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CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD. II

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1. Introduction

A submanifold N in a Kaehler manifold $\tilde{M}$ is called a CR-submanifold if (1) the maximal complex subspace $\mathfrak{D}_x$ of the tangent space $T_x\tilde{M}$ containing in T_xN, $x \in N$, defines a differentiable distribution on N, and (2) the orthogonal complementary distributiion $\mathfrak{D}^\perp$ of $\mathfrak{D}$ is a totally real distribution, i.e., $J\mathfrak{D}_x^\perp \subseteq T_x^\perp N$, $x \in N$, where J denotes the almost complex structure of $\tilde{M}$, and $T_x^\perp N$ the normal space of N in $\tilde{M}$ at x.

In the first part of this series, we have obtained several fundamental results for CR-submanifolds. In the present part, we shall continue our study on such submanifolds. In particular, we prove that (a) the holomorphic distribution $\mathfrak{D}$ of any CR-submanifold in a Kaehler manifold is minimal (Proposition 3.9); (b) every leaf of the holomorphic distribution of a mixed foliate proper CR-submanifold in a complex hyperbolic space H^m is Einstein-Kaehlerian (Proposition 4.4); and (c) every CR-submanifold with semi-flat normal connection in $\mathbf{C}P^m$ is either an anti-holomorphic submanifold in some totally geodesic $\mathbf{C}P^{h+p}$ of $\mathbf{C}P^m$ or a totally real submanifold (Theorem 5.11).

2. Preliminaries

Let $\tilde{M}^m$ be a complex m-dimensional Kaehler manifold with complex structure J, and N be a real n-dimensional ($n \geqslant 2$) Riemannian manifold isometrically immersed in $\tilde{M}^m$. We denote by $\langle\,,\rangle$ the metric tensor of $\tilde{M}^m$ as well as that induced on N. Let ∇ and $\tilde{\nabla}$ be the covariant differentiations on N and $\tilde{M}$ respectively. Then the Gauss and Weingartan formulas for N are given respectively by

$$\tilde{\nabla}_XY = \nabla_XY + \sigma(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X\xi = -A_\xi X + D_X\xi, \tag{2.2}$$

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for any vector fields X, Y tangent to N, and ξ normal to N, where σ is the second fundamental form, and D the normal connection.

For any vector X tangent to N and ξ normal to N we put

$$JX = PX + FX, \tag{2.3}$$

$$J\xi = t\xi + f\xi \tag{2.4}$$

where PX and $t\xi$ (respectively, FX and $f\xi$) are the tangential (respectively, normal) components of JX and $J\xi$ respectively.

In the following we shall denote by $\tilde{M}^m(c)$ a complex m-dimensional complex-space-form of constant holomorphic sectional curvature c. We have

$$\begin{aligned}\tilde{R}(X, Y)Z = \tfrac{c}{4}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX \\ -\langle JX, Z\rangle JY + 2\langle X, JY\rangle JZ\}.\end{aligned} \tag{2.5}$$

We denote by R and $R^{\perp}$ the curvature tensors associated with ∇ and D respectively. A submanifold N is said to be flat (respectively, to have flat normal connection) if $R \equiv 0$ (respectively, $R^{\perp} \equiv 0$). For any vector fields X, Y, Z, W in the tangent bundle TN, and ξ, η in the normal bundle $T^{\perp}N$, the equations of Gauss, Codazzi and Ricci are given respectively by

$$\begin{aligned}R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \langle\sigma(X, W), \sigma(Y, Z)\rangle \\ -\langle\sigma(X, Z), \sigma(Y, W)\rangle,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\tilde{R}(X, Y; Z, \xi) = \langle D_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \xi\rangle \\ -\langle D_Y\sigma(X, Z) - \sigma(\nabla_Y X, Z) - \sigma(X, \nabla_Y Z), \xi\rangle,\end{aligned} \tag{2.7}$$

$$\tilde{R}(X, Y; \xi, \eta) = R^{\perp}(X, Y; \xi, \eta) - \langle[A_\xi, A_\eta]X, Y\rangle, \tag{2.8}$$

where $R(X, Y; Z, W) = \langle R(X, Y)Z, W\rangle, \cdots$, etc.

Definition 2.1. A submanifold N of a Kaehler manifold $\tilde{M}$ is called a *CR-submanifold* if there is a differentiable distribution $\mathfrak{D}: x \to \mathfrak{D}_x \subseteq T_xN$ on N satisfying the following conditions:

(a) $\mathfrak{D}$ is holomorphic, i.e., $J\mathfrak{D}_x = \mathfrak{D}_x$ for each $x \in N$, and

(b) the complementary orthogonal distribution $\mathfrak{D}^{\perp}: x \to \mathfrak{D}_x^{\perp} \subseteq T_xN$ is totally real, i.e., $J\mathfrak{D}_x^{\perp} \subseteq T_x^{\perp}N$ for each $x \in N$.

If $\dim \mathfrak{D}_x^{\perp} = 0$ (respectively, $\dim \mathfrak{D}_x = 0$), N is called a *complex* (respectively, *totally real*) submanifold. A CR-submanifold is said to be *proper* if it is neither complex nor totally real.

For a CR-submanifold N we shall denote by ν the orthogonal complementary subbundle of $J\mathfrak{D}^{\perp}$ in $T^{\perp}N$. We have

$$T^{\perp}N = J\mathfrak{D}^{\perp} \oplus \nu, \quad \nu_x = T_x^{\perp}N \cap J(T_x^{\perp}N). \tag{2.9}$$

A subbundle μ of the normal bundle is said to be parallel if $D_x\xi \in \mu$ for any vector $X \in TN$ and section ξ in μ.

A CR-submanifold N in a Kaehler manifold $\tilde{M}$ is said to be anti-holomorphic if $T_x^\perp N = J\mathcal{D}_x^\perp$, $x \in N$.

3. Some basic lemmas

First we recall some basic lemmas for later use.

Lemma 3.1 [4]. *Let N be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Then we have*

$$\langle \nabla_U Z, X\rangle = \langle JA_{JZ}U, X\rangle, \tag{3.1}$$

$$A_{JZ}W = A_{JW}Z, \tag{3.2}$$

$$A_{J\xi}X = -A_\xi JX \tag{3.3}$$

for any vector fields U tangent to N, X in $\mathcal{D}$, Z, W in $\mathcal{D}^\perp$, and ξ in ν.

