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HYPER-q-CONVEX DOMAINS IN KÄHLER MANIFOLDS

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Dedicated to Professor Buchin Su on his 80th birthday

1. In a recent paper [2], C. Badji used harmonic theory to prove that if D is a C^∞ strongly pseudoconvex domain in a Kähler manifold of nonnegative bisectional curvature, then the Dolbeault group $H^{1,1}(D)$ vanishes. Now it is implicit in the arguments of [4] that under the same hypothesis, $-\log \rho$ ($\rho =$ distance to the boundary ∂D) is strictly plurisubharmonic so that D is in fact a Stein manifold and hence $H^{p,q}(D) = 0$ for all $q \geqslant 1$. With the availability of [7], even more general statements can be made under weaker hypotheses. It therefore seems worth while to explicitly write down some of these theorems for future references. Deferring the technical definitions to the next section, we may state the main theorem as follows.

Theorem. *Let M be an n-dimensional Kähler manifold (not necessarily complete), and D be a relatively compact domain in M. Then D is strongly q-pseudoconvex $(1 \leqslant q \leqslant n)$, if for some neighborhood W of ∂D in M the bisectional curvature of M is q-positive in $W \cap D$, and D is weakly hyper-q-convex. Furthermore, D is q-complete if any of the following holds:*

(i) *The bisectional curvature of M is q-nonnegative in all of D and is q-positive in $W \cap D$ for some neighborhood W of ∂D in M, and D is weakly hyper-q-convex.*

(ii) *The bisectional curvature of M is q-nonnegative in all of D, and D is C^∞ hyper-q-convex.*

(iii) *The bisectional curvature of M is q-nonnegative in all of D, D is weakly hyper-q-convex, and there exists a continuous function f: $D \to \mathbf{R}$ which is in $\Psi(q)$ on D.*

Corollary. *Let M be a Kähler manifold whose bisectional curvature is q-nonnegative. Then M has no exceptional analytic sets of dimension not less than q.*

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In this theorem there is no need to make any assumption on the bisectional curvature of M outside D. However, in part (iii) above the existence of such an f would follow, if M is complete and its bisectional curvature is everywhere q-nonnegative and is q-positive outside a compact set [7, Theorem 3]. Note that many variations on this theorem are possible. Indeed, the crucial argument of such a theorem in its various guises always involves showing that a certain sum of second variations of arc length is positive (cf. [4, pp. 177–178] or [7, Lemma 6]). Since the second variation formula is itself the sum of a boundary term involving the Levi form of ∂D and an integral involving the bisectional curvature of D, suitable assumptions on ∂D and on the bisectional curvature of D balancing one against the other will always insure the positivity of the resulting sum. Thus while [4, Theorem 1] assumes the pseudoconvexity of D and the *positivity* of the bisectional curvature in D, it is obvious that it could have assumed instead the *strict* pseudoconvexity of ∂D and the nonnegativity of the bisectional curvature. This observation explains the remark in the opening paragraph as well as the proofs to be given below. We leave the precise enumeration of the other possibilities to the reader.

The above theorem has been known to the author for some time (when $q = 1$ it was of course also known to R. E. Greene), but the thought of actually writing down the details came only after the receipt of the Badji preprint [2]. The author wishes to thank Dr. Badji for the courtesy of sending this preprint. In the meantime, the author received the preprint [5] which contains among other things the case $q = 1$ in (ii) and (iii) of the above theorem as well as the case $q = 1$ in the Corollary. It should be made clear however that the Corollary was added only after reading [5], and that its proof uses the argument in [5].

