mu \leqslant \frac{1}{2} l$, be positive numbers whose values are to be chosen later in the proof. Let

$$\psi = \sum_{0 \leqslant \mu \leqslant \frac{1}{2} l} c_\mu L^\mu v_{l-2\mu}.$$

Then ψ is a harmonic l-form on M. As before, $*$ denotes the Hodge star operator with respect to the Kähler metric of M, and $\bar{*}$ denotes the composite of $*$ and the complex conjugation. Since $\bar{*} L\psi$ is a harmonic form on M, it follows that

$$\int_M \langle u \wedge \varphi, L\psi\rangle = \int_M u \wedge \varphi \wedge \bar{*} L\psi = 0.$$

On the other hand, by Lemmas 9.5, 9.6, 9.7, and 9.8, we have

$$\begin{aligned}\langle u \wedge \varphi, L\psi\rangle = \sum_{\mu \geq 0} \{ & c_\mu \langle u \wedge L^\mu v_{l-2\mu}, L^{\mu+1} v_{l-2\mu}\rangle \\ & + c_\mu \langle u \wedge L^\mu v_{l-2\mu}, L^{\mu+2} v_{l-2\mu-2}\rangle \\ & + c_{\mu+1}\langle u \wedge L^{\mu+1} v_{l-2\mu-2}, L^{\mu+1} v_{l-2\mu}\rangle\} \\ = \sum_{\mu \geq 0} \Big\{ & c_\mu \frac{(\mu+1)(n-l+\mu)}{n-l} \langle L^\mu v_{l-2\mu}, L^\mu v_{l-2\mu}\rangle' \\ & + c_\mu \frac{\mu+2}{2} \langle L^\mu v_{l-2\mu}, L^{\mu+1} v_{l-2\mu-2}\rangle' \\ & - c_{\mu+1} \frac{n-l+\mu}{2} \langle L^{\mu+1} v_{l-2\mu-2}, L^\mu v_{l-2\mu}\rangle' \Big\}.\end{aligned}$$

We now choose $c_0 = 1$ and inductively c_μ so that

$$c_{\mu+1} = \frac{c_\mu(\mu+2)}{n-l+\mu}.$$

Then

$$\langle u \wedge \varphi, L\psi\rangle = \sum_{0 \leq \mu \leq \frac{1}{2}l} c_\mu \frac{(\mu+1)(n-l+\mu)}{n-l} \langle L^\mu v_{l-2\mu}, L^\mu v_{l-2\mu}\rangle'.$$

From the positive definiteness of $\langle \cdot , \cdot \rangle'$ and the vanishing of the integral of $\langle u \wedge \varphi, L\psi\rangle$ over M it follows that each $L^\mu v_{l-2\mu}$ is identically zero on M. Hence φ is identically zero on M.

9.10. We now proceed to prove Theorem 9.2. Let M and E be as in the assumptions of Theorem 9.2. Let F be the tautological line bundle over the projective bundle $\mathbf{P}(E^*)$ associated to the dual bundle E^* of E. The curvature form of the dual bundle F^* of F with the metric induced from that of E is a $(1, 1)$-form on $\mathbf{P}(E^*)$, which is positive semidefinite and has $n - k + r$ positive eigenvalues at every point of $\mathbf{P}(E^*)$ (see Lemma 4.5). Denote this curvature form by u. By Lemma 9.9, the map Φ_l: $H^l(\mathbf{P}(E^*), \mathbf{C}) \to H^{l+2}(\mathbf{P}(E^*), \mathbf{C})$ defined by cupping with u is injective for $l < n - k + r$.

We now use the argument of Bloch-Gieseker [9, p. 113, Prop. 1.1] to get the injectivity of f_i for $i \leq n - k + 1 - r$. By [27, p. 144], we have the following ring-isomorphism:

$$\Psi: H^*(M, \mathbf{C})[T]/I \xrightarrow{\approx} H^*(\mathbf{P}(E^*), \mathbf{C}).$$

Here I is the ideal generated by

$$T^r - c_1 T^{r-1} + \cdots + (-1)^{r-1} c_{r-1} T + (-1)^r c_r,$$

where $c_i \in H^{2i}(M, \mathbf{C})$ is the ith Chern class of E. Moreover, this ring-isomorphism sends T to the Chern class of F^*.

