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TIGHT TOPOLOGICAL IMMERSIONS OF SURFACES IN EUCLIDEAN SPACE

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Introduction

Tight submanifolds of Euclidean space R^n were originally looked at by Chern and Lashof [1] as submanifolds minimizing the total absolute Lipschitz-Killing curvature. However, unlike other problems of minimizing integrals in mathematics, this one leads not to differential equations, but to a geometry based on certain notions of convexity. This in turn led to the elimination of differentiability hypotheses and the emergence of a kind of differential geometry without differentiability assumptions.

Lucio Rodríguez [10, p. 236] has presented three main results on the classification of surfaces with boundary, the first two of which involve the two-piece property and the third tightness. It is the principal aim of the present work to show that the first two results are true with the differentiability hypothesis removed. That the third result is not true without the differentiability hypothesis we show by a counterexample. We also show that a theorem of Kuiper [7, p. 275] on tight embeddings of spheres can be generalized to immersions, thus giving a complete generalization of a theorem of Chern and Lashof [1, p. 307] with all differentiability assumptions removed.

We next give some definitions. Let M be a compact manifold without boundary, and f: $M \to R^n$ a *topological immersion*, that is, a continuous map such that for every point $p \in M$ there is a neighborhood U_p in M such that f restricted to U_p is one-to-one. The following definitions of *tightness* have been given.

(1) f is smooth and has minimal total absolute Lipschitz-Killing curvature among all smooth immersions of M in all Euclidean spaces R^n.

(2) f is smooth, and for almost every unit vector $v \in R^n$ the function $v \cdot f$ on M has the least number of critical points among all Morse functions on M.

(3) For every unit vector $v \in R^n$ and every real number c, the inclusion $M(v, c) = \{x \in M\colon\ v \cdot f(x) < c\} \subset M$ induces an injection $H_*(M(v, c)) \to H_*(M)$ on the homology R-modules (in §4 we show that this condition

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depends on the coefficient ring so that the remarks below hold only for suitable R).

(4) $M(v, c)$ is connected for all v and c (the two-piece property, TPP).

Conditions (1) and (2) are equivalent [5, p. 149]. For surfaces and smooth f all four conditions are equivalent, the latter two having the advantage of being formulated without differentiability assumptions (consult [6, p. 219] for the equivalence of (2) and (4) in this context). Conditions (3) and (4) are equivalent for surfaces without boundary, as we show in §4, while not for higher dimensional M, as shown by an example of Kuiper [6, p. 221] which shows also that condition (4) does not imply (1) or (2) for smooth f. Each of conditions (1), (2), and (3) implies (4); condition (3) (for smooth f) implies (1) and (2).

Conditions (3) and (4) may be applied to manifolds with boundary, though they are no longer equivalent even for surfaces, as we show below (Theorem 6 of §4). With Rodríguez we adopt the following.

Definition. A topological immersion of a compact manifold with or without boundary into R^n is *tight* if condition (3) holds.

We now list some of the main results obtained in this work. Although we state the first two theorems as they would appear in Rodríguez without differentiability assumptions, the results we present in §3 provide a more complete and general description of these matters. Specifically, the improvement is two-fold: we treat in detail the case wherein $f(M)$ is planar, and show that some of the assertions (including the second theorem) are true for more general subsets of S^2. In what follows, the convex hull of a subset A of R^n, denoted by κA, is the intersection of all convex sets which contain A.

Theorem. *If M is a manifold with boundary topologically equivalent to S^2 with p disjoint discs removed, then for any topological immersion f: $M \to R^3$ with the two-piece property we have*

(a) *$f|_{\partial M}$ consists of planar convex curves,*

(b) *f is an embedding into $\partial\kappa f(M)$ or $f(M)$ is planar.*

Theorem. *If f: $M \to R^n$ is a topological immersion with the two-piece property where M is as above, then $f(M)$ is not substantial for $n \geqslant 4$.*

The main theorem of §2 represents a generalization of results obtained by Kuiper [7, p. 275] and by Chern and Lashof [1, p. 307].

Theorem. *If f: $S^k \to R^n$ is a topological immersion such that $f^{-1}(H)$ is $(k \dot{-} 1)$-connected for each open half-space H, then f is an embedding, and $f(S^k)$ is the boundary of a $(k + 1)$-dimensional convex set.*

The principal result of §4 is the following characterization of tight topological immersions of compact connected surfaces with boundary.

Theorem. *If M is a compact connected surface with nonvoid boundary, and f is a topological immersion, then f is tight if and only if f has the two-piece property and $f(M) \subset \kappa f(\partial M)$.*

Using this theorem, we generalize the results of Kuiper [7] on tight bands, and construct an example which shows that the smoothness assumption in the third theorem of Rodríguez [10, p. 236] cannot be dropped.

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1. Preliminary definitions and results

We begin with some basic terminology and notation. If m and n are integers such that $m \leqslant n$, then $\langle m, n \rangle = \{i \in \mathbf{Z}\colon m \leqslant i \leqslant n\}$ where $\mathbf{Z}$ is the set of integers. If Y is a subset of a topological space, then the complement of Y, the interior of Y, and the boundary of Y will be denoted by Y^c, Y^0, and ∂Y, respectively; the closure of Y will be denoted by $\overline{Y}$ or $\operatorname{Cl} Y$. A subset of a finite dimensional vector space V is said to be a *k-plane* if it is a translate of a k-dimensional subspace of V. A k-plane is said to be a *hyperplane* if $k = \dim V - 1$. If H is a component of $V \setminus h$ where h is a hyperplane, we say that H is an open half-space of V. The phrase "*h bounds H*" will always refer to a hyperplane h and an open half-space H related as above. Finally, R^n is the n-dimensional number space, and $S^{n-1} = \{x \in R^n\colon |x| = 1\}$ is the $(n-1)$-dimensional sphere.

Definition 1. If $A \subset R^n$, then a hyperplane h in R^n is said to be a *support hyperplane* of A, or is said to support A, provided that $h \cap A \neq \varnothing$ and $A \subset \overline{H}$ where h bounds H. More generally, a k-plane Q which is contained in a $(k+1)$-plane P is said to support a set B in P if $B \cap Q \neq \varnothing$ and B is contained in the closure of one of the components of $P \setminus Q$.

Proposition 2. *If A is a subset of R^n which is contained in a k-plane F, and if h is a support hyperplane of A such that $F \setminus h \neq \varnothing$, then $F \cap h$ is a $(k-1)$-plane which supports A in F.*

Proof. We prove the assertion under the assumption that $0 \in A \cap h$; the general result can then be obtained through the use of translations. There is a linear functional $\chi\colon R^n \to R$ such that $h = \ker \chi$ and $A \subset \chi^{-1}((-\infty, 0])$. Note that

(1) $\chi|_F \neq 0$ (since $F \setminus h \neq \varnothing$),
(2) $F \cap h = \ker(\chi|_F)$,
(3) $A \subset (\chi|_F)^{-1}((-\infty, 0])$,
(4) $A \cap \ker(\chi|_F) \neq \varnothing$ (since $0 \in A \cap h$).

Now (1) and (2) imply that $F \cap h$ is a $(k-1)$-plane, and (3) and (4) imply that $F \cap h$ supports A in F.

Proposition 3. *Let A be a subset of R^n which is contained in a k-plane F, and suppose that Q is a $(k-1)$-plane contained in F which supports A in F. Then there is a support hyperplane h of A such that $h \cap F = Q$.*

Proof. If it happens that $0 \in Q \cap A$, then $h = Q + F^{\perp}$ meets the requirements where $F^{\perp} = \{x \in R^n: x \cdot y = 0 \text{ for all } y \in F\}$. In the general case we apply a suitable translation and refer to the above special case.

Definition 4. A sequence $h_1, \cdots, h_p$ of hyperplanes is said to be a *support sequence* of a subset A of R^n if

(1) h_1 supports A,

(2) h_i supports $h_{i-1} \cap \cdots \cap h_1 \cap A$ for each $i \in \langle 2, p \rangle$,

(3) $h_i \cap \cdots \cap h_1 \neq h_{i-1} \cap \cdots \cap h_1$ for each $i \in \langle 2, p \rangle$.

Defintion 5. Consider a set T and a function $f\colon T \to R^n$. T is defined to be the only *top°-set of f*. A subset S of T is said to be a *top¹-set of f* if $S = f^{-1}(h)$ where h is a support hyperplane of $f(T)$. Assuming that we have defined for all $i \in \langle 1, k \rangle$ what is meant by a *top^i-set* of a function with values in R^n, we define a top^{k+1}-set of $f\colon T \to R^n$ to be a top¹-set of $f|_S$ where S is a top^k-set of f. A top^k-set is said to be *proper* if it is not a top^{k-1}-set of f; a *top*-set of f* is a top^k-set of f for some $k \geqslant 0$. Finally, if $T \subset R^n$ and $f\colon T \to R^n$ is the inclusion map, then top*-sets of f are also called top*-sets of T.

Proposition 6. *If S is a proper top^k-set of a map $f\colon T \to R^n$ where $k > 0$, then there is a support sequence $h_1, \cdots, h_k$ of $f(T)$ and a sequence of open half-spaces $H_1, \cdots, H_k$ such that*

(1) *h_i bounds H_i for each $i \in \langle 1, k \rangle$,*

(2) *$f(T) \subset \overline{H}_1$,*

(3) *$h_{i-1} \cap \cdots \cap h_1 \cap f(T) \subset \overline{H}_i$ for each $i \in \langle 2, k \rangle$,*

(4) *$S = f^{-1}(h_1 \cap \cdots \cap h_k)$,*

(5) *$f^{-1}(H_1) \cap \cdots \cap f^{-1}(H_k) \neq \varnothing$.*

Proof. If S is a proper top¹-set of f, it is true that $S = f^{-1}(h_1)$ and $f(T) \subset \overline{H}_1$ where h_1 is a support hyperplane of $f(T)$ which bounds H_1. Since S is proper, $f^{-1}(H_1) \neq \varnothing$. Therefore the assertion is true for $k = 1$. Now suppose the result is valid for some $k \geqslant 1$, and let S be a proper top^{k+1}-set of f.

According to the definitions, there is a top^k-set S' of f such that S is a top¹-set of $f|_{S'}$, and there is a support hyperplane h of $f(S')$ such that $S = (f|_{S'})^{-1}(h)$. It is clear that S' is a proper top^k-set of f so that by the induction hypothesis we can find relative to S' sequences $h_1, \cdots, h_k$ and $H_1, \cdots, H_k$ as in the statement of the proposition. Since S is proper and

$S = (f|_{S'})^{-1}(h) = S' \cap f^{-1}(h) = f^{-1}(h_1 \cap \cdots \cap h_k \cap h)$, it follows that $h \neq h_k$. Therefore $h \cap h_k$ is an $(n-2)$-plane which supports $f(S')$ in h_k (Proposition 2).

Applying a rigid motion if necessary, we assume that $h_k = \{z^n = 0\}$, $H_k = \{z^n > 0\}$, $h_k \cap h = \{z^{n-1} = z^n = 0\}$, and $f(S') \subset \{z^{n-1} \leqslant 0 \text{ and } z^n = 0\}$; furthermore, we define P: $R^n \to R^2$ by $P(z^1,\cdots,z^n) = (z^{n-1}, z^n)$. Picking $x \in \bigcap_{i=1}^{k} f^{-1}(H_i)$ and bearing in mind the above assumption, we see that there is a line l and an open half-plane L in R^2 such that l passes through the origin, l has a positive slope, l bounds L, $P(f(x)) \in L$, and $P(f(S')) \subset \bar{L}$. Straightforward arguments reveal that $h_{k+1} = P^{-1}(l)$ is a hyperplane, $H_{k+1} = P(L)$ is an open half-space, and h_{k+1} bounds H_{k+1}.

To finish the proof we show that the sequences $h_1,\cdots,h_{k+1}$ and $H_1,\cdots,H_{k+1}$ meet all the requirements. First, we observe that (3) holds since the inclusion $P(f(S')) \subset \bar{L}$ implies that $h_k \cap \cdots \cap h_1 \cap f(T) = f(S') \subset P^{-1}(\bar{L}) = \bar{H}_{k+1}$. Second, as a consequence of the equality $h \cap h_k = h_{k+1} \cap h_k$, we see that (4) holds: $S = f^{-1}(h \cap h_k \cap \cdots \cap h_1) = f^{-1}(h_{k+1} \cap h_k \cap \cdots \cap h_1)$. Since $S \neq \varnothing$ and $S \neq S'$, (4) implies that $h_{k+1} \cap \cdots \cap h_1 \cap f(T) \neq \varnothing$ and $h_{k+1} \cap \cdots \cap h_1 \neq h_k \cap \cdots \cap h_1$ which, along with (3), imply that $h_1,\cdots,h_{k+1}$ is a support sequence of $f(T)$. Finally, by the above paragraph h_{k+1} bounds H_{k+1} and $x \in \bigcap_{i=1}^{k+1} f^{-1}(H_i)$. Thus (1) and (5) hold. q.e.d.