Lemma 3.2 [4]. *The totally real distribution $\mathcal{D}^\perp$ of any CR-submanifold in a Kaehler manifold is integrable.*

Lemma 3.3 [1], [2], [4]. *Let N be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Then the holomorphic distribution $\mathcal{D}$ is integrable if and only if*

$$\langle \sigma(X, JY), JZ\rangle = \langle \sigma(JX, Y), JZ\rangle \tag{3.4}$$

for any vectors X, Y in $\mathcal{D}$, and Z in $\mathcal{D}^\perp$.

Lemma 3.4 [2]. *Let N be a CR-submanifold in a Kaehler manifold $\tilde{M}$. Then the leaves of $\mathcal{D}^\perp$ are totally geodesic in $\tilde{M}$ if and only if*

$$\langle \sigma(\mathcal{D}, \mathcal{D}^\perp), J\mathcal{D}^\perp\rangle = \{0\}. \tag{3.5}$$

Lemma 3.5. *Let N be a CR-submanifold in a Kaehler manifold $\tilde{M}$. We have the following statements*:

(a) *If the leaves of $\mathcal{D}^\perp$ are totally geodesic in $\tilde{M}$, then*

$$\sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = \{0\}, \quad \langle \sigma(\mathcal{D}, \mathcal{D}^\perp), J\mathcal{D}^\perp\rangle = \{0\}, \tag{3.6}$$

$$\tilde{H}_B(X, Z) = 2\|\sigma(X, Z)\|^2 + 2\langle A_{JZ}JX, JA_{JZ}X\rangle \tag{3.7}$$

for any unit vectors X in $\mathcal{D}$, and Z in $\mathcal{D}^\perp$, where $\tilde{H}_B$ denotes the holomorphic bisectional curvature of $\tilde{M}$.

(b) *If* (3.6) *holds, the leaves of $\mathcal{D}^\perp$ are totally geodesic in $\tilde{M}$.*

Proof. Let N be a CR-submanifold in a Kaehler manifold $\tilde{M}$. Then $\mathcal{D}^\perp$ is integrable (Lemma 3.2). Let $N^\perp$ be a leaf of $\mathcal{D}^\perp$. We denote by $\sigma^\perp$ and σ'' the second fundamental form of $N^\perp$ in $\tilde{M}$ and N, respectively. We have

$$\sigma^\perp(Z, W) = \sigma''(Z, W) + \sigma(Z, W)$$

for any vectors Z, W in $\mathfrak{D}^\perp$. Thus, by Lemma 3.4, the leaves of $\mathfrak{D}^\perp$ are totally geodesic in $\tilde{M}$, if and only if (3.6) holds.

Assume that the leaves of $\mathfrak{D}^\perp$ are totally geodesic in $\tilde{M}$. For any vector fields X, Y in $\mathfrak{D}$ and Z, W in $\mathfrak{D}^\perp$, equation (2.7) of Codazzi and (3.5) give

$$\begin{aligned}\tilde{R}(X, Y; Z, JW) &= \langle D_X\sigma(Y, Z) - \sigma(Y, \nabla_X Z), JW\rangle \\ &\quad -\langle D_Y\sigma(X, Z) - \sigma(X, \nabla_Y Z), JW\rangle, \\ &= \langle\sigma(X, Z), J\tilde{\nabla}_Y W\rangle - \langle\sigma(Y, Z), J\tilde{\nabla}_X W\rangle \\ &\quad +\langle A_{JW}X, \nabla_Y Z\rangle - \langle A_{JW}Y, \nabla_X Z\rangle \\ &= \langle\sigma(X, Z), J\sigma(Y, W)\rangle - \langle\sigma(Y, Z), J\sigma(X, W)\rangle \\ &\quad +\langle A_{JW}X, \nabla_Y Z\rangle - \langle A_{JW}Y, \nabla_X Z\rangle.\end{aligned}$$

Thus by applying (3.5) and Lemma 4.1 we find

$$\begin{aligned}\tilde{R}(X, Y; Z, JW) &= \langle\sigma(X, Z), \sigma(JY, W)\rangle - \langle\sigma(Y, Z), \sigma(JX, W)\rangle \\ &\quad +\langle A_{JW}X, JA_{JZ}Y\rangle - \langle A_{JW}Y, JA_{JZ}X\rangle,\end{aligned}$$

from which we obtain (3.7).

Corollary 3.6. *Let N be a proper anti-holomorphic submanifold in $\mathbf{C}P^{h+p}$. If the leaves of $\mathfrak{D}^\perp$ are totally geodesic in $\mathbf{C}P^{h+p}$, then the holomorphic distribution is not integrable.*

This corollary follows from Lemmas 3.4 and 3.5.

For the holomorphic distribution $\mathfrak{D}$, we have

Lemma 3.7. *Let N be a CR-submanifold in a Kaehler manifold $\tilde{M}$. Then*

(1) *the holomorphic distribution is integrable, and its leaves are totally geodesic in N if and only if*

$$\langle\sigma(\mathfrak{D}, \mathfrak{D}), J\mathfrak{D}^\perp\rangle = \{0\}, \tag{3.8}$$

(2) *the holomorphic distribution is integrable, and its leaves are totally geodesic in $\tilde{M}$ if and only if*

$$\sigma(\mathfrak{D}, \mathfrak{D}) = \{0\}. \tag{3.9}$$

Proof. Let N be a CR-submanifold in a Kaehler manifold $\tilde{M}$. If (3.8) holds, then also (3.4). Thus the holomorphic distribution $\mathfrak{D}$ is integrable (Lemma 3.3). Moreover, from (2.1), (2.2) and (2.3) we have

$$\begin{aligned}\langle\nabla_X Z, JY\rangle &= \langle\tilde{\nabla}_X Z, JY\rangle = -\langle\tilde{\nabla}_X JZ, Y\rangle \\ &= -\langle A_{JZ}X, Y\rangle = -\langle\sigma(X, Y), JZ\rangle = 0\end{aligned}$$

for any vector fields X, Y in $\mathfrak{D}$, and Z in $\mathfrak{D}^\perp$. Thus the leaves of $\mathfrak{D}$ are totally geodesic in N. The converse of this has been proved in [4].

Statement (2) follows from statement (1) and the following identity

$$\sigma^T(X, Y) = \sigma'(X, Y) + \sigma(X, Y)$$

for any vectors X, Y in $\mathfrak{D}$, where σ' and σ^T are the second fundamental forms of any leaf N^T of $\mathfrak{D}$ in N and $\tilde{M}$ respectively.