Fix $i \leqslant n - k + 1 - r$. Assume that some nonzero element ξ of $H^i(M, \mathbf{C})$ is in the kernel of f_i. We want to derive a contradiction. Since $\xi c_r(E) = 0$, it follows that the nonzero element

$$\eta := \Psi\left(\xi\left(T^{r-1} - c_1 T^{r-2} + \cdots + (-1)^{r-2} c_{r-2} T + (-1)^{r-1} c_{r-1}\right)\right)$$

of $H^{i+2r-2}(\mathbf{P}(E^*), \mathbf{C})$ satisfies $\Phi_{i+2r-2}(\eta) = 0$. This is a contradiction because $i + 2r - 2 < n - k + r$.

We now use Poincaré duality to get the surjectivity of f_i for $i \geqslant n + k - 1 - r$. Let v be the real harmonic $2r$-form on M representing c_r. Suppose f_i is not surjective for some $i \geqslant n + k - 1 - r$, and we want to derive a contradiction. Since f_i is not surjective, there exists some nonzero harmonic $(i + 2r)$-form φ on M which is orthogonal to $v \wedge \psi$ for all harmonic i-forms ψ on M. That is,

$$\int_M v \wedge \psi \wedge \bar{*}\varphi = 0,$$

or equivalently

$$\int_M v \wedge (\bar{*}\varphi) \wedge \bar{*}(\bar{*}\psi) = 0.$$

Hence the closed $(2n - i)$-form $v \wedge \bar{*}\varphi$ is orthogonal to $\bar{*}\psi$ for all harmonic i-forms ψ on M. That is, $v \wedge \bar{*}\varphi$ is orthogonal to all harmonic $(2n - i)$-forms. It follows that $v \wedge \bar{*}\varphi$ is d-exact. The cohomology class represented by $\bar{*}\varphi$ is mapped to zero by $f_{2n-i-2r}$. This contradicts the injectivity of $f_{2n-i-2r}$.

9.11. Theorem. *Let M be a compact Kähler manifold of complex dimension n, whose holomorphic bisectional curvature is nonnegative and k-nondegenerate. Let V be a complex submanifold of complex dimension d. Then the restriction map Φ_l: $H^l(M, \mathbf{C}) \to H^l(V, \mathbf{C})$ is surjective for $l \geqslant 2n - k - 1 - d$.*

Proof. Let $[V]$ denote the closed $(n - d, n - d)$-current defined by integration over V. This current $[V]$ defines an element $v \in H^{2(n-d)}(M, \mathbf{C})$. Let $j: V \hookrightarrow M$ be the inclusion map. Then $j^* v \in H^{2(n-d)}(V, \mathbf{C})$ is equal to the $(n - d)$th Chern class $c_{n-d}(N_V)$ of the normal bundle N_V of V in M. Fix $l \geqslant 2n - k - 1 - d$ and take $\alpha \in H^l(V, \mathbf{C})$. By the assumption on the holomorphic bisectional curvature of M, the tangent bundle T_M of M is Griffiths 1-semipositive and Griffiths k-positive. Being a quotient bundle of T_M, N_V is also Griffiths 1-semipositive and k-positive. By Theorem 9.2, there exists $\beta \in H^{l-2(n-d)}(V, \mathbf{C})$ such that α equals the cup product $\beta \cup c_{n-d}(N_V)$. Let G be an open tubular neighborhood of V in M such that V is differentiably a deformation retract of G. Then β can be extended to an element γ of

$H^{l-2(n-d)}(G, \mathbf{C})$. Let φ be a closed $(l-2(n-d))$-form on G representing γ. Then $\varphi \wedge [V]$ is a closed l-current on M which defines an element $\tilde{\alpha} \in H^l(M, \mathbf{C})$ with $\Phi_l(\tilde{\alpha}) = \alpha$.

10. A vanishing theorem for semipositive line bundles

10.1. We conclude this paper by giving a vanishing theorem for semipositive line bundles over non-Kähler manifolds. It is obtained by combining the ∇ and $\overline{\nabla}$ Bochner-Kodaira techniques described in §1 and using some simple estimates from elementary real analysis. This vanishing theorem is motivated by the conjecture of Grauert-Riemenschneider [23, p. 277], [47, Conjecture I] which is still an open problem. The difficulty with the conjecture is how to prove the following special case.

