J. DIFFERENTIAL GEOMETRY
16 (1981) 305–322

CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD. I

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1. Introduction

Let $\tilde{M}$ be a Kaehler manifold with complex structure J, N a Riemannian manifold isometrically immersed in $\tilde{M}$, and $\mathfrak{D}_x$ the maximal holomorphic subspace of the tangent space T_xN of N. If the dimension of $\mathfrak{D}_x$ is the same for all x in N, $\mathfrak{D}_x$ gives a holomorphic distribution $\mathfrak{D}$ on N.

Recently, A. Bejancu [1] introduced the notion of a CR-submanifold of $\tilde{M}$ as follows. A submanifold N in a Kaehler manifold $\tilde{M}$ is called a CR-submanifold if there exists on N a differentiable holomorphic distribution $\mathfrak{D}$ such that its orthogonal complement $\mathfrak{D}^\perp$ is a totally real distribution, i.e., $J\mathfrak{D}_x^\perp \subseteq T_x^\perp N$.

In this series of papers, we shall obtain some fundamental properties of CR-submanifolds in Kaehler manifolds.

2. Preliminaries

Let $\tilde{M}$ be a complex m-dimensional Kaehler manifold with complex structure J, and N a real n-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$. We denote by $\langle\ ,\ \rangle$ the metric tensor of $\tilde{M}$ as well as that induced on N. Let ∇ and $\tilde{\nabla}$ be the covariant differentiations on N and $\tilde{M}$, respectively. Then the Gauss and Weingarten formulas for N are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for any vector fields X, Y tangent to N and any vector field ξ normal to N, where σ denotes the second fundamental form, and D the linear connection, called the normal connection, induced in the normal bundle $T^\perp N$. The second fundamental tensor A_ξ is related to σ by

$$\langle A_\xi X, Y\rangle = \langle \sigma(X, Y), \xi\rangle. \tag{2.3}$$

Communicated by K. Yano, September 9, 1980.

For any vector field X tangent to N, we put

$$JX = PX + FX, \tag{2.4}$$

where PX and FX are the tangential and normal components of JX, respectively. Then P is an endomorphism of the tangent bundle TN, and F is a normal-bundle-valued 1-form on TN.

For any vector field ξ normal to N, we put

$$J\xi = t\xi + f\xi, \tag{2.5}$$

where $t\xi$ and $f\xi$ are the tangential and normal components of $J\xi$, respectively. Then f is an endomorphism of the normal bundle $T^{\perp}N$, and t is a tangent-bundle-valued 1-form on $T^{\perp}N$.

A Kaehler manifold N is called a *complex-space-form* if it is of constant holomorphic sectional curvature. We denote by $\tilde{M}(c)$ (or $\tilde{M}^m(c)$) a complex m-dimensional complex-space-form of constant holomorphic sectional curvature c. Then the curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ is given by

$$\begin{aligned}\tilde{R}(X, Y)Z = \frac{c}{4}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX \\ -\langle JX, Z\rangle JY + 2\langle X, JY\rangle JZ\}\end{aligned} \tag{2.6}$$

for any vector fields X, Y and Z tangent to $\tilde{M}(c)$. We denote the curvature tensors associated with ∇ and D by R and $R^{\perp}$ respectively.

For the second fundamental form σ, we define the covariant differentiation $\overline{\nabla}$ with respect to the connection in $(TN) \oplus (T^{\perp}N)$ by

$$(\overline{\nabla}_X\sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{2.7}$$

for any vector fields X, Y and Z tangent to N.

The equations of Gauss, Codazzi, and Ricci are then given respectively by [4]

$$\begin{aligned}R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \langle\sigma(X, W), \sigma(Y, Z)\rangle \\ -\langle\sigma(X, Z), \sigma(Y, W)\rangle.\end{aligned} \tag{2.8}$$

$$(\tilde{R}(X, Y)Z)^{\perp} = (\overline{\nabla}_X\sigma)(Y, Z) - (\overline{\nabla}_Y\sigma)(X, Z), \tag{2.9}$$

$$\tilde{R}(X, Y; \xi, \eta) = R^{\perp}(X, Y; \xi, \eta) - \langle[A_\xi, A_\eta]X, Y\rangle, \tag{2.10}$$

where $R(X, Y; Z, W) = \langle R(X, Y)Z, W\rangle, \cdots$, etc., X, Y, Z, W are tangent to N, ξ and η are normal to N, and $^{\perp}$ in (2.9) denotes the normal component.

Definition 2.1. A submanifold N of Kaehler manifold $\tilde{M}$ is called a *CR-submanifold* if there is a differentiable distribution $\mathfrak{D}: x \to \mathfrak{D}_x \subseteq T_xN$ on N satisfying the following conditions:

(a) $\mathfrak{D}$ is holomorphic, i.e., $J\mathfrak{D}_x = \mathfrak{D}_x$ for each $x \in N$, and

(b) the complementary orthogonal distribution $\mathfrak{D}^\perp: x \to \mathfrak{D}_x^\perp \subseteq T_xN$ is totally real, i.e., $J\mathfrak{D}_x^\perp \subset T_n^\perp N$ for each $x \in N$.

If $\dim \mathfrak{D}_x^\perp = 0$ (respectively, $\dim \mathfrak{D}_x = 0$), then the CR-submanifold N is a *holomorphic* submanifold [11] (respectively, *totally real submanifold* [8]). If $\dim \mathfrak{D}_x^\perp = \dim T_x^\perp N$, then the CR-submanifold is an *anti-holomorphic submanifold* [3] (or generic submanifold [12]). A CR-submanifold is called a *proper CR-submanifold* if it is neither holomorphic nor totally real.

We shall always denote by h the *complex dimension* of $\mathfrak{D}_x$ and by p the *real dimension* of $\mathfrak{D}_x^\perp$, i.e., $h = \dim_C \mathfrak{D}_x$ and $p = \dim_{\mathbf{R}} \mathfrak{D}_x^\perp$.

We denote by ν the complementary orthogonal subbundle of $J\mathfrak{D}^\perp$ in $T^\perp N$. Hence we have

$$T^\perp N = J\mathfrak{D}^\perp \oplus \nu, \quad J\mathfrak{D}^\perp \perp \nu. \tag{2.11}$$

3. Some basic lemmas

In this section we shall give some basic lemmas for later use.

Let $\tilde{M}$ be a Kaehler manifold. Then we have $\tilde{\nabla} J = 0$. If N is a CR-submanifold of $\tilde{M}$, then (2.1) and (2.2) give

$$J\nabla_U Z + J\sigma(U, Z) = -A_{JZ}U + D_U JZ \tag{3.1}$$

for U tangent to N and Z in $\mathfrak{D}^\perp$.