2. We first recall some definitions, A C^∞ function τ on an n-dimension complex manifold M is *strongly q-pseudoconvex* if its Levi form $L\tau$ has at least $n - q + 1$ positive eigenvalues at each point of M; M is *strongly q-pseudoconvex* (resp. *q-complete*) if it possesses a C^∞ exhaustion function which is strongly q-pseudoconvex outside a compact set (resp. strongly q-pseudoconvex everywhere), [1]. A domain D in M is said to have C^∞ *boundary* ∂D if ∂D is an imbedded C^∞ real hypersurface of M; in this case, D is said to be a C^∞ *domain*. Let M be Kähler. Then D or ∂D is said to be C^∞ *hyper-q-convex* if D is C^∞, and each $x \in \partial D$ admits a local defining function ϕ of ∂D at x (i.e., locally $\partial D = \phi^{-1}(0)$, $\phi|_D < 0$ and $|d\phi(x)| = 1$) such that the eigenvalues $\lambda_1, \cdots, \lambda_{n-1}$ of the restriction of the Levi form $L\phi$ to the maximal complex subspace of the tangent space $T_x(\partial D)$ of ∂D at x satisfy $\Sigma_{i=1}^q \lambda_{j_i} > 0$ for all $1 \leqslant j_i \leqslant n-1$, [3]. It is natural to consider the case of a C^∞ domain D which merely satisfies $\Sigma_{i=1}^q \lambda_{j_i} \geqslant 0$ for all $1 \leqslant j_i \leqslant n-1$; for a reason which will become obvious, we adopt the ad hoc terminology that such a D or ∂D is C^∞ *q-convex*. Next, recall the class of continuous functions $\Psi(q)$ from [7]. A set of vectors $\{Z_1, \cdots, Z_q\}$ in T_xM is *ε-orthonormal* if $|G(Z_i, Z_j) - \delta_{ij}| < \varepsilon$ for $i, j = 1, \cdots, q$, where G is the Hermitian inner product on T_xM given by the Kähler metric. Given $K \subset M$ and positive constants ε and η, define $\mathcal{L}(K, \varepsilon, \eta)$ to be the set of all C^∞ functions f defined on K such that if $\{Z_1, \cdots, Z_q\}$ is an ε-orthonormal set in T_xM ($x \in K$), then $\Sigma_{i=1}^q Lf(Z_i, Z_i) \geqslant \eta$. Now let U be an open set in M. Then a function $F \in \Psi(q; U)$ iff for each compact subset K of U, there exist positive constants ε and η and a sequence $\{f_i\} \subset \mathcal{L}(K, \varepsilon, \eta)$ such that f_i converges uniformly to F on K. If $U = M$ or there is no danger of confusion, we will simply write $\Psi(q)$ in place of $\Psi(q; U)$. It follows easily from the considerations in [7, §2] that this definition of $\Psi(q)$ coincides with that given in [7]. Note that by definition, $\Psi(q)$ consists of continuous functions, and that if $\mathcal{C}^\infty$ (or in case there is any confusion, $\mathcal{C}^\infty(U)$) denotes the C^∞ functions on U, then $\mathcal{C}^\infty \cap \Psi(q; U)$ consists of exactly those C^∞ functions f on U such that the sum of any q eigenvalues of Lf at any point of U is positive. In particular, any function in $\mathcal{C}^\infty \cap \Psi(q)$ is strongly q-pseudoconvex. Moreover, [7, Proposition 1] shows that $\mathcal{C}^\infty \cap \Psi(q)$ is dense in $\Psi(q)$ in the C^0-topology.

A domain D in a Kähler manifold M is said to be *weakly hyper-q-convex* if for every $x \in \partial D$ there exists a neighborhood V_x of x in M such that $V_x \cap D$ admits an exhaustion function which belongs to $\Psi(q; V_x \cap D)$. The following lemma gives the relationship among the various domains; its proof will be given at the end of §3.