The following statement follows in a straightforward way from Proposition 6(4). If S is a top*-set of a function f: $T \to R^n$, then $S = f^{-1}(f(S))$. This fact will be used tacitly several times in what follows.

Proposition 7. *The following assertions hold for any function f: $T \to R^n$:*

(a) *If S is a top^k-set of f, then $f(S)$ is a top^k-set of $f(T)$.*

(b) *If B is a top^k-set of $f(T)$, then $f^{-1}(B)$ is a top^k-set of f.*

Proof. As both results are clear for top°-sets, we can disregard that case. In what follows i: $f(T) \to R^n$ is the inclusion map.

If S is a top^1-set of f, then $S = f^{-1}(h)$ where h is a support hyperplane of $f(T)$. Since $f(S) = h \cap f(T) = i^{-1}(h)$ and since h supports $f(T)$, $f(S)$ is a top^1-set of $f(T)$. Now suppose the assertion is true for some $k \geqslant 1$, and let S be a top^{k+1}-set of f. By definition $S = (f|_{S'})^{-1}(h)$ where S' is a top^k-set of f, and h is a support hyperplane of $f(S')$. Clearly, $f(S) = h \cap f(S') = (i|_{B'})^{-1}(h)$ where $B' = f(S')$. Since B' is a top^k-set of $f(T)$ and h supports $f(S') = i(B')$, $f(S)$ is a top^{k+1}-set of $f(T)$. This proves (a).

If B is a top^1-set of $f(T)$, then $B = i^{-1}(h) = h \cap f(T)$ where h is a support hyperplane of $f(T)$. Since $f^{-1}(B) = f^{-1}(h)$, $f^{-1}(B)$ is a top^1-set of f. Now assume the result is true for some $k \geqslant 1$, and let B be a top^{k+1}-set of $f(T)$. By definition $B = (i|_{B'})^{-1}(h)$ where B' is a top^k-set of $f(T)$, and h is a support hyperplane of B'. Observe that $B = h \cap B'$, and $f^{-1}(B) = (f|_{S'})^{-1}(h)$ where

$S' = f^{-1}(B')$. Since S' is a top^k-set of f and h supports $f(S') = B'$, $f^{-1}(B)$ is a top^{k+1}-set of f. This proves (b). q.e.d.

It should be noted that S is not necessarily a top*-set of f if $f(S)$ is a top*-set of $f(T)$. To see this we need only consider a constant map on a two-point space.

We now consider two standard notions concerning convexity in R^n. If $A \subset R^n$, the *convex hull of* A, denoted by κA, is the intersection of all convex subsets of R^n which contain A. An element x of R^n is said to be a *convex combination* of elements of A if there exists a positive integer k; $\alpha_1, \cdots, \alpha_k \in (0, 1]$; and $a_1, \cdots, a_k \in A$ such that $x = \sum_{i=1}^k \alpha_i a_i$ and $\sum_{i=1}^k \alpha_i = 1$. We will need the following three facts.

Fact 1. *Let $A \subset R^n$. Then κA is the set of convex combinations of members of A* [9, *p*. 12].

Fact 2. *Let A be a compact subset of R^n. Then κA is compact* [9, *p*. 158].

Fact 3. *Let A be a closed convex subset of R^n, and let $x \in \partial A$. Then there is a support hyperplane of A which contains x* [9, *p*. 100].

Definition 8. If $A \subset R^n$ and $j \in \langle 0, n \rangle$, then A is said to be an *E^j-set* if it is contained in a j-plane while it is contained in no $(j-1)$-plane. An E^n-set is said to be *substantial*. If f: $T \to R^n$ is any function and $S \subset T$, then S is said to be an E^j-set (substantial set) of f if $f(S)$ is an E^j-set (substantial).

Theorem 9. *If A is a compact subset of R^n which is contained in a k-plane F, then $\partial_F \kappa A = \bigcup \kappa B$ where B ranges over all E^j-top^1-sets of A where $0 \leqslant j < k$.*

Proof. As the assertion is clear if A is not substantial in F, we can disregard that case.

Let $x \in \kappa B$ where B is an E^j-top^1-set of A with $0 \leqslant j < k$. Choose a support hyperplane h of A such that $B = h \cap A$, and observe that $F \setminus h \neq \varnothing$—for otherwise $B = A$ so that B is not an E^j-set. By Proposition 2 the $(k-1)$-plane $Q = F \cap h$ supports A, and hence κA, in F. Since $B = Q \cap A$, we see that $x \in \kappa B = \kappa(Q \cap A) \subset Q \cap \kappa A \subset \partial_F \kappa A$. As x was arbitrary, $\bigcup \kappa B \subset \partial_F \kappa A$.

Next, let y be an arbitrary element of $\partial_F \kappa A$. By Facts 2 and 3 there is a $(k-1)$-plane Q contained in F which supports κA in F and contains y. It follows from Proposition 3 that there is a support hyperplane h of κA such that $Q = h \cap F$.

We claim that $h \cap \kappa A = \kappa(h \cap A)$. Applying a rigid motion if necessary, we can assume that $h = \{z^n = 0\}$ and $\kappa A \subset \{z^n \geqslant 0\}$. Since $h \cap \kappa A$ is a convex set containing $h \cap A$, $\kappa(h \cap A) \subset h \cap \kappa A$. In order to prove the reverse inclusion, we choose $x \in h \cap \kappa A$ arbitrarily. By Fact 1 there exist $a_1, \cdots, a_j \in A$ and $\alpha_1, \cdots, \alpha_j \in (0, 1]$ such that $\sum_{i=1}^j \alpha_i = 1$ and $\sum_{i=1}^j \alpha_i a_i = x$. But $x \in h$ so that $x^n = \sum_{i=1}^j \alpha_i a_i^n = 0$. As $\alpha_1, \cdots, \alpha_j > 0$ and $a_1^n, \cdots, a_j^n \geqslant 0$, we can conclude that $a_1^n = \cdots = a_j^n = 0$. Consequently, $a_1, \cdots, a_j \in h \cap A$. Therefore $x \in \kappa(h \cap A)$ by Fact 1. Since x was arbitrary, the inclusion $h \cap \kappa A \subset \kappa(h \cap A)$ and the claim have been proven.

We can now state that $y \in \kappa(h \cap A)$. But $h \cap A$ is a top^1-set of A contained in Q, a $(k - 1)$-plane. Thus $y \in \bigcup \kappa B$. Since y was arbitrary, $\partial_F \kappa A \subset \bigcup \kappa B$.

Definition 10. If X is a topological space, and if f: $X \to R^n$ is a function, then f is said to have the two-piece property (TPP for short) if $f^{-1}(H)$ is connected for each open half-space H in R^n. If X is a subset of R^n with the relative topology and if f is the inclusion map, then we say that X has the TPP if f does.

Proposition 11. *If X is topological space, and f: $X \to R^n$ is a function with the TPP, then X is connected.*

Proof. Pick $z \in f(X)$ and choose open half-spaces H_1 and H_2 such that $z \in H_1 \cap H_2$ and $R^n = H_1 \cup H_2$. Then $f^{-1}(H_1) \cap f^{-1}(H_2)$ is nonempty and $X = f^{-1}(H_1) \cup f^{-1}(H_2)$. Since $f^{-1}(H_1)$ and $f^{-1}(H_2)$ are connected, X is connected.

Proposition 12. *Let k and n be integers such that $1 \leqslant k < n$, and let i: $R^k \to R^n$ be the map defined by $i(x^1, \cdots, x^k) = (x^1, \cdots, x^k, 0, \cdots, 0)$. Then a function f: $X \to R^k$ has the TPP if and only if $i \circ f$: $X \to R^n$ has the TPP where X is a topological space.*

Proof. Assume that f has the TPP and let H be an arbitrary open half-space in R^n. Observe that if $i^{-1}(H)$ is not an open half-space in R^k, then $i^{-1}(H) = \varnothing$ or $i^{-1}(H) = R^k$. It follows from Proposition 11 and our assumption about f that $(i \circ f)^{-1}(H) = f^{-1}(i^{-1}(H))$ is connected. Thus $i \circ f$ has the TPP.

Next assume that $i \circ f$ has the TPP, and let K be an arbitrary open half-space in R^k. It is clear that there is an open half-space H in R^n such that $i^{-1}(H) = K$. Then $f^{-1}(K) = f^{-1}(i^{-1}(H)) = (i \circ f)^{-1}(H)$ is connected. Therefore f has the TPP.

Proposition 13. *If X is a topological space, and f: $X \to R^n$ is a continuous map with the TPP, then $f(X)$ has the TPP.*

Proof. Let i: $f(X) \to R^n$ be the inclusion map. If H is an arbitrary open half-space in R^n, then we have $i^{-1}(H) = f(X) \cap H = f(f^{-1}(H))$. But the latter set is connected, as f is a continuous map with the TPP. Therefore i has the TPP. q.e.d.

The converse of Proposition 13 is false as the following example shows. Let $X = \{p, q\}$ have the discrete topology, and let $f(p) = f(q) \in R^n$. Then $f(X)$ has the TPP while f does not.

Proposition 14. *Suppose that f: $X \to R^n$ is a continuous map where X is a compact Hausdorff space. Then f has the TPP if and only if $f^{-1}(\overline{H})$ is connected for each open half-space H in R^n.*

Proof. Assume that f has the TPP, and let H be any open half-space in R^n. If $f^{-1}(\overline{H})$ is not connected, there are disjoint nonempty relatively closed subsets A and B of $f^{-1}(\overline{H})$ such that $f^{-1}(\overline{H}) = A \cup B$. Since $f^{-1}(\overline{H})$ is closed, A and B are closed in X. Therefore by the normality of X there are disjoint open sets V and W such that $A \subset V$ and $B \subset W$. By the continuity of f and the compactness of X we can find an open half-space K such that $\overline{H} \subset K$ and $f^{-1}(K) \subset V \cup W$. Hence $f^{-1}(K)$ is not connected. This contradiction proves that $f^{-1}(\overline{H})$ is connected.

To prove the converse we choose any open half-space H in R^n. Since $f^{-1}(H)$ is connected if $f^{-1}(H) = \varnothing$, we can assume that $f^{-1}(H)$ contains an element x. Pick a sequence $\{H_k\}$ of open half-spaces with the following properties: $x \in f^{-1}(H_k)$ for each k, $\overline{H}_k \subset H$ for each k, and $d(\partial H_k, \partial H) \to 0$. Then $\bigcap f^{-1}(\overline{H}_k) \neq \varnothing$ and $f^{-1}(H) = \bigcup f^{-1}(\overline{H}_k)$ so that $f^{-1}(H)$ is connected. Since H was arbitrary, f has the TPP.

Proposition 15. *If X is a topological space, and f: $X \to R^n$ is a function with the TPP, then $X \setminus Z$ is connected for each top*-set Z of f (Note: f is not necessarily continuous).*

Proof. As the assertion is clear for top°-sets, we can assume that Z is a proper top^k-set of f for some $k \geqslant 1$. Let $h_1, \cdots, h_k$ and $H_1, \cdots, H_k$ be sequences as in Proposition 6. Because $\bigcap_{i=1}^k f^{-1}(H_i) \neq \varnothing$, we need only show $X \setminus Z = \bigcup_{i=1}^k f^{-1}(H_i)$.

To this end let $x \in X \setminus Z$, and let p be the smallest integer in the set $\{i \in \langle 1, k\rangle: x \notin f^{-1}(h_1 \cap \cdots \cap h_i)\}$. If $p = 1$, then $x \in f^{-1}(H_1)$, since $f(X) \subset \overline{H}_1$. If $p > 1$, then $x \in f^{-1}(h_1 \cap \cdots \cap h_{p-1})$. Thus $x \in f^{-1}(H_p)$, since $h_{p-1} \cap \cdots \cap h_1 \cap f(X) \subset \overline{H}_p$. We conclude that $X \setminus Z \subset \bigcup_{i=1}^k f^{-1}(H_i)$. The reverse inclusion is clear.