Lemma 3.1. *Let N be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Then we have*

$$\langle \nabla_U Z, X\rangle = \langle JA_{JZ}U, X\rangle, \tag{3.2}$$

$$A_{JZ}W = A_{JW}Z, \tag{3.3}$$

$$A_{J\xi}X = -A_\xi JX, \tag{3.4}$$

for U tangent to N, X in $\mathfrak{D}$, Z and W in $\mathfrak{D}^\perp$, and ξ in ν.

Proof. (3.2) and (3.3) follow immediately from (3.1).

(3.4) follows from the fact that $\langle \sigma(JX, Y), \xi\rangle = \langle \tilde{\nabla}_Y JX, \xi\rangle = \langle J\sigma(X, Y), \xi\rangle$.

Lemma 3.2. *Let N be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Then for any Z, W in $\mathfrak{D}^\perp$ we have*

$$D_W JZ - D_Z JW \in J\mathfrak{D}^\perp. \tag{3.5}$$

Proof. For any ξ in ν and Z, W in $\mathfrak{D}^\perp$, we have

$$\langle A_{J\xi}Z, W\rangle = -\langle \tilde{\nabla}_Z J\xi, W\rangle = \langle D_Z\xi, JW\rangle = -\langle \xi, D_Z JW\rangle.$$

Thus we obtain

$$\langle \xi, D_W JZ - D_Z JW\rangle = \langle A_{J\xi}Z, W\rangle - \langle A_{J\xi}W, Z\rangle = 0.$$

Since this is true for all ξ in ν, (3.5) holds.

From Lemma 3.1 it follows that we have $J[Z, W] = J(\nabla_Z W - \nabla_W Z) = D_Z JW - D_W JZ$. Thus by Lemma 3.2 we obtain

Lemma 3.3. *The totally real distribution* $\mathfrak{D}^\perp$ *of a CR-submanifold in a Kaehler manifold is integrable.*

This theorem has been generalized to CR-submanifolds in a locally conformal almost Kaehler manifolds in [3].

For the holomorphic distribution $\mathfrak{D}$ we have, [1], [3],

Lemma 3.4. *Let N be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Then $\mathfrak{D}$ is integrable if and only if*

$$\langle\sigma(X, JY), JZ\rangle = \langle\sigma(JX, Y), JZ\rangle$$

for any vectors X, Y in $\mathfrak{D}$, and Z in $\mathfrak{D}^\perp$.

From (3.2) we obtain, [2],

Lemma 3.5. *For a CR-submanifold N in a Kaehler manifold $\tilde{M}$, the leaf $N^\perp$ of $\mathfrak{D}^\perp$ is totally geodesic in N if and only if*

$$\langle\sigma(\mathfrak{D}, \mathfrak{D}^\perp), J\mathfrak{D}^\perp\rangle = 0. \tag{3.6}$$

The following lemma can be obtained easily from Lemma 3.4.

Lemma 3.6. *If* (3.6) *holds and $\mathfrak{D}$ is integrable, then for any X in $\mathfrak{D}$ and ξ in $J\mathfrak{D}^\perp$, we have*

$$A_\xi JX = -JA_\xi X. \tag{3.7}$$

Let P, F, t and f be the endomorphisms and vector-valued 1-forms defined by (2.4) and (2.5). Put

$$(\overline{\nabla}_U P)V = \nabla_U(PV) - P\nabla_U V, \tag{3.8}$$

$$(\overline{\nabla}_U F)V = D_U(FV) - F(\nabla_U V), \tag{3.9}$$

$$(\overline{\nabla}_U t)\xi = \nabla_U(t\xi) - tD_U\xi, \tag{3.10}$$

$$(\overline{\nabla}_U f)\xi = D_U(f\xi) - fD_U\xi \tag{3.11}$$

for U, V tangent to N, and ξ normal to N. Then the endomorphism P (respectively, endomorphism f, 1-forms F or t) is *parallel* if $\overline{\nabla}P = 0$ (respectively, $\overline{\nabla}f = 0$, $\overline{\nabla}F = 0$, or $\overline{\nabla}t = 0$).

From (2.1), (2.2) and (2.4) we obtain

$$(\overline{\nabla}_U P)V = t\sigma(U, V) + A_{FV}U. \tag{3.12}$$

4. CR-products in Kaehler manifolds

According to Lemma 3.3, every CR-submanifold N of a Kaehler manifold is foliated by totally real submanifolds. In §4–§7 we shall study the problem when a CR-submanifold N is a Riemannian product of a holomorphic submanifold and a totally real submanifold?

Definition 4.1. A CR-submanifold N of a Kaehler manifold $\tilde{M}$ is called a *CR-product* if it is locally a Riemannian product of a holomorphic submanifold N^T and a totally real submanifold $N^\perp$ of $\tilde{M}$.

First we give the following *characterization* of CR-products.

Theorem 4.1. *A CR-submanifold of a Kaehler manifold $\tilde{M}$ is a CR-product if and only if P is parallel, i.e., $\overline{\nabla} P = 0$.*

Proof. If P is parallel, (3.12) gives

$$t\sigma(U, V) = -A_{FV}U \tag{4.1}$$

for any vectors U, V tangent to N. In particular, if $X \in \mathfrak{D}$, then $FX = 0$. Hence (4.1) implies $t\sigma(U, X) = 0$, i.e.,

$$A_{JZ}X \equiv 0, \tag{4.2}$$

for any Z in $\mathfrak{D}^\perp$, and X in $\mathfrak{D}$. Thus by Lemmas 3.4 and 3.5 we know that $\mathfrak{D}$ is integrable and the leaf $N^\perp$ of $\mathfrak{D}^\perp$ is totally geodesic in N. Let N^T be a leaf of $\mathfrak{D}$. For any X, Y in $\mathfrak{D}$, and Z in $\mathfrak{D}^\perp$, (4.2) and Lemma 3.1 give

$$0 = \langle A_{JZ}Y, X\rangle = \langle JA_{JZ}Y, JX\rangle = \langle \nabla_Y Z, JX\rangle = -\langle Z, \nabla_Y JX\rangle.$$

From this we may conclude that N^T is totally geodesic in N, and N is a CR-product in $\tilde{M}$.

Conversely, if N is a CR-product, then $\nabla_U Y \in \mathfrak{D}$ for any Y in $\mathfrak{D}$ and U tangent to N. Thus by (2.1) and (2.2), we may obtain $J\sigma(U, Y) = \sigma(U, JY)$. From this, together with (2.1) and (3.8), we may prove that $(\overline{\nabla}_U P)Y = 0$. Similarly, from $\nabla_U Z \in \mathfrak{D}^\perp$ for any Z in $\mathfrak{D}^\perp$ and U tangent to N, we may also prove that $(\overline{\nabla}_U P)Z = 0$.

From the proof of Theorem 4.1 we have the following.

