J. DIFFERENTIAL GEOMETRY
16 (1981) 577–593

DIVISION ALGEBRAS AND FIBRATIONS OF SPHERES BY GREAT SPHERES

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Dedicated to Professor Buchin Su on his 80th birthday

Smooth fibrations of spheres by *great spheres* occur naturally in the study of the Blaschke conjecture. In fact, if M is a Blaschke manifold, m is a point of M, $T_m M$ is the tangent space of M at m, $\exp_m\colon T_m M \to M$ is the exponential map at m, and $\mathrm{Cut}(m)$ is the cut locus of m in M, then $\exp_m^{-1}(\mathrm{Cut}(m))$ is a sphere S_m in $T_m M$ of center 0, and $\exp_m\colon S_m \to \mathrm{Cut}(m)$ is a smooth *great sphere fibration* of the sphere S_m. For general information of the Blaschke conjecture, see [2].

If $\mathbf{K}$ is the real, complex, quaternionic or Cayley algebra, n is the dimension of $\mathbf{K}$ as a euclidean space, which is $1, 2, 4$ or 8, and S^{2n-1} is the unit $(2n-1)$-sphere in the euclidean $2n$-space $\mathbf{K} \times \mathbf{K}$, then there is a natural smooth great $(n-1)$-sphere fibration of S^{2n-1} such that any $(u, w), (u', w') \in S^{2n-1}$ belong to the same fibre iff either $w = w' = 0$ or $uw^{-1} = u'w'^{-1}$. When $n > 1$, this fibration, as well as isomorphic ones, is often referred as the *Hopf fibration*. Related to this result, Adams' theorem [1] says that a smooth fibration of S^{2n-1} by $(n-1)$-spheres can occur only when $n = 1, 2, 4$ or 8, and a classical theorem of Hurwitz [4] says that any division algebra $\mathbf{K}$, which possesses a norm such that for any $v, w \in \mathbf{K}$, $|vw| = |v|\,|w|$, must be the real, complex, quaternionic or Cayley algebra. If $n = 1$ or 2, then any n-dimensional division algebra is the real or complex algebra, and any fibration of S^{2n-1} by $(n-1)$-spheres is unique up to an isomorphism. Hence in these cases, the correspondence between n-dimensional division algebras and smooth great $(n-1)$-sphere fibrations of S^{2n-1} is trivial.

In this paper, we show that for $n = 4$ or 8, each n-dimensional division algebra $\mathbf{K}$ determines a smooth great $(n-1)$-sphere fibration of S^{2n-1}, and every smooth great $(n-1)$-sphere fibration of S^{2n-1}, up to an isomorphism, is determined by an n-dimensional division algebra $\mathbf{K}$. However, it is possible

Received December 23, 1981. The author is supported in part by the National Science Foundation.

that two division algebras, not isomorphic to each other, may determine isomorphic smooth great $(n-1)$-sphere fibrations of S^{2n-1}. Such an example can be found using division algebras constructed in Bruck [3].

We also show that any division algebra of dimension > 1 contains the complex algebra as a subalgebra. Results of a subsequent paper of the author's joint work with Herman Gluck and Frank Warner will be used to show that any smooth great 3-sphere fibration of S^7 is isomorphic to the Hopf fibration, and hence any Blaschke manifold which has the integral cohomology ring of the quaternionic projective 2-space $\mathbf{H}P^2$ is homeomorphic to $\mathbf{H}P^2$.

The author wishes to express his gratitude to many colleagues of his for numerous dicussions, and especially to McKenzie Y. Wang for bringing Bruck's paper to his attention, and to Stephen S. Shatz for showing him an algebraic proof of the result that any division algebra of dimension > 1 contains the complex algebra.

Throughout this paper, $\mathbf{R}$ denotes the real algebra, and $\mathbf{C}$ the complex algebra. Let $\mathbf{K}$ be the euclidean n-space, $n \geqslant 1$, which is often regarded as a vector space over $\mathbf{R}$. By a *regular multiplication* on $\mathbf{K}$, we mean a bilinear function

$$m\colon \mathbf{K} \times \mathbf{K} \to \mathbf{K}$$

such that for any $a, b \in \mathbf{K}$ with $a \neq 0$, each of

$$m(v, a) = b, \quad m(a, w) = b$$

has a unique solution in $\mathbf{K}$. $\mathbf{K}$ together with a regular multiplication on $\mathbf{K}$ is called a *regular algebra* which we also denote by $\mathbf{K}$. If m is the only regular multiplication on $\mathbf{K}$ under our consideration, we often write vw in place of $m(v, w)$. We note that a regular multiplication may not be associative, and a regular algebra may have no identity, and that a regular algebra may not have a norm such that the norm of a product is equal to the product of the norms. On the other hand, it can be shown that any 1-dimensional regular algebra must be $\mathbf{R}$, and that the dimension of any regular algebra is $1, 2, 4$ or 8. A *division algebra* is defined to be a regular algebra having an identity. Notice that the real, complex, quaternionic and Cayley algebras are division algebras.

Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathbf{K}$ as a vector space over $\mathbf{R}$. Then for any bilinear function $m\colon \mathbf{K} \times \mathbf{K} \to \mathbf{K}$, there are n^3 real numbers a_{ijk}, $i, j, k = 1, \cdots, n$, such that

$$m\left(\sum_{i=1}^{n} v_i e_i, \sum_{k=1}^{n} w_k e_k\right) = \sum_{j=1}^{n}\left(\sum_{i,k=1}^{n} v_i a_{ijk} w_k\right) e_j.$$

Hence regular multiplications are always smooth.

Proposition 1. *Any* 1*-dimensional regular algebra is the real algebra.*

Proof. Let $\mathbf{K}$ be a 1-dimensional regular algebra, and let a be an element of $\mathbf{K}$ different from the zero of $\mathbf{K}$. By definition, $ae = a$ for some $e \in \mathbf{K}$. e is different from the zero of $\mathbf{K}$; otherwise, $a = ae = a(0e) = 0(ae) = 0e = e$, contradicting to our assumption.

Let $a = te$, $t \in \mathbf{R}$. Then $t \neq 0$, and $te = (te)e = te^2$ so that $e^2 = e$. Hence e is the identity of $\mathbf{K}$, and $\mathbf{K}$ can be naturally identified with $\mathbf{R}$ by setting $re = r$ for all $r \in \mathbf{R}$.