Proposition 16. *Suppose that Y is a top^1-set of a continuous map f: $X \to R^n$ with the TPP where X is a compact Hausdorff space. Then $f|_Y$ has the TPP.*

Proof. In the light of Proposition 12 we can assume that $n \geqslant 2$. If h is a support hyperplane of $f(X)$ such that $Y = f^{-1}(h)$, then $Y = f^{-1}(\overline{H})$ where H is that open half-space bounded by h which does not meet $f(X)$. Therefore Y is a compact Hausdorff space which is, by Proposition 14, connected. Let $\hat{f} = f|_Y$.

Suppose that $\hat{f}$ does not have the TPP. Then by Proposition 14 there is an open half-space K bounded by a hyperplane k such that $\hat{f}^{-1}(\overline{K}) = Y_1 \cup Y_2$ where Y_1 and Y_2 are nonempty disjoint relatively closed sets. Since X is normal and $\hat{f}^{-1}(\bar{K})$ is closed in X, there exist disjoint open sets V_1 and V_2 containing Y_1 and Y_2, respectively. Also, since Y is connected, k cannot be parallel to h; therefore $h \cap k$ is a $(n-2)$-plane.

Applying a rigid motion if necessary, we can assume that $h = \{z^n = 0\}$, $k \cap h = \{z^{n-1} = z^n = 0\}$, $f(X) \subset \{z^n \leqslant 0\}$, and $\bar{K} \cap h = \{z^{n-1} \leqslant 0, z^n = 0\}$. Define P: $R^n \to R^2$ by $P(z^1, \cdots, z^n) = (z^{n-1}, z^n)$, and for each positive integer j, let L_j be the upper of the two half-planes which are bounded by the line l_j through the origin with slope j^{-1}. It follows that $h_j = P^{-1}(l_j)$ is a hyperplane which bounds the open half-space $H_j = P^{-1}(L_j)$.

We claim that $f^{-1}(\bar{H}_q) \subset V_1 \cup V_2$ for some q. To prove this we assume the contrary and choose $x_j \in f^{-1}(\bar{H}_j) \setminus (V_1 \cup V_2)$ for each j. Since X is compact, we can find a subnet $\{x_{j_\lambda}\}$ which converges to some element $x_0 \in X$. It is clear that $x_0 \notin V_1 \cup V_2$. However, using the compactness of $P(f(X))$ and the fact that $P(f(x_{j_\lambda})) \to P(f(x_0))$, we see that $f(x_0)^{n-1} \leqslant 0$ and $f(x_0)^n = 0$. Thus $x_0 \in f^{-1}(\bar{K}) \cap Y = \hat{f}^{-1}(\bar{K}) \subset V_1 \cup V_2$. This contradiction establishes the claim.

With q as above, it is clear that $Y_1 \subset f^{-1}(\bar{H}_q) \cap V_1$, $Y_2 \subset f^{-1}(\bar{H}_q) \cap V_2$, and $f^{-1}(\bar{H}_q) = [f^{-1}(\bar{H}_q) \cap V_1] \cup [f^{-1}(\bar{H}_q) \cap V_2]$. Therefore $f^{-1}(\bar{H}_q)$ is not connected—which violates Proposition 14. We conclude that $\hat{f}$ has the TPP.

Corollary 17. *If Y is a top*-set of a continuous map f: $X \to R^n$ with the TPP where X is a compact Hausdorff space, then $f|_Y$ has the TPP.*

Proof. The result follows from Proposition 16 by a routine induction argument.

Definition 18. If f: $S \to T$ is a function and $\alpha \in S$, then α is said to be a *simple point of f* if $f^{-1}(f(\{\alpha\})) = \{\alpha\}$.

Defintion 19. Let X and Y be topological spaces, and let f: $X \to Y$ be a continuous map. Then f is said to be a *topological immersion* if for each $x \in X$ there is a neighborhood N of x such that f maps N homeomorphically onto $f(N)$ where $f(N)$ has the relative topology.

Proposition 20. *If f: $X \to R^n$ is a topological immersion where X is a compact Hausdorff space, then the following assertions hold:*

(a) *$f^{-1}(\{\alpha\})$ is finite for each $\alpha \in R^n$.*

(b) *The set of simple points of f is open.*

Proof. Suppose $f^{-1}(\{\alpha\})$ is infinite for some $\alpha \in R^n$. Then there is an infinite sequence of distinct members of $f^{-1}(\{\alpha\})$. Since X is compact, there is a convergent subnet which converges to a point x in X. But then x has no neighborhood V such that $f|_V$ is injective. This contradiction establishes (a).

Suppose x_0 is a simple point, and $\{x_\beta\}$ is a net of nonsimple points which converges to x_0. Clearly we can assume $\{x_\beta\}$ is contained in a neighborhood V

of x_0 such that $f|_V$ is injective. For each β pick $y_\beta \in f^{-1}(\{f(x_\beta)\})\setminus V$. Since X is compact, we can find a convergent subnet $\{y_{\beta_\lambda}\}$ with limit y_0 in X. Clearly $y_0 \notin V$ so that $x_0 \neq y_0$. Yet $f(x_0) = f(y_0)$ since $f(y_{\beta_\lambda}) = f(x_{\beta_\lambda})$ for each λ. Therefore x_0 is not a simple point of f. Hence a simple point cannot be the limit of a net of nonsimple points. This establishes (b).

Definition 21. If f: $T \to R^n$ is a function, and S is a subset of T, then S is said to be a *convex set of* f if $f(S)$ is a convex subset of R^n. If a top*-set of f is not convex, we say that it is *essential.*

Theorem 22. *If* f: $X \to R^n$ *is a topological immersion with the TPP where* X *is a compact Hausdorff space, then convex top*-sets of* f *consist of simple points.*

Proof. Since top*-sets are connected (Proposition 11 and Corollary 17) and since E^0-top*-sets are finite (Proposition 20 (a)), the result holds for E^0-top*-sets.

Now suppose that $0 < k \leqslant n$ and that the result is true for all j such that $0 \leqslant j < k$. Let Z be a convex E^k-top*-set of f, and let F be the k-plane containing $f(Z)$.

We know by Theorem 9 and the convexity of $f(Z)$ that $\partial_F f(Z) = \bigcup \kappa B$ where B ranges over the E^j-top^1-sets of $f(Z)$ with $0 \leqslant j < k$. Moreover the convexity of $f(Z)$ implies that each B is convex; therefore $\partial_F f(Z) = \bigcup B$. By the induction hypothesis and Proposition 7, $f^{-1}(B)$ consists of simple points for each B so that $f^{-1}(\partial_F f(Z))$ consists of simple points.

In order to prove that Z consists of simple points it is enough to show that $f|_Z$ is an injection. To simplify matters we assume, applying a rigid motion if necessary, that $F = \{x^{k+1} = \cdots = x^n = 0\}$, and we let $\hat{f}$ denote $f|_Z$ considered as a map into R^k. Observe that $\hat{f}$ is a topological immersion with the TPP (Proposition 12 and Corollary 17) and that $\hat{f}(Z)$ is a substantial convex subset of R^k. Furthermore, since $\hat{f}^{-1}(\partial \hat{f}(Z)) = f^{-1}(\partial_F f(Z))$, $\hat{f}^{-1}(\partial \hat{f}(Z))$ consists of simple points of $\hat{f}$. By Proposition 20(b) the set of simple points of $\hat{f}$ is open in Z; thus the image N under $\hat{f}$ of the set of nonsimple points is a compact set which does not meet $\partial \hat{f}(Z)$. This implies that κN is a compact set contained in $\hat{f}(Z)^0$.

Suppose that $N \neq \varnothing$. We can then choose $z_0 \in N$, a hyperplane h, and an open half-space H bounded by h such that h supports κN, $h \cap \kappa N = \{z_0\}$, and $H \cap \kappa N = \varnothing$. This choice can be made as follows: pick $y \in N^c$, let z_0 be a point in N such that $d(z_0, y)$ is maximal, and let h be the tangent hyperplane at z_0 of the sphere with radius $d(z_0, y)$ centered at y. By Proposition 20 and the fact that $z_0 \in N$, $\hat{f}^{-1}(\{z_0\})$ is a finite set with members $x_1, \cdots, x_p$ where $p \geqslant 2$.

Now let $U_1, \cdots, U_p$ by mutually disjoint open sets containing $x_1, \cdots, x_p$, respectively, and choose an open ball B centered at z_0 such that $B \subset \hat{f}(Z)$ and

$\hat{f}^{-1}(B) \subset \bigcup_{i=1}^{p} U_i$. As $B \cap H$ is connected and $B \cap H \cap N = \varnothing, \hat{f}^{-1}(B \cap H) \subset U_q$ for some q. Clearly $x_1, \cdots, x_{q-1}, x_{q+1}, \cdots, x_p$ are not cluster points of $\hat{f}^{-1}(H)$. It follows from the normality of Z that $\hat{f}^{-1}(\overline{H}) \subset V \cup W$ where V and W are disjoint open sets containing $\mathrm{Cl}_Z \hat{f}^{-1}(H)$ and $\{x_i: i \neq q\}$, respectively. Therefore $\hat{f}^{-1}(\overline{H})$ is not connected. This violation of Proposition 14 proves that $N = \varnothing$. Thus $\hat{f}$ is injective so that $f|_Z$ is also. This completes the proof.

Definition 23. Consider a compact two-dimensional manifold with boundary M, a topological immersion f: $M \to R^2$, and a line l in R^2 parametrized by $\theta(t) = x + tz$. Then $\Gamma \subset f^{-1}(l)$ is said to be an *l-arc of f with respect to θ* if there is a continuous map χ: $[a, b] \to M$ such that $\chi([a, b]) = \Gamma$, $\{a, b\} = \chi^{-1}(\partial M)$, and $\theta|_{[a,b]} = f \circ \chi$. The interior of Γ, denoted by $\mathrm{INT}\Gamma$, is $\Gamma \setminus \partial M$.

Proposition 24. *If M, f, l, and θ are as in Definition* 23, *and Λ_l is the set of l-arcs of f with respect to θ, then the following assertions hold*:

(1) $f^{-1}(l) \setminus \partial M = \bigcup \mathrm{INT}\Gamma$ *where Γ ranges over Λ_l.*

(2) *If* $\Gamma_1, \Gamma_2 \in \Lambda_l$ *and* $\mathrm{INT}\Gamma_1 \cap \mathrm{INT}\Gamma_2 \neq \varnothing$, *then* $\Gamma_1 = \Gamma_2$.

Proof. Let $y \in f^{-1}(l) \setminus \partial M$ be arbitrary, and pick an open neighborhood B of y such that $B \subset \mathrm{INT}M$ and such that $f|_B$ maps B homeomorphically onto $f(B)$. By Invariance of Domain, $f(B)$ is an open subset of R^2. Therefore, if $\theta(s_0) = f(y)$, there exist real numbers α and β such that $\alpha < s_0 < \beta$ and $\theta((\alpha, \beta)) \subset f(B)$; moreover $(f|_B)^{-1} \circ (\theta|_{(\alpha,\beta)})$: $(\alpha, \beta) \to \mathrm{INT}M$ is a continuous map which takes s_0 into y and yields $\theta|_{(\alpha,\beta)}$ when composed with f. Thus the collection E of continuous maps γ: $(c, e) \to \mathrm{INT}M$ such that $s_0 \in (c, e)$, $\gamma(s_0) = y$, and $f \circ \gamma = \theta|_{(c,e)}$ is nonempty.

If γ_1 and γ_2 are members of E, we write $\gamma_1 \leqslant \gamma_2$ if γ_2 is an extension of γ_1. Clearly "$\leqslant$" is a partial order on E. Now let L be a nonempty linearly ordered subset of E, and observe that the union of the domains of the members of L is an open interval (c, e) which is bounded by the compactness of M. For $s \in (c, e)$ let $\gamma_0(s) = \gamma(s)$ where γ is any member of L with s in its domain. It follows that γ_0: $(c, e) \to \mathrm{INT}M$ is a well-defined member of E and an upper bound of L. By Zorn's Lemma we can find a maximal element χ_0: $(a, b) \to \mathrm{INT}M$ of E.

Next, choose two sequences $\{t_n\}$ and $\{u_n\}$ contained in (a, b) such that $t_n \to a$ and $u_n \to b$. By the compactness of M we might just as well assume that there are points p and q in M which are the limits of $\{\chi_0(t_n)\}$ and $\{\chi_0(u_n)\}$, respectively. Extend χ_0 to a map χ: $[a, b] \to M$ by setting $\chi(a) = p$ and $\chi(b) = q$. We see that $\theta(a) = \lim \theta(t_n) = \lim f(\chi(t_n)) = f(p) = f(\chi(a))$ and, similarly, that $\theta(b) = f(\chi(b))$. Therefore $f \circ \chi = \theta|_{[a,b]}$.