Lemma. *Let D be a C^∞ domain in a Kähler manifold M. Then the following hold*:

(i) *If D is C^∞ hyper-q-convex, then it is weakly hyper-q-convex.*

(ii) *Suppose in addition the bisectional curvature is q-nonnegative in a neighborhood of ∂D, then D is C^∞ q-convex iff it is weakly hyper-q-convex.*

We conclude this section by defining the q-positivity of the bisectional curvature. Let M be Kähler and let $x \in M$. If X and Y are nonzero vectors in T_xM, the bisectional curvature determined by X and Y is by definition $H(X, Y) \equiv R(X, JX, Y, JY)/(|X|^2 \cdot |Y|^2)$, where R is the curvature tensor of M. The bisectional curvature of M is *q-nonnegative* (resp., *q-positive*) in an open set U if for every $x \in U$ and for every orthonormal basis $\{e_1, Je_k, \cdots, e_n, Je_n\}$ of T_xM and $0 \neq X \in T_xM$, $\Sigma_{i=1}^q H(X, e_i) \geqslant 0$ (resp., $\Sigma_{i=1}^q H(X, e_i) > 0$). See [7] for further details.

3. We now supply the proofs of the preceding theorem, corollary, and lemma. The reader is assumed to be acquainted with [4] and [7].

We first prove the first assertion of the theorem concerning the strong q-pseudoconvexity of D. Since D is weakly hyper-q-convex, each $x \in \partial D$ has a neighborhood V_x such that $V_x \cap D$ admits an exhaustion function ϕ which belongs to $\Psi(q; V_x \cap D)$. By the density $\mathcal{C}^\infty \cap \Psi(q; V_x \cap D)$ in $\Psi(q; V_x \cap D)$, we may further assume that ϕ is $\mathcal{C}^\infty$. If $\{r_i\}$ is a sequence of regular values of ϕ such that $r_i \uparrow \infty$, then $\{\phi^{-1}(r_i)\}$ is a sequence of $\mathcal{C}^\infty$ real hypersurfaces in $V_x \cap D$ which are $\mathcal{C}^\infty$ hyper-q-convex and approximate $V_x \cap \partial D$ from within. The proof of Theorem 1(A) in [4] coupled with the technique in §3 of [7] now yields the following assertion: Let ρ: $D \to [0, \infty)$ denote the distance from the boundary ∂D. Then there exist a neighborhood W of ∂D in M and a $\mathcal{C}^\infty$ increasing convex function $\chi: (-\infty, 0) \to \mathbf{R}$ such that $\chi(-\rho) \uparrow \infty$ near ∂D and $\chi(-\rho) \in \Psi(q; W \cap D)$. Again using the density of $\mathcal{C}^\infty \cap \Psi(q; W \cap D)$ in $\Psi(q; W \cap D)$ in the C^0-topology, we obtain a $\tau \in \mathcal{C}^\infty \cap \Psi(q; W \cap D)$ such that $\tau \uparrow \infty$ uniformly near ∂D. By shrinking W if necessary, we may assume τ is a $\mathcal{C}^\infty$ function defined on D. This proves that D is strongly q-pseudoconvex.

Continuing with the same notation, we shall go on to prove part (i) of the theorem. Indeed, let $\{t_i\}$ be a sequence of regular values of τ such that $t_i \uparrow \infty$. Then $\{\tau^{-1}(t_i)\}$ is a sequence of $\mathcal{C}^\infty$ real hypersurfaces in D, which are $\mathcal{C}^\infty$ hyper-q-convex and uniformly approximate ∂D from within. The arguments used for the proofs of Theorems 2 and 4 in [7] are now applicable; they prove that for some $\mathcal{C}^\infty$ increasing convex function $\chi_0: (-\infty, 0) \to \mathbf{R}$ such that $\chi_0 \uparrow \infty$ near 0, the function $\chi_0(-\rho)$, is an exhaustion function of D and is in $\Psi(q; D)$ because the bisectional curvature is now everywhere q-nonnegative in D. The density of $\mathcal{C}^\infty \cap \Psi(q; D)$ in $\Psi(q; D)$ allows us to replace $\chi_0(-\rho)$ by a $\mathcal{C}^\infty$ exhaustion function of D, which belongs to $\Psi(q; D)$. This proves (i).

The proof of (ii) is essentially identical with the proof of Theorem 2 of [7], the only necessary change being in the proof of Lemma 6 of [7]; the latter has to do with showing a certain sum of second variations of arc length is positive. That this is so is guaranteed by the $\mathcal{C}^\infty$ hyper-q-convexity of ∂D (in the presense of everywhere q-nonnegative bisectional curvature; see [7, (15)]). See the discussion after the corollary in §1.