Theorem 1. *Any division algebra of dimension* > 1 *contains a subalgebra isomorphic to the complex algebra.*

Corollary 1. *Any* 2*-dimensional division algebra is the complex algebra.*

Let $\mathbf{K}$ be a division algebra of dimension $n > 1$, and let S^{2n-1} be the unit $(n-1)$-sphere in $\mathbf{K}$. We may assume that the identity e of $\mathbf{K}$ is contained in S^{n-1}; otherwise all we have to do is to use a new norm on $\mathbf{K}$ which is equal to $|e|^{-1}$ times the old one.

Lemma 1. *The map* $f\colon S^{n-1} \to S^{n-1}$ *defined by* $f(x) = x^2/|x^2|$ *is of degree* 2.

Proof. Let

$$\phi\colon S^{n-1} \times S^{n-1} \to S^{n-1}$$

be the map defined by

$$\phi(x, y) = xy/|xy|\,.$$

Notice that ϕ is well-defined and continuous, since $xy \in \mathbf{K} - \{0\}$ for any $x, y \in \mathbf{K} - \{0\}$.

Let Δ be the diagonal set in $S^{n-1} \times S^{n-1}$. Let S^{n-1} be oriented, and let $S^{n-1} \times \{e\}$, $\{e\} \times S^{n-1}$ and Δ be so oriented that the natural projection of each of them onto S^{n-1} is orientation-preserving. Let

$$\phi_*\colon H_{n-1}(S^{n-1} \times S^{n-1}) \to H_{n-1}(S^{n-1})$$

be the induced homomorphism of integral homology groups by ϕ. Then

$$\phi_*[S^{n-1} \times \{e\}] = [S^{n-1}] = \phi_*[\{e\} \times S^{n-1}],$$

$$[\Delta] = [S^{n-1} \times \{e\}] + [\{e\} \times S^{n-1}],$$

so that

$$\phi_*[\Delta] = 2[S^{n-1}].$$

Since $\phi(x, x) = f(x)$ for any $x \in S^{n-1}$, our assertion follows.

Proof of Theorem 1. By Lemma 1, the map

$$g\colon \mathbf{K} \to \mathbf{K}$$

defined by $g(x) = x^2$ is onto. Therefore there is an element i of $\mathbf{K} - \{0\}$ such that

$$i^2 = g(i) = -e.$$

The linear 2-subspace of $\mathbf{K}$ having $\{e, i\}$ as a basis is clearly a subalgebra of $\mathbf{K}$ isomorphic to $\mathbf{C}$.

As mentioned earlier, Stephen S. Shatz has an algebraic proof of Lemma 1, and hence Theorem 1 can be proved algebraically.

Theorem 2. *Let $\mathbf{K}$ be a regular algebra of dimension $n > 1$, and let S^{2n-1} be the unit $(2n - 1)$-sphere in the euclidean $2n$-space $\mathbf{K} \times \mathbf{K}$. Then $\mathbf{K}$ determines a smooth great $(n - 1)$-sphere fibration of S^{2n-1} such that any (u, w), $(u', w') \in S^{2n-1}$ belong to the same fibre iff either $w = w' = 0$ or $u = vw$ and $u' = vw'$ for some $v \in \mathbf{K}$. Moreover, the fibrations determined by two isomorphic regular algebras are smoothly isomorphic.*

Notice that if $\mathbf{K}$ is the complex, quaternionic or Cayley algebra, then the fibration determined by $\mathbf{K}$ is the Hopf fibration.

Proof. Let $\Sigma^n = \mathbf{K} \cup \{\infty\}$ be the one-point compactification of $\mathbf{K}$. Then Σ^n can be made a smooth manifold as follows. For any $u \in \mathbf{K} - \{0\}$, we let

$$\lambda_u \colon \Sigma^n - \{0\} \to \mathbf{K}$$

be the homeomorphism such that $\lambda_u(\infty) = 0$ and $v\lambda_u(v) = u$ for any $v \in \mathbf{K} - \{0\} = \Sigma^n - \{0, \infty\}$. Since $\lambda_u \colon \mathbf{K} - \{0\} \to \mathbf{K} - \{0\}$ is a diffeomorphism, there is a smooth structure on Σ^n such that the inclusion map of $\mathbf{K}$ into Σ^n is a smooth imbedding, and λ_u is a diffeomorphism for some $u \in \mathbf{K} - \{0\}$. The smooth structure on Σ^n is independent of the choice of u. In fact, for any $u, u' \in \mathbf{K} - \{0\}$, u and u' can be joined by a smooth path in $\mathbf{K} - \{0\}$, and hence $\lambda_u, \lambda_{u'} \colon \mathbf{K} - \{0\} \to \mathbf{K} - \{0\}$ are isotopic.

Let

$$\pi \colon S^{2n-1} \to \Sigma^n$$

be the map such that $\pi(u, 0) = \infty$ for any $u \in S^{n-1}$, and $\pi(u, w)w = u$ for any $(u, w) \in S^{2n-1}$ with $w \neq 0$. Since the multiplication on $\mathbf{K}$ is bilinear, it follows that $\pi^{-1}v$ is a great $(n - 1)$-sphere in S^{2n-1} for any $v \in \Sigma^n$.

There is a smooth imbedding

$$g_0 \colon \mathbf{K} \times S^{n-1} \to S^{2n-1}$$

given by

$$g_0(v, w) = \left(vw/\sqrt{|vw|^2 + 1}\,,\ w/\sqrt{|vw|^2 + 1} \right),$$

and for any $v \in \mathbf{K}$, $\pi g_0(\{v\} \times S^{n-1}) = v$. Also there is a smooth imbedding

$$g_1 \colon S^{n-1} \times (\Sigma^n - \{0\}) \to S^{2n-1}$$

given by

$$g_1(u, v) = \left(u/\sqrt{1+|\lambda_u(v)|^2}, \lambda_u(v)/\sqrt{1+|\lambda_u(v)|^2}\right),$$

and for any $v \in \Sigma^n - \{0\}$, $\pi g_1(S^{n-1} \times \{v\}) = v$. Hence

$$\pi \colon S^{2n-1} \to \Sigma^n$$

is a smooth great $(n-1)$-sphere fibration.