Let $\mathcal{H}$ be a differentiable distribution on a CR-submanifold N ($\mathcal{H}$: $x \to \mathcal{H}_x \subseteq T_xN$, $x \in N$). We put

$$\mathring{\sigma}(X, Y) = (\nabla_X Y)^\perp \tag{3.10}$$

for any vector fields X, Y in $\mathcal{H}$, where $(\nabla_X Y)^\perp$ denotes the component of $\nabla_X Y$ in the orthogonal complementary distribution $\mathcal{H}^\perp$ in N. Then the Frobenius theorem gives the following

Lemma 3.8. *The distribution $\mathcal{H}$ is integrable if and only if $\mathring{\sigma}$ is a symmetric on $\mathcal{H} \times \mathcal{H}$.*

Let $X_1, \cdots, X_r$ be an orthonormal basis in $\mathcal{H}$. We put

$$\mathring{H} = \frac{1}{r} \sum_{i=1}^{r} \mathring{\sigma}(X_i, X_i).$$

Then $\mathring{H}$ is a well-defined vector field on N (up to sign). We call $\mathring{H}$ the *mean-curvature vector of the distribution* $\mathcal{H}$.

A distribution $\mathcal{H}$ on N is said to be *minimal* if the mean curvature vector $\mathring{H}$ of $\mathcal{H}$ vanishes identically, and $\mathcal{H}$ is said to be totally geodesic if $\mathring{\sigma} \equiv 0$.

Proposition 3.9. *Let N be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Then*

(a) *the holomorphic distribution $\mathfrak{D}$ is minimal, and*

(b) *the distribution $\mathfrak{D}$ is totally geodesic if and only if $\mathfrak{D}$ is integrable, and its leaves are totally geodesic in N.*

Proof. Let N be a CR-submanifold of a Kaehler manifold $\tilde{M}$. For any vector fields X in $\mathfrak{D}$, and Z in $\mathfrak{D}^\perp$, Lemma 3.1 gives

$$\langle Z, \nabla_X X \rangle = \langle A_{JZ}X, JX \rangle. \tag{3.11}$$

Thus we have

$$\langle Z, \nabla_{JX} JX \rangle = -\langle A_{JZ}X, JX \rangle. \tag{3.12}$$

Combining (3.11) and (3.12) we obtain

$$\langle \nabla_X X + \nabla_{JX} JX, Z \rangle = 0. \tag{3.13}$$

This implies statement (a). Statement (b) follows from (3.10) and Lemma 3.8.

4. Mixed foliate CR-submanifolds

Definition 4.1. A CR-submanifold is said to be *mixed totally geodesic* if $\sigma(\mathfrak{D}, \mathfrak{D}^\perp) = \{0\}$.

Definition 4.2. A CR-submanifold N in a Kaehler manifold $\tilde{M}$ is said to be *mixed foliate*, if it is mixed totally geodesic, and its holomorphic distribution is integrable.

In [2], Bejancu, Kon and Yano proved that there is no mixed foliate proper CR-submanifold in $\tilde{M}^m(c)$ with $c > 0$. In [4] the author proved that a CR-submanifold in C^m is mixed foliate if and only if N is a CR-product (for anti-holomorphic case, see [2]).

In this section, we shall study mixed foliate CR-submanifolds in a complex hyperbolic space H^m. For simplicity, we assume that H^m is a complex m-dimensional complex hyperbolic space with constant holomorphic sectional curvature -4.

Lemma 4.1. *Let N be a mixed foliate CR-submanifold in H^m. Then for any unit vectors $X \in \mathfrak{D}$ and $Z \in \mathfrak{D}^\perp$,*

$$\|A_{JZ}X\| = 1, \tag{4.1}$$

$$\|\sigma\|^2 \geqslant 2hp, \tag{4.2}$$

where $h = \dim_{\mathbf{C}} \mathfrak{D}$, and $p = \dim_{\mathbf{R}} \mathfrak{D}^\perp$. The equality sign in (4.2) *holds if and only if* (a) *the leaves of $\mathfrak{D}^\perp$ are totally geodesic in H^m, and* (b) $\operatorname{Im}\sigma = J\mathfrak{D}^\perp$.

Proof. Let N be a mixed foliate CR-submanifold in H^m. Then Lemma 9.1 of [4] gives

$$\tilde{H}_B(X, Z) = -2\|A_{JZ}X\|^2, \tag{4.3}$$

for any unit vectors X in $\mathfrak{D}$, and Z in $\mathfrak{D}^\perp$. This gives (4.1).

Inequality (4.2) follows immediately from (4.1). From (4.1) it is clear that $\|\sigma\| = 2hp$ if and only if we have

$$\operatorname{Im}\sigma = J\mathfrak{D}^\perp, \tag{4.4}$$

$$A_{J\mathfrak{D}^\perp}\mathfrak{D}^\perp = \{0\}. \tag{4.5}$$

The lemma thus follows from Lemma 3.5.

Let N be a mixed foliate CR-submanifold in H^m, and N^T a leaf of the holomorphic distribution $\mathfrak{D}$. Then N^T is a Kaehler submanifold of H^m. We denote by σ^T, $D^T, \cdots$, etc. the second fundamental form, the normal connection, $\cdots$, etc. for N^T in H^m, and by σ', $D', \cdots$, etc. the corresponding quantities for N^T in N. Then we have

$$\sigma^T(X, Y) = \sigma'(X, Y) + \sigma(X, Y) \tag{4.6}$$

for X, Y in TN^T. For any Z in $\mathfrak{D}^\perp$, this implies

$$\langle A_Z^T X, Y\rangle = \langle J\sigma^T(X, Y), JZ\rangle = \langle \sigma(JX, Y), JZ\rangle = \langle A_{JZ}JX, Y\rangle, \tag{4.7}$$

$$\langle A_{JZ}^T X, Y\rangle = \langle \sigma(X, Y), JZ\rangle = \langle A_{JZ}X, Y\rangle. \tag{4.8}$$