Conjecture of Grauert-Riemenschneider. *Let M be a compact complex manifold which admits a Hermitian holomorphic line bundle L whose curvature form is positive definite on a dense open subset G of M. Then M is Moishezon.*

An equivalent form is that if L is a Hermitian holomorphic line bundle over a compact complex manifold M whose curvature form is positive definite on a dense open subset G of M, then $H^1(M, L^\nu \otimes K_M) = 0$ for ν sufficiently large. Here K_M is the canonical line bundle of M. (See [47, Conjecture II]).

When M is Kähler, the above equivalent form follows from the $\overline{\nabla}$ Bochner-Kodaira technique and the identity theorem for solutions of second-order elliptic partial differential equations. It holds with $\nu \geqslant 1$, [47]. However, when M is not assumed to be Kähler, no proof is known even for the special case where $M - G$ is a subvariety except when it is a subvariety of dimension zero [46] or one [52].

The conjecture of Grauert-Riemenschneider was originally introduced for the purpose of characterizing Moishezon spaces by quasi-positive torsion-free sheaves. Since then a number of other characterizations of Moishezon spaces have been obtained [20], [44], [57], [63] which circumvent the difficulty of proving the Grauert-Riemenschneider conjecture by stating the characterizations in such a way that a proof can be obtained by using blow-ups, Kodaira's vanishing and embedding theorems, or L^2 estimates of $\bar{\partial}$ for complete Kähler manifolds.

We now state our vanishing theorem for semipositive line bundles over non-Kähler manifolds.

10.2. Theorem. *Let M be a compact complex manifold of complex dimension n with a Hermitian metric, and L be a Hermitian holomorphic line bundle over M whose curvature form u is positive definite on an open subset G of M. Suppose* *there exists a positive number A such that at every point P of M the eigenvalues $\lambda_1(P) \leqslant \cdots \leqslant \lambda_n(P)$ of u at P with respect to the Hermitian metric of M satisfy the condition $\lambda_n(P) \leqslant A\lambda_1(P)$. Suppose $M - G$ can be covered by a finite number of differentiable coordinate charts U_κ, $x_\kappa^1, \cdots, x_\kappa^{2n}$ $(1 \leqslant \kappa \leqslant k)$ such that $(M - G) \cap \{x_\kappa^i = \text{constant}, 1 \leqslant i \leqslant 2n - 1\}$ is always a finite set. Then for any holomorphic line bundle F over M, $H^1(M, L^\nu \otimes F)$ vanishes for ν sufficiently large.*

The rest of this section will be used for the proof of this theorem.

10.3. We use the notations of Theorem 10.2. Let $u = \sqrt{-1}\, u_{i\bar{j}} dz^i \wedge d\overline{z^j}$. Let $\sqrt{-1}\, z^i \wedge dz^{\bar{j}}$ be the curvature form of the canonical line bundle K_M of M with the Hermitian metric induced from that of M. The raising and lowering of indices will be done with respect to the Hermitian metric of M. Choose a Hermitian metric for F, and let $\sqrt{-1}\, v_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ be its curvature form. Though M is in general not Kähler, for $L^\nu \otimes F$ and M one can derive formulas analogous to (1.3.3) and (1.3.5). The only difference is the presence of an additional term coming from the torsion tensor of the Hermitian metric of M (cf. [25, p. 429, Theorem 7.2]). Take a positive integer ν and an L^ν-valued harmonic form φ on M. We have the following two formulas corresponding to (1.3.3) and (1.3.5).

$$(10.3.1) \quad 0 = \|\overline{\nabla}\varphi\|_M^2 + \nu \int_M u^{i\bar{j}} \varphi_{\bar{i}} \overline{\varphi_{\bar{j}}} + \int_M \left(v^{i\bar{j}} - R^{i\bar{j}}\right) \varphi_{\bar{i}} \overline{\varphi_{\bar{j}}} + T_1,$$

$$(10.3.2) \quad \begin{aligned} 0 = \|\nabla\varphi\|_M^2 &+ \nu \int_M u^{i\bar{j}} \varphi_{\bar{i}} \overline{\varphi_{\bar{j}}} - \nu \int_M u^{i}{}_{\bar{i}} \varphi_{\bar{j}} \overline{\varphi^{\bar{j}}} + \int_M v^{i\bar{j}} \varphi_{\bar{i}} \overline{\varphi_{\bar{j}}} \\ &- \int_M v^{i}{}_{\bar{i}} \varphi_{\bar{j}} \overline{\varphi^{\bar{j}}} + T_2, \end{aligned}$$

where T_1, T_2 are terms from the torsion tensor of the Hermitian metric of M, and can be estimated by

$$|T_i| \leqslant C_1 \|\varphi\|_M \left(\|\varphi\|_M + \|\nabla\varphi\|_M + \|\overline{\nabla}\varphi\|_M\right) \quad (i = 1, 2),$$