Let $\mathbf{K}_1$ be a regular algebra isomorphic to $\mathbf{K}$, and let

$$\pi_1 \colon S_1^{2n-1} \to \Sigma^n$$

be the smooth great $(n-1)$-sphere fibration determined by $\mathbf{K}_1$, where S_1^{2n-1} is the unit $(2n-1)$-sphere in $\mathbf{K}_1 \times \mathbf{K}_1$. Then $\pi_1 \colon S_1^{2n-1} \to \Sigma_1^n$ is smoothly isomorphic to $\pi \colon S^{2n-1} \to \Sigma^n$. In fact, if $f \colon \mathbf{K}_1 \to \mathbf{K}$ is an isomorphism, then

$$f \times f \colon \mathbf{K}_1 \times \mathbf{K}_1 \to \mathbf{K} \times \mathbf{K}$$

defined by $(f \times f)(u_1, w_1) = (fu_1, fw_1)$ is a nonsingular linear map so that

$$h \colon S_1^{2n-1} \to S^{2n-1}$$

defined by $h(u_1, w_1) = (fu_1, fw_1)/|(fu_1, fw_1)|$ is a diffeomorphism. It is easy to see that h maps fibres of $\pi_1 \colon S_1^{2n-1} \to \Sigma_1^n$ into fibres of $\pi \colon S^{2n-1} \to \Sigma^n$. Hence the proof is completed.

As a consequence of Theorem 2 and Adams' theorem, we have

Corollary 2. *The dimension of any regular algebra is* 1, 2, 4 *or* 8.

Let $GL(\mathbf{K})$ be the group of nonsingular linear maps of $\mathbf{K}$ into $\mathbf{K}$. Two regular multiplications m and m_1 on $\mathbf{K}$ are said to be *equivalent* if there exist $\mu, \nu, \omega \in GL(\mathbf{K})$ such that $m_1(\nu \times \omega) = \mu m$, that means, the diagram

$$\begin{array}{ccc} \mathbf{K} \times \mathbf{K} & \xrightarrow{m_1} & \mathbf{K} \\ \uparrow {\scriptstyle \nu \times \omega} & & \uparrow {\scriptstyle \mu} \\ \mathbf{K} \times \mathbf{K} & \xrightarrow{m} & \mathbf{K} \end{array}$$

is commutative.

Proposition 2. *Let m and m_1 be equivalent regular multiplications on the euclidean n-space* $\mathbf{K}$. *Then the smooth great $(n-1)$-sphere fibrations of S^{2n-1} determined by the regular algebras* $(\mathbf{K}, m)$ *and* $(\mathbf{K}, m_1)$ *are smoothly isomorphic.*

Proof. Let

$$\pi \colon S^{2n-1} \to \Sigma^n, \quad \pi_1 \colon S^{2n-1} \to \Sigma_1^n$$

be the smooth great $(n-1)$-sphere fibrations determined by $(\mathbf{K}, m)$ and $(\mathbf{K}, m_1)$. Since m and m_1 are equivalent, there are $\mu, \nu, \omega \in GL(\mathbf{K})$ such that

$m_1(\nu \times \omega) = \mu m$. Then $\mu \times \omega: \mathbf{K} \times \mathbf{K} \to \mathbf{K} \times \mathbf{K}$ is a nonsingular linear map so that $h: S^{2n-1} \to S^{2n-1}$ defined by $h(u, w) = (\mu u, \omega w)/|(\mu u, \omega w)|$ is a diffeomorphism. It is easily seen that h maps fibers of $\pi: S^{2n-1} \to \Sigma^n$ into fibres of $\pi_1: S^{2n-1} \to \Sigma_1^n$.

Proposition 3. *On the euclidean n-space* $\mathbf{K}$, *any regular multiplication is equivalent to one having an identity.*

Proof. Let m be a regular multiplication on $\mathbf{K}$, and let

$$\Phi, \Psi: \mathbf{K} - \{0\} \to GL(\mathbf{K})$$

be the smooth maps such that

$$\Phi(v)w = m(v, w), \quad \Psi(w)v = m(v, w).$$

Let $e \in \mathbf{K} - \{0\}$, let

$$\mu, \nu, \omega: \mathbf{K} \to \mathbf{K}$$

be the elements of $GL(\mathbf{K})$ given by

$$\mu(u) = \Psi(e)^{-1}u, \quad \nu(v) = v, \quad \omega(w) = \Psi(e)^{-1}\Phi(e)w,$$

and let m' be the regular multiplication on $\mathbf{K}$ such that

$$m'(\nu \times \omega) = \mu m.$$

Then for any $v', w' \in \mathbf{K} - \{0\}$,

$$m'(v', w') = \Psi(e)^{-1}m\left(v', \Phi(e)^{-1}\Psi(e)w'\right) = \begin{cases} \Psi(e)^{-1}\Phi(v')\Phi(e)^{-1}\Psi(e)w', \\ \Psi(e)^{-1}\Psi\left(\Phi(e)^{-1}\Psi(e)w'\right)v'. \end{cases}$$

Therefore

$$m'(e, w') = \Psi(e)^{-1}\Phi(e)\Phi(e)^{-1}\Psi(e)w' = w',$$

so that

$$e = m'(e, e) = \Psi(e)^{-1}\Psi\left(\Phi(e)^{-1}\Psi(e)e\right)e.$$

From the last equality, we infer that $\Phi(e)^{-1}\Psi(e)e = e$ and hence

$$m'(v', e) = \Psi(e)^{-1}\Psi\left(\Phi(e)^{-1}\Psi(e)e\right)v' = v'.$$

As a consequence of Propositions 2 and 3, we have

Corollary 3. *Any smooth great* $(n-1)$*-sphere fibration of* S^{2n-1} *determined by a regular algebra is smoothly isomorphic to one determined by a division algebra.*

Now we are in a position to construct, from a given smooth great $(n-1)$-sphere fibration of S^{2n-1}, an n-dimensional division algebra $\mathbf{K}$ such that the smooth great $(n-1)$-sphere fibration of S^{2n-1} determined by $\mathbf{K}$ is smoothly isomorphic to the given one. Since it is trivial for $n = 1$ or 2, in the following we assume that

$$n = 4 \text{ or } 8.$$

Let $\mathbf{K}$ be the euclidean n-space, and S^{n-1} the unit $(n-1)$-sphere in $\mathbf{K}$. Let $GL(\mathbf{K})$ be the group of all nonsingular linear maps of $\mathbf{K}$ into $\mathbf{K}$, and $SL(\mathbf{K})$ the subgroup of $GL(\mathbf{K})$ consisting of all the $g \in GL(\mathbf{K})$ with $\det g = 1$.