Because N is mixed foliate, these give

$$A_Z^T X = A_{JZ}JX, \quad A_{JZ}^T X = A_{JZ}X. \tag{4.9}$$

Moreover, for any unit vector fields X in $\mathfrak{D}$, and Z in $\mathfrak{D}^\perp$, we have that

$$J\nabla_X Z = \tilde{\nabla}_X JZ = -A_{JZ}X + D_X JZ, \tag{4.10}$$

so that

$$D_X JZ = F\nabla_X Z. \tag{4.11}$$

From

$$\tilde{\nabla}_X JZ = -A^T_{JZ}X + D^T_X JZ$$

we also get

$$D^T_X JZ = D_X JZ. \tag{4.12}$$

Let η be any normal vector field in ν (for the definition of ν, see (2.9)) and X, Y any tangent vector fields in $\mathfrak{D}$, (2.5), (4.11) and (4.12) imply

$$\tilde{R}(X, Y; JZ, \eta) = 0, \tag{4.13}$$

$$R^\perp_T(X, Y; JZ, \eta) = 0. \tag{4.14}$$

Combining these with equation (2.7) of Codazzi we obtain

$$\left[A^T_{JZ}, A^T_\eta\right] = 0 \quad \text{for } \eta \in \nu, z \in \mathfrak{D}^\perp. \tag{4.15}$$

Because N^T is a Kaehler submanifold, $A^T_{J\eta} = JA^T_\eta = -A^T_\eta J$. Thus by using (4.15) we have

$$0 = A^T_{JZ}A^T_\eta - A^T_\eta A^T_{JZ} = J\left(A^T_\eta A^T_{JZ} + A^T_{JZ}A^T_\eta\right).$$

Since J is nonsingular, this gives

$$A^T_\eta A^T_{JZ} + A^T_{JZ}A^T_\eta = 0. \tag{4.16}$$

Combining (4.15) and (4.16) we have

$$A^T_{JZ}A^T_\eta = 0. \tag{4.17}$$

Because N is mixed foliate, $A_{JZ}\mathfrak{D} \subseteq \mathfrak{D}$ for any Z in $\mathfrak{D}^\perp$. Thus using Lemma 4.1 and (4.9) we get

$$\|A^T_Z X\| = \|A^T_{JZ}X\| = 1 \tag{4.18}$$

for any unit vectors X in TN^T, and Z in $\mathfrak{D}^\perp$. By linearity, this implies

$$\langle A^T_{JZ}X, A^T_{JZ}Y\rangle = 0 \tag{4.19}$$

for orthogonal vectors X, Y in TN^T. From (4.18) and (4.19) we find

$$A^T_Z, A^T_{JZ} \in \in O(2h). \tag{4.20}$$

In particular, A^T_{JZ} is nonsingular. Thus we have, in consequence of (4.17), $A^T_\eta = 0$ for any vector η in ν. Since N is mixed foliate, (2.1) and (2.2) give

$$-A^T_Z X + D^T_X Z = \tilde{\nabla}_X Z = \nabla_X Z = -A'_Z X + D'_X Z.$$

from which we find $D_X^T Z = D'_X Z$. This shows that the normal subbundle $\mathfrak{D}^\perp|_{N^T}$ is a parallel subbundle of the normal bundle of N^T in H^m. Therefore we have

$$R_T^\perp(X, Y; Z, JW) = 0 \tag{4.21}$$

for any vector fields X, Y in TN^T, and Z, W in $\mathfrak{D}^\perp|_{N^T}$. Let $Z_1, \cdots, Z_p$ be an orthonormal basis of $\mathfrak{D}_x^\perp$, $x \in N^T$. (2.5), (4.21) and the Ricci equation for N^T in H^m give

$$\left[A_{Z_\alpha}^T, A_{JZ_\beta}^T\right] = 0 \text{ for } \alpha \neq \beta,\ \alpha, \beta = 1, \cdots, p. \tag{4.22}$$

Since $A_{JZ}^T J = -J A_{JZ}^T$, (4.20) shows that A_{JZ}^T has two eigenvalues 1 and -1 with the same multiplicity h. We put

$$V_1 = \left\{X \in T_x N \mid A_{JZ_1}^T X = X\right\}.$$

Thus, for any $X \in V_1$, (4.22) gives

$$A_{JZ_1}^T A_{Z_\alpha}^T X = A_{Z_\alpha}^T A_{JZ_1}^T X = A_{Z_\alpha}^T X, \quad \alpha = 2, \cdots, p.$$

Moreover, for any unit vector X in V_1, (4.22) implies that $A_{Z_\alpha}^T X$, $\alpha = 2, \cdots, p$ lie in V_1, which are orthonormal by (4.18). Consequently, we obtain $p \leqslant h + 1$.

From (4.22), we may also get

$$A_{Z_\alpha}^T A_{Z_\beta}^T + A_{Z_\beta}^T A_{Z_\alpha}^T = 0 \quad \text{for } \alpha \neq \beta.$$

From the equation of Gauss and (2.5), the sectional curvature K of N satisfies

$$K(X, Z) = -1 + \langle \sigma(X, X), \sigma(Z, Z) \rangle \tag{4.23}$$

for any unit vectors X in $\mathfrak{D}$, and Z in $\mathfrak{D}^\perp$. Since N is mixed foliate, we also have

$$K(JX, Z) = -1 - \langle \sigma(X, X), \sigma(Z, Z) \rangle.$$

Combining this with (4.23) gives

$$K(X, Z) + K(JX, Z) = -2.$$

By summarizing the above facts we can state the next lemma.

Lemma 4.2. *Let N be a mixed foliate CR-submanifold in H^m. Then*

(a) $D_X^T JZ = D_X JZ = F \nabla_X Z$,

(b) $D_X^T Z = D'_X Z = -t D_X JZ$,

(c) $\operatorname{Im} \sigma^T = \mathfrak{D}^\perp \oplus J\mathfrak{D}^\perp$,

(d) $A_Z^T, A_{JZ}^T \in O(2h)$,

(e) $p \leqslant h + 1$,

(f) $A_Z^T A_W^T + A_W^T A_Z^T = 0$,

(g) $K(X, Z) + K(JX, Z) = -2$, *for any unit vector field X in TN^T, and orthonormal vector fields Z, W in $\mathfrak{D}^\perp$.*

From Lemma 4.2 and Proposition 3 of [2] we have the following.

Lemma 4.3. *Let N be a mixed foliate proper CR-submanifold of* $\tilde{M}^m(c)$, $c \neq 0$. *Then* $c < 0$ *and* $p > 1$.

Proof. Let N be a mixed foliate proper CR-submanifold of $\tilde{M}^m(c)$, $c \neq 0$. Then Proposition 3 of [2] implies $c < 0$. If $p = 1$, then, for any unit vector field Z in $\mathfrak{D}^\perp$, statement (b) of Lemma 4.2 implies $D_X^T Z = D'_X Z = 0$. Hence, Z is a parallel normal vector field of the complex submanifold N^T in $\tilde{M}^m(c)$, $c < 0$. This contradicts a theorem of Chen and Ogiue [5].