C_1 being a constant independent of ν. Multiplying (10.3.1) by nA and adding the resulting equation to (10.3.2) we obtain

$$\begin{aligned} 0 &= \|\overline{\nabla}\varphi\|_M^2 + \|\nabla\varphi\|_M^2 + \nu \int_M (nA + 1) u^{i\bar{j}} \varphi_{\bar{i}} \overline{\varphi_{\bar{j}}} - \nu \int_M u^{i}{}_{\bar{i}} \varphi_{\bar{j}} \overline{\varphi^{\bar{j}}} \\ &\quad + \int_M \left((nA + 1) v^{i\bar{j}} - nAR^{i\bar{j}}\right) \varphi_{\bar{i}} \overline{\varphi_{\bar{j}}} - \int_M v^{i}{}_{\bar{i}} \varphi_{\bar{j}} \overline{\varphi^{\bar{j}}} + nAT_1 + T_2 \\ &\geqslant \|\overline{\nabla}\varphi\|_M^2 + \|\nabla\varphi\|_M^2 + \nu \int_M u^{i\bar{j}} \varphi_{\bar{i}} \overline{\varphi_{\bar{j}}} + S, \end{aligned}$$

where

$$|S| \leqslant C_2 \|\varphi\|_M \big(\|\varphi\|_M + \|\overline{\nabla}\varphi\|_M + \|\nabla\varphi\|_M\big),$$

with C_2 independent of ν. Since

$$\|\varphi\|_M \big(\|\overline{\nabla}\varphi\|_M + \|\nabla\varphi\|_M\big) \leqslant C_2 \|\varphi\|_M^2 + \frac{1}{2C_2}\big(\|\overline{\nabla}\varphi\|_M^2 + \|\nabla\varphi\|_M^2\big),$$

it follows that

$$|S| \leqslant C_2(1 + C_2)\|\varphi\|_M^2 + \frac{1}{2}\|\overline{\nabla}\varphi\|_M^2 + \frac{1}{2}\|\nabla\varphi\|_M^2,$$

$$(10.3.3) \qquad \|\overline{\nabla}\varphi\|_M^2 + \|\nabla\varphi\|_M^2 + 2\nu\int_M u^{i\bar{j}}\varphi_{\bar{i}}\overline{\varphi_{\bar{j}}} \leqslant 2C_2(1 + C_2)\|\varphi\|_M^2.$$

10.4. By replacing the family of charts $\{U_\kappa\}$ by another family if necessary, we can assume without loss of generality that in addition

(i) $U'_\kappa := = \{|x_\kappa^1| < 1, \cdots, |x_\kappa^{2n}| < 1\}$ is relatively compact in U_κ, $1 \leqslant \kappa \leqslant k$, and

(ii) $M - G \subset \bigcup_{\kappa=1}^k U'_\kappa$.

Take any $\varepsilon > 0$. Since $(M - G) \cap \{x_\kappa^i = \text{constant}, \ 1 \leqslant i \leqslant 2n - 1\}$ is always a finite set, we can cover U'_κ by a finite number of open subsets $W_{\kappa\lambda}$ of U'_κ such that

(i) each $W_{\kappa\lambda}$ is of the form $D_{\kappa\lambda} \times J_{\kappa\lambda}$ with respect to the coordinate system $x_\kappa^1, \cdots, x_\kappa^{2n}$ with $D_{\kappa\lambda} \subset \mathbf{R}^{2n-1}$ and $J_{\kappa\lambda} \subset \mathbf{R}$,

(ii) $J_{\kappa\lambda}$ is a finite union of open intervals $I_{\kappa\lambda\mu}$ with center $c_{\kappa\lambda\mu}$ and length $< \varepsilon$,

(iii) the interval $\tilde{I}_{\kappa\lambda\mu}$ with center $c_{\kappa\lambda\mu}$ and length twice that of $I_{\kappa\lambda\mu}$ lies in $\{|x_\kappa^{2n}| < 1\}$, and for fixed κ, λ the intervals $\tilde{I}_{\kappa\lambda\mu}$ are pairwise disjoint.