Lemma 4.2. *A CR-submanifold N in a Kaehler manifold $\tilde{M}$ is a CR-product if and only if $A_{J\mathfrak{D}^\perp}\mathfrak{D} = 0$.*

Remark 4.1. In [2] Bejancu-Kon-Yano proved that if N is an *anti-holomorphic submanifold* and $\overline{\nabla} P = 0$, then N is a CR-product.

Lemma 4.3. *Let N be a CR-product of a Kaehler manifold $\tilde{M}$. Then for any unit vectors X in $\mathfrak{D}$ and Z in $\mathfrak{D}^\perp$ we have*

$$\tilde{H}_B(X, Z) = 2\|\sigma(X, Z)\|^2,$$

where $\tilde{H}_B(X, Z) = \tilde{R}(X, JX; JZ, Z)$ is the holomorphic bisectional curvature of $X \wedge Z$.

Proof. Let N be a CR-product in $\tilde{M}$. Then we have (4.2) for any Z in $\mathfrak{D}^\perp$ and X in $\mathfrak{D}$. Thus by equation (2.9) of Codazzi we obtain

$$\tilde{R}(X, JX; Z, JZ) = \langle D_X\sigma(JX, Z) - D_{JX}\sigma(X, Z), JZ\rangle, \tag{4.3}$$

where we have used the fact that N^T is totally geodesic in N. Since $\langle\sigma(\mathfrak{D}, \mathfrak{D}^\perp), J\mathfrak{D}^\perp\rangle = 0$, (4.2) and (4.3) imply

$$
\begin{aligned}
\tilde{R}(X, JX; Z, JZ) &= \langle\sigma(X, Z), D_{JX}JZ\rangle - \langle\sigma(JX, Z), D_XJZ\rangle \\
&= \langle\sigma(X, Z), J\tilde{\nabla}_{JX}Z\rangle - \langle\sigma(JX, Z), J\tilde{\nabla}_XZ\rangle \qquad (4.4)\\
&= \langle\sigma(X, Z), J\sigma(JX, Z)\rangle - \langle\sigma(JX, Z), J\sigma(X, Z)\rangle.
\end{aligned}
$$

Thus by (4.2) and Lemma 3.6 we obtain the lemma.

Theorem 4.4. *Let $\tilde{M}$ be a Kaehler manifold with negative holomorphic bisectional curvature. Then every CR-product in $\tilde{M}$ is either a holomorphic submanifold or a totally real submanifold. In particular, there exists no proper CR-product in any complex hyperbolic space $\tilde{M}^m(c)$ $(c < 0)$.*

Corollary 4.5. *Let $\tilde{M}^m$ be a Kaehler manifold with $\tilde{H}_B > 0$, and N a proper CR-product in $\tilde{M}$. Then* (1) *N is not an anti-holomorphic submanifold, and* (2) *$\sigma(\mathfrak{D}, \mathfrak{D}^\perp) \neq 0$; hence N is not totally geodesic in $\tilde{M}$.*

Theorem 4.4 and Corollary 4.5 follow immediately from Lemma 4.3.

Theorem 4.6. *Every CR-product N in $\mathbf{C}^m$ is the Riemannian product of a holomorphic submanifold in a linear complex subspace $\mathbf{C}^N$ and a totally real submanifold of a $\mathbf{C}^{m-N}$ locally, i.e.,*

$$N = N^T \times N^\perp \subset \mathbf{C}^N \times \mathbf{C}^{m-N} = \mathbf{C}^m.$$

Proof. Since N is a CR-product in $\mathbf{C}^m$, Lemma 4.3 implies

$$\sigma(\mathfrak{D}, \mathfrak{D}^\perp) = 0. \qquad (4.5)$$

Thus by applying a lemma of Moore [10] we see that $N = N^T \times N^\perp$ is a product submanifolds in $\mathbf{R}^r \times \mathbf{R}^{2m-r}$. Since N^T is a holomorphic submanifold of $\mathbf{C}^m$, we may choose $\mathbf{R}^r$ to be a complex linear subspace of $\mathbf{C}^m$.

5. Standard CR-products

In this section we shall derive the smallest codimension of CR-product in complex projective spaces and classify CR-product in complex projective spaces with smallest codimension.

First we shall give examples of CR-products in $\mathbf{C}P^m$. Let $\mathbf{C}P^m$ denote the complex m-dimensional complex projective space with constant holomorphic sectional curvature 4. We define a mapping

$$S_{hp}\colon \mathbf{C}P^h \times \mathbf{C}P^p \to \mathbf{C}P^{h+p+hp}$$

by

$$(z_0, \cdots, z_h; \eta_0, \cdots, \eta_p) \to (z_0\eta_0, \cdots, z_i\eta_j, \cdots, z_h\eta_p),$$

where $(z_0, \cdots, z_h)$ (respectively, $(\eta_0, \cdots, \eta_p)$) are the homogeneous coordinates of $\mathbf{C}P^h$ (respectively, $\mathbf{C}P^p$). It is easy to see that $S_{h,p}$ is a Kaehler imbedding of the Riemannian product $\mathbf{C}P^h \times \mathbf{C}P^p$ into $\mathbf{C}P^{h+p+hp}$. Let $N^\perp$ be a p-dimensional totally real submanifold of $\mathbf{C}P^p$. Then $\mathbf{C}P^h \times N^\perp$ induced a natural CR-product in $\mathbf{C}P^{h+p+hp}$ via $S_{h,p}$, in which $N^T = \mathbf{C}P^h$ is a totally geodesic submanifold, and $N^\perp$ is a totally real submanifold of $\mathbf{C}P^{h+p+hp}$.

Definition 5.1. A CR-product $N = N^T \times N^\perp$ in $\mathbf{C}P^m$ is called a *standard CR-product* if

(1) $m = h + p + hp$ and

(2) N^T is a totally geodesic holomorphic submanifold of $\mathbf{C}P^m$, where $h = \dim_{\mathbf{C}} \mathfrak{D}_x$ and $p = \dim_{\mathbf{R}} \mathfrak{D}_x^\perp$.

We shall prove that $m = h + p + hp$ is in fact the smallest dimension of $\mathbf{C}P^m$ for admitting a CR-product. First we shall prove the following lemma.

Lemma 5.1. *Let N be a CR-product in* $\mathbf{C}P^m$. *Then*

$$\{\sigma(X_i, Z_\alpha)\} \quad i = 1, \cdots, 2h, \alpha = 1, \cdots, p$$

are orthonormal vectors in ν $(T^\perp N = J\mathfrak{D}^\perp \oplus \nu)$, *where* $X_1, \cdots, X_{2h}$ *and* $Z_1, \cdots, Z_p$ *are orthonormal bases for* $\mathfrak{D}_x$ *and* $\mathfrak{D}_x^\perp$ *respectively.*

Proof. Since $\mathbf{C}P^m$ is of constant holomorphic sectional curvature 4 and N is a CR-product, Lemma 4.2 gives

$$\|\sigma(X, Z)\| = 1, \tag{5.2}$$

for any unit vector X in $\mathfrak{D}$ and Z in $\mathfrak{D}^\perp$. Hence we may obtain by linearity that

$$\langle \sigma(X_i, Z), \sigma(X_j, Z)\rangle = 0, \quad i \neq j. \tag{5.3}$$

Moreover, by Lemma 4.2 we see that $\sigma(X, Z)$ is a normal vector in ν. Hence, if $\dim_{\mathbf{R}} \mathfrak{D}_x^\perp = 1$, the lemma is proved.