Finally to prove (iii), we have to invoke the generalized Levi form Pf of a continuous function F, [6], [7]. Under the assumption of (iii), the by-now familiar arguments of [4] and [7] show that for some $\mathcal{C}^\infty$ increasing convex function $\chi_1: (-\infty, 0) \to \mathbf{R}$, $\chi_1(-\rho)$ is an exhaustion function of D, and the following holds for $\chi_1(-\rho)$. Let δ be a given continuous positive function on D. Then there exists a continuous positive function ε_1 on D such that whenever $x \in D$ and $\{Z_1, \cdots, Z_q\}$ is any $\varepsilon_1(x)$-orthonormal set in T_xM, $\sum_{i=1}^q P(\chi_1(-\rho))(x, Z_i) > -\delta(x)$. By hypothesis, there exists an $f \in \Psi(q; D)$. By the density of $\mathcal{C}^\infty \cap \Psi(q; D)$ in $\Psi(q; D)$, f may be assumed to be actually $\mathcal{C}^\infty$. Replacing f by e^f if necessary, we may also assume $f > 0$. Now let δ be any continuous positive function in D such that for some positive continuous function ε_2 on D, $\Sigma_{i=1}^q Lf(Z_i, Z_i) > 2\delta(x)$ for all $x \in D$ and all $\varepsilon_2(x)$-orthonormal sets $\{Z_1,\cdots,Z_q\}$ in T_xM. Consequently, $\tau \equiv \chi_1(-\rho) + f$ is an exhaustion function of D, and if we denote $\min\{\varepsilon_1, \varepsilon_2\}$ by ε, then $\Sigma_{i=1}^q P\tau(x, Z_i) > \delta(x)$ for all $x \in D$ and all $\varepsilon(x)$-orthonormal sets $\{Z_1,\cdots,Z_q\}$ in T_xM. In particular, $\tau \in \Psi(q; D)$. By the usual reasoning, the q-completeness of D follows. q.e.d.

Next, we prove the corollary using the idea of [5]. If the corollary is false, there would exist a compact subvariety S in M of dimension $s > q$ and a holomorphic map π: $M \to M'$ into a complex space M' such that $\pi(S)$ is a point $x' \in M'$ and π: $M - S \to M' - \{x'\}$ is biholomorphic. Let Σ' be the boundary of some ε-ball B' relative to some coordinate system centered at $x' \in M'$. B' is then a $\mathcal{C}^\infty$ strictly pseudoconvex domain. Let $B = \pi^{-1}(B')$ and $\Sigma = \pi^{-1}(\Sigma')$. Then B is a $\mathcal{C}^\infty$ strictly pseudoconvex domain because π is biholomorphic in a neighborhood of Σ. By part (ii) of the theorem, there is an exhaustion function τ of B, which belongs to $\Psi(q; B)$. By the density of $\mathcal{C}^\infty \cap \Psi(q; B)$ in $\Psi(q; B)$, we may assume τ is $\mathcal{C}^\infty$. Since $s \geqslant q$, it is clear that the restriction $\tau|_S$ is a $\mathcal{C}^\infty$ strictly subharmonic function on the regular points $\mathfrak{R}S$ of S and is a continuous function on S. Moreover, since S is compact, $\Delta\tau$ is bounded below by a positive constant on $\mathfrak{R}S$, where Δ denotes the Laplacian of the Kähler manifold $\mathfrak{R}S$. Since $\tau|_S$ must attain an absolute maximum on S, a standard argument shows that such an S does not exist. q.e.d.