(iv) $M - G$ is disjoint from $D_{\kappa\lambda} \times (\tilde{I}_{\kappa\lambda\mu} - I_{\kappa\lambda\mu})$.

Let $\tilde{J}_{\kappa\lambda} = \bigcup_\mu \tilde{I}_{\kappa\lambda\mu}$ and $\tilde{W}_{\kappa\lambda} = D_{\kappa\lambda} \times \tilde{J}_{\kappa\lambda}$.

When j is sufficiently large, for each $(2n - 1)$-tuple of integers $(p_1, \cdots, p_{2n-1})$ with $1 \leqslant p_i \leqslant 2^j - 1$, the cube

$$\left\{\frac{p_i - 1}{2^j} < x_\nu^i < \frac{p_i + 1}{2^j}, \quad 1 \leqslant i \leqslant 2n - 1\right\}$$

is contained in some $D_{\kappa\lambda}$. By replacing the sets $D_{\kappa\lambda}$ by these cubes, we can assume without loss of generality that

(10.4.1) the intersection of $2k + 1$ distinct members of $\{\tilde{W}_{\kappa\lambda}\}_{\kappa,\lambda}$ is empty.

Choose a smooth function $-\varepsilon \leqslant \rho_{\kappa\lambda\mu}(x_\kappa^{2n}) \leqslant \varepsilon$ with compact support in $\tilde{I}_{\kappa\lambda\mu}$ so that $\rho_{\kappa\lambda\mu} = x_\kappa^{2n} - c_{\kappa\lambda\mu}$ on $I_{\kappa\lambda\mu}$ and $|(\partial/\partial x_\kappa^{2n})\rho_{\kappa\lambda\mu}| \leqslant 2$ on $\tilde{I}_{\kappa\lambda\mu}$. We can naturally regard $\rho_{\kappa\lambda\mu}$ as a function on $D_{\kappa\lambda} \times \tilde{I}_{\kappa\lambda\mu}$. Write

$$\int_{D_{\kappa\lambda} \times I_{\kappa\lambda\mu}} |\varphi|^2 \omega_\kappa = V_{\kappa\lambda\mu} - V'_{\kappa\lambda\mu}, \tag{10.4.2}$$

where

$$\omega_\kappa = dx_\kappa^1 \wedge \cdots \wedge dx_\kappa^{2n},$$

$$V_{\kappa\lambda\mu} = \int_{D_{\kappa\lambda\mu} \times \tilde{I}_{\kappa\lambda\mu}} \left(\frac{\partial}{\partial x_\kappa^{2n}} \rho_{\kappa\lambda\mu} \right) |\varphi|^2 \omega_\kappa,$$

$$V'_{\kappa\lambda\mu} = \int_{D_{\kappa\lambda} \times (\tilde{I}_{\kappa\lambda\mu} - I_{\kappa\lambda\mu})} \left(\frac{\partial}{\partial x_\kappa^{2n}} \rho_{\kappa\lambda\mu} \right) |\varphi|^2 \omega_\kappa.$$

Integration by parts yields

$$V_{\kappa\lambda\mu} = -\int_{D_{\kappa\lambda} \times \tilde{I}_{\kappa\lambda\mu}} \rho_{\kappa\lambda\mu} \left(\frac{\partial}{\partial x_\kappa^{2n}} |\varphi|^2 \right) \omega_\kappa.$$

We can find constants E_κ depending only on κ and independent of ε and ν such that

$$\int_{D_{\kappa\lambda} \times I_{\kappa\lambda\mu}} |\varphi|^2 \leqslant E_\kappa \int_{D_{\kappa\lambda} \times I_{\kappa\lambda\mu}} |\varphi|^2 \omega_\kappa, \tag{10.4.3}$$

$$|V_{\kappa\lambda\mu}| \leqslant \varepsilon E_\kappa \int_{D_{\kappa\lambda} \times \tilde{I}_{\kappa\lambda\mu}} |\varphi| \left(|\nabla\varphi| + |\overline{\nabla}\varphi| \right), \tag{10.4.4}$$