In order to show that χ is continuous we need only show that χ is continuous at a and b. We concentrate on the first case, the argument in the second case being similar. Begin by defining $p = p_0, \cdots, p_k$ as the members of $f^{-1}(\{f(p)\})$ (see Proposition 20(a)) and letting $N_0, \cdots, N_k$ be disjoint open neighborhoods of $p_0, \cdots, p_k$, respectively, such that f maps N_0 homeomorphically onto $f(N_0)$. Choose $c \in (a, b)$ such that $\chi([a, c]) \subset \bigcup_{i=0}^{k} N_i$. This selection is possible by the following argument. If there were no such c, we could produce, using the compactness of M, a sequence $\{c_n\}$ in (a, b) which converges to a and is such that $\{\chi(c_n)\}$ is a sequence converging to a point r not in $\bigcup_{i=0}^{k} N_i$. Then $f(r) = \lim f(\chi(c_n)) = \lim \theta(c_n) = \theta(a) = f(\chi(a)) = f(p)$. Therefore $r \in f^{-1}(\{f(p)\}) \setminus \{p_1, \cdots, p_k\}$—an impossibility. Since χ is continuous on $(a, c]$, $\chi((a, c]) \subset N_j$ for some j. But $\lim t_n = a$, and $\lim \chi(t_n) = p$ so that $\chi([a, c]) \subset N_0$. Noting that $(f|_{N_0})^{-1} \circ (\theta|_{[a,c]}) = \chi|_{[a,c]}$, we conclude that χ is continuous at a.

We next assert that $\chi(a), \chi(b) \in \partial M$. The argument is as follows. If $\chi(a) \in \mathrm{INT}M$, we can find an open neighborhood $D \subset \mathrm{INT}M$ of $\chi(a)$ such that f maps D homeomorphically onto $f(D)$. Since $f(D)$ is open in R^2, there exists $e \in R^1$ such that $e < a$ and $\theta((e, a]) \subset f(D)$. Define $\chi_1: (e, b) \to \mathrm{INT}M$ by $\chi_1(s) = \chi(s)$ if $s \in [a, b)$ and $\chi_1(s) = (f|_D)^{-1}(\theta(s))$ if $s \in (e, a)$. Clearly χ_1 is a member of E which properly extends χ_0. Since this is a violation of the maximality of χ_0, we conclude that $\chi(a) \in \partial M$. Similarly, $\chi(b) \in \partial M$.

We can now conclude that $\chi([a, b])$ is a l-arc of f with respect to θ with $y = \chi(s_0)$ in its interior. This establishes (1).

Let Γ_1 and Γ_2 be l-arcs of f with respect to θ such that $\mathrm{INT}\Gamma_1 \cap \mathrm{INT}\Gamma_2 \neq \varnothing$, and let $\chi_1: [a, b] \to M$ and $\chi_2: [c, e] \to M$ be parametrizations, as per the definition of l-arcs, of Γ_1 and Γ_2, respectively. By assumption there is a pair (s_0, t_0) in $(a, b) \times (c, e)$ such that $\chi_1(s_0) = \chi_2(t_0)$. Note that $\theta(s_0) = f(\chi_1(s_0)) = f(\chi_2(t_0)) = \theta(t_0)$. Therefore $s_0 = t_0$.

We claim that $a = c$. Supposing that $a < c$, we pick δ so that $s_0 < \delta < \mathrm{MIN}\{b, e\}$, and let $F = \{s\colon\ s \in (c, \delta) \text{ and } \chi_1(s) = \chi_2(s)\}$. Since $s_0 \in F$, $F \neq \varnothing$; furthermore it is clear that F is closed in (c, δ). We wish to show that F is open in (c, δ). To this end let $s \in F$ be arbitrary, and pick an open neighborhood G of $\chi_1(s) = \chi_2(s)$ such that f maps G homeomorphically onto $f(G)$. Choose $\varepsilon > 0$ such that $c < s - \varepsilon < s + \varepsilon < \delta$ and $\chi_1((s - \varepsilon, s + \varepsilon)) \cup \chi_2((s - \varepsilon, s + \varepsilon)) \subset G$. For $t \in (s - \varepsilon, s + \varepsilon)$ we have $f|_G(\chi_1(t)) = \theta(t) = f|_G(\chi_2(t))$ and, since $f|_G$ is injective, $\chi_1(t) = \chi_2(t)$. Therefore $(s - \varepsilon, s + \varepsilon) \subset F$. We conclude that F is open in (c, δ) so that $F = (c, \delta)$. It follows that $\chi_1(c) = \chi_2(c)$. But this is impossible since our assumption that $a < c$ implies that $\chi_1(c) \in \mathrm{INT}M$ and $\chi_2(c) \in \partial M$. This contradiction implies that $a \geqslant c$. Similarly $c \geqslant a$. Hence $a = c$.

In the same way it can be shown that $b = e$. If we let $H = \{s: s \in (a, b)$ and $\chi_1(s) = \chi_2(s)\}$, we conclude by the argument above that $H = (a, b)$. It follows that $\chi_1 = \chi_2$ so that $\Gamma_1 = \Gamma_2$. This completes the proof.

2. Spheres

In this section we consider topological immersions of spheres of various dimensions into R^n. Theorem 6 will be useful in the next section, and Theorem 4 is a generalization of results obtained by Chern and Lashof [1, p. 307] and Kuiper [7, p. 275].

Definition 1. Let k be a nonnegative integer, and X a topological space. Then X is said to be *k-connected* if X is pathwise connected and, if $k > 0$, the first k homotopy groups of X are trivial ($\pi_1(X) = \cdots = \pi_k(X) = 0$).

Proposition 2. *If X is a compact subset of R^n, Y is a top*-set of X, and U is a neighborhood of Y in X, then there is an open half-space H of R^n such that $Y \subset H \cap X \subset U$.*

Proof. Kuiper and Pohl [8, p. 180].

Proposition 3. *Suppose that X is a compact Hausdorff space, and $f: X \to R^n$ is a topological immersion such that $f^{-1}(H)$ is k-connected for each open half-space H. If Y is an E^j-top*-set of f, then*

(1) *Y is convex if $j \leq k + 1$,*

(2) *$\partial_Q \kappa f(Y) \subset f(Y)$ if $j = k + 2$ where Q is the j-plane containing $f(Y)$.*

Proof. We proceed by induction on j. Since f has the TPP, it is clear that the assertion is true for $j = 0, 1$. Assume that the result is true for all i such that $1 \leq i < j \leq n$, and let Y be an E^j-top*-set of f with $j \leq k + 2$.

If Q is the j-plane containing $f(Y)$, we have the equation $\partial_Q \kappa f(Y) = \bigcup \kappa Z$ where Z ranges over the E^l-top^1-sets of $f(Y)$ with $0 \leq l < j$ (Theorem 9 of §1). The inequality $j - 1 \leq k + 1$, the induction hypothesis, and Proposition 7 of §1 now imply that $S = \partial_Q \kappa f(Y) \subset f(Y)$. If $j = k + 2$, this is all we need to show. Therefore assume that $j < k + 2$ (and note that $f^{-1}(S)$ is a homeomorph of S^{j-1} by Theorem 22 of §1).

We suppose that $f(Y) \neq \kappa f(Y)$. Letting $e_1, \cdots, e_n$ be the standard basis of R^n and applying a rigid motion if necessary, we assume that $Q = \operatorname{span}\{e_1, \cdots, e_j\}$ and that $0 \in \kappa f(Y) \setminus f(Y)$. Moreover letting $P = \operatorname{span}\{e_{j+1}, \cdots, e_n\}$ and noting that $f(Y) \cap P = \varnothing$, we can choose a neighborhood U in $f(X)$ of $f(Y)$ such that $U \cap P = \varnothing$. By Proposition 2 there is an open half-space H such that $f(Y) \subset H \cap f(X) \subset U$. Since $H \cap f(X) \subset P^c$ and S is not contractible in P^c, S is not contractible in $H \cap f(X)$. It follows that $f^{-1}(S)$ is not contractible in $f^{-1}(H)$ since any homotopy $F: f^{-1}(S) \times [0,1] \to f^{-1}(H)$ determines a homotopy $G: S \times [0,1] \to H \cap f(X)$ defined by $G(y,t) = f(F(f^{-1}(y),t))$. But this violates $\pi_{j-1}(f^{-1}(H)) = 0$.

Theorem 4. *If f: $S^k \to R^n$ is a topological immersion such that $f^{-1}(H)$ is $(k-1)$-connected for each open half-space H, then f is an embedding, and $f(S^k)$ is the boundary of a convex E^{k+1}-set.*

Proof. Since S^k is not homeomorphic to a convex set, we can conclude by Theorem 22 of §1 that $f(S^k)$ is not convex. Thus there is at least one essential top*-set of $f(S^k)$. Let Y be an essential E^j-top*-set of $f(S^k)$ where j is minimal, and note that $j > (k-1)+1 = k$ by Proposition 3.

Letting Q be the j-plane containing Y, we observe that $\partial_Q \kappa Y = \bigcup B$ where B ranges over the E^l-top^1-sets of Y with $0 \leqslant l < j$ (Theorem 9 of §1 and our choice of j). Since $f^{-1}(\partial_Q \kappa Y)$ consists of simple points by Proposition 7 and Theorem 22 of §1, $f^{-1}(\partial_Q \kappa Y)$ is a homeomorph of S^{j-1} so that $j - 1 \leqslant k$. But we have seen that $j - 1 \geqslant k$. Therefore $j - 1 = k$ and $f^{-1}(\partial_Q \kappa Y) = S^k$. It follows that f is an embedding and $f(S^k) = \partial_Q \kappa Y$.

Remark. The hypothesis of Theorem 4 is equivalent to the TPP if $k = 1$ or 2. As this is clear in the first case, we focus our attention on a topological immersion f: $S^2 \to R^n$ with the TPP. If H is an open half-space, $f^{-1}(H)$ is pathwise connected because f is a continuous map with the TPP; furthermore, since $S^2 \setminus f^{-1}(H)$ is connected (Proposition 14 of §1), $f^{-1}(H)$ is simply connected [12, p. 292]. Thus the inverse image of any open half-space is 1-connected. That the hypothesis of Theorem 4 implies the TPP is immediate.

An *arc* is a nonempty connected open subset of S^1. If a, b, and c are distinct points of S^1, the arc "from a to c through b" is denoted by abc; removing b from abc yields two arcs denoted by ab and bc.

Definition 5. Let g: $S^1 \to R^2$ be an arbitrary function, and l a line in R^2. Then $u \in S^1$ is said to be an *l-crossing point of g* if for each arc zuw centered at u there is a pair $(y_1, y_2) \in zu \times uw$ such that y_1 and y_2 are mapped into different open half-planes bounded by l.

Theorem 6. *If g: $S^1 \to R^2$ is a topological immersion such that there are at most three l-crossing points of g for each line l in R^2, then g is an embedding, and $g(S^1)$ bounds a convex body.*

Proof. We begin by noting that if l is a line in R^2 such that $g(S^1) \cap l$ contains a segment, then $g^{-1}(l)$ contains an arc. The argument is as follows. Pick an open segment s contained in $g(S^1) \cap l$, and cover S^1 with closed arcs $A_1, \cdots, A_n$ such that $g|_{A_i}$ is injective for each $i \in \langle 1, n \rangle$. Since $\bigcap_{i=1}^n (s \setminus g(A_i)) = s \setminus g(S^1) = \varnothing$, there is a minimal k such that $\bigcap_{i=1}^k (s \setminus g(A_i)) = \varnothing$. If $k > 1$, we use the minimality of k to find an open segment $s_0 \subset \bigcap_{i=1}^{k-1} (s \setminus g(A_i))$

and to conclude that $s_0 \subset g(A_k)$. In this case $(g|_{A_k})^{-1}(s_0)$ is the arc we seek. If $k = 1$, then $s \subset g(A_1)$, and $(g|_{A_1})^{-1}(s)$ is an arc contained in $g^{-1}(l)$. Next, we simplify our language by calling a line l cleanly cutting if $g(S^1) \cap l$ contains no segment. From the above observation, the local injectiveness of g, and the second countability of S^1, it follows that there are at most countably many lines which are not cleanly cutting.