We finally give the proof of the lemma. To prove part (1), let $x \in \partial D$ and let ϕ be a local defining function at X satisfying the hyper-q-convex condition at X. By a standard argument, there exists a $\mathcal{C}^\infty$ strictly increasing, strictly convex function $\chi(t)$ such that $\chi(\phi)$ has the property that $\Sigma_{i=1}^q L\chi(\phi)(Z_i, Z_i) > 0$ for any orthonormal basis $\{Z_1, JZ_1,\cdots,Z_n, JZ_n\}$ in T_xM. Thus for a sufficiently small neighborhood V_x of x, $\chi(\phi) \in \mathcal{C}^\infty \cap \Psi(q; V_x)$. We may assume that relative to some coordinate system $\{z_1,\cdots,z_n\}$ centered at x, V_x is given by $\{\Sigma_{i=1}^n |z_i|^2 < b\}$ for some $b > 0$. Let $\tau_1 = -1/\chi(\phi)$ with $\chi(0) = 0$, and let $\tau_2 = 1/(b - \Sigma_{i=1}^n |z_i|^2)$. Both τ_1 and τ_2 are in $\mathcal{C}^\infty \cap \Psi(q; V_x \cap D)$ so that also $\max\{\tau_1, \tau_2\} \equiv \tau$ belongs to $\Psi(q; V_x \cap D)$ (cf. [7, Lemma 2(d)]). τ is clearly an exhaustion function on $V_x \cap D$. This proves (i).

To prove (ii), first assume D is $\mathcal{C}^\infty$ q-convex. We know from the proof of the theorem that, in the presence of q-nonnegative bisectional curvature near ∂D, the following holds. There exist a neighborhood W of ∂D in M and a $\mathcal{C}^\infty$ increasing convex function $\chi: (-\infty, 0) \to \mathbf{R}$ such that $\chi(-\rho) \uparrow \infty$ near ∂D (ρ