$$|V'_{\kappa\lambda\mu}| \leqslant E_\kappa \int_{D_{\kappa\lambda} \times (\tilde{I}_{\kappa\lambda\mu} - I_{\kappa\lambda\mu})} |\varphi|^2, \tag{10.4.5}$$

where the integration of a function means with respect to the volume form of M. Let

$$Q_\varepsilon = \left(\cup_{\kappa,\lambda} W_{\kappa\lambda} \right) - \left(\cup_{\kappa,\lambda,\mu} D_{\kappa\lambda} \times \left(\tilde{I}_{\kappa\lambda\mu} - I_{\kappa\lambda\mu} \right) \right)^-,$$

and $E = \sup_\kappa (E_\kappa)^2$. From (10.4.1)–(10.4.5) it follows that

$$\int_{Q_\varepsilon} |\varphi|^2 \leqslant 2kE\varepsilon \int_M |\varphi| \left(|\nabla\varphi| + |\overline{\nabla}\varphi| \right) + 2kE \int_{M - Q_\varepsilon} |\varphi|^2.$$

Choose ε so small that $kE\varepsilon \leqslant \frac{1}{2}$. Then

$$\frac{1}{2} \int_{Q_\varepsilon} |\varphi|^2 \leqslant kE(2 + \varepsilon) \int_{M - Q_\varepsilon} |\varphi|^2 + 2kE\varepsilon \left(\|\nabla\varphi\|_M^2 + \|\overline{\nabla}\varphi\|_M^2 \right). \tag{10.4.6}$$

Combining this with (10.3.3), we obtain

$$\begin{aligned}
&\|\overline{\nabla}\varphi\|_M^2 + 2\|\nabla\varphi\|_M^2 + 2\nu\int_M u^{i\bar{j}}\varphi_{\bar{i}}\overline{\varphi_{\bar{j}}} \\
&\leqslant 2C_2\left(1+\frac{1}{2}C_2\right)(1+2kE(2+\varepsilon))\int_{M-Q_\varepsilon}|\varphi|^2 \qquad (10.4.7)\\
&\quad +8C_2\left(1+\frac{1}{2}C_2\right)kE\varepsilon\left(\|\nabla\varphi\|_M^2 + \|\overline{\nabla}\varphi\|_M^2\right).
\end{aligned}$$

We choose ε so small that $8C_2(1+\frac{1}{2}C_2)kE\varepsilon < 1$, and ν so large that

$$2\nu u^{i\bar{j}}\varphi_{\bar{i}}\overline{\varphi_{\bar{j}}} > 2C_2\left(1+\frac{1}{2}C_2\right)(1+2kE(2+\varepsilon))|\varphi|^2$$

on $M - Q_\varepsilon$. It follows from (10.4.7) that $\varphi \equiv 0$ on $M - Q_\varepsilon$ and $\|\nabla\varphi\|_M = \|\overline{\nabla}\varphi\|_M = 0$, and it follows from (10.4.6) that $\varphi \equiv 0$ on M. This concludes the proof of Theorem 10.2.

10.5. Remarks. 1. To get the vanishing of $H^q(M, L^\nu \otimes F)$ for ν sufficiently large, the assumptions of Theorem 10.2 concerning the positivity of u and its eigenvalues can be weakened to $\Sigma_{i=1}^q \lambda_i(P) > 0$ for $P \in G$ and $\Sigma_{i=n-q+1}^n \lambda_i(P) \leqslant A\Sigma_{i=1}^q \lambda_i(P)$ for $P \in M$. The proof is completely analogous to that of Theorem 10.2.

2. A corollary to Theorem 10.2 is that M is Moishezon. For, one can use Kodaira's method [34, §3] of blowing up points to show that for ν sufficiently large $\Gamma(M, L^\nu)$ separates points of G and gives local coordinates at points of G.

3. Theorem 10.2 cannot be used to characterize Moishezon manifolds because the pullback of L to a blow-up of M in general fails to satisfy the assumption on the eigenvalues of the curvature form. In this regard Theorem 10.2 is highly unsatisfactory.

References

[1] Y. Akizuki & S. Nakano, *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc. Japan Acad. **30** (1954) 266–272.

[2] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. **36** (1957) 235–249.

[3] M. Atiyah, R. Bott & V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973) 279–330.

[4] W. Barth, *Transplanting cohomology classes in complex-projective space*, Amer. J. Math. **92** (1970) 951–967.