Let L_i be a normed real vector n-space, and S_i^{n-1} the unit $(n-1)$-sphere in L_i, $i = 1, 2$. A diffeomorphism $f\colon S_1^{n-1} \to S_2^{n-1}$ is called a *linear* diffeomorphism if there is a nonsingular linear map $g\colon L_1 \to L_2$ such that for any $x \in S_1^{n-1}, f(x) = g(x)/|g(x)|$.

Lemma 2. *Whenever $g \in GL(\mathbf{K})$, we have a linear diffeomorphism*

$$\bar{g}\colon S^{n-1} \to S^{n-1}$$

defined by $g(x) = g(x)/|g(x)|$. Conversely, whenever $f\colon S^{n-1} \to S^{n-1}$ is a linear diffeomorphism, there is a unique $g \in GL(\mathbf{R})$ such that $\bar{g} = f$ and $\det g = \pm 1$, *and $g'g^{-1}$ is in the center of $GL(\mathbf{K})$ for any $g' \in GL(\mathbf{K})$ with $\bar{g}' = f$. Hence*

$$\overline{SL}(\mathbf{K}) = \{\bar{g} \mid g \in SL(\mathbf{K})\}$$

acts on S^{n-1} as a smooth transformation group.

For any map $\alpha\colon S^{n-1} \to SL(\mathbf{K})$, we have a map $\bar{\alpha}\colon S^{n-1} \to \overline{SL}(\mathbf{K})$ defined by $\bar{\alpha}(v) = \overline{\alpha(v)}$, called the *associated map* of α.

Lemma 3. *Let S_i^{n-1} be S^{n-1} or a great $(n-1)$-sphere in S^{2n-1}, $i = 1, 2$. Then any linear diffeomorphism $f\colon S_1^{n-1} \to S_2^{n-1}$ maps great circles into great circles, and any map $f\colon S_1^{n-1} \to S_2^{n-1}$ which maps great circles into great circles is a linear diffeomorphism.*

Lemma 2 is quite obvious and Lemma 3 is a consequence of the well-known theorem in projective geometry that any map of a projective space of dimension > 1 into itself which maps projective lines into projective lines is a projective transformation.

Let

$$\pi\colon S^{2n-1} \to \Sigma^n$$

be a given smooth great $(n-1)$-sphere fibration of S^{2n-1}. We first observe that Σ^n is homeomorphic to the n-sphere. In fact, if S^n is a great n-sphere in S^{2n-1} containing a fibre F, then F is a great $(n-1)$-sphere in S^n, and Σ^n is obtained from a closed hemisphere in S^n with boundary F by identifying F to a single point.

Let F_0 and F_1 be two distinct fibres. Whenever x is a point of $S^{2n-1} - F_i$, F_i and x determine a great n-sphere in S^{2n-1}. The closed hemisphere in this great n-sphere of boundary F_i containing x will be denoted by $F_i x$.

Let

$$h_0: S^{2n-1} - F_1 \to F_0, h_1: S^{2n-1} - F_0 \to F_1$$

be the smooth maps such that for any $x \in S^{2n-1} - F_{1-i}$, $h_i(x)$ is the point of intersection of $F_{1-i}x$ with F_i, $i = 0, 1$. Let

$$x_0 = \pi F_0, \quad x_1 = \pi F_1.$$

Then

$$\pi \times h_0: S^{2n-1} - F_1 \to (\Sigma^n - \{x_1\}) \times F_0,$$
$$h_1 \times \pi: S^{2n-1} - F_0 \to F_1 \times (\Sigma^n - \{x_0\})$$

are diffeomorphisms, which are local trivializations of the fibration over $\Sigma^n - \{x_1\}$ and $\Sigma^n - \{x_0\}$ respectively.

Let S be the $(n-1)$-sphere of unit tangent vectors of Σ^n at x_0 with respect to any preassigned Riemannian metric on Σ^n. Then for any $(v, w) \in S \times F_0$, there is a tangent vector $\tau(v, w)$ of $F_1 w$ at w such that

$$d\pi(\tau(v, w)) = v.$$

Now we define a smooth map

$$\xi: S \times F_0 \to F_1$$

as follows. Let $(v, w) \in S \times F_0$. Then there is a smooth map $f: [0, 1] \to F_1 w$ such that $f(t) = w$ iff $t = 0$, and $f'(0) = \tau(v, w)$. It is not hard to see that $\lim_{t \to 0} F_0 f(t)$ exists and is a closed hemisphere of boundary F_0 with $\tau(v, w)$ as a tangent vector at w. $\xi(v, w)$ is defined to be the point of intersection of $\lim_{t \to 0} F_0 f(t)$ with F_1.

The following lemma plays a key role in our paper.

Lemma 4. *For any $v \in S$, $w \to \xi(v, w)$ is a linear diffeomorphism of F_0 onto F_1, and for any $w \in F_0$, $v \to \xi(v, w)$ is a linear diffeomorphism of S onto F_1.*

Proof. Let $v \in S$ and let $f: [0, 1] \to \Sigma^n - \{x_1\}$ be a smooth map such that $f(t) = x_0$ iff $t = 0$, and $f'(0) = v$. Then for any $w \in F_0$, we have a smooth map $f_w: [0, 1] \to F_1 w$ such that $\pi f_w = f$. Clearly $f_w(t) = w$ iff $t = 0$, and $f_w'(0) = \tau(v, w)$. Moreover,

$$\xi(v, w) = \lim_{t \to 0} h_1 f_w(t).$$

Let C be a great circle in F_0. Then for any $t \in (0, 1]$, $C_t = \{f_w(t) \mid w \in C\}$ is the intersection of $\pi^{-1} f(t)$ with the great $(n+1)$-sphere in S^{2n-1} determined by F_1 and C, so that it is a great circle in $\pi^{-1} f(t)$. Therefore $h_1(C_t)$, which is the intersection of F_1 with the great $(n+1)$-sphere in S^{2n-1} determined by F_0 and C_t, is a great circle in F_1. Hence $\xi(v, C) = \lim_{t \to 0} h_1(C_t)$ is a great circle in F_1. From this result and Lemma 3 we conclude that $w \to \xi(v, w)$ is a linear diffeomorphism of F_0 onto F_1.