If $\dim_{\mathbf{R}} \mathfrak{D}_x^\perp = p \geqslant 2$, then from (5.3) it follows that

$$\langle \sigma(X_i, Z_\alpha), \sigma(X_j, Z_\beta)\rangle + \langle \sigma(X_i, Z_\beta), \sigma(X_j, Z_\alpha)\rangle = 0 \tag{5.4}$$

for $i \neq j$, $\alpha \neq \beta$.

On the other hand, because $N = N^T \times N^\perp$ is a CR-product, we have

$$R(X_i, X_j; Z_\alpha, Z_\beta) = 0. \tag{5.5}$$

Moreover, by Theorem 6.1 of [3] we obtain

$$\tilde{R}(X_i, X_j; Z_\alpha, Z_\beta) = 0. \tag{5.6}$$

Therefore by (2.8), (5.5) and (5.6) we get

$$\langle \sigma(X_i, Z_\alpha), \sigma(X_j, Z_\beta)\rangle = \langle \sigma(X_i, Z_\beta), \sigma(X_j, Z_\alpha)\rangle. \tag{5.7}$$

Combining (5.4) with (5.7) gives the lemma.

As immediate consequence of Lemma 5.1 we obtain the following.

Theorem 5.2. *Let N be a CR-product in $\mathbf{C}P^m$. Then*

$$m \geqslant h + p + hp. \tag{5.8}$$

Remark 5.1. Since the standard CR-products in $\mathbf{C}P^m$ satisfies the equality sign of (5.8). The estimate of m given in (5.8) is best possible.

The following result seem to be remarkable.

Theorem 5.3. *Every CR-product N in $\mathbf{C}P^m$ with $m = h + p + hp$ is a standard CR-product.*

Proof. Let N be a CR-product in $\mathbf{C}P^m$ with $m = h + p + hp$. Then for any X, Y, Z in $\mathfrak{D}$ and W in $\mathfrak{D}^\perp$ we have, from (2.8),

$$0 = \tilde{R}(X, Y; Z, W) + \langle\sigma(X, W), \sigma(Y, Z)\rangle - \langle\sigma(X, Z), \sigma(Y, W)\rangle. \tag{5.9}$$

On the other hand, by (2.6) we obtain

$$\tilde{R}(X, Y; Z, W) = 0.$$

Thus (5.9) gives

$$\langle\sigma(X, W), \sigma(Y, Z)\rangle = \langle\sigma(X, Z), \sigma(Y, W)\rangle. \tag{5.10}$$

In particular, if $Y = JX$, then Lemmas 3.1, 3.6, 4.2, and (5.10) imply

$$\begin{aligned} \langle\sigma(X, Z), \sigma(JX, W)\rangle &= \langle\sigma(JX, Z), \sigma(X, W)\rangle \\ &= \langle J\sigma(X, Z), \sigma(X, W)\rangle \\ &= -\langle\sigma(X, Z), \sigma(JX, W)\rangle, \end{aligned} \tag{5.11}$$

from which we get

$$\langle\sigma(X, Z), \sigma(JX, W)\rangle = 0. \tag{5.12}$$

Combining (5.11) with (5.12) yields

$$\langle\sigma(X, Z), \sigma(X, W)\rangle = 0$$

for any X, Z in $\mathfrak{D}$ and W in $\mathfrak{D}^\perp$. Thus by linearity we have

$$\langle\sigma(X, Z), \sigma(Y, W)\rangle + \langle\sigma(Y, Z), \sigma(X, W)\rangle = 0.$$

Combining this with (5.10), we obtain, for any X, Y, Z in $\mathfrak{D}$ and W in $\mathfrak{D}^\perp$,

$$\langle\sigma(X, Z), \sigma(Y, W)\rangle = 0. \tag{5.13}$$

Now since $m = h + p + hp$, Lemma 5.1 and (5.13) show that $\sigma(X, Z)$ lies in $J\mathfrak{D}^\perp$ for any X, Z in $\mathfrak{D}$. On the other hand, Lemma 4.2 shows that $\sigma(X, Z)$ must lie in ν. Consequently, we have $\sigma(\mathfrak{D}, \mathfrak{D}) = 0$. Therefore by the fact that N is a CR-product, N^T must be totally geodesic in N. Thus N is totally geodesic in $\mathbf{C}P^m$.

As an immediate consequence of Theorem 5.3 we have the following.

Theorem 5.4. *Let $M = M_1 \times M_2$ be the Riemannian product of two Kaehler manifolds with $\dim_{\mathbf{C}} M_1 = h$ and $\dim_{\mathbf{C}} M_2 = p$. Then $M_1 \times M_2$ admits a Kaehler immersion in $\mathbf{C}P^{h+p+hp}$ if and only if both M_1 and M_2 are* *complex-space-forms of constant holomorphic sectional curvature* 4. *Moreover* $h + p + hp$ *is the smallest dimension of* $\mathbf{C}P^m$ *which admits such Kaehler immersions, and* M_1 *and* M_2 *are both totally geodesic in* $\mathbf{C}P^m$, *and any such immersion is obtained by the Segre imbedding.*

6. Length of second fundamental form

The main purpose of this section is to prove the following.

Theorem 6.1. *Let* N *be a CR-product in* $\mathbf{C}P^m$. *Then we have*

$$\|\sigma\|^2 \geqslant 4hp, \tag{6.1}$$

where $h = \dim_{\mathbf{C}} \mathfrak{D}_x$ *and* $p = \dim_{\mathbf{R}} \mathfrak{D}_x^{\perp}$. *If the equality sign of* (6.1) *holds, then* N^T *and* $N^{\perp}$ *are both totally geodesic in* $\mathbf{C}P^m$. *Moreover, the immersion is rigid. In this case* N^T *is a complex-space-form of constant holomorphic sectional curvature* 4, *and* $N^{\perp}$ *is a real-space-form of constant sectional curvature* 1.