Proposition 4.4. *Let N be a mixed foliate proper CR-submanifold of* H^m. *Then*

(a) *each leaf* N^T *of* $\mathfrak{D}$ *lies in a complex* $(h + p)$*-dimensional totally geodesic complex submanifold* H^{h+p} *of* H^m,

(b) *each leaf* N^T *is an Einstein-Kaehler submanifold of* H^{h+p} *with Ricci tensor given by*

$$S^T(X, Y) = -2(h + p + 1)\langle X, Y\rangle, \tag{4.24}$$

(c) $h + 1 \geqslant p \geqslant 2$; $h \geqslant 2$, *and*

(d) *the leaves of* $\mathfrak{D}^\perp$ *are totally geodesic in N.*

Proof. Lemma 4.2 implies that the first normal space $\operatorname{Im}\sigma^T$ is nothing but $\mathfrak{D}^\perp \oplus J\mathfrak{D}^\perp$. Since $\mathfrak{D}^\perp \oplus J\mathfrak{D}^\perp$ is a parallel normal subbundle of the normal bundle of N^T in H^m, by a theorem of Chen and Ogiue [5], N^T lies in a complex $(h + p)$-dimensional totally geodesic submanifold H^{h+p} of H^m. Thus (a) is proved.

Since N^T is a Kaehler submanifold of H^m, equation (2.8) of Gauss gives

$$S^T(X, Y) = -2(h + 1)\langle X, Y\rangle - \sum \langle A^T_{\xi_\alpha} X, A^T_{\xi_\alpha} Y\rangle,$$

where ξ_α's form an orthonormal basis of $T^\perp N^T$. Thus by Lemmas 4.1 and 4.2 we obtain

$$S^T(X, X) = -2(h + p + 1)\langle X, X\rangle,$$

which implies (4.24).

If $h = \dim_{\mathbf{C}} \mathfrak{D} = 1$, then from statement (b) it follows that N^T is of constant curvature $-2(p + 2)$. Since N^T is a Kaehler submanifold of H^m, a theorem of Calabi [4] gives that $p = 0$. This is a contradiction. The remaining part of this proposition follows from Lemmas 3.4 and 4.3.

Theorem 4.5. *Let N be a mixed foliate CR-submanifold of* H^m. *If* $\dim_{\mathbf{R}} N \leqslant 5$, *then N is either a complex submanifold or a totally real submanifold.*

This theorem follows immediately from statement (c) of Proposition 4.4.

Remark 4.1. The author believes that Theorem 4.5 holds for any mixed foliate CR-submanifold of H^m. However, he is unable to prove it at this moment.

5. Semi-flat normal connection

First we recall the following definition [6].

Definition 5.1. A CR-submanifold N in a complex-space-form $\tilde{M}^m(c)$ is said to have *semi-flat normal connection* if its normal curvature tensor $R^\perp$ satisfies

$$R^\perp(X, Y; \xi, \eta) = \tfrac{c}{2}\langle X, PY\rangle\langle J\xi, \eta\rangle \tag{5.1}$$

for any vectors X, Y in TN, and ξ, η in $T^\perp N$.

The main purpose of this section is to classify CR-submanifolds with semi-flat normal connection.

Lemma 5.1. *A CR-submanifold N in a complex-space-form $\tilde{M}^m(c)$ has semi-flat normal connection if and only if*

$$\langle [A_\xi, A_\eta]X, Y\rangle = \tfrac{c}{4}\{\langle JX, \xi\rangle\langle JY, \eta\rangle - \langle JX, \eta\rangle\langle JY, \xi\rangle\} \tag{5.2}$$

for any vectors X, Y in TN, and ξ, η in $T^\perp N$.

This lemma follows from Definition 5.1 and the equation of Ricci.

From Lemma 5.1 we obtain the following.

Lemma 5.2. *Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$. Then*

$$\langle [A_\xi, A_\eta]X, U\rangle = 0, \tag{5.3}$$

$$\langle [A_\xi, A_\eta]Z, W\rangle = \tfrac{c}{4}\{\langle JZ, \zeta\rangle\langle JW, \eta\rangle - \langle JZ, \eta\rangle\langle JW, \xi\rangle\} \tag{5.4}$$

for any vectors U in TN, X in $\mathfrak{D}$, Z, W in $\mathfrak{D}^\perp$, and ξ, η in $T^\perp N$.

Moreover, we also have

Lemma 5.3. *Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$. Then*

$$A_\nu \mathfrak{D} = \{0\}, \tag{5.5}$$

$$\langle A_{J\mathfrak{D}^\perp}\mathfrak{D}, A_\nu \mathfrak{D}^\perp\rangle = \{0\}, \tag{5.6}$$

where $\nu_x = T_x^\perp N \cap J(T_x^\perp N)$, $x \in N$.

Proof. From Lemmas 3.1 and 5.2 we have

$$0 = \langle [A_\xi, A_{J\xi}]X, JX\rangle = -\|A_\xi JX\|^2 - \|A_\xi X\|^2$$

for any vectors X in $\mathfrak{D}$, and ξ in ν. Thus we get (5.5). Formula (5.6) follows from (5.4) and (5.5).

Lemma 5.4 is an immediate consequence of Lemma 5.3.

Lemma 5.4. *Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$. If there is a ξ in ν such that $A_\xi \mathfrak{D}^\perp = \mathfrak{D}^\perp$, then N is mixed totally geodesic.*

From Lemma 5.2 we have

Lemma 5.5. *Let N be a CR-submanifold with semi-flat normal connection. Then*

$$\|A_{JZ}W\|^2 = \tfrac{c}{4} + \langle A_{JZ}Z, A_{JW}W\rangle \tag{5.7}$$

for orthonormal vectors Z, W in $\mathfrak{D}^\perp$.

Proof. For orthonormal vectors Z and W in $\mathfrak{D}^\perp$, Lemma 5.2 gives

$$\tfrac{c}{4} = \langle [A_{JZ}, A_{JW}]Z, W\rangle = \langle A_{JZ}W, A_{JW}W\rangle - \langle A_{JZ}Z, A_{JW}W\rangle.$$

Thus by using Lemma 3.1 we obtain (5.7).

Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$. By Lemma 5.3 we obtain $A_\nu\mathfrak{D} = \{0\}$. Define an endomorphism

$$\tilde{A}_\xi\colon \mathfrak{D}_x^\perp \to \mathfrak{D}_x^\perp$$

by

$$\tilde{A}_\xi Z = A_\xi Z \tag{5.8}$$

for any vectors ξ in ν_x, and Z in $\mathfrak{D}_x^\perp$. Then $\tilde{A}_\xi$ is self-adjoint.

Let $\lambda_1,\cdots,\lambda_r$ be the distinct eigenvalues of $\tilde{A}_\xi$, and $V_1,\cdots,V_r$ the corresponding eigenspaces. Then we have

$$\mathfrak{D}_x^\perp = V_1 \oplus \cdots \oplus V_r, \langle V_i, V_j\rangle = 0 \quad \text{for } i \neq j. \tag{5.9}$$

Lemma 5.6. *Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$, $c \neq 0$. Then, for any ξ in ν, $\tilde{A}_\xi$ is proportional to the identity endomorphism.*

Proof. Under the hypothesis, Lemma 5.2 implies

$$\langle A_\xi W, A_{JZ}Y\rangle = \langle A_\xi Y, A_{JZ}W\rangle \tag{5.10}$$

for any vectors ξ in ν, and Y, Z, W in $\mathfrak{D}^\perp$. If $\tilde{A}_\xi$ is not proportional to the identity endomorphism, $r \geqslant 2$. Let $Z = W = Z_i \in V_i$, $Y = Z_j \in V_j$, for $i \neq j$. Then (5.10) and Lemma 3.1 imply

$$\langle A_{JZ_j}Z_i, Z_i\rangle = 0. \tag{5.11}$$

By linearity we have

$$\langle A_{JZ_j}V_i, V_i\rangle = \{0\} \quad \text{for } i \neq j. \tag{5.12}$$

Putting $W = Z_i \in V_i$, $Y = Z_j \in V_j$ and $Z = Z_k \in V_k$ for $i \neq j$, (5.10) gives

$$\lambda_i\langle A_{JZ_k}Z_j, Z_i\rangle = \lambda_j\langle A_{JZ_k}Z_j, Z_i\rangle \quad \text{for } i \neq j,$$

which implies

$$A_{JZ_k}V_j \subseteq \mathfrak{D} \oplus V_j. \tag{5.13}$$

On the other hand, by Lemma 5.3 we obtain

$$0 = \langle A_{JZ_k}X, A_\xi Z_j\rangle = \lambda_j\langle A_{JZ_k}Z_j, X\rangle$$

for any vectors X in $\mathfrak{D}$, $Z_j \in V_j$, and $Z_k \in V_k$. This shows that $A_{JZ_k}V_j \subseteq \mathfrak{D}^\perp$ if $\lambda_j \neq 0$. Combining this with (5.13) yields

$$A_{JZ_k}V_j \subseteq V_j \quad \text{whenever } \lambda_j \neq 0. \tag{5.14}$$

From (5.12) and (5.14) we get

$$A_{JZ_k}V_i = 0 \text{ if } j \neq i \text{ and } \lambda_i \neq 0. \tag{5.15}$$

Since A_ξ has at least two distinct eigenvalues, we may assume that $\lambda_1 \neq 0$. From (5.7) of Lemma 5.5 and (5.15) we have

$$0 = \|A_{JZ_2}Z_1\|^2 = \tfrac{c}{4} + \langle A_{JZ_2}Z_2, A_{JZ_1}Z_1\rangle. \tag{5.16}$$

On the other hand, Lemma 3.1 and (5.12) imply

$$0 = \langle A_{JZ_j}Z_i, Z_i\rangle = \langle A_{JZ_i}Z_i, Z_j\rangle \quad \text{for } i \neq j.$$

Combining this with (5.14) we find

$$A_{JZ_i}Z_i \in \mathfrak{D} \oplus V_i. \tag{5.17}$$

Since $A_{JZ_1}Z_1 \in V_1$ by (5.14), equations (5.16) and (5.17) give $c = 0$. This is a contradiction.

From Lemmas 5.3 and 5.6 we immediately have the following.

Lemma 5.7. *Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$, $c \neq 0$. Then for any $x \in N$, there is a unit normal vector $\bar{\eta} \in \nu_x$ such that*

$$A_{\bar{\eta}}X = 0, \quad A_{\bar{\eta}}Z = \lambda Z, \tag{5.18}$$

$$A_\xi = 0 \tag{5.19}$$

for any vectors X in $\mathfrak{D}_x$, Z in $\mathfrak{D}_x$, and ξ in ν_x with $\langle \xi, \bar{\eta}\rangle = 0$.

Lemma 5.8. *Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$, $c \neq 0$. If λ is nowhere zero on N, then N is mixed foliate.*

Proof. Under the hypothesis, Lemmas 5.4 and 5.7 imply that N is mixed totally geodesic.

For any vector fields X, Y in $\mathfrak{D}$, Z in $\mathfrak{D}^\perp$, and ξ in $T^\perp N$, equation (2.9) of Codazzi gives

$$\tilde{R}(X, Y; Z, \xi) = \langle \sigma([X, Y], Z), \xi\rangle + \langle \sigma(X, \nabla_Y Z) - \sigma(Y, \nabla_X Z), \xi\rangle.$$

In particular, if we choose ξ to be the vector $\bar{\eta}$ of Lemma 5.7, we can reduce this to

$$0 = \langle \sigma([X, Y], Z), \bar{\eta}\rangle = \lambda\langle [X, Y], Z\rangle$$

by applying (2.6) and Lemma 5.7. Since $\lambda \neq 0$, this shows that the holomorphic distribution is integrable.