Let $w \in F_0$. For any great circle C in S we have a great $(n+1)$-sphere S^{n+1} in S^{2n-1} containing F_0 such that for any $v \in C$, $\tau(v, w)$ is a tangent vector of S^{n+1} at w. It can be seen that $\xi(C, w)$ is the intersection of F_1 and S^{n+1} so that it is a great circle in F_1. Hence by Lemma 3, $v \to \xi(v, w)$ is a linear diffeomorphism of S onto F_1.

Since Σ^n is 1-connected, we may assume that $\pi\colon S^{2n-1} \to \Sigma^n$ is oriented. Then for any $v \in S$, $w \to \xi(v, w)$ is an orientation-preserving linear diffeomorphism of F_0 onto F_1. We let S be so oriented that for any $w \in F_0$, $v \to \xi(v, w)$ is also an orientation-preserving linear diffeomorphism of S onto F_1.

Let S^{n-1} be naturally oriented, and let us identify F_0, F_1 and S with S^{n-1} by orientation-preserving linear diffeomorphisms. Then $\xi\colon S \times F_0 \to F_1$ becomes a smooth map

$$\xi\colon S^{n-1} \times S^{n-1} \to S^{n-1}$$

such that for some smooth maps

$$\phi, \psi\colon S^{n-1} \to SL(\mathbf{K}),$$

we have

$$\xi(v, w) = \bar{\phi}(v)w = \bar{\psi}(w)v,$$

where $\bar{\phi}, \bar{\psi}\colon S^{n-1} \to \overline{SL}(\mathbf{K})$ are the associated maps of ϕ and ψ.

The following result can be proved in the same way as Proposition 3.

Lemma 5. *For any $e \in S^{n-1}$, we let*

$$\mu_e = \psi(e)^{-1}, \quad \nu_e = \textit{identity}, \quad \omega_e = \psi(e)^{-1}\phi(e),$$

let

$$\phi_e, \psi_e\colon S^n \to SL(\mathbf{K})$$

be the smooth maps defined by

$$\phi_e(v) = \mu_e\phi(\nu_e^{-1}v)\omega_e^{-1},$$
$$\psi_e(w) = \mu_e\psi(\omega_e^{-1}w)\nu_e^{-1},$$

and let

$$\xi_e\colon S^{n-1} \times S^{n-1} \to S^{n-1}$$

be the smooth map defined by

$$\xi_e(v, w) = \bar{\mu}_e\xi(\bar{\nu}_e^{-1}v, \bar{\omega}_e^{-1}w).$$

Then

$$\phi_e(e) = \psi_e(e) = \textit{identity},$$
$$\xi_e(v, w) = \bar{\phi}_e(v)w = \bar{\psi}_e(w)v,$$

where $\bar{\phi}_e, \bar{\psi}_e\colon S^{n-1} \to \overline{SL}(\mathbf{K})$ are the associated maps of ϕ_e and ψ_e.

Lemma 6. **K** *can be made a division algebra with identity e such that for any* $v, w \in S^{n-1}$,

$$\xi_e(v, w) = vw/|vw|\,.$$

The following results are needed in the proof of Lemma 6.

Sublemma 1. *Let U be a nonnull open subset of* **R**, *and let* $\nu, \omega\colon U \to \mathbf{R}$ *and* $\alpha\colon \mathbf{R} \to \mathbf{R}$ *be smooth maps such that*

$$\alpha(r) > 0$$

for any $r \in \mathbf{R}$, *and*

$$\alpha(r) = \frac{1 + \nu(s)r}{1 + \omega(s)r}$$

for any $r \in \mathbf{R}$ *and* $s \in U$. *Then*

$$\nu = \omega, \quad \alpha = 1.$$

Proof. By hypothesis,

$$\alpha(r)(1 + \omega(s)r) = 1 + \nu(s)r.$$

Partially differentiating the equality with respect to s, we obtain

$$\alpha(r)\omega'(s)r = \nu'(s)r.$$

Therefore

$$\alpha(r)\omega'(s) = \nu'(s).$$

If $\omega'(s) \not\equiv 0$, then $\alpha(r) = \nu'(s)/\omega'(s)$ which is independent of the choice of r. Therefore $\alpha(r) = \alpha(0) = 1$ and hence

$$\alpha = 1.$$

If $\omega'(s) \equiv 0$, then $\nu'(s) \equiv 0$. Therefore there are $\nu, \omega \in \mathbf{R}$ such that

$$\alpha(r) = \frac{1 + \nu r}{1 + \omega r}.$$

Since $\alpha(r) > 0$ for all $r \in \mathbf{R}$, it follows that $\nu = \omega$. Hence $\alpha(r) = 1$ for all $r \in \mathbf{R}$ or

$$\alpha = 1.$$

Sublemma 2. *Let* $\lambda_1, \lambda_2, \mu_1, \mu_2, \alpha\colon \mathbf{R} \to \mathbf{R}$ *be smooth maps such that*

$$\alpha(r) > 0$$

for any $r \in \mathbf{R}$, *and*

$$\alpha(r) = \frac{1 + \lambda_1(s)r + \lambda_2(s)r^2}{1 + \mu_1(s)r + \mu_2(s)r^2}$$

for any $r, s \in \mathbf{R}$. *Then either* $\lambda_1, \lambda_2, \mu_1, \mu_2$ *are constant maps or* $\alpha = 1$.