Proof. Since $\mathbf{C}P^m$ is of constant holomorphic sectional curvature 4 and N is a CR-product, Lemma 4.3 gives

$$\|\sigma(X, Z)\| = 1 \tag{6.2}$$

for any unit vectors X in $\mathfrak{D}$ and Z in $\mathfrak{D}^{\perp}$. Thus we have

$$\|\sigma\|^2 = 4hp + \sum_{A,B=1}^{2h} \|\sigma(X_A, X_B)\|^2 + \sum_{\alpha,\beta=1}^{p} \|\sigma(Z_\alpha, Z_\beta)\|^2, \tag{6.3}$$

where $\{X_1, \cdots, X_{2h}\}$ (respectively, $\{Z_1, \cdots, Z_p\}$) is an orthonormal basis of $\mathfrak{D}$ (respectively, $\mathfrak{D}^{\perp}$). From (6.3) we obtain (6.1).

If the equality sign of (6.1) holds, (6.3) implies

$$h(\mathfrak{D}, \mathfrak{D}) = 0 \quad \text{and} \quad h(\mathfrak{D}^{\perp}, \mathfrak{D}^{\perp}) = 0. \tag{6.4}$$

Since N^T and $N^{\perp}$ are both totally geodesic in N, (6.4) implies that N^T and $N^{\perp}$ are both totally geodesic in $\mathbf{C}P^m$. Consequently, equation (2.8) of Gauss shows that N^T is a complex-space-form of constant holomorphic sectional curvature 4, and $N^{\perp}$ is a real-space-form of constant sectional curvature 1.

Now we shall prove that the immersion is rigid if $\|\sigma\|^2 = 4hp$.

Since $N = N^T \times N^{\perp}$ is a Riemannian product of N^T and $N^{\perp}$, we may assume that $\{X_1, \cdots, X_h, JX_1, \cdots, JX_h\}$ is an orthonormal basis of $\mathfrak{D}$ such that $X_1, \cdots, X_h, JX_1, \cdots, JX_h$ are parallel along $\mathfrak{D}^{\perp}$, and $\{Z_1, \cdots, Z_p\}$ is an orthonormal basis of $\mathfrak{D}^{\perp}$, which are parallel along $\mathfrak{D}$. Thus we have

$$\begin{aligned} &\nabla_{X_A} Z_\alpha = 0, \nabla_{Z_\alpha} X_A = 0, \\ &A, B, C, \cdots = 1, \cdots, 2h; \alpha, \beta, \gamma = 1, \cdots, p, \end{aligned} \tag{6.5}$$

where $X_{h+i} = JX_i$, $i = 1, \cdots, h$. If we put

$$\xi_{(A,\alpha)} = \sigma(X_A, Z_\alpha), \tag{6.6}$$

then Lemma 5.1 implies that

$$JZ_\alpha,\ \xi_{(A,\beta)},\quad \alpha, \beta = 1, \cdots, p;\ A = 1, \cdots, 2h, \tag{6.7}$$

are orthonormal vectors in $T^\perp N$. Let

$$\mu = \operatorname{Span}\{\xi_{(A,\alpha)} | A = 1, \cdots, 2h;\ \alpha = 1, \cdots, p\}.$$

Then μ_x, $x \in N$, is a $2hp$-dimensional linear subspace of ν_x.

Denote by $\mu_x^\perp$ the complementary orthogonal subspace of μ_x in ν_x. Then

$$T^\perp N = J\mathfrak{D}^\perp \oplus \mu \oplus \mu^\perp,$$

where $J\mathfrak{D}^\perp$, μ and $\mu^\perp$ are mutually orthogonal.

From Lemma 4.2, (6.4) and (6.6) it follows that

$$\operatorname{Im} \sigma = \mu, \tag{6.8}$$

and $TN \oplus J\mathfrak{D}^\perp \oplus \mu$ is a complex vector bundle over N.

We prove the following lemmas.

Lemma 6.2. *$J\mathfrak{D}^\perp \oplus \mu$ is a parallel normal subbundle, i.e., $D_U\xi \in J\mathfrak{D}^\perp \oplus \mu$ for any U in TN and any ξ in $J\mathfrak{D}^\perp \oplus \mu$.*

Proof. Let U be any vector in TN, Z in $\mathfrak{D}$ and η in $\mu^\perp$. Then (6.8) implies

$$0 = \langle \sigma(U, Z), \eta \rangle = \langle \tilde{\nabla}_U JZ, J\eta \rangle = \langle D_U JZ, J\eta \rangle.$$

Hence

$$D_U(J\mathfrak{D}^\perp) \subseteq J\mathfrak{D}^\perp \oplus \mu, \tag{6.9}$$

for any U tangent to N.

From (2.6), (2.9), (6.6) and (6.8) we have

$$0 = \tilde{R}(X_A, Z_\alpha; Z_\beta, \eta) = -\langle D_{Z_\alpha}\xi_{(A,\beta)}, \eta \rangle,$$
$$0 = \tilde{R}(X_A, Z_\alpha; X_B, \eta) = \langle D_{X_A}\xi_{(B,\alpha)}, \eta \rangle.$$

Consequently, we have $D_U\mu \subseteq J\mathfrak{D}^\perp \oplus \mu$, which together with (6.9) gives the lemma.

Combining (6.8) with Lemma 6.2 we may conclude that N is in fact lies in a totally geodesic $\mathbf{C}P^{h+p+hp}$ of $\mathbf{C}P^m$.

Now we put

$$\nabla_{X_A} X_B = \sum \Gamma_{AB}^C X_C,\ \nabla_{Z_\alpha} Z_\beta = \sum \Gamma_{\alpha\beta}^\gamma Z_\gamma. \tag{6.10}$$

Then from (2.9), (6.4), (6.5), (6.6) and (6.8) we have that

$$\begin{aligned} 0 &= \tilde{R}(X_A, Z_\alpha; Z_\beta, \xi_{(\beta,\gamma)}) \\ &= \langle \sigma(X_A, \nabla_{Z_\alpha} Z_\beta) - D_{Z_\alpha}\xi_{(A,\beta)}, \xi_{(B,\gamma)} \rangle, \end{aligned}$$

which implies that

$$\langle D_{Z_\alpha}\xi_{(A,\beta)}, \xi_{(B,\gamma)}\rangle = \Gamma^{\gamma}_{\alpha\beta}\delta_{AB}. \tag{6.11}$$

Similarly, by considering $\tilde{R}(Z_\alpha, X_A; X_B, \xi_{(C,\beta)})$, $\tilde{R}(X_A, Z_\alpha; Z_\beta, JZ_\gamma)$, and $\tilde{R}(Z_\alpha, X_A; X_B, JZ_\beta)$, respectively, we obtain

$$\langle D_{X_A}\xi_{(B,\alpha)}, \xi_{(C,\beta)}\rangle = \Gamma^{C}_{AB}\delta_{\alpha\beta}, \tag{6.12}$$

$$\langle D_{Z_\alpha}\xi_{(A,\beta)}, JZ_\gamma\rangle = 0, \tag{6.13}$$

$$\langle D_{X_A}\xi_{(B,\alpha)}, JZ_\beta\rangle = \langle X_A, JX_B\rangle\delta_{\alpha\beta}. \tag{6.14}$$

Moreover, from (6.8) it follows that

$$D_U JZ_\alpha = J\tilde{\nabla}_U Z_\alpha = J\nabla_U Z_\alpha + J\sigma(U, Z_\alpha). \tag{6.15}$$

Since $N = N^T \times N^\perp$ is a CR-product of a complex-space-form N^T of constant holomorphic sectional curvature 4 and a real-space-form $N^\perp$ of constant sectional curvature 1, the Riemannian structure of N is completely determined. From (6.4), (6.6), and (6.8), the second fundamental form of N in $\mathbf{C}P^m$ is also determined completely. Moreover, from (6.11)–(6.15) and Lemma 6.2, we see that the normal connection D on $T^\perp N$ is also completely determined. Hence the immersion is rigid.

