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MONOTONICITY OF INTEGRAL GAUSS CURVATURE

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1. The results

1.1. Spaces, surfaces, curves, boundaries of domains and so on are supposed to be of class C^∞ unless otherwise stated or subtended. A set S in a Riemannian space will be said to be convex if for any two points of S there exists a geodesic in S not longer than any other arc in S between the two points.

A Riemannian space M will be said to be *enlarging* (*reducing*) if for any two convex compact domains D_0 and $D_1 \subset M$ homeomorphic to a ball, the integral Gauss curvatures G_0 and G_1 of their boundaries with respect to the interior normals satisfy $G_0 \leqslant G_1$ ($G_0 \geqslant G_1$) when $D_0 \subset D_1$. (By Gauss curvature we mean product of principal normal curvatures.)

A space M either enlarging or reducing will be said to be *monotonic*.

1.2. For dimension $n = 2$, Gauss-Bonnet theorem yields a clear idea about monotonic space: those of nonpositive (nonnegative) curvatures are enlarging (reducing) and those of alternating curvature are not monotonic. Moreover, for $n = 2$, the requirement of convexity and homeomorphism to a ball can be omitted in the definition above.

We study in this paper to what extent this situation survives for $n \geqslant 3$.

One can hardly expect that a space M can be enlarging (reducing) if it contains a point p and a 2-dimensional direction σ at p where the sectional curvature is positive (negative). If for example the geodesics emanating from p and tangent to σ form a geodesic 2-dimensional surface S in a neighborhood of p, then for any distinct compact convex domains $D_0 \subset D_1 \subset S$ close to p and homeomorphic to a circle, the total curvatures C_0 and C_1 of their boundaries satisfy $C_0 > C_1$ ($C_0 < C_1$). But D_0 and D_1 can be treated as (degenerate) convex domains in M described in §1.1. Then $C_0 > C_1$ implies $G_0 > G_1$, and $C_0 < C_1$ implies $G_0 < G_1$.

For this reason, we consider only spaces of nonpositive (nonnegative) curvature for the purpose of studying enlarging (reducing) spaces. In contrast

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to the 2-dimensional case, this condition (nonalternating curvature) is far from being sufficient for the monotonicity even though only convex domains are involved. In §4 we construct a space M_1 of dimension $n > 2$ which is arbitrarily close to a part of a sphere by its metric and curvature but is not reducing. At the same time, we prove that spaces of constant positive (negative) curvature are reducing (enlarging). This fact arises easily from Gauss-Bonnet type inequalities (1.3.1) and (1.3.2) of the following theorem.

1.3. Theorem. *Let M be an n-dimensional, $n \geqslant 2$, space of constant curvature k, and compact domains $D_0 \subset D_1 \subset M$ be homeomorphic to a ball. Suppose the normal curvatures of their boundaries on the side of the interior normals are not less than some $\kappa \geqslant 0$. Then the volumes V_0, V_1 of the domains and the integral Gauss curvatures G_0 and G_1 of their boundaries satisfy*

$$G_1 - G_0 \geqslant -(n-1)\kappa^{n-2}k(V_1 - V_0) \quad \text{if } k < 0, \tag{1.3.1}$$

$$G_1 - G_0 \leqslant -(n-1)\kappa^{n-2}k(V_1 - V_0) \quad \text{if } k > 0. \tag{1.3.2}$$

The theorem is proved in §2. The equalities hold when $n = 2$ (by Gauss-Bonnet theorem) and when $k = 0$.

Later on we denote by P_k^n an n-dimensional sphere, or a Euclidean, or hyperbolic space of curvature k. For $k > 0$, P_k^1 is a closed curve of the length $2\pi\sqrt{k}$. For $k \leqslant 0$, we put $P_k^1 = R$. P_k^0 will denote a point.

The equality in (1.3.2) holds also in P_k^n, $k > 0$, $n > 2$ for domains D_0 and D_1 bounded by halves of different spheres P_k^{n-1} spanning the same $(n-2)$-dimensional equator P_k^{n-2}. In this case $G_0 = G_1 = \kappa = 0$.

1.4. The following example shows that the assumptiion of convexity of D_0 in Definition 1.1 is important, i.e., that without the assumption the monotonicity can fail even in spaces of constant curvature.

Let D_1 be, say, a convex ball in P_1^3 (P_{-1}^3), and $D_0 \subset D_1$ be an arbitrary domain homeomorphic to a ball. Denote by S the area of the boundary ∂D_0 of D_0. Integration of Gauss "Theorema Egregium" over ∂D_0 and application of Gauss-Bonnet theorem yield $G_0 = 4\pi - S$ ($G_0 = 4\pi + S$). So for sufficiently large S one has $G_1 > G_0$ ($G_1 < G_0$).

The assumption of convexity of D_1 is also important in the same sense. An appropriate example is an odd-dimensional sphere where D_0 is its half and D_1 is some larger ball.

1.5. The inequalities (1.3.1) and (1.3.2) allow us to establish inequalities similar to Gauss-Bonnet type for a convex domain D, homeomorphic to a ball, in a space of constant curvature. Let D_1 be as in §1.3, and $D_0 \subset D_1$ be a compact metric ball. When D_0 is sufficiently small, the normal curvatures of its boundary $\geqslant \kappa$, and the inequalities (1.3.1) and (1.3.2) hold. Passing now to the limit as D_0 contracts to its center (and omitting the index 1 at D_1, G_1, V_1)

one obtains

$$G - c_n \geqslant -(n-1)\kappa^{n-2}kV \quad \text{if } k < 0, \tag{1.5.1}$$

$$c_n - G \geqslant -(n-1)\kappa^{n-2}kV \quad \text{if } k > 0, \tag{1.5.2}$$

where c_n is the volume of P_1^{n-1}. The equalities hold when $n = 2$ (by Gauss-Bonnet theorem) and when $k = 0$.