Proof. By hypothesis,

$$\alpha(r)\big(1+\mu_1(s)r+\mu_2(s)r^2\big)=1+\lambda_1(s)r+\lambda_2(s)r^2.$$

Partially differentiating the equality with respect to s, we obtain

$$\alpha(r)\big(\mu_1'(s)r+\mu_2'(s)r^2\big)=\lambda_1'(s)r+\lambda_2'(s)r^2.$$

Therefore

$$\alpha(r)\big(\mu_1'(s)+\mu_2'(s)r\big)=\lambda_1'(s)+\lambda_2'(s)r.$$

Assume first that $\mu_1'(s)\equiv 0$. Then

$$\lambda_1'(s)=\alpha(0)\mu_2'(s)0\equiv 0,$$

so that

$$\alpha(r)\mu_2'(s)=\lambda_2'(s).$$

If $\mu_2'(s)\equiv 0$, then $\lambda_2'(s)\equiv 0$. Hence $\lambda_1, \lambda_2, \mu_1, \mu_2$ are constant maps. If $\mu_2'(s)\not\equiv 0$, then there is a nonnull open subset U of $\mathbf{R}$ such that for any $s\in U$, $\mu_2'(s)\neq 0$. Therefore for any $r\in\mathbf{R}$ and $s\in U$, $\alpha(r)=\lambda_2'(s)/\mu_2'(s)$ which is independent of the choice of r. Hence $\alpha(r)=\alpha(0)=1$ or $\alpha=1$.

Assume next that $\mu_1'(s)\not\equiv 0$. Then there is a nonnull open subset U of $\mathbf{R}$ such that $\mu_1'(s)\neq 0$ and

$$\lambda_1'(s)=\alpha(0)\mu_1'(s)=\mu_1'(s)$$

for any $s\in U$. Therefore for any $r\in\mathbf{R}$ and $s\in U$,

$$\alpha(r)=\frac{1(\lambda_2'(s)/\mu_1'(s))r}{1+(\mu_2'(s)/\mu_1'(s))r}.$$

Hence by Sublemma 1,

$$\alpha=1.$$

Proof of Lemma 6. In this proof, we drop the subscript e from ξ_e, ϕ_e, ψ_e so that ξ, ϕ, ψ are actually ξ_e, ϕ_e, ψ_e of Lemma 5.

Let

$$\Phi, \Psi\colon \mathbf{K}-\{0\}\to GL(\mathbf{K})$$

be the maps such that for any $v, w\in\mathbf{K}-\{0\}$,

$$\Phi(v)=\frac{|v|}{|\phi(v/|v|)e|}\phi(v/|v|),\quad \Psi(w)=\frac{|w|}{|\psi(w/|w|)e|}\psi(w/|w|).$$

Then for any $v, w\in\mathbf{K}-\{0\}$,

$$\Phi(v)e=v=\Psi(e)v,\quad \Psi(w)e=w=\Phi(e)w,$$
$$\Phi(v)w/|\Phi(v)w|=\Psi(w)v/|\Psi(w)v|.$$

If we are able to show that for any $v, w \in \mathbf{K} - \{0\}$,

$$\Phi(v)w = \Psi(w)v,$$

then $\mathbf{K}$ can be made a division algebra such that for any $v, w \in \mathbf{K} - \{0\}$, $vw = \Phi(v)w = \Psi(w)v$ so that for any $v, w \in S^{n-1}$, $\xi(v, w) = vw/|vw|$.

In the following, we let v and w be two fixed elements of $\mathbf{K} - \{0\}$. If $v = re$ for some $r \in \mathbf{R}$, then

$$\Phi(v)w = r\Phi(e)w = r\Psi(w)e = \Psi(w)v.$$

If $w = re$ for some $r \in \mathbf{R}$, then $\Phi(v)w = r\Phi(v)e = r\Psi(e)v = \Psi(w)v$. Hence we may assume that $v, w \notin \mathbf{R}e$. Let γ be the real number such that

$$\Phi(v)w = \gamma\Psi(w)v.$$

We claim that $\gamma = 1$.

Assume first that e, v, w are not linearly independent. Then for some $t, t' \in \mathbf{R}$,

$$w = te + t'v, t' \neq 0$$

Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathbf{K}$ such that

$$e_1 = e, e_2 = v,$$

and let $\gamma_1, \cdots, \gamma_n \in \mathbf{R}$ be such that

$$\Psi(e_2)e_2 = \gamma_1 e_1 + \cdots + \gamma_n e_n.$$

If $\gamma_1 = \gamma_3 = \gamma_4 = \cdots = \gamma_n = 0$, then

$$\Psi(e_2)(e_2 - \gamma_2 e_1) = \Psi(e_2)e_2 - \gamma_2\Psi(e_2)e_1 = \gamma_2 e_2 - \gamma_2 e_2 = 0,$$

which is impossible. Therefore $\gamma_k \neq 0$ for some $k \neq 2$. We may assume that

$$\gamma_1 \neq 0.$$

In fact, if $\gamma_1 = 0$, then $\gamma_k \neq 0$ for some $k > 2$, so that by replacing e_k by $e_k + e_1$ we obtain a new γ_1 different from 0.

For any $r, s \in \mathbf{R}$, there are smooth real valued functions

$$\alpha = \alpha(r), \quad \beta = \beta(s)$$

such that

$$\Phi(e_1 + re_2)e_2 = \alpha\Psi(e_2)(e_1 + re_2), \quad \Psi(e_1 + se_2)e_2 = \beta\Phi(e_2)(e_1 + se_2).$$

Clearly $\alpha(0) = 1$ and $\alpha(r) \neq 0$ for all $r \in \mathbf{R}$. Hence $\alpha(r) > 0$ for all $r \in \mathbf{R}$. Similarly $\beta(0) = 1$ and $\beta(s) > 0$ for all $s \in \mathbf{R}$. Now

$$\begin{aligned}\Phi(e_1 + re_2)(e_1 + se_2) &= \Phi(e_1 + re_2)e_1 + s\Phi(e_1 + re_2)e_2 \\ &= e_1 + re_2 + s\alpha\Psi(e_2)(e_1 + re_2) \\ &= e_1 + re_2 + s\alpha e_2 + rs\alpha(\gamma_1 e_1 + \cdots + \gamma_n e_n).\end{aligned}$$

$$\Psi(e_1 + se_2)(e_1 + re_2) = e_1 + r\beta e_2 + se_2 + rs\beta\gamma(\gamma_1 e_1 + \cdots + \gamma_n e_n).$$