Remark 6.1. Let $\mathbf{R}P^p$ be a totally geodesic, totally real submanifold of $\mathbf{C}P^p$. Then the composition of the immersions

$$\mathbf{C}P^h \times \mathbf{R}P^p \to \mathbf{C}P^h \times \mathbf{C}P^p \xrightarrow{S_{h,p}} \mathbf{C}P^{h+p+hp} \to \mathbf{C}P^m$$

gives a CR-product in $\mathbf{C}P^m$ with $\|\sigma\|^2 = 4hp$. Theorem 6.1 tells us that it is in fact the only CR-product in $\mathbf{C}P^m$ with $\|\sigma\|^2 = 4hp$.

As a consequence of Theorem 6.1, we have

Corollary 6.2. *If N is a minimal CR-product in $\mathbf{C}P^m$, then the scalar curvature ρ of N satisfies*

$$\rho \leqslant 4h^2 + 4h + p^2 - p, \tag{6.16}$$

where the equality sign holds if and only if $\|\sigma\|^2 = 4hp$.

Proof. Since N is a minimal CR-product, the Ricci tensor S of N satisfies

$$S(X, X) = (2h + p + 2)\|X\|^2 - \sum \|A_\xi X\|^2, \quad X \in \mathfrak{D},$$

$$S(Z, Z) = 2h + p - 1 - \sum \|A_\xi Z\|^2, \quad Z \in \mathfrak{D}^\perp,$$

which imply that

$$\rho = 4h^2 + 4h + p^2 - p - \|\sigma\|^2 + 4hp. \tag{6.17}$$

Combining (6.1) with (6.17) we obtain (6.16). It is clear that the equality sign of (6.16) holds if and only if $\|\sigma\| = 4hp$.

7. CR-products in Hermitian symmetric spaces

First we give the following.

Lemma 7.1. *Let $\tilde{M}^m$ be a nonpositively curved Kaehler manifold, and N a CR-product in $\tilde{M}^m$. If N is anti-holomorphic, then*

(1) *the Ricci tensors of $\tilde{M}$ and N^T satisfy*

$$\tilde{S}(X, Y) = S^T(X, Y) \tag{7.1}$$

for any vectors X, Y tangent to N^T, and

(2) *N^T is totally geodesic in $\tilde{M}^m$.*

Proof. Since N is a CR-product and $\tilde{M}$ is nonpositively curved, Lemma 4.3 implies

$$\tilde{K}(X, Z) = \tilde{K}(X, JZ) = \sigma(X, Z) = 0 \tag{7.2}$$

for any X in $\mathfrak{D}$ and Z in $\mathfrak{D}^{\perp}$. On the other hand, since N is anti-holomorphic, Lemma 4.2 gives

$$\sigma(\mathfrak{D}, \mathfrak{D}) = 0. \tag{7.3}$$

Thus N^T is a totally geodesic submanifold of $\tilde{M}$. Consequently, we have

$$\tilde{K}(X, Y) = K^T(X, Y). \tag{7.4}$$

for any orthonormal vectors X, Y tangent to N^T, where K^T denotes the sectional curvature of N^T. Combining (7.2) with (7.4), we obtain (7.1).

Theorem 7.2. *Let $\tilde{M}^m$ be a Hermitian symmetric space of noncompact type, and N a complete CR-product in $\tilde{M}^m$. If N is anti-holomorphic, then*

(1) *N^T is also a Hermitian symmetric space of noncompact type.*

(2) *there exists another Hermitian symmetric space $M^{\perp}$ of noncompact type such that*

(2a) *$\tilde{M}$ is the Riemannian product of N^T and $M^{\perp}$, and*

(2b) *N^T is a totally real submanifold of $M^{\perp}$.*

Proof. Since $\tilde{M}^m$ is a Hermitian symmetric space of noncompact type, $\tilde{M}^m$ is nonpositively curved [9]. By Lemma 7.1, we have (7.1) and N^T is totally geodesic in $\tilde{M}^m$. Hence from equation (2.8) of Gauss we have

$$S^T(X, X) = \tilde{S}(X, X) - \sum_{\alpha=1}^{p} \{\tilde{R}(Z_\alpha, X; X, Z_\alpha) + \tilde{R}(JZ_\alpha, X; X, JZ_\alpha)\} \tag{7.5}$$

for any orthonormal basis $Z_1, \cdots, Z_p, JZ_1, \cdots, JZ_p$ of $T^{\perp}N^T$ in $\tilde{M}^m$. Since (7.1) holds and $\tilde{M}$ is nonpositively curved, (7.5) implies

$$\tilde{K}(X, Z) = \tilde{K}(X, JZ) = 0, \tag{7.6}$$

for any X in TN^T and Z in $T^{\perp}N^T$. From these we may obtain the theorem by using the same argument as we gave in the last part of the proof of Theorem 1 of [5].

As an immediate consequence of Theorem 7.2 we have the following.

Theorem 7.3. *Let $\tilde{M}^m$ be an irreducible Hermitian symmetric space of noncompact type. If $\tilde{M}^m$ admits a proper CR-product N, then N is not anti-holomorphic.*

Remark 7.1. Although the complex hyperbolic space admits no proper CR-product (Theorem 4.3), other irreducible Hermitian symmetric spaces of noncompact type admit CR-products in general (see, Chen-Nagano [7]). For example, the rank 2 irreducible Hermitian symmetric space $SU(2, m)/S(U_2 \times U_m)$, admits a proper CR-product N for any $h = \dim_{\mathbf{C}} \mathfrak{D}$ satisfying $0 < h < m$.

8. CR-submanifolds with $\overline{\nabla} F = 0$

Let N be a CR-submanifold in a Kaehler manifold $\tilde{M}$. Then it associates a canonical normal-bundle-valued 1-form F on TN and a tangent-bundle-valued 1-form t on $T^{\perp}N$. In this section we shall classify CR-submanifolds with parallel F (or t).