Suppose that $w \in S^1$ is not a l-crossing point of g where l is cleanly cutting, and choose an arc awb which violates the l-crossing condition. As g is an immersion and l is cleanly cutting, we can find $p \in aw \cap g^{-1}(l)^c$ and $q \in wb \cap g^{-1}(l)^c$. Our assumption on awb implies that $g(aw)$ and $g(q)$ are contained in the same closed half-plane bounded by l; likewise $g(wb)$ and $g(p)$ are on the same side of l. Hence awb is mapped into one of the closed half-planes bounded by l.

Now consider a cleanly cutting line l and an arc A which is free of l-crossing points. We claim that A is mapped into one of the closed half-planes bounded by l. To see this we appeal to the above paragraph and choose for each $w \in A$ an arc $A(w)$ which is contained in A, contains w, and is mapped into one of the closed half-planes bounded by l. If two of these arcs intersect, they are mapped into the same closed half-plane bounded by l, since their intersection is an arc (or arcs) and l is cleanly cutting. The claim now follows from the fact that $\{A(w)\colon w \in A\}$ is an open cover of A.

Let L be an arbitrary open half-plane, and assume that $g^{-1}(L)$ has more than one component, two of which are xyz and abc. Clearly $x, z, a, c \in g^{-1}(\partial L)$. By the remark at the end of the first paragraph we can find an open half-plane L' bounded by a cleanly cutting line l' such that $\bar{L}' \subset L$ and $y, b \in g^{-1}(L')$. It follows from the above claim that there are l'-crossing points of g in xy, yz, ab, and bc. But this is impossible by hypothesis. Therefore $g^{-1}(L)$ is connected, and since L was arbitrary, g has the TPP. We now apply Theorem 4 to complete the proof.

3. The surfaces S_k

The three theorems presented in this section give a satisfactory characterization of the TPP topological immersions of S_k (see Definition 1) into R^n for all k and n. Moreover, Theorems 6 and 7 are generalizations of results obtained by Rodríguez [10, p. 236] concerning these surfaces under differentiability assumptions.

All homology considered in this section is over $\mathbf{Z}$ with $H^{\#}$ denoting reduced homology. The symbol $\cong$ is used to indicate that two groups are isomorphic.

Finally we state a theorem [13, p. 47] which is of fundamental importance in what follows.

Schoenflies' theorem. *If f: $S^1 \to R^2$ is an embedding, then there is a homeomorphism F: $R^2 \to R^2$ which is such that $f = F|_{S^1}$.*

Definition 1. S_k is the space obtained from S^2 by removing a collection Ω_k of k open circular discs with disjoint closures.

We remark that S_k is a compact connected surface with boundary which is topologically independent of the collection Ω_k.

Proposition 2. *If f: $S_k \to S^2$ is a topological immersion such that $f|_{\partial H}$ is an embedding for each $H \in \Omega_k$, then f is an embedding.*

Proof. Choose $H \in \Omega_k$ arbitrarily, and let $N_\varepsilon = \{x \in S_k\colon d(x, \partial H) < \varepsilon\}$ for each $\varepsilon > 0$. We claim that $f|_{N_\varepsilon}$ is an injection for sufficiently small ε. If we assume the contrary, we can find two sequences $\{x_n\}$ and $\{y_n\}$ in S_k which approach ∂H in such a way that $f(x_n) = f(y_n)$ and $x_n \neq y_n$ for each n. By the compactness of S_k we might just as well assume that $\{x_n\}$ converges; furthermore we can find a convergent subsequence $\{y_{n_k}\}$ of $\{y_n\}$. Denoting these limits by x_0 and y_0, we see that both are contained in ∂H. If $x_0 = y_0$, we quickly arrive at a violation of the local injectiveness of f. If $x_0 \neq y_0$, we get a violation of the injectiveness of $f|_{\partial H}$ since $f(x_0) = \lim_k f(x_{n_k}) = \lim_k f(y_{n_k}) = f(y_0)$. This proves the claim. Now let $N_H = N_\varepsilon$ where ε is so small that $f|_{N_\varepsilon}$ is injective and $N_\varepsilon \setminus \partial H$ is connected.

For each $H \in \Omega_k$ let C_H be the component of $S^2 \setminus f(\partial H)$ which does not meet $f(N_H)$. It follows from Schoenflies' theorem that there exists, for each $H \in \Omega_k$, a homeomorphism g_H: $S^2 \to S^2$ which extends $f|_{\partial H}$ in such a way that $g_H(\overline{H}) = \overline{C}_H$. If we define G: $S^2 \to S^2$ by $G(x) = f(x)$ for $x \in S_k$ and $G(x) = g_H(x)$ for $x \in H$, we see that G is a continuous map. In fact, G is a topological immersion. In proving this we can concentrate on points in ∂S_k, since it is clear that $G|_{S^2 \setminus \partial S_k}$ is an immersion. Let $H \in \Omega_k$ and $x \in \partial H$ be chosen arbitrarily, and pick a closed neighborhood B of x in S^2 so that $B \cap S_k \subset N_H$ and $B \setminus S_k \subset H$. It follows that $G|_B$ is an injection and hence a homeomorphism onto $G(B)$. We conclude that G is a topological immersion; furthermore G is an open map by Invariance of Domain. Therefore G is onto as $G(S^2)$ is both open and closed.

Now let $z \in S^2$, and let $x_1, \cdots, x_m$ be an enumeration of $G^{-1}(\{z\})$ (see Proposition 20 of §1). Choose mutually disjoint open neighborhoods $U_1, \cdots, U_m$ of $x_1, \cdots, x_m$, respectively, such that G maps U_i homeomorphically onto $G(U_i)$ for each i. We claim that there is an open neighborhood V of z such that $V \subset \bigcap G(U_i)$ and $G^{-1}(V) \subset \bigcup U_i$. If we cannot find such a neighborhood, we can find a sequence $\{w_n\}$ contained in $S^2 \setminus (\bigcup U_i)$ such that $\lim G(w_n) = z$.

Since S^2 is compact, there is a convergent subsequence of $\{w_n\}$ whose limit is clearly a member of $G^{-1}(\{z\})\setminus(\bigcup U_i)$. This contradiction establishes the claim. Letting $V_i = (G|_{U_i})^{-1}(V)$ for each $i \in \langle 1, m\rangle$, we see that $G^{-1}(V) = \bigcup V_i$ and that G maps V_i homeomorphically onto V for each i. Since z was arbitrary, we conclude that $G\colon S^2 \to S^2$ is a covering space of S^2. Since id: $S^2 \to S^2$ is also a covering space of S^2, the simple connectivity of S^2 implies that G is a homeomorphism [3, p. 23]. Therefore $f = G|_{S_k}$ is an injection.

Proposition 3. *Suppose that A and B are closed subsets of S^2 such that $H_0^{\#}(S^2\setminus A) = 0$, $S^2 \neq A \cup B$, and $A \cap B$ is either empty or a singleton. Then $H_0^{\#}(S^2\setminus(A \cup B)) \cong H_0^{\#}(S^2\setminus B)$.*

Proof. Since $A \cap B$ is either empty or a singleton, $(S^2\setminus A) \cup (S^2\setminus B)$ is either S^2 or S^2 minus a point. Also $(S^2\setminus A) \cap (S^2\setminus B) = S^2\setminus(A \cup B)$ is nonempty as $A \cup B \neq S^2$. Let $T = (S^2\setminus A) \cup (S^2\setminus B)$.

Using the information in the above paragraph, we conclude that the reduced Mayer-Vietoris sequence is applicable to the triad $(T; S^2\setminus A, S^2\setminus B)$ and yields the exact sequence $0 = H_1(T) \to H_0^{\#}(S^2\setminus(A \cup B)) \to H_0^{\#}(S^2\setminus A) \oplus H_0^{\#}(S^2\setminus B) \to H_0^{\#}(T) = 0$. Therefore $H_0^{\#}(S^2\setminus(A \cup B)) \cong H_0^{\#}(S^2\setminus A) \oplus H_0^{\#}(S^2\setminus B)$. As $H_0^{\#}(S^2\setminus A) = 0$, the result follows.

Theorem 4. *If $f\colon S_k \to R^2$ is a topological immersion with the TPP, then the following assertions hold:*

(1) f is an embedding.

(2) $f|_{\partial H}$ is a convex curve for each $H \in \Omega_k$.

(3) There is an enumeration $H_1,\cdots,H_k$ of Ω_k such that $f(S_k) = \overline{\mathrm{INT}\, f(\partial H_1)\setminus(\bigcup_{i=2}^{k} \mathrm{INT}\, f(\partial H_i))}$ where $\mathrm{INT}\, f(\partial H_i)$ is the bounded component of $R^2\setminus f(\partial H_i)$.

Proof. Let l be an arbitrary line in R^2, and Λ_l the set of l-arcs of f with respect to some parametrization of l (see Definition 23 and Proposition 24 of §1). We begin the proof by establishing four claims concerning l and Λ_l.

Claim 1. *If $\Lambda_l = \varnothing$, then the following assertions hold:*

(a) $H_0^{\#}(\mathrm{INT} S_k\setminus \bigcup_{\Gamma\in\Lambda_l}\Gamma) \cong Z$.

(b) $H_0^{\#}(\mathrm{INT} S_k\setminus \bigcup_{\Gamma\in\Lambda'}\Gamma) = 0$ *if Λ' is a finite proper subset of Λ_l.*

(c) *Λ_l is finite.*

Proof. We introduce the following notation. If $\Lambda \subset \Lambda_l$, we set $N(\Lambda) = \mathrm{INT} S_k\setminus \bigcup_{\Gamma\in\Lambda}\Gamma$ and $T(\Lambda) = \bigcup_{\Gamma\in\Lambda_l\setminus\Lambda}$.

It is a fact that a manifold with boundary and its interior have the same number of components (which are in fact the path components). In the present context we apply this fact to $f^{-1}(l)^c$ and its interior $N(\Lambda_l)$. Since f is a topological immersion and $\Lambda_l = \varnothing$, the inverse images of both open half-planes bounded by l are nonempty. As f has the TPP, these sets are the components of $f^{-1}(l)^c$. Thus (a) holds.

We begin the proof of (b) by noting that, since Λ' is finite, $N(\Lambda')$ is an open subset of S^2 so that its components are open in S^2. Let A' and B' be the components of $N(\Lambda')$ which contain the two components A and B of $N(\Lambda_l)$. If C' is another component of $N(\Lambda')$, then $C' \setminus T(\Lambda') = \varnothing$ so that $C' \subset T(\Lambda')$. This, however, is impossible since $f^{-1}(l)$ contains no open set. If $A' \neq B'$, it follows that $A = A' \setminus T(\Lambda')$ and $B = B' \setminus T(\Lambda')$. Now choose $\Gamma_0 \in \Lambda_l \setminus \Lambda'$ and observe that $\mathrm{INT}\Gamma_0 \subset A'$ or $\mathrm{INT}\Gamma_0 \subset B'$. Since A' and B' are open sets, we can find $(z, w) \in (A' \times A') \cup (B' \times B')$ such that z and w are mapped into different open half-planes bounded by l. But then $(z, w) \in (A \times A) \cup (B \times B)$, which is an impossibility. Thus $A' = B'$ and $N(A')$ is connected. This proves (b).

Next suppose that Λ_l is an infinite set. If there is a member Γ of Λ_l both of whose endpoints lie on some component of ∂S_k, we arrive at a contradiction of (b) by setting $\Lambda' = \{\Gamma\}$. If we can find no such l-arc, there must be distinct members Γ_1 and Γ_2 of Λ_l which have their endpoints on the same two components of ∂S_k. In this case we obtain a contradiction of (b) by setting $\Lambda' = \{\Gamma_1, \Gamma_2\}$, and thus establish (c).

Claim 2. *If $\Gamma_1 \in \Lambda_l$ and C is a component of ∂S_k such that $\Gamma_1 \cap C$ is a singleton, then there exists $\Gamma_2 \in \Lambda_l$ such that $\Gamma_2 \cap C \neq \varnothing$ and $\Gamma_1 \neq \Gamma_2$.*

Proof. Let H_1 be the member of Ω_k such that $C = \partial H_1$, and note that $\overline{H}_1$ and Γ_1 are closed cells so that $H_0^\#(S^2 \setminus \Gamma_1) = 0$ and $H_0^\#(S^2 \setminus \overline{H}_1) = 0$ [3, p. 78]. Therefore, since $\Gamma_1 \cap \overline{H}_1$ is a singleton, we can conclude by Proposition 3 that $H_0^\#(S^2 \setminus A) = 0$ where $A = \Gamma_1 \cup \overline{H}_1$.