Since the coefficients of $e_1, \cdots, e_n$ in $\Phi(e_1 + re_2)(e_1 + se_2)$ and those in $\Psi(e_1 + se_2)(e_1 + re_2)$ are proportional, we infer that

$$\frac{1 + rs\alpha\gamma_1}{1 + rs\beta\gamma\gamma_1} = \frac{r + s\alpha + rs\alpha\gamma_2}{r\beta + s + rs\beta\gamma\gamma_2} = \frac{\alpha\gamma_k}{\beta\gamma\gamma_k}, \quad k > 2.$$

Therefore

$$\alpha = \frac{1 + ((\beta - 1)/s + \beta\gamma\gamma_2)r - (\beta\gamma\gamma_1)r^2}{1 + (\gamma_2 + s\gamma_1 - s\beta\gamma\gamma_1)r - (\beta\gamma_1)r^2}.$$

By Sublemma 2, either $\alpha = 1$ or

$$((\beta - 1)/s + \beta\gamma\gamma_2)' = (\beta\gamma\gamma_1)' = (\gamma_2 + s\gamma_1 - s\beta\gamma\gamma_1)' = (\beta\gamma_1)' = 0.$$

In the first case,

$$(\beta - 1)/s + \beta\gamma\gamma_2 = \gamma_2 + s\gamma_1 - s\beta\gamma\gamma_1, \quad \beta\gamma\gamma_1 = \beta\gamma_1.$$

Since $\beta\gamma_1 \neq 0$, it follows from the second equality that $\gamma = 1$. Then the first equality becomes

$$(\beta - 1)(1/s + \gamma_2 + s\gamma_1) = 0.$$

Therefore $\beta - 1 = 0$ or $\beta = 1$. In the second case, $\beta'(s) = 0$ so that $\beta(s) = \beta(0) = 1$. Then

$$0 = (\gamma_2 + s\gamma_1 - s\beta\gamma\gamma_1)' = \gamma_1(1 - \gamma),$$

so that $\gamma = 1$. Therefore $\alpha = 1$. Hence we always have

$$\alpha = 1, \beta = 1, \gamma = 1.$$

Since $v = e_2$ and $w = te_1 + t'e_2$, it follows that when $t = 0$,

$$\Phi(v)w = t'\Phi(e_2)e_2 = t'\Psi(e_2)e_2 = \Psi(w)v,$$

and when $t \neq 0$,

$$\Phi(v)w = t\Phi(e_2)(e_1 + (t'/t)e_2) = t\Psi(e_1 + (t'/t)e_2)e_2 = \Psi(w)v.$$

Assume now that e, v, w are linearly independent. Then there is a basis $\{e_1, \cdots, e_n\}$ of $\mathbf{K}$ such that

$$e_1 = e, e_2 = v, e_3 = w.$$

Let $\gamma_1, \cdots, \gamma_n \in \mathbf{R}$ be such that

$$\Psi(e_3)e_2 = \gamma_1 e_1 + \cdots + \gamma_n e_n.$$

Then

$$\Phi(e_1 + re_2)(e_1 + se_3) = e_1 + re_2 + s\alpha e_3 + rs\alpha(\gamma_1 e_1 + \cdots + \gamma_n e_n),$$
$$\Psi(e_1 + se_3)(e_1 + re_2) = e_1 + r\beta e_2 + se_3 + rs\beta\gamma(\gamma_1 e_1 + \cdots + \gamma_n e_n).$$

Therefore

$$\frac{1 + rs\alpha\gamma_1}{1 + rs\beta\gamma\gamma_1} = \frac{r + rs\alpha\gamma_2}{r\beta + rs\beta\gamma\gamma_2} = \frac{s\alpha + rs\alpha\gamma_3}{s + rs\beta\gamma\gamma_3} = \frac{\alpha\gamma_k}{\beta\gamma\gamma_k}, \quad k > 3.$$

We may assume that one of $\gamma_1, \gamma_2, \gamma_3$ is not 0. In fact, if $\gamma_1 = \gamma_2 = \gamma_3 = 0$, then for some $k > 0$, $\gamma_k \neq 0$. By replacing e_k by $e_k + e_1$ we obtain a new γ_1 different from 0.

If either γ_1 or γ_3 is not 0, from

$$\frac{1 + rs\alpha\gamma_1}{1 + rs\beta\gamma\gamma_1} = \frac{s\alpha + rs\alpha\gamma_3}{s + rs\beta\gamma\gamma_3}$$

we obtain that

$$\alpha = \frac{1 + (\beta\gamma\gamma_3)r}{1 + (\gamma_3 + s\beta\gamma\gamma_1 - s\gamma_1)r}.$$

By Sublemma 1, $\alpha = 1$, so that

$$\beta\gamma\gamma_3 = \gamma_3 + s\beta\gamma\gamma_1 - s\gamma_1,$$

or

$$(\beta\gamma - 1)(\gamma_3 - s\gamma_1) = 0.$$

Since either γ_1 and γ_3 is not 0, it follows that $\beta\gamma - 1 = 0$. Hence

$$\gamma = 1, \beta = 1, \alpha = 1.$$

If γ_2 is not 0, we have

$$\beta = \frac{1 + (\alpha\gamma_2)s}{1 + (\gamma\gamma_2 + r\alpha\gamma_1 - r\gamma\gamma_1)s}.$$

Similarly,

$$\gamma = 1, \alpha = 1, \beta = 1.$$

Since $v = e_2$ and $w = e_3$, it follows that

$$\Phi(v)w = \Phi(e_2)e_3 = \Psi(e_3)e_2 = \Psi(w)v.$$

Hence the proof of Lemma 6 is completed.

Theorem 3. *Let* **K** *be a division algebra of dimension* n, $n = 4$ *or* 8, *and let*

$$\pi: S^{2n-1} \to \Sigma^n$$

be the smooth great $(n - 1)$*-sphere fibration determined by* **K** *as seen in Theorem* 2. *Then* **K** *can be recovered from the fibration by the construction given above.*

Proof. Let

$$F_0 = \{0\} \times S^{n-1}, \quad F_1 = S^{n-1} \times \{0\},$$

and let Σ^n be assigned a Riemannian metric such that the smooth imbedding ρ of $D^n = \{x \in \mathbf{K} \mid |x| \leq 1\}$ into Σ^n given by

$$\rho(v) = \pi\left(vw/\sqrt{|vw|^2+1}\,,\, w/\sqrt{|vw|^2+1}\right)$$

is isometric. Then we have natural linear diffeomorphisms of F_0, F_1 and S onto S^{n-1}, of which the first two are projections and the last is $(d\rho)^{-1}$.