Lemma 5.9. *Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$, $c \neq 0$.*

(1) *Then λ is constant, and for any vectors X, Y in TN and Z in $\mathfrak{D}^\perp$ we have*

$$F(R(X,Y)Z) = \sigma(X, P\nabla_Y Z) - \sigma(Y, P\nabla_X Z) + \lambda^2\{\langle Y, Z\rangle FX - \langle X, Z\rangle FY\}, \tag{5.20}$$

$$D_X JZ = F\nabla_X Z + \lambda\langle X, Z\rangle J\bar{\eta}, \tag{5.21}$$

(2) *If $\lambda = 0$, then N lies in a totally geodesic complex submanifold $\tilde{M}^{h+p}(c)$ of $\tilde{M}^m(c)$ as an anti-holomorphic submanifold.*

(3) *If $\lambda \neq 0$, then N is a mixed foliate CR-submanifold with $fD\bar{\eta} = 0$.*

Proof. For any vectors X, Y in TN, and Z in $\mathfrak{D}^\perp$, we have

$$-A_{JZ}X + D_X JZ = J\nabla_X Z + \sigma(X, Z).$$

Thus

$$D_X JZ = F\nabla_X Z + f\sigma(X, Z). \tag{5.22}$$

By applying Lemma 5.7, this gives

$$D_X JZ = F\nabla_X Z + \lambda\langle X, Z\rangle J\bar{\eta}. \tag{5.23}$$

Therefore by considering the normal component of $\tilde{\nabla}_X D_Y JZ$ we obtain

$$D_X D_Y JZ = D_X(F\nabla_Y Z) + X(\lambda\langle Y, Z\rangle)J\bar{\eta} - \lambda^2\langle Y, Z\rangle FX + \lambda\langle Y, Z\rangle fD_X\bar{\eta}. \tag{5.24}$$

On the other hand, by equation (3.9) of [4] and Lemma 8.1 of [4] we have

$$D_X(F\nabla_Y Z) = f\sigma(X, \nabla_Y Z) - \sigma(X, P\nabla_Y Z) + F(\nabla_X\nabla_Y Z).$$

Substituting this into (5.24) we obtain

$$D_X D_Y JZ = f\sigma(X, \nabla_Y Z) - \sigma(X, P\nabla_Y Z) + F(\nabla_X\nabla_Y Z) + X(\lambda\langle Y, Z\rangle)J\bar{\eta} - \lambda\langle Y, Z\rangle\{FX - fD_X\bar{\eta}\}.$$

Thus the normal curvature tensor $R^\perp$ is given by

$$\begin{aligned} R^\perp(X,Y)JZ = {} & F(R(X,Y)Z) + f\sigma(X, \nabla_Y Z) - f\sigma(Y, \nabla_X Z) \\ & - \sigma(X, P\nabla_Y Z) + \sigma(Y, P\nabla_X Z) - \lambda\langle [X,Y], Z\rangle J\bar{\eta} \\ & + \{X(\lambda\langle Y, Z\rangle) - Y(\lambda\langle X, Z\rangle)\}J\bar{\eta} \\ & - \lambda^2\{\langle Y, Z\rangle FX - \langle X, Z\rangle FY\} \\ & + \lambda\{\langle Y, Z\rangle fD_X\bar{\eta} - \langle X, Z\rangle fD_Y\bar{\eta}\}. \end{aligned}$$

By applying Lemma 5.7 this gives

$$\begin{aligned} R^{\perp}(X,Y)JZ = {} & F(R(X,Y)Z) - \lambda\{\langle PX, P\nabla_Y Z\rangle - \langle PY, P_X Z\rangle\}J\bar{\eta} \\ & -\sigma(X, P\nabla_Y Z) + \sigma(Y, P\nabla_Y Z) \\ & + \{(X\lambda)\langle Y,Z\rangle - (Y\lambda)\langle X,Z\rangle\}J\bar{\eta} \\ & - \lambda^2\{\langle Y,Z\rangle FX - \langle X,Z\rangle FY\} \\ & + \lambda\{\langle Y,Z\rangle fD_X\bar{\eta} - \langle X,Z\rangle fD_Y\bar{\eta}\}. \end{aligned} \tag{5.25}$$

It follows from Lemma 5.7 that both $\sigma(X, P\nabla_Y Z)$ and $\sigma(Y, P\nabla_X Z)$ lie in $J\mathfrak{D}^{\perp}$. Since $R^{\perp}(X,Y)JZ = 0$ by (5.1), equation (5.25) gives (5.20) and

$$(X\lambda)\langle Y,Z\rangle - (Y\lambda)\langle X,Z\rangle = \lambda\{\langle PX, P\nabla_Y Z\rangle - \langle PY, P\nabla_X Z\rangle\}, \tag{5.26}$$

$$\lambda\{\langle Y,Z\rangle fD_X\bar{\eta} - \langle X,Z\rangle fD_Y\bar{\eta}\} = 0. \tag{5.27}$$

If N is a complex submanifold of $\tilde{M}^m(c)$, then $\mathfrak{D} = TN$ and $\nu = T^{\perp}N$. Lemma 5.5 shows that N is a totally geodesic complex submanifold of $\tilde{M}^m(c)$.

Now we assume that N is *not* a complex submanifold. We have $\dim_{\mathbf{R}} \mathfrak{D}^{\perp} = p > 0$.

Case (a). If $\mu \equiv 0$, then we have $\operatorname{Im}\sigma \subseteq J\mathfrak{D}^{\perp}$. Moreover, for any vector fields X in TN, Z in $\mathfrak{D}^{\perp}$, and ξ in ν, Lemma 5.7 gives

$$0 = \langle \sigma(X,Z), \xi\rangle = \langle \tilde{\nabla}_X JZ, J\xi\rangle = \langle D_X JZ, J\xi\rangle.$$

Since this is true for all ξ in ν, $J\mathfrak{D}^{\perp}$ is a parallel normal subbundle. Because the first normal spaces of N lie in $J\mathfrak{D}^{\perp}$, the fundamental theorem of submanifolds shows that N lies in a totally geodesic complex submanifold $\tilde{M}^{h+p}(c)$ of $\tilde{M}^m(c)$. In this case, N is an anti-holomorphic submanifold of $\tilde{M}^{h+p}(c)$.

Case (b). If $\lambda \not\equiv 0$, then $N' = \{x \in N \mid \lambda(x) \neq 0\}$ is an open nonempty subset of N. Lemma 5.8 tells us that each component of N' is a mixed foliate CR-submanifold $\tilde{M}^m(c)$, $c \neq 0$.

If $c > 0$, then N is totally real (Lemma 4.3). Thus (5.26) gives

$$(X\lambda)\langle Y,Z\rangle - (Y\lambda)\langle X,Z\rangle = 0, \tag{5.28}$$

for any vectors X, Y in TN, and Z in $\mathfrak{D}^{\perp}$. Because $\dim_{\mathbf{R}} \mathfrak{D}^{\perp}_x = \dim_{\mathbf{R}} N \geqslant 2$ and λ^2 is differentiable, (5.28) implies that λ is a nonzero constant on N. Thus by (5.27) we get $fD\bar{\eta} = 0$.