Assume that the claim is false. Letting H range over Ω_k and Γ over Λ_l, we set $B = (\bigcup_{H \neq H_1} \overline{H}) \cup (\bigcup_{\Gamma \neq \Gamma_1} \Gamma)$, a closed subset of S^2 by claim 1(c). It is clear from our assumption that $A \cap B$ is the singleton consisting of the endpoint of Γ_1 not on ∂H_1 and that $B \cap \overline{H}_1 = \varnothing$. Thus $H_0^\#(S^2 \setminus (A \cup B)) \cong H_0^\#(S^2 \setminus B)$ and $H_0^\#(S^2 \setminus B) \cong H_0^\#(S^2 \setminus (B \cup \overline{H}_1))$ by Proposition 3. Combining this information with Claim 1 gives $\mathbf{Z} \cong H_0^\#(S^2 \setminus (A \cup B)) \cong H_0^\#(S^2 \setminus (B \cup \overline{H}_1)) = 0$. This contradiction establishes Claim 2.

Claim 3. *No three distinct l-arcs can meet a given component of ∂S_k.*

Proof. Suppose that there are three distinct l-arcs Γ_1, Γ_1', and Γ_1'' which meet a component C_1 of ∂S_k. It follows from Claim 1(b) that there is a component C_2 of ∂S_k such that $C_1 \neq C_2$ and $C_2 \cap \Gamma_1 \neq \varnothing$. We extend to a sequence $\Gamma_1, \cdots, \Gamma_n$ of l-arcs and a sequence $C_1, \cdots, C_{n+1}$ of components of ∂S_k such that the C_i's are distinct, Γ_i joins C_i to C_{i+1} for each $i \in \langle 1, n \rangle$, and n is maximal with respect to these properties. Observe that Γ_1', Γ_1'', $\Gamma_1, \cdots, \Gamma_n$ are distinct.

It follows from Claim 2 that there is an l-arc Γ_{n+1}, distinct from Γ_n, which joins C_{n+1} to a component C_{n+2} of ∂S_k. Since n is maximal, $C_{n+2} = C_p$ for some $p \in \langle 1, n+1 \rangle$ so that $\mathrm{INT} S_k \setminus \bigcup_{i=p}^{n+1} \Gamma_i$ is not connected by a Jordan curve theorem argument. But this is a violation of Claim 1(b) since $\Gamma_{n+1} \neq \Gamma_1'$ or $\Gamma_{n+1} \neq \Gamma_1''$. Thus Claim 3 is proved.

Claim 4. *If C is a component of ∂S_k, and u is an l-crossing point of $f|_C$ (see Definition 5 of §2), then there exists $\Gamma \in \Lambda_l$ such that $u \in \Gamma$.*

Proof. Let N be an arbitrary neighborhood of u in S_k, and choose an arc $zuw \subset N \cap C$ which is centered at u. Since u is an l-crossing point, there is a pair $(y_1, y_2) \in zu \times uw$ such that y_1 and y_2 are mapped into different open half-planes bounded by l. It is easy to see that there is a path $\chi\colon [0, 1] \to S_k$ such that $\chi(0) = y_1$, $\chi(1) = y_2$, and $\chi((0, 1))$ is contained in $N \cap \mathrm{INT} S_k$; moreover it is clear that $f \circ \chi((0, 1)) \cap l$ is nonempty. Therefore N meets $\bigcup_{\Gamma \in \Lambda_l} \Gamma$ by Proposition 24 of §1. Since N was arbitrary, u is in the closure of $\bigcup_{\Gamma \in \Lambda_l} \Gamma$, a closed set by Claim 1(c). Thus Claim 4 is proved.

Now suppose that C is a component of ∂S_k for which there are three distinct l-crossing points of $f|_C$ for some line l. Each of these points must, by Claim 4, lie on an l-arc; furthermore two of them lie on the same l-arc by Claim 3. We are led therefore to a violation of Claim 1(b). This contradiction ensures that there are at most two l-crossing points of $f|_C$ for any component C of ∂S_k and any line l. Therefore parts (1) and (2) of the theorem follow from Theorem 6 of §2 and Proposition 2.

Next let $R_*^2 = R^2 \cup \{*\}$ be the one-point compactification of R^2. Since f is an injection and $S_k \setminus \partial S_k$ is connected, there is for each $H \in \Omega_k$ a component A_H of $R_*^2 \setminus f(\partial H)$ which does not meet $f(S_k \setminus \partial S_k)$; furthermore it follows from Schoenflies' theorem that there is for each $H \in \Omega_k$ a homeomorphic extension $g_H\colon \overline{H} \to \overline{A}_H$ of $f|_{\partial H}$. We define $g\colon S^2 \to R_*^2$ by $g(x) = f(x)$ if $x \in S_k$ and $g(x) = g_H(x)$ if $x \in H$. Observing that g is a topological immersion, we conclude that g is a homeomorphism by an argument given in the proof of Proposition 2. As a result, $R_*^2 = f(S_k) \cup A_{H_1} \cup \cdots \cup A_{H_k}$ is a disjoint union where $H_1, \cdots, H_k$ is an enumeration of Ω_k such that $* \in A_{H_1}$. It follows that $A_{H_1} = \mathrm{EXT} f(\partial H_1)$, $A_{H_i} = \mathrm{INT} f(\partial H_i)$ if $i \in \langle 2, k \rangle$, and $f(S_k) = \overline{\mathrm{INT} f(\partial H_1)}(\bigcup_{i=2}^{k} \mathrm{INT} f(\partial H_i))$. Thus (3) is proved.

Fact. *Suppose that X is a compact subset of R^n. If Y is an E^k-top*-set of X with containing k-plane P and Z is an E^j-top*-set of X with containing j-plane Q, then $Z \subset Y$ in case $P \cap \mathrm{INT}_Q \kappa Z \neq \varnothing$* [8, *p.* 182].

Proposition 5. *Let X be a compact subset of R^n, Y an E^k-top*-set of X, and Z an E^j-top*-set of X. If P is the k-plane containing Y, and Q is the j-plane containing Z, then the following assertions hold:*

(1) *$P \neq Q$ in the event that $Y \neq Z$.*

(2) *$\kappa Y \cap \kappa Z \subset \partial_Q \kappa Z$ if $k < j$.*

(3) $\kappa Y \cap \kappa Z \subset \partial_P \kappa Y \cap \partial_Q \kappa Z$ *if* $k = j$ *and* $Y \neq Z$.

Proof. If $Y = X$, it is clear that $Y = P \cap X$. If $Y \neq X$, there is, by Proposition 6 of §1, a support sequence $h_1, \cdots, h_s$ of X such that $Y = h_s \cap \cdots \cap h_1 \cap X$. Since $Y \subset P$, $Y = P \cap h_s \cap \cdots \cap h_1 \cap X$. But this means that $P \subset h_s \cap \cdots \cap h_1$ for otherwise Y is contained in a plane $P \cap h_s \cap \cdots \cap h_1$ of dimension less than k. Therefore $Y = P \cap X$, and similarly $Z = Q \cap X$. These equations imply (1).

To prove (2) we note that if $\kappa Y \cap \kappa Z$ meets $\mathrm{INT}_Q \kappa Z$, then $Z \subset Y$ by the Fact cited above. This, however, implies that Z is contained in the k-plane P, which is impossible by the condition $k < j$. Thus (2) holds.

If (3) fails to hold, $Z \subset Y$ or $Y \subset Z$ by the Fact. As Y and Z are both E^k-top*-sets of X, this leads to the equality $P = Q$. This violation of (1) establishes (3).

Theorem 6. *If A is a nonempty closed subset of S^2 such that $S^2 \setminus A$ has finitely many components, and if f: $A \to R^n$ is a topological immersion with the TPP, then the following assertions are valid*:

(1) *If* $n \geqslant 4$, $f(A)$ *is not substantial.*

(2) *If* $n = 3$ *and* $f(A)$ *is substantial, then* $f(A) \subset \partial \kappa f(A)$.

Proof. If Y is an essential E^2-top*-set of f, and P is the 2-plane containing $f(Y)$, then we set $\gamma(Y) = f^{-1}(\partial_P \kappa f(Y))$. It follows from Theorem 9, Proposition 7, and Theorem 22 of §1 that $\partial_p \kappa f(Y) \subset f(Y)$ and that $\gamma(Y)$ consists of simple points. Thus, since $\partial_P \kappa f(Y)$ is a homeomorph of S^1, $\gamma(Y)$ is topologically a circle. Because $A \setminus Y$ is connected (Proposition 15 of §1) and $\gamma(Y) \subset Y$, $A \setminus Y$ is contained in one of the components of $S^2 \setminus \gamma(Y)$. Thus, if $C(Y)$ is the other component, $C(Y) \cap A \subset Y$. Furthermore $C(Y)$ contains a component of $S^2 \setminus A$; if not, we conclude serially that $C(Y) \subset A$ (recall $S^2 \setminus A \subset S^2 \setminus \gamma(Y)$), $C(Y) \subset Y$, and Y is not essential.

We claim that there are at most finitely many essential E^2-top*-sets of f. To prove this we consider two distinct essential E^2-top*-sets Y_1 and Y_2 of f. It is a consequence of Proposition 5(3) that $\gamma(Y_1) \cap C(Y_2) = \varnothing$ so that $C(Y_2) \subset C(Y_1)$ or $C(Y_2) \cap C(Y_1) = \varnothing$. Since the former leads to a violation of Proposition 5(1), we conclude that $C(Y_2) \cap C(Y_1) = \varnothing$ and, as a result, that $C(Y_1)$ and $C(Y_2)$ contain different components of $S^2 \setminus A$. The claim now follows from the fact that $S^2 \setminus A$ has finitely many components.

We consider next an E^3-top*-set Z of f. The contention here is that $f(A) \subset \partial_Q \kappa f(Z)$ where Q is the 3-plane containing $f(Z)$. To prove this we will need the following equation (Theorem 9 of §1):

$$(1) \qquad \partial_Q \kappa f(Z) = \bigcup \kappa B,$$

where B ranges over the E^j-top[1]-sets of $f(Z)$ with $0 \leqslant j < 3$.

If there are no essential E^2-top^1-sets of $f(Z)$, we conclude from equation (1) that $\partial_Q \kappa f(Z) = \bigcup B \subset f(Z)$. Also it follows from Proposition 7 and Theorem 22 of §1 that $f^{-1}(\partial_Q \kappa f(Z))$ consists of simple points. Therefore $f^{-1}(\partial_Q \kappa f(Z))$ is a homeomorph of S^2 so that this set actually is S^2. Thus the claim holds in the case under consideration.

If there are essential E^2-top^1-sets of $f(Z)$, they are finite in number by Proposition 7 of §1 and the finite nature of the set of essential E^2-top*-sets of f. Choose an enumeration $B_1, \cdots, B_k$ of the essential E^2-top^1-sets of $f(Z)$, and let $Y_i = f^{-1}(B_i)$ for each $i \in \langle 1, k\rangle$. From equation (1) we obtain the equality $\partial_Q \kappa f(Z) = K \cup \mathrm{INT}_{P_1} \kappa B_1 \cup \cdots \cup \mathrm{INT}_{P_k} \kappa B_k$ where

(a) P_i is the 2-plane containing B_i for each $i \in \langle 1, k\rangle$,

(b) $K = \partial_Q \kappa f(Z) \setminus \bigcup_{i=1}^k \mathrm{INT}_{P_i} \kappa B_i$,

(c) $K \subset f(Z)$ and $f^{-1}(K)$ consists of simple points,

(d) $f^{-1}(K)$, $C(Y_1), \cdots, C(Y_k)$ are disjoint sets,

(e) K, $\mathrm{INT}_{P_1} \kappa B_1, \cdots, \mathrm{INT}_{P_k} \kappa B_k$ are disjoint sets.