[5] ______, *Larsen's theorem on the homotopy groups of projective manifolds of small embedding codimension*, Proc. Sympos. Pure Math. **29** (1975) 307–313.

[6] W. Barth & M. E. Larsen, *On the homology groups of complex projective manifolds*, Math. Scand. **30** (1972) 88–94.

[7] W. Barth & A. Van de Ven, *On the geometry in codimension 2 of Grassmann manifolds*, Lecture Notes in Math. Vol. 412, Springer, Berlin, 1974.

[8] R. L. Bishop & S. I. Goldberg, *On the second cohomology group of a Kähler manifold of positive curvature*, Proc. Amer. Math. Soc. **16** (1965) 119–122.

[9] S. Bloch & D. Gieseker, *The positivity of the Chern classes of an ample vector bundle*, Invent. Math. **12** (1971) 112–117.

[10] S. Bochner, *Curvature and Betti numbers*. I, II, Ann. of Math. **49** (1948) 379–390, **50** (1949) 77–93.

[11] A. Borel, *On the curvature tensor of the Hermitian symmetric manifolds*, Ann. of Math. **71** (1960) 508–521.

[12] E. Calabi & E. Vesentini, *On compact locally symmetric Kähler manifolds*, Ann. of Math. **71** (1960) 472–507.

[13] J. Cheeger & D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.

[14] J. P. Demailly, *Relations entre les différents notions de fibrés et de courants positifs*, Sèminaire Lelong-Skoda (Analyse), 20^e et 21^e année, 1980–81, Springer Lecture Notes (to appear).

[15] J. P. Demailly & H. Skoda, *Relations entre les notions de positivités de P. A. Griffiths et de S. Nakano pour les fibrés vectoriels*, Sèminaire Lelong-Skoda (Analyse), 18^e et 19^e année, 1978–79, Lecture Notes in Math. Vol. 822, Springer, Berlin, 1980.

[16] G. De Rham, *Sur la reductibilité d'un espace de Riemann*, Comment. Math. Helv. **26** (1952) 328–344.

[17] J. Eells & L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978) 1–68.

[18] J. Eells & J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964) 109–160.

[19] A. Futaki, *On compact Kähler manifolds with semipositive bisectional curvature*, Preprint, 1980.

[20] R. Frankel, *A differential geometric criterion for Moishezon spaces*, Math. Ann. **241** (1979) 107–112; *Erratum*, Math. Ann. **252** (1980) 259–260.

[21] S. I. Goldberg & S. Kobayashi, *Holomorphic bisectional curvature*, J. Differential Geometry **1** (1967) 225–233.

[22] H. Grauert & O. Riemenschneider, *Kählersche Mannifätigkeiten mit hyper-q-konvexem Rand*, A sympos. in Honor of S. Bochner, Princeton University Press, Princeton, 1970, 61–76.

[23] ______, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent Math. **11** (1970) 263–292.

[24] R. E. Greene & H. Wu, *C^∞ approximations of convex, subharmonic, and plurisubharmonic functions*, Ann. Sci. École Norm. Sup. **12** (1979) 47–84.

[25] P. A. Griffiths, *The extension problem in complex analysis*. II: *embeddings with positive normal bundle*, Amer. J. Math. **88** (1966) 366–446.

[26] ______, *Hermitian differential geometry, Chern classes, and positive vector bundles*, papers in honor of K. Kodaira, Princeton University Press, Princeton, 1969, 185–251.

[27] A. Grothendieck, *La théorie des classes de Chern*, Bull. Soc. Math. France **86** (1958) 137–184.

[28] R. Hamilton, *Harmonic maps of manifolds with boundary*, Lecture Notes in Math. Vol. 471, Springer, Berlin, 1975.

[29] R. Hartshorne, *Varieties of small codimension in projective space*, Bull. Amer. Math. Soc. **80** (1974) 1017–1032.

[30] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

[31] A. Howard, B. Smyth & H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature*. I.

[32] M. Kalka, *Deformation of submanifolds of strongly negatively curved manifolds*, Math. Ann. **251** (1980) 243–248.

[33] K. Kodaira, *On a differential-geometric method in the theory of analytic stacks*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953) 1268–1273.

[34] ______, *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)*, Ann. of Math. **60** (1954) 28–48.