If $c < 0$, then Proposition 4.4 and Lemma 5.8 show that $\dim_{\mathbf{R}} \mathfrak{D}^{\perp}_x = p > 1$. Thus for any unit vector Z in $\mathfrak{D}^{\perp}$ there exists a unit vector W in $\mathfrak{D}^{\perp}$ so that $\langle Z, W\rangle = 0$. From (5.26) we find

$$Z(\lambda^2) = 0 \quad \text{for } Z \in \mathfrak{D}^{\perp}. \tag{5.29}$$

Let X and Z be any unit vector fields in $\mathfrak{D}$ and $\mathfrak{D}^\perp$ respectively. Then (5.26) gives

$$X(\lambda)^2 = 2\lambda^2\langle X, \nabla_Z Z\rangle. \tag{5.30}$$

On the other hand, for such X and Z we have

$$\langle X, \nabla_Z Z\rangle = \langle JX, \tilde{\nabla}_Z JZ\rangle = -\langle A_{JZ}Z, JX\rangle = -\langle \sigma(Z, JX), JZ\rangle.$$

Thus by using (5.30), Lemma 5.8, and the continuity of λ^2 we get $X(\lambda^2) \equiv 0$ for any vector X in $\mathfrak{D}$. Combining this with (5.29), we conclude that λ is a nonzero constant on N. The equation $fD\bar{\eta} = 0$ then follows from (5.27).

Lemma 5.10. *Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$, $c \neq 0$. If $\lambda \neq 0$, then the sectional curvature of N satisfies*

$$K(Z \wedge W) = \lambda^2 \tag{5.31}$$

for any orthonormal vectors Z, W in $\mathfrak{D}^\perp$.

Proof. Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$, $c \neq 0$. If $\lambda \neq 0$, then N is mixed foliate (Lemma 5.8). For any vector U in TN, $PU \in \mathfrak{D}$. Thus for any orthonormal vectors Z, W in $\mathfrak{D}^\perp$, (5.20) of Lemma 5.9 gives

$$F(R(Z, W)Z) = -\lambda^2 FW.$$

From this we obtain (5.31).

Now we give the following classification theorem.

Theorem 5.11. *Let N be a CR-submanifold in a complex-space-form $\tilde{M}^m(c)$, $c \neq 0$. Then N has semi-flat normal connection in $\tilde{M}^m(c)$ if and only if N is one of the following:*

(1) *a totally geodesic complex submanifold $\tilde{M}^h(c)$,*

(2) *a flat totally real submanifold of a totally geodesic complex submanifold $\tilde{M}^p(c)$ of $\tilde{M}^m(c)$,*

(3) *a proper anti-holomorphic submanifold with flat normal connection in a totally geodesic complex submanifold $\tilde{M}^{h+p}(c)$ of $\tilde{M}^m(c)$,*

(4) *a space of positive constant sectional curvature immersed in a totally geodesic complex submanifold $\tilde{M}^{p+1}(c)$ of $\tilde{M}^m(c)$ with flat normal connection as a totally real submanifold.*

Proof. Let N be a CR-submanifold with semi-flat normal connection in $\tilde{M}^m(c)$, $c \neq 0$. If N is a complex submanifold of $\tilde{M}^m(c)$, N is a totally geodesic complex submanifold of $\tilde{M}^m(c)$ (Lemma 5.5). Thus N is itself a complex-space-form $\tilde{M}^h(c)$.

Assume that N is *not* a complex submanifold of $\tilde{M}^m(c)$. Then $p > 0$, and there exists a unit normal vector field $\bar{\eta}$ satisfies (5.18) and (5.19) for some constant λ (Lemmas 5.7 and 5.8).

If $\lambda = 0$ and N is totally real, (5.20) shows that N is flat.

If $\lambda = 0$ and N is neither complex nor totally real, then N lies in a totally geodesic complex submanifold $\tilde{M}^{h+p}(c)$ as an anti-holomorphic submanifold (Lemma 5.9). In this case, (5.1) implies that N has flat normal connection.

If $\lambda \neq 0$, Lemma 5.9 gives

$$D_X\bar{\eta} \in J\mathfrak{D}^{\perp} \tag{5.32}$$

for any vector X in TN. On the other hand, Lemma 5.7 also gives

$$D_X J\bar{\eta} = \tilde{\nabla}_X J\bar{\eta} = -JA_{\bar{\eta}}X + JD_X\bar{\eta}. \tag{5.33}$$

From Lemma 5.7 and (5.32) we see that $A_{\bar{\eta}}X \in \mathfrak{D}^{\perp}$, $JD_X\bar{\eta} \in TN$. Thus (5.33) gives

$$D\bar{\eta} \equiv 0. \tag{5.34}$$

Now, since N is mixed foliate (Lemma 5.8), the holomorphic distribution is integrable. Let N^T be a leaf of $\mathfrak{D}$. Denote by A^T and D^T the second fundamental tensor and normal connection of N^T in $\tilde{M}^m(c)$ as before. Then we have

$$-A^T_{\bar{\eta}}X + D^T_X\bar{\eta} = \tilde{\nabla}_X\bar{\eta} = -A_{\bar{\eta}}X + D_X\bar{\eta} = 0 \text{ for } X \in TN^T$$

by virtue of (5.34) and Lemma 5.7. This shows that $\bar{\eta}|_{N^T}$ is parallel in the normal bundle of N^T in $\tilde{M}^m(c)$. This contradicts a theorem of [5] unless N is totally real in $\tilde{M}^m(c)$. If N is totally real, N is of positive constant sectional curvature λ^2 (Lemma 5.10), and N has flat normal connection (Definition 5.1).

From (5.33) and (5.34) we find

$$D_X J\bar{\eta} = -JA_{\bar{\eta}}X \in J\mathfrak{D}^{\perp} \tag{5.35}$$

for any vector X in TN. Therefore by (5.21) of Lemma 5.9, (5.34) and (5.35), we see that $\mu = J\mathfrak{D}^{\perp} \oplus \mathrm{Span}\{\bar{\eta}, J\bar{\eta}\}$ is a parallel normal subbundle, and $\mu \supseteq \mathrm{Im}\,\sigma$. From these we conclude that N lies in a totally geodesic complex submanifold $M^{p+1}(c)$ of $\tilde{M}^m(c)$ as a totally real submanifold with flat normal connection.

The converse of this is trivial.

Remark 5.1. From Lemma 5.9 it follows that the assumption of compactness in Theorem 2 of [7] can be omitted.

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