The last three statements are consequences of Proposition 7 of §1, Theorem 22 of §1, the results of the second paragraph, and Proposition 5. Next we employ Schoenflies' theorem to choose for each $i \in \langle 1, k\rangle$ a homeomorphic extension $g_i\colon \kappa B_i \to \overline{C(Y_i)}$ of f^{-1} restricted to $\partial_{P_i} \kappa B_i$. Finally we define $g\colon \partial_Q \kappa f(Z) \to S^2$ as follows: $g(x) = f^{-1}(x)$ if $x \in K$, and $g(x) = g_i(x)$ if $x \in \mathrm{INT}_{P_i} \kappa B_i$. Using the properties listed above, we see that g is a continuous injection and, since $\partial_Q \kappa f(Z)$ is topologically S^2, a surjection. To complete the proof of the claim we let x be any member of A. If $x \in \bigcup_{i=1}^k Y_i$, then $f(x) \in \bigcup_{i=1}^k B_i$ so that $f(x) \in \partial_Q \kappa f(Z)$. In the event that $x \in A \setminus \bigcup_{i=1}^k Y_i$, we choose $z \in \partial_Q \kappa f(Z)$ such that $g(z) = x$. We see that $z \in K$, for otherwise $x = g(z) \in A \cap (\bigcup_{i=1}^k C(Y_i)) \subset \bigcup_{i=1}^k Y_i$, which is contrary to the assumption. Therefore $x = f^{-1}(z)$, and consequently $f(x) = z \in \partial_Q \kappa f(Z)$. Since x was arbitrary, the claim has been established.

We now note that assertion (2) follows from the above claim with $Z = A$. Furthermore to prove assertion (1) we can assume that there are no E^3-top*-sets of f. We complete the proof of the theorem by deriving a contradiction from the assumption that $f(A)$ is substantial in R^n where $n \geqslant 4$. Let k be the smallest integer in $\langle 4, n\rangle$ for which there is an E^k-top*-set of D of $f(A)$, and let Q be the k-plane containing D. By Theorem 9 of §1 $\partial_Q \kappa D = \bigcup \kappa B$ where B ranges over the E^j-top^1-sets of D with $0 \leqslant j \leqslant 2$. As we have seen that there are finitely many essential E^2-top*-sets of f, we conclude from Proposition 7 of §1 that there are finitely many essential E^2-top^1-sets of $B_1, \cdots, B_s$ of D. Since $\partial_Q \kappa D$ is a homeomorph of S^{k-1}, $G = \partial_Q \kappa D \setminus \bigcup_{i=1}^s \kappa B$ is topologically the result of removing finitely many closed 2-cells from a sphere of dimension at least three and, as such, is a nonempty open subset of $\partial_Q \kappa D$. Moreover

$G \subset f(A)$ and $f^{-1}(G)$ consists of simple points. It now follows readily that an open subset of R^3 can be embedded in S^2. This however is impossible by Invariance of Dimension [2, p. 60].

Theorem 7. *If $f\colon S_k \to R^3$ is a topological immersion with the TPP such that $f(S_k)$ is substantial, then*

(1) f is an embedding,

(2) for each $H \in \Omega_k$, $f|_{\partial H}$ is a convex curve whose image is contained in a support plane P_H of $f(S_k)$,

(3) $f(S_k) = \partial\kappa f(S_k) \setminus \cup \operatorname{INT} f(\partial H)$ where H ranges over Ω_k, and $\operatorname{INT} f(\partial H)$ is the bounded component in P_H of $P_H \setminus f(\partial H)$.

Proof. Since S_k and $f\colon S_k \to R^3$ satisfy the hypothesis of Theorem 6, the arguments employed in the proof of that theorem apply here. We gather together in the following list some of the pertinent notation and results from that argument.

(a) There are finitely many essential E^2-top*-sets (which, since $n = 3$, are top^1-sets) $Y_1, \cdots, Y_s$ of f, and these are mapped into the 2-planes $P_1, \cdots, P_s$.

(b) For each $i \in \langle 1, s\rangle$, $\gamma(Y_i) = f^{-1}(\partial_{P_i}\kappa f(Y_i))$ and $C(Y_i)$ is that component of $S^2 \setminus \gamma(Y_i)$ such that $C(Y_i) \cap S_k \subset Y_i$.

(c) $K = \partial\kappa f(S_k) \setminus \cup_{i=1}^{s} \operatorname{INT}_{P_i}\kappa f(Y_i) \subset f(S_k)$.

(d) $S^2 = f^{-1}(K) \cup C(Y_1) \cup \cdots \cup C(Y_s)$ and the union is disjoint.

As consequences, we have the following two results:

(e) For each $H \in \Omega_k$, there exists $i \in \langle 1, s\rangle$ such that $H \subset C(Y_i)$.

(f) For each $i \in \langle 1, s\rangle$, $Y_i = \overline{C(Y_i)} \setminus \cup H$ where H ranges over those members of Ω_k which are contained in $C(Y_i)$.

Statement (e) is immediate from (d). As for (f), suppose that $x \in Y_i \setminus \overline{C(Y_i)}$. Appealing to (b)–(d), we conclude successively that $x \in Y_i \setminus \gamma(Y_i)$, $f(x) \in \operatorname{INT}_{P_i}\kappa f(Y_i)$, $x \notin f^{-1}(K)$, $x \in C(Y_j)$ for some $j \neq i$, and $f(x) \in \operatorname{INT}_{P_j}\kappa f(Y_j)$. But the second and fifth conclusions together constitute a violation of Proposition 5(3). Thus there can be no member of $Y_i \setminus \overline{C(Y_i)}$ so that $Y_i \subset \overline{C(Y_i)}$. Statement (f) is now obvious.

We now wish to enlarge each Y_i and extend each $f|_{Y_i}$ in such a way that Theorem 4 applies. To this end we regard $\overline{C(Y_i)}$ as lying in R^2, and we employ Schoenflies' theorem to obtain a homeomorphism $G_i\colon \overline{\operatorname{EXT}\gamma(Y_i)} \to \overline{\operatorname{EXT}\partial_{P_i}\kappa f(Y_i)}$ which extends $f|_{\gamma(Y_i)}$ (EXT refers to the unbounded component of the complement of a simple closed plane curve). Now take a large circle L in P_i which contains $\kappa f(Y_i)$ in its interior, let $J = G_i^{-1}(L)$, and note that $\overline{C(Y_i)}$ is contained in the interior of J. Finally, define a map F_i: $(\operatorname{INT} J \setminus \overline{C(Y_i)}) \cup Y_i \to P_i$ by $F_i(x) = f(x)$ if $x \in Y_i$ and $F_i(x) = G_i(x)$ otherwise. It is clear that F_i is a topological immersion, and it follows readily from statement (f) that the domain of F_i is topologically S_n for some n. Furthermore F_i has the TPP. The proof is as follows. Let $A = \overline{\mathrm{INT}\,J} \setminus C(Y_i)$, and note that $F_i(A)$ is topologically an annulus whose boundary curves are convex. Therefore $F_i(A)$ has the TPP and, since F_i maps A homeomorphically onto $F_i(A)$, $F_i|_A$: $A \to P_i$ has the TPP. Now let M be an arbitrary open half-plane of P_i, and make the following three observations:

(A) $F_i^{-1}(M) = (f|_{Y_i})^{-1}(M) \cup (F_i|_A)^{-1}(M)$.

(B) $\gamma(Y_i) \cap f^{-1}(M) \neq \varnothing$ if $(f|_{Y_i})^{-1}(M) \neq \varnothing$.

(C) $\gamma(Y_i) \cap f^{-1}(M) \subset (f|_{Y_i})^{-1}(M) \cap (F_i|_A)^{-1}(M)$.

Since $f|_{Y_i}$ and $F_i|_A$ have the TPP, the sets on the righthand side of equation (A) are connected; moreover, if $(f|_{Y_i})^{-1}(M) \neq \varnothing$, their intersection is nonempty by (B) and (C). We conclude that $F_i^{-1}(M)$ is connected and, since M was arbitrary, that F_i has the TPP.

Next let H be an arbitrary member of Ω_k, and let $P_H = P_i$ where i is such that $H \subset C(Y_i)$ (see statement (e)). Since ∂H is a boundary component of the domain of F_i, we can apply Theorem 4 to the results of the above paragraph to conclude that $f|_{\partial H} = F_i|_{\partial H}$ is an injective convex curve whose image is contained in P_H, a support plane of $f(S_k)$. As H was arbitrary, assertion (2) holds; furthermore assertion (1) holds by Proposition 2 and Theorem 6(2).

It remains to show assertion (3). For each $H \in \Omega_k$ we choose, employing Schoenflies' theorem once again, a homeomorphism χ_H: $\overline{H} \to \overline{\mathrm{INT}\, f(\partial H)}$ which extends $f|_{\partial H}$. Since $f(S_k) \subset \partial\kappa f(S_k)$ by Theorem 6(2), we can define a mapping F: $S^2 \to \partial\kappa f(S_k)$ by $F(x) = f(x)$ if $x \in S_k$, and $F(x) = \chi_H(x)$ if $x \in H$. Clearly F is continuous; moreover since $f(S_k)$ is substantial, $F(S_k \setminus \partial S_k) \cap \mathrm{INT}\, f(\partial H) = \varnothing$ for each $H \in \Omega_k$. This implies that F is an immersion and hence a homeomorphism (see the proof of Proposition 2). The result now follows.

4. Tightness

In this section we introduce the notion of tightness for functions mapping a topological space into a Euclidean space. It is the content of Theorems 3 and 4 that the TPP and tightness (with respect to a suitable coefficient ring) are equivalent for continuous maps defined on compact connected surfaces without boundary, and Theorem 6 provides a useful characterization of tight topological immersions defined on compact connected surfaces with nonvoid boundary. The latter result is then used to prove Theorem 7, which generalizes the work of Kuiper on tight bands [7], and to construct a counter-example which demonstrates that the third theorem of Rodríquez [10, p. 236] fails to hold without differentiability assumptions.

Definition 1. If X is a topological space, $f\colon X \to R^n$ is a function, and A is a ring with identity, then f is said to be *A-tight* if the inclusion map $i\colon f^{-1}(H) \to M$ induces an injection $i_*\colon H_k(f^{-1}(H); A) \to H_k(M; A)$ for each nonnegative integer k and each open half-space H. If the above condition is satisfied at the kth homology level, we say that f is *A-k-tight*.

In what follows A will be an integral domain. Recall that the *characteristic of* A (char(A)) is the nonnegative generator of the kernel of the homomorphism $n \to ne$ where e is the identity of A.

Proposition 2. *If M is a compact connected nonorientable surface without boundary, and* char$(A) \neq 2$, *then no topological immersion mapping M into R^n is A-tight.*

Proof. Assume that $f\colon M \to R^n$ is an A-tight topological immersion. We first observe that the A-0-tightness of f implies, since M is connected, that f has the TPP. Now choose a support hyperplane h of $f(M)$ such that $h \cap f(M)$ is a singleton, and let H be that open half-space bounded by h which contains $f(M)$ in its closure. It follows from Theorem 22 of §1 that $f^{-1}(H) = M \setminus p$ for some $p \in M$. Since char$(A) \neq 2$ and $H_2(M; A) \cong \{x \in A\colon 2x = 0\}$ [2, p. 260], $H_2(M; A) = 0$ so that the homology sequence of the pair $(M, f^{-1}(H))$ gives the exact sequence $0 \to H_2(M, f^{-1}(H); A) \to H_1(f^{-1}(H); A) \xrightarrow{i_*} H_1(M; A)$. Therefore $A \cong H_2(M, M \setminus p; A) = H_2(M, f^{-1}(H); A) \cong \ker i_*$. This however is a violation of the A-1-tightness of f. Thus no A-tight topological immersion $f\colon M \to R^n$ can exist.

Theorem 3. *Suppose that A is a field of characteristic 2. If M is a compact connected surface without boundary, and $f\colon M \to R^n$ is a continuous map, then f has the TPP if and only if f is A-tight.*

Proof. Suppose that f has the TPP. Choose any open half-space H such that $\phi \neq f^{-1}(H) \neq M$ and observe that, since $f^{-1}(H)$ is noncompact and connected (TPP), $H_2(f^{-1}(H); A) = 0$. Therefore the homology sequence of the pair $(M, f^{-1}(H))$ yields the exact sequence

$$0 \to H_2(M; A) \xrightarrow{k} H_2(M, f^{-1}(H); A) \xrightarrow{j} H_1(f^{-1}(H); A) \xrightarrow{i_*} H_1(M; A).$$

Since $f^{-1}(H) = M \setminus f^{-1}(\overline{H}_0)$ where H_0 is the open half-space opposite H, since $f^{-1}(\overline{H}_0)$ is connected (Proposition 14 of §1), and since $\{x \in A\colon 2x = 0\} = A$ (char$(A) = 2$), it follows that $H_2(M; A) \cong A \cong H_2(M, f^{-1}(H); A)$, [2, p. 260]. But then k, as a linear injection between one dimensional vector spaces, is a surjection so that $0 = \operatorname{im}(j) = \ker(i_*)$. We conclude that f is A-1-tight.