Lemma 8.1. *For any vectors U, V tangent to N and ξ normal to N, we have*

$$(\overline{\nabla}_U t)\xi = A_{f\xi}U - PA_{\xi}U, \tag{8.1}$$

$$(\overline{\nabla}_U f)\xi = -FA_{\xi}U - \sigma(U, t\xi), \tag{8.2}$$

$$(\overline{\nabla}_U F)V = f\sigma(U, V) - \sigma(U, PV). \tag{8.3}$$

Proof. From (2.1) and (2.2), we have

$$\begin{aligned}\tilde{\nabla}_U J\xi &= \tilde{\nabla}_U t\xi + \tilde{\nabla}_U f\xi = \nabla_U t\xi + \sigma(U, t\xi) - A_{f\xi}U + D_U f\xi \\ &= J\tilde{\nabla}_U \xi = -PA_{\xi}U - FA_{\xi}U + tD_U\xi + fD_U\xi.\end{aligned} \tag{8.4}$$

Comparing the tangential and normal components of both sides of (8.4) yields

$$\nabla_U(t\xi) - tD_U\xi = A_{f\xi}U - PA_{\xi}U, \tag{8.5}$$

$$D_U f\xi - fD_U\xi = -FA_{\xi}U - \sigma(U, t\xi). \tag{8.6}$$

Since the left-hand sides of (8.5) and (8.6) are nothing but $(\overline{\nabla}_U t)\xi$ and $(\overline{\nabla}_U f)\xi$, respectively, we have (8.1) and (8.2).

Similarly, for any U, V tangent to N, (2.1) and (2.2) give

$$\begin{aligned}\tilde{\nabla}_U JV &= \nabla_U PV + \sigma(U, PV) - A_{FV}U + D_U FV \\ &= P\nabla_U V + F\nabla_U V + t\sigma(U, V) + f\sigma(U, V).\end{aligned}$$

Thus

$$D_U FV - F(\nabla_U V) = f\sigma(U, V) - \sigma(U, PV),$$

which is nothing but (8.3).

Proposition 8.2. *Let N be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Then $\overline{\nabla} F = 0$ if and only if $\overline{\nabla} t = 0$.*

Proof. From Lemma 8.1, we see that $\overline{\nabla} t = 0$ if and only if, for any U, V tangent to N, ξ normal to N,

$$\langle A_{f\xi} U, V \rangle = \langle PA_\xi U, V \rangle,$$

i.e.,

$$\langle \sigma(U, V), J\xi \rangle = -\langle \sigma(U, PV), \xi \rangle,$$

which is equivalent to

$$\sigma(U, PV) = f\sigma(U, V), \text{ i. e., } \overline{\nabla} F = 0.$$

Lemma 8.3. *Let N be a CR-submanifold in a Kaehler manifold. Then $\overline{\nabla} F = 0$ if and only if*

(1) N is a CR-product, and

(2) $A_\nu \mathfrak{D}^\perp = 0$.

Proof. By Lemma 8.1, $\overline{\nabla} F = 0$ if and only if

$$\sigma(U, PV) = f\sigma(U, V). \tag{8.7}$$

Thus for any Z in $\mathfrak{D}^\perp$ we have $f\sigma(U, Z) = 0$, which is equivalent to (2). Moreover, for any X in $\mathfrak{D}$, (8.7) gives

$$\sigma(U, JX) = f\sigma(U, X) \in \nu.$$

Thus $A_{J\mathfrak{D}^\perp}\mathfrak{D} = 0$. Consequently by Lemma 4.2, N is a CR-product. Conversely, if $A_{J\mathfrak{D}^\perp}\mathfrak{D} = 0$ and $A_\nu \mathfrak{D}^\perp = 0$, then (8.7) holds by Lemma 3.6.

Lemma 8.4. *Let N be a CR-submanifold in a Kaehler manifold. If $\overline{\nabla} F = 0$, then*

$$\tilde{H}_B(\mathfrak{D}, \mathfrak{D}^\perp) = 0, \tag{8.8}$$

$$\sigma(\mathfrak{D}, \mathfrak{D}^\perp) = 0. \tag{8.9}$$

Proof. (8.9) follows from Lemmas 4.2 and 8.3, and (8.8) from Lemmas 4.3 and (8.9).

From Theorem 4.6 and Lemma 8.4 we obtain the following.

Theorem 8.5. *Let N be a proper CR-submanifold in a complete simply-connected complex-space-form $\tilde{M}^m(c)$. If $\overline{\nabla} F = 0$, then $c = 0$, i.e., $\tilde{M}^m(c) = \mathbf{C}^m$. Moreover, N is the Riemannian product of a holomorphic submanifold N^T of a $\mathbf{C}^{m-p}$ and a totally real submanifold $N^\perp$ of a $\mathbf{C}^p$ locally, where $p =$* $\dim_{\mathbf{R}} N^\perp$.

Proof. From Lemma 8.4 it follows that $\tilde{M}^m(c)$ is $\mathbf{C}^m$. Thus by Theorem 4.6 we see that N is the Riemannian product of a holomorphic submanifold N^T in a $\mathbf{C}^N$ and a totally real submanifold in a $\mathbf{C}^{m-N}$ locally.

Using Lemma 8.3 and the fact that $N^\perp$ is totally geodesic in N, we obtain from Theorem 2 of [6] that $N^\perp$ is in fact lies in a complex p-dimensional linear complex subspace $\mathbf{C}^p$ of $\mathbf{C}^m$ as a totally real submanifold.

Remark 8.1. Let N^T be a holomorphic submanifold of a Kaehler manifold M^T, and $N^\perp$ a totally real submanifold of a Kaehler manifold $M^\perp$. Then it is easy to verify that $N = N^T \times N^\perp$ is a *CR*-submanifold in $\tilde{M} = M^T \times M^\perp$ with $\overline{\nabla} F = 0$. From this we may conclude that the rank-2 irreducible Hermitian symmetric spaces $SU(2, m)/S(U_2 \times U_m)$ and $SU(2 + m)/S(U_2 \times U_m)$ both admit proper *CR*-submanifolds with $\overline{\nabla} F = 0$.

Remark 8.2. From (8.2) it is easy to see that a proper *CR*-submanifold N satisfies $\overline{\nabla} f = 0$ if and only if $J\mathfrak{D}^\perp$ is parallel (or $A_\nu \mathfrak{D}^\perp = 0$).