Let us use these diffeomorphisms to identify F_0, F_1 and S with S^{n-1}. Then $\xi\colon S \times F_0 \to F_1$ becomes

$$\xi\colon S^{n-1} \times S^{n-1} \to S^{n-1}$$

defined by

$$\xi(v, w) = vw/|vw|\,.$$

Hence the regular multiplication constructed in Lemma 6 is the same as that in **K**.

Theorem 4. *Let*

$$\pi\colon S^{2n-1} \to \Sigma^n$$

be a given smooth great $(n-1)$-sphere fibration, $n = 4$ or 8, and let **K** *be the n-dimensional division algebra constructed from the fibration as seen earlier. Then the fibration is smoothly isomorphic to that determined by* **K**.

Proof. With respect to a preassigned Riemannian metric on Σ^n, there is a $\delta > 0$ such that if D_δ is the closed n-disk in the tangent space of Σ^n at x_0 of center 0 and radius δ, then the exponential map exp imbeds D_δ smoothly into $\Sigma^n - \{x_1\}$. Let D be the compact smooth n-manifold obtained from the disjoint union of $\Sigma^n - \{x_0\}$ and $S \times [0, \delta)$ by identifying every $(v, t) \in S \times (0, \delta)$ with $\exp tv \in \Sigma^n \times \{x_0\}$. It is clear that D is a smooth closed n-disk, and its boundary is $S \times \{0\} = S$.

Let

$$\lambda\colon D_\delta \times F_0 \to S^{2n-1}$$

be the smooth imbedding such that $\lambda(v, w) \in F_1 w$ and $\pi\lambda(v, w) = \exp v$ for any $(v, w) \in D_\delta \times F_0$. Then we have a compact smooth $(2n-1)$-manifold W obtained from the disjoint union of $S^{2n-1} - F_0$ and $S \times [0, \delta) \times F_0$ by identifying every $(v, t, w) \in S \times (0, \delta) \times F_0$ with $\lambda(tv, w) \in S^{2n-1} - F_0$. It is clear that the boundary of W is $S \times \{0\} \times F_0 = S \times F_0$, and that $\pi\colon S^{2n-1} - F_0 \to \Sigma^n - \{x_0\}$ can be naturally extended to a smooth fibration

$$\pi\colon W \to D.$$

From the construction of $\xi\colon S \times F_0 \to F_1$, it can be shown that ξ can be naturally extended to a smooth fibration

$$\xi\colon W \to F_1$$

such that for any $x \in W - (S \times F_0)$, $\xi(x)$ is the point of intersection of F_0x with F_1. Hence

$$h_1 = (\xi \times \pi)^{-1}: F_1 \times D \to W$$

is a diffeomorphism.

The inclusion map of $S^{2n-1} - F_0$ into S^{2n-1} can be extended to a smooth map

$$h_2: W \to S^{2n-1}$$

such that $h_2(v, w) = w$ for any $(v, w) \in S \times F_0 = \partial W$. Therefore we have a smooth map

$$h = h_2 h_1: F_1 \times D \to S^{2n-1}$$

such that the fibration π: $S^{2n-1} \to \Sigma^n$ is induced by the projection fibration $F_1 \times D \to D$. Moreover, whenever $(u, v), (u', v') \in F_1 \times D$, $h(u, v) = h(u', v')$ iff either $(u, v) = (u', v')$ or $u = u'$, $v, v' \in S = \partial D$ and for some $w, w' \in F_0$. $u = \xi(v, w) = \xi(v', w') = u'$.

In the construction of the division algebra $\mathbf{K}$, we identify F_0, F_1 and S with $S^{n-1} \subset \mathbf{K}$. Then we have a smooth map

$$h': F_1 \times D \to S^{2n-1}$$

given as follows. Let us regard $D - \{x_1\}$ as $\{v \in \mathbf{K} \mid 0 < |v| \leqslant 1\}$. Then for any $(u, v) \in F_1 \times (D - \{x_1\})$ there is a unique $w(u, v) \in \mathbf{K}$ with $vw(u, v) = u$. The map h' is given by

$$h'(u, v) = \begin{cases} (u, 0) & \text{if } v = x_1, \\ \dfrac{u}{\sqrt{1 + |w(u, v)|^2}}, \dfrac{w(u, v)}{\sqrt{1 + |w(u, v)|^2}} & \text{otherwise.} \end{cases}$$

Now it is not hard to see that the identity map of $F_1 \times D$ induces a smooth isomorphism between the fibration π: $S^{2n-1} \to \Sigma^n$ and that determined by the division algebra $\mathbf{K}$.

Corollary 4. *Up to a smooth isomorphism, every smooth great* $(n - 1)$*-sphere fibration of* S^{2n-1} *is determined by an n-dimensional division algebra.*

Remark. It is possible to have many n-dimensional division algebras, not isomorphic to one another but determining isomorphic smooth great $(n - 1)$-sphere fibrations of S^{2n-1}. In fact, whenever $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive real numbers satisfying

$$\alpha + \beta + \gamma - \alpha\beta\gamma = \alpha' + \beta' + \gamma' - \alpha'\beta'\gamma',$$

there is a 4-dimensional division algebra which, as the quaternionic algebra, has $\{e, i, j, k\}$ as a basis, but in which the multiplication is given by:

	e	i	j	k
e	e	i	j	k
i	i	$-e$	γk	$-\beta' k$
j	j	$-\gamma' k$	$-e$	αi
k	k	βj	$-\alpha' i$	$-e$

Also for any $\theta \in [0, \pi/2]$, there is a 4-dimensional division algebra which has $\{e, i, j, k\}$ as a basis and in which the multiplication is given by:

	e	i	j	k
e	e	i	j	k
i	i	$-e$	k	$-j$
j	j	$-k$	$-e\cos\theta + i\sin\theta$	$i\cos\theta + e\sin\theta$
k	k	j	$-i\cos\theta - e\sin\theta$	$-e\cos\theta + i\sin\theta$

For details, see Bruck [4]. Since all these division algebras are homotopic to the quaternionic algebra, the smooth great 3-sphere fibrations of S^7 determined by them are smoothly isomorphic to the Hopf fibration.

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