If H is an open half-space such that $f^{-1}(H) \neq \varnothing$, the homology sequence of the pair $(M, f^{-1}(H))$ gives the exact sequence $H_0(f^{-1}(H); A) \xrightarrow{i_*} H_0(M; A) \to 0$

where both vector spaces are one-dimensional (TPP). Therefore, if i_* is to be surjective, $\ker(i_*)$ must be zero. Hence f is A-0-tight. Finally f is A-n-tight for all $n \geqslant 2$ since $H_n(f^{-1}(H); A) = 0$ whenever H is an open half-space such that $f^{-1}(H) \neq M$.

Next suppose that f is A-tight. If H is an open half-space such that $f^{-1}(H) \neq \varnothing$, we have the exact sequence $H_0(f^{-1}(H); A) \xrightarrow{i_*} H_0(M; A) \to 0$. Since f is A-0-tight, we conclude that i_* is an isomorphism. The connectedness of $f^{-1}(H)$ now follows from the fact that M is connected. Therefore f has the TPP.

Theorem 4. *Suppose that A is $\mathbf{Z}$ or any field. If M is a compact connected orientable surface without boundary, and f: $M \to R^n$ is a continuous map, then f has the TPP if and only if f is A-tight.*

Proof. If A is a field, the proof is the same as that given above. We assume therefore that $A = \mathbf{Z}$.

Suppose that f has the TPP, and let H be any open half-space such that $\varnothing \neq f^{-1}(H) \neq M$. As above, we obtain the relationships $H_2(M) \cong \mathbf{Z} \cong H_2(M, f^{-1}(H))$ and an exact sequence $0 \to H_2(M) \to H_2(M, f^{-1}(H)) \xrightarrow{j} H_1(f^{-1}(H)) \xrightarrow{i_*} H_1(M)$. It follows that $\ker(i_*) = \operatorname{im}(j) \cong H_2(M, f^{-1}(H))/\ker(j)$ and, since $\ker(j) \cong \mathbf{Z}$, that $\ker(i_*)$ is a torsion group. Therefore $\ker(i_*) = 0$ [2, p. 261], and we conclude that f is $\mathbf{Z}$-1-tight.

Consider next an open half-space H such that $f^{-1}(H) \neq \varnothing$. Since $H_0(f^{-1}(H))/\ker(i_*) \cong H_0(M)$ (from the homology sequence of the pair) and $H_0(M) \cong \mathbf{Z} \cong H_0(f^{-1}(H))$ (TPP), it follows that $\ker(i_*) = 0$. Therefore f is $\mathbf{Z}$-0-tight. The remainder of the proof is as in the proof of Theorem 3.

Proposition 5. *If M is an n-dimensional manifold with nonvoid boundary, then the following statements are true*:

(a) *If C is a closed connected subset of M which meets ∂M, then $H_k(M, M \setminus C; A) = 0$ for all $k \geqslant n$.*

(b) *$H_n(M, M \setminus p; A) \cong A$ if $p \in M \setminus \partial M$.*

Proof. Let $\tilde{M}$ be the n-dimensional manifold without boundary which is obtained from M by attaching a collar: $\tilde{M} = (M \times (0)) \cup (\partial M \times [0, 1))$. In addition, let $M' = M \times (0)$, $C' = C \times (0)$, and $\tilde{C} = C' \cup [(C \cap \partial M) \times [0, 1)]$.

The two sets whose union constitutes $\tilde{C}$ are closed in M' and $\partial M \times [0, 1)$, respectively. Since the latter sets are closed in $\tilde{M}$, $\tilde{C}$ is also closed in $\tilde{M}$. If $p \in C \cap \partial M$, the connected set $(p) \times [0, 1)$ meets C'; therefore it follows that $\tilde{C}$ is a connected set. Finally $\tilde{C}$ is not compact because it contains the closed, noncompact set $(p) \times [0, 1)$ where $p \in C \cap \partial M$.

Now consider the map $H\colon \tilde{M} \times [0, 1] \to \tilde{M}$ which is defined by $H((p, s), t) = (p, (1-t)s)$. Observing that $H_t(\tilde{M} \setminus \tilde{C}) \subset \tilde{M} \setminus \tilde{C}$ for each $t \in [0, 1]$, $H_0\colon \tilde{M} \to \tilde{M}$ is the identity map, $H_1(\tilde{M}) \subset M'$, $H_1(\tilde{M} \setminus \tilde{C}) \subset M' \setminus C'$, and $H_1|_{M'}\colon M' \to \tilde{M}$ is the inclusion map, we conclude that $(M', M' \setminus C')$ is a deformation retract of $(\tilde{M}, \tilde{M} \setminus \tilde{C})$. Therefore $H_k(M, M \setminus C; A) \cong H_k(M', M' \setminus C'; A) \cong H_k(\tilde{M}, \tilde{M} \setminus \tilde{C}; A)$. This proves part (a), [2, p. 260].

If $p \in M \setminus \partial M$, then $H_t(\tilde{M} \setminus p) \subset \tilde{M} \setminus p$ for each $t \in [0, 1]$ and $H_1(\tilde{M} \setminus p) \subset M' \setminus p$. Thus, as above, $H_n(M, M \setminus p; A) \cong H_n(\tilde{M}, \tilde{M} \setminus p; A) \cong A$.

Theorem 6. *Suppose that A is $\mathbf{Z}$ or any field. If M is a compact connected surface with nonvoid boundary, and $f\colon M \to R^n$ is a topological immersion, then f is A-tight if and only if f has the TPP and $f(M) \subset \kappa f(\partial M)$.*

Proof. Assume that f is A-tight. Since f has the TPP (as in previous arguments) we concentrate on showing that $f(M) \subset \kappa f(\partial M)$. To this end we assume the contrary, and choose an open half-space H which meets $f(M)$ and misses $\kappa f(\partial M)$. By the Krein-Milman theorem [11, p. 70] H contains an extreme point of $\kappa f(M)$ so that by Straszewicz's theorem [9, p. 167] H contains an extreme point x of $\kappa f(M)$ which constitutes the intersection of $\kappa f(M)$ with one of its support hyperplanes. It follows that $x \in f(M)$ and (Theorem 22 of §1) that there is an open half-space K such that $f^{-1}(K) = M \setminus p$ where p is the unique point in M satisfying $f(p) = x$. As $p \in M \setminus \partial M$, $H_2(M, f^{-1}(K); A) = H_2(M, M \setminus p; A) \cong A$ by Proposition 5(b); moreover, applying Proposition 5(a) to M (with $C = M$) yields $H_2(M; A) = 0$. It follows from the homology sequence of the pair $(M, f^{-1}(K))$ that $i_*\colon H_1(f^{-1}(K); A) \to H_1(M; A)$ is not an injection. This contradiction forces the conclusion that $f(M) \subset \kappa f(\partial M)$.

To begin the proof of the converse we let H be an arbitrary open half-space, and H_0 the open half-space which is opposite H. If $f^{-1}(\overline{H}_0) = \varnothing$, then $f^{-1}(H) = M$ and $i_*\colon H_1(f^{-1}(H); A) \to H_1(M; A)$ is an injection. If, on the other hand, $f^{-1}(\overline{H}_0) \neq \varnothing$, then $f^{-1}(\overline{H}_0)$ is a closed connected (TPP and Proposition 14 of §1) set which meets $\partial M (f(M) \subset \kappa f(\partial M))$. Therefore $H_2(M, f^{-1}(H); A) = H_2(M, M \setminus f^{-1}(\overline{H}_0); A) = 0$ by Proposition 5(a). It now follows from the homology sequence of the pair $(M, f^{-1}(H))$ that $i_*\colon H_1(f^{-1}(H); A) \to H_1(M; A)$ is an injection. Since H was arbitrary, f is A-1-tight.

Since $H_n(f^{-1}(H); A) = 0 = H_n(M; A)$ for all open half-spaces H and all $n \geqslant 2$ (Proposition 5(a)), f is A-n-tight for all $n \geqslant 2$. Finally, since f has the TPP, f is A-0-tight (see the proof of Theorem 4).

Theorem 7. *Suppose that A is $\mathbf{Z}$ or any field, and let S_k be as in §3. If $f\colon S_k \to R^3$ is an A-tight topological immersion such that $f(S_k)$ is substantial, then*

(a) *f is an embedding,*

(b) *for each* $H \in \Omega_k$, $f|_{\partial H}$ *is a convex curve whose image is contained in a support plane* P_H *of* $f(S_k)$,

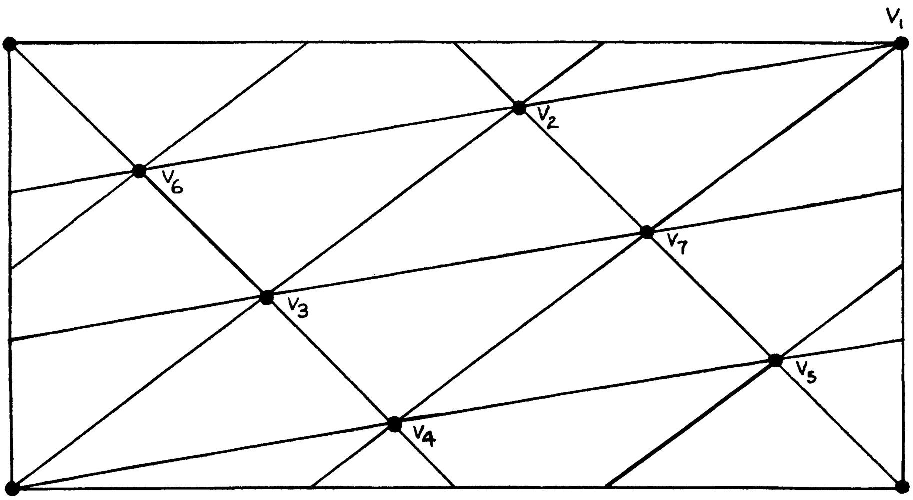
(c) $f(S_k) = \partial\kappa f(\partial S_k) \setminus \bigcup \mathrm{INT} f(\partial H)$ *where H ranges over* Ω_k *and* $\mathrm{INT} f(\partial H)$ *is the bounded component in* P_H *of* $P_H \setminus f(\partial H)$.

Proof. It follows from Theorem 6 that f has the TPP and that $f(S_k) \subset \kappa f(\partial S_k)$. Since this inclusion implies that $\kappa f(S_k) = \kappa f(\partial S_k)$, the result follows from Theorem 7 of §3. q.e.d.

Finally we construct an example which shows that the following theorem of Rodríguez [10, p. 236] is not true if the smoothness assumption on f is dropped.

Theorem. *If A is a field and* $f\colon M \to R^n$ *is a smooth A-tight immersion of a compact connected surface with nonvoid boundary, then* $f(M)$ *is not substantial for* $n \geqslant 4$.

Consider the following triangulation (with vertices $v_1, \cdots, v_7$) of the two-dimensional torus.



Removing the star of v_7 (that is, v_7 together with the interiors of those triangles and edges which contain v_7) leaves us with a compact connected surface M with nonvoid boundary. We define $f\colon M \to R^7$ as follows. If v_i, v_j, and v_k are the vertices of a triangle of the given triangulation which is contained in M, then f is to map that triangle linearly onto $\kappa\{e_i, e_j, e_k\}$ in such a way that $f(v_i) = e_i$, $f(v_j) = e_j$, and $f(v_k) = e_k$ ($e_1, \cdots, e_7$ is the standard basis of R^7). Clearly f is well-defined continuous and injective.

We claim that f has the TPP. Checking the above diagram reveals that the segment joining any two of the points $e_1, \cdots, e_6$ lies in $f(M)$. Now suppose that x and y are contained in $f(M) \cap H$ where H is an open half-space in R^7. At least one vertex e_i of the image triangle containing x must lie in H. In a like manner, we choose a vertex e_j relative to y. Clearly the path consisting of the (possibly degenerate) segments $\overline{xe_i}$, $\overline{e_ie_j}$, and $\overline{e_jy}$ lies entirely in $f(M) \cap H$. It follows that $f^{-1}(H)$ is connected and, consequently, that f has the TPP. Moreover, since $f(M) \subset \kappa\{e_1, \cdots, e_6\} \subset \kappa f(\partial M)$, f is A-tight (A a field) by Theorem 6. Lastly $f(M)$ is substantial in the 5-plane containing $e_1, \cdots, e_6$ just because $f(M)$ contains these points.

As a final note, we suggest that the reader interested in tight and 0-tight polyhedral embeddings $M \to R^n$ of surfaces (with and without boundary) consult [4], where necessary and sufficient conditions, relating n and the Euler characteristics of various M, are given for the existence of substantial embeddings of this type.

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