9. Mixed foliate CR-submanifolds

Definition 9.1. A *CR*-submanifold N in a Kaehler manifold is said to be *mixed foliate* if

(1) $\mathfrak{D}$ is integrable and

(2) $\sigma(\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

Lemma 9.1. *Let N be a mixed foliate CR-submanifold in a Kaehler manifold $\tilde{M}$. Then for any unit vectors X in $\mathfrak{D}$ and Z in $\mathfrak{D}^\perp$ we have*

$$\tilde{H}_B(X, Z) = -2\|A_{JZ}X\|^2. \tag{9.1}$$

Proof. If N is mixed foliate *CR*-submanifold, then

$$\sigma(\mathfrak{D}, \mathfrak{D}^\perp) = 0, [\mathfrak{D}, \mathfrak{D}] \subset \mathfrak{D}, \sigma(X, JY) = \sigma(JX, Y) \tag{9.2}$$

for any X, Y in $\mathfrak{D}$. Thus for any X in $\mathfrak{D}$ and Z in $\mathfrak{D}^\perp$, the equation of Codazzi gives

$$\begin{aligned} \tilde{H}_B(X, Z) &= \langle \sigma(JX, \nabla_X Z), JZ \rangle - \langle \sigma(X, \nabla_{JX} Z), JZ \rangle. \\ &= \langle A_{JZ}JX, \nabla_X Z \rangle - \langle A_{JZ}X, \nabla_{JX} Z \rangle. \end{aligned}$$

Hence by Lemma 3.1 we have

$$\begin{aligned} \tilde{H}_B(X, Z) &= \langle A_{JZ}JX, JA_{JZ}X \rangle - \langle A_{JZ}X, JA_{JZ}JX \rangle \\ &= -2\|A_{JZ}X\|^2. \end{aligned}$$

Theorem 9.2. *Let $\tilde{M}$ be a Kaehler manifold with $\tilde{H}_B > 0$. Then $\tilde{M}$ admits no mixed foliate proper CR-submanifolds.*

This theorem follows immediately from Lemma 9.1.

Corollary 9.3 (*Bejancu-Kon-Yano* [2]). *A complex-space-form $\tilde{M}^m(c)$ with $c > 0$ admits no mixed foliate proper CR-submanifolds.*

Remark 9.1. Geodesic spheres of CP^m are real hypersurfaces with $\sigma(\mathcal{D}, \mathcal{D}^{\perp}) = 0$.

Theorem 9.3. *Let N be a CR-submanifold in $\mathbf{C}^m$. Then N is mixed foliate if and only if N is a CR-product.*

Proof. Let N be a CR-submanifold in $\mathbf{C}^m$. If N is mixed foliate, Lemma 9.1 implies

$$A_{J\mathcal{D}^{\perp}}\mathcal{D} = 0. \tag{9.3}$$

So by Lemma 4.2, N is a CR-product.

Conversely, if N is a CR-product, then (9.3) holds. Thus by Lemma 4.2 and 4.3, we get $\sigma(\mathcal{D}, \mathcal{D}^{\perp}) = 0$. Hence N is mixed foliate.

Remark 9.2. For an anti-holomorphic submanifold N, Theorem 9.3 is due to Bejancu-Kon-Yano [2].

10. CR-submanifolds in Hermitian symmetric spaces of compact type

Using Lemma 9.1 we obtain

Theorem 10.1. *Let $\tilde{M}$ be a compact (type) Hermitian symmetric space and N a mixed foliate CR-submanifold in $\tilde{M}$. Then*

(1) *N is CR-product,*

(2) $\tilde{K}(\mathcal{D}, \mathcal{D}^{\perp}) = 0, \tilde{K}(\mathcal{D}, J\mathcal{D}^{\perp}) = 0,$

(3) $A_{J\mathcal{D}^{\perp}}\mathcal{D} = 0.$

Proof. If N is a mixed foliate CR-submanifold in a compact Hermitian symmetric space $\tilde{M}$, then $\tilde{M}$ is nonnegatively curved [9], and hence by Lemma 9.1 we have

$$\tilde{H}_B(X, Z) = \tilde{K}(X, Z) + \tilde{K}(X, JZ) = 0, \quad A_{JZ}X = 0,$$

which imply (2) and (3). Statement (1) follows from (3) and Lemma 4.2.

Remark 10.1. Although CP^m admits no mixed foliate proper CR-submanifolds (Corollary 9.3), other irreducible compact Hermitian symmetric spaces admit mixed foliate proper CR-submanifolds in general. For example the rank-2 irreducible compact Hermitian symmetric space $SU(2 + m)/S(U_2 \times U_m)$ admit such submanifolds N for any $h = \dim_{\mathbf{C}} \mathcal{D}$ satisfying $0 < h < m$.

In view of Remark 10.1 it seems to be interesting to give the following.

Theorem 10.2. *Let N be a complete mixed foliate CR-submanifold in a compact type Hermitian symmetric space $\tilde{M}^m$. If N is anti-holomorphic, then*

(1) *N is a CR-product $N^T \times N^{\perp}$,*

(2) *N^T is also a compact type Hermitian symmetric space,*

(3) *there is another compact type Hermitian symmetric space $M^\perp$ such that*

(3.1) *$\tilde{M}$ is the Riemannian product $N^T \times M^\perp$,*

(3.2) *$N^\perp$ is a totally real submanifold of $M^\perp$.*

Proof. (1) follows from Theorem 10.1. Let N^T be a leaf of the holomorphic distribution $\mathfrak{D}$. Then, since N^T is totally geodesic in N, Theorem 10.1 implies that N^T is a totally geodesic holomorphic submanifold of $\tilde{M}^m$. Thus N^T is also a compact type Hermitian symmetric space, and (2) is proved.

Using (2) of Theorem 10.1, the equation of Gauss, and the fact that N^T is a totally geodesic submanifold of $\tilde{M}^m$, we obtain that

$$S^T(X, Y) = \tilde{S}(X, Y) \tag{10.1}$$

for any vectors X, Y tangent to N^T. Thus by applying Theorem 2 of [5] we arrive at (3).

The following is an immediate consequence of Theorem 10.2.

Corollary 10.3. *Every irreducible compact type Hermitian symmetric space admits no mixed foliate proper anti-holomorphic submanifolds.*

Remark 10.2. Although Theorem 10.1 shows that every mixed foliate CR-submanifold in a compact Hermitian symmetric space is a CR-product, and Theorem 9.3 shows that a mixed foliate CR-submanifold in $\mathbf{C}^m$ is nothing but a CR-product, CR-products in compact Hermitian symmetric spaces are not mixed foliate in general. For example, the standard CR-products in $\mathbf{C}P^m$ are *not* mixed foliate.

11. Remarks

11.1. The classification of mixed foliate CR-submanifolds in complex hyperbolic spaces and the classification of CR-submanifolds with semi-flat normal connection together with other results on CR-submanifolds will be given in the second part of this series.

11.2. A portion of this paper was done while the author was a visiting professor at the University of Granada, Spain. The author would like to express his hearty thanks to his colleagues there for their hospitality. Moreover, he would like to express his thanks for the valuable discussions with Professors Barros and Urbano on this subject.

Added in Proof. Recently A. Bejancu informed me that he also obtained Theorem 4.1.

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