denotes the distance from ∂D as usual) and such that if δ is a given continuous positive function on $W \cap D$, then there exists a continuous positive function ε_1 on $W \cap D$ with the property that whenever $x \in W \cap D$ and $\{Z_k, \cdots, Z_q\}$ is any $\varepsilon_1(x)$-orthonormal set in $T_x M$, $\Sigma_{i=1}^q P\chi(-\rho)(x, Z_i) > -\delta(x)$. Now fix an $x_0 \in \partial D$ and let $\{z_1, \cdots, z_n\}$ be a coordinate system centered at x_0. Choose a positive number b so small that if $V_0 \equiv \{\Sigma_{i=1}^n |z_i|^2 < b\}$, then $V_0 \cap D \subset W \cap D$. Let δ be a positive continuous function on V_0, so that for some positive continuous function ε_2 on V_0, every $\varepsilon_2(x)$-orthonormal set $\{Z_1, \cdots, Z_q\}$ in $T_x M$ ($x \in V_0$) satisfies $\Sigma_{i=1}^q Lf(Z_i, Z_i) > 2\delta(x)$, where $f \equiv 1/(b - \Sigma_{i=1}^n |z_i|^2)$. Let $\tau_1 = \chi(-\rho) + f$. Then $\tau_1 \in \Psi(q; V_0 \cap D)$ and $\tau_1 \uparrow \infty$ near ∂D. Therefore the function $\tau \equiv \max\{f, \tau_1\}$ is an exhaustion function of $V_0 \cap D$, which belongs to $\Psi(q; V_0 \cap D)$. This shows D is weakly hyper-q-convex. To prove the converse, let D be C^∞ and weakly hyper-q-convex. Fix an $x_0 \in \partial D$, and let V_0 be a neighborhood of x_0 such that on $V_0 \cap D$ there exists an exhaustion function ϕ of $V_0 \cap D$, which is in $\Psi(q; \phi)$. Let $\{r_i\}$ be a sequence of regular values of ϕ such that $r_i \uparrow \infty$, and let $D_i \equiv \{\phi < r_i\}$. Thus each $D_i \subset\subset V_0 \cap D$, and each D_i is $\mathcal{C}^\infty$ hyper-q-convex; furthermore, each ∂D_i approximates $\partial D \cap V_0$ (we ignore the portion of ∂D_i which approximates $\partial V_0 \cap D$). In D_i, let ρ_i denote the distance from ∂D_i, and let ρ be the usual function on D denoting the distance from ∂D. Let V_1 be a sufficiently small neighborhood of x_0 such that $V_1 \subset V_0$ and such that for all $y \in V_1 \cap D$, $\rho(y)$ is realized as the length of a unique geodesic from y to $\partial D \cap V_1$. Fix such a $y \in V_1 \cap D$, and consider only i so large that $y \in V_1 \cap D_i$. For each such i, there exist a $p_i \in \partial D$ and a geodesic of unit speed ζ_i joining y to p_i with length $\rho_i(y)$. Elementary considerations show that, after passing to a subsequence if necessary, $p_i \in V_1 \cap \partial D_i$ for all large i and that p_i converges to some $p \in V_1 \cap \partial D$, and ζ_i converges to a minimizing geodesic ζ joining y to p. We may as well assume at this point that V_0 is so small that the bisectional curvature is q-nonnegative in V_0, and ρ is $\mathcal{C}^\infty$ in V_0. Now at y, let C_i (resp., C) be the complex subspace of dimension $n - 1$ in $T_y M$ orthogonal to $\dot{\zeta}_i$ (resp., $\dot{\zeta}$). Then the standard second variation argument (see especially the proof of Lemma 6 in [7]) shows that for any orthonormal basis $\{e_1, Je_1, \cdots, e_{n-1}, Je_{n-1}\}$ in C_i, $-\Sigma_{j=1}^q P\rho_i(e_j e_j) > 0$. Since the C_i's converge to C and ρ_i converges uniformly in a neighborhood of y to ρ, we obtain in the limit: $-\Sigma_{j=1}^q P\rho(e_j, e_j) \geqslant 0$ for all orthonormal bases $\{e_1, Je_1, \cdots, e_{n-1}, Je_{n-1}\}$ in C. This is equivalent to $\Sigma_{j=1}^q L(-\rho)(e_j, e_j) \geqslant 0$ at y, since ρ is $\mathcal{C}^\infty$ near y. By letting y approach x_0, we obtain the following: Let $\tilde{\rho}$: $V_1 \to \mathbf{R}$ be the function $\tilde{\rho} = -\rho$ on $V_1 \cap D$ and $\tilde{\rho} =$ the distance from ∂D on $V_1 - D$. Then $\tilde{\rho}$ is a $\mathcal{C}^\infty$ local defining function of ∂D at x_0 such that $\Sigma_{j=1}^q L\tilde{\rho}(e_j, e_j) \geqslant 0$ at x_0 for any orthonormal basis $\{e_1, Je_1, \cdots, e_{n-1}, Je_{n-1}\}$ of $T_{x_0}M$, which lies in ∂D. This is clearly equivalent to the C^∞ q-convexity of ∂D at x_0.

Bibliography

[1] A. Andreotti & H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962) 193–259.

[2] C. Badji, *Sur la $\bar{\partial}$-cohomologies de bidegre* (1, 1) *de certaines variétés Kähleriennes à bord, à courbure holomorphe bisectionnelle non negative: Un théorème de nullité*, Proc. Sem. Complex Analysis, Intern. Centre Theor. Phys., Trieste, 1980.

[3] H. Grauert & O. Riemenschneider, *Kählersche Mannifaltigkeiten mit hyper-q-konvexen Rand*, Problems in Analysis, A Sympos. in Honor of Salomon Bochner, Princeton University Press, Princeton, 1970, 61–79.

[4] R. E. Greene & H. Wu, *On Kähler manifolds of positive bisectional curvature and a theorem of Hartogs*, Abh. Math. Sem. Univ. Hamburg **47** (1978) 171–185.

[5] A. Kasue, *A note on Kähler manifolds of nonnegative holomorphic bisectional curvature*, to appear in Osaka J. Math.

[6] H. Wu, *An elementary method in the study of nonnegative curvature*, Acta Math. **142** (1979) 57–78.

[7] ______, *On certain Kähler manifolds which are q-complete*, to appear.

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