[35] J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds*, I, II, Ann. of Math. **78** (1963) 112–148, **79** (1964) 450–472.

[36] M. E. Larsen, *On the topology of complex projective manifolds*, Invent. Math. **19** (1973) 251–260.

[37] D. Meyer, *Sur les variétés riemanniennes à opérateur de courbure positif*, C. R. Acad. Sci. Paris Ser. A-B, **272** (1971) 482–485.

[38] J. Milnor, *Morse theory*, Annals of Math. Studies No. 51, Princeton University Press, Princeton, 1963.

[39] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. **110** (1979) 593–606.

[40] C. B. Morrey, *The analytic embedding of abstract real analytic manifolds*, Ann. of Math. **68** (1958) 159–201.

[41] S. Nakano, *On complex analytic vector bundles*, J. Math. Soc. Japan **7** (1955) 1–12.

[42] S. Nichikawa & K. Shiga, *On the holomorphic equivalence of bounded domains in complete Kähler manifolds of nonpositive curvature*, Nagoya University Preprint, No. 8, 1981.

[43] K. Ogiue & S. Tachibana, *Kähler manifolds of positive curvature operator*, Proc. Amer. Math. Soc. **78** (1980) 548–550.

[44] J. H. Rabinowitz, *Positivity notions for coherent sheaves over compact complex spaces*, Invent. Math. **62** (1980) 79–87; *Erratum*, Invent. Math. **63** (1981) 355.

[45] R. Richberg, *Stetige streng pseudokonvexe Funktionen*, Math. Ann. **175** (1968) 257–286.

[46] O. Riemenschneider, *Characterizing Moishezon spaces by almost positive coherent analytic sheaves*, Math. Z. **123** (1971) 263–284.

[47] ______, *A generalization of Kodaira's embedding theorem*, Math. Ann. **200** (1973) 99–102.

[48] M. Schneider, *Ein einfacher Beweise des Verschwindungssatzes für positive holomorphe Vektorraumbündel*, Manuscripta Math. **11** (1974) 95–101.

[49] ______, *Lefschetzsätze und Hyperkonvexität*, Invent. Math. **31** (1975) 183–192.

[50] R. Schoen, *Existence and regularity theorems for some geometric variational problems*, Ph.D. thesis, Stanford University, 1977.

[51] J. Simons, *On the transitivity of holonomy systems*, Ann. of Math. **76** (1962) 213–234.

[52] Y. T. Siu, *The Levi problem*, Proc. Sympos. Pure Math. **30** (1977) 45–48.

[53] ______, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. of Math. **112** (1980) 73–111.

[54] ______, *Curvature characterization of hyperquadrics*, Duke Math. J. **47** (1980) 641–654.

[55] ______, *Strong rigidity of compact quotients of exceptional bounded symmetric domains*, Duke Math. J. **48** (1981) 857–871.

[56] Y. T. Siu & S. T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980) 189–204.

[57] A. J. Sommese, *Criteria for quasi-projectivity*, Math. Ann. **217** (1975) 247–256; *Addendum*, Math. Ann. **221** (1976) 95–96.

[58] ______, *Theorems of Barth-Lefschetz type for complex subspaces of homogeneous complex manifolds*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977) 1332–1333.

[59] ______, *Submanifolds of Abelian varieties*, Math. Ann. **233** (1978) 229–256.

[60] ______, *Concavity theorems*, Math. Ann. **235** (1978) 37–53.

[61] ______, *Complex subspaces of homogeneous complex manifolds*. I. *Transplanting theorems*, Duke Math. J. **46** (1979) 527–548; II, *Homotopy results*, to appear.

[62] A. Weil, *Variétés Kaehlériennes*, Hermann, Paris, 1958.

[63] R. O. Wells, *Moishezon spaces and the Kodaira embedding theorem, value distribution theory*, Part A, Marcel Dekker, New York, 1974, 29–42.

[64] H. Weyl, *The classical groups*, Princeton University Press, Princeton, 1946.

[65] J. C. Wood, *An extension theorem for holomorphic mappings*, Math. Proc. Cambridge Philos. Soc. **88** (1980) 125–127.

[66] J. C. Zhong, *The degree of strong nondegeneracy of the bisectional curvature of exceptional bounded symmetric domains*, Proc. Intern. Conf. Several Complex Variables, Hangzhou, China, 1981.

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