1.6. Let C_ε, $0 \leqslant \varepsilon \leqslant 1$, be the class of spaces with sectional curvatures varying in the segment $[1 - \varepsilon, 1 + \varepsilon]$. The example M_1 (see §§1.2, 3.1–3.4) shows that C_ε contains nonreducing spaces (M_1 and close ones) for any $\varepsilon \neq 0$, while C_0 consists entirely of reducing spaces.

We do not have a similar example for spaces with negative sectional curvature. However, we still can show that for any negative numbers $-u^2$ and $-v^2$ satisfying $u^2/v^2 > 2$ and for any $n > 2$, there exists an n-dimensional space M_2 with sectional curvatures varying in the segment $[-u^2, -v^2]$ which is not enlarging. M_2 is constructed in §§3.5–3.8.

In connection with a possible interest in the set of all monotonic spaces, we would like to mention the following observation.

Let M be an n-dimensional, $n \geqslant 3$, space of nonnegative (nonpositive) sectional curvature. Suppose in M there is a point p such that the minimum a and the maximum b of absolute values of the sectional curvatures at the point p satisfy

$$b > 0, \quad \frac{a}{b} < \frac{1}{2}\,\frac{\sqrt{2}-1}{\sqrt{2}+1} \approx 0.085.$$

Then M is not reducing (is not enlarging) and hence not monotonic. The appropriate domains $D_0 \subset D_1$ illustrating this statement are constructed in a small neighborhood of the point p. Their exact description and the calculations involved are rather long and not produced here. The constant 0.085 is certainly not exact.

1.7. One can hardly generalize the inequality (1.3.2) to spaces of positive, other than constant positive, curvature. A "reasonable" generalization would look like

$$G_1 - G_0 \leqslant -(n-1)\kappa^{n-2}\int_{D_1 \setminus D_0} X\,dV,$$

where the quantity X turns into k if the space approaches a space of constant curvature $k > 0$. Then on a certain stage of the approximation there should be $X \geqslant 0$ and $G_1 - G_0 \leqslant 0$ which is not always true according to the example M_1.

1.8. If $k \neq 0$ and $\kappa > 0$, the inequalities (1.5.1) and (1.5.2) yield the proper upper estimates of the volume V of the compact domain D in §1.5. When $\kappa = 0$, the established monotonicity of the space still allows us to estimate V in terms of G if the domain D is not "too degenerate". Denote by b and B the inscribed and circumscribed closed metric balls of the domain D with radii r and R, respectively. Suppose that $r/R \geqslant \varepsilon > 0$, and also that B is compact and homeomorphic to a ball. Then the same is true of b. One can easily see that B and b are convex (2-dimensional sections help here a lot).

For shorter calculation, assume $|k| = 1$. Due to the monotonicity of spaces of constant sectional curvature (see §§1.2, 1.3), the integral Gauss curvatures G_b and G_B of b and B satisfy

$$G_b = c_n \cos^{n-1} r \geqslant G, \quad G_B = c_n \cos^{n-1} R \leqslant G \quad \text{for } k = 1$$

$$G_b = c_n \cosh^{n-1} r \leqslant G, \quad G_B = c_n \cosh^{n-1} R \geqslant G \quad \text{for } k = -1,$$

which together with $r/R \geqslant \varepsilon > 0$ imply

$$\varepsilon f(G) \leqslant \varepsilon R \leqslant r \leqslant R \leqslant r/\varepsilon \leqslant f(G)/\varepsilon,$$

where

$$f(G) = \begin{cases} \cos^{-1}(G/c_n)^{\frac{1}{n-1}} & \text{if } k = 1, \\ \cosh^{-1}(G/c_n)^{\frac{1}{n-1}} & \text{if } k = -1. \end{cases}$$

Therefore $V(\varepsilon f(G)) \leqslant V \leqslant V(f(G)/\varepsilon)$ where $V(x)$ denotes the volume of a ball of radius x in a sphere or hyperbolic space with $k = \pm 1$ respectively.

1.9. Notice that for $n = 3$, the volume V of an arbitrary domain D homeomorphic to a ball in an arbitrary space of sectional curvature $\leqslant -\delta^2 < 0$ can be easily estimated from above in terms of integral Gauss curvature G of ∂D. Indeed, integrating Gauss "Theoreme Egregium" over ∂D and applying then Gauss-Bonnet theorem, one has $\iint_{\partial D} K\, dS = 4\pi - G$ where K is the sectional curvature in the direction tangent to ∂D. Since $K \leqslant -\delta^2 < 0$, the area S of ∂D satisfies now $\delta^2 S \leqslant G - 4\pi$. According to [1, (5.26)], $S \geqslant 2\delta V$. The last two inequalities yield

$$V \leqslant \frac{G - 4\pi}{2\delta^3}. \tag{1.9.1}$$

2. Proof of Theorem 1.3

2.1. Since the case $k = 0$ is trivial, we assume further $k \neq 0$. By a simple limit reasoning, one may assume that $D_0 \subset \operatorname{int} D_1$, and the normal curvatures of D_0, D_1 are greater than κ. We may also assume that $\kappa > 0$ because if $\kappa = 0$

then the boundaries of D_0, D_1 can be approached by equidistant surfaces of positive normal curvature. The surfaces should be inside D_0, D_1 when $k > 0$, and outside when $k < 0$.

The local convexity of ∂D_1 implies that the diameter of D_1 is less than $\pi/\sqrt{k}$ in the case $k > 0$. As D_0 is homeomorphic to a ball, one can easily imbed D_1 (with D_0 inside) into P_k^n. An exponential type of mapping with a pole inside D_1 will realize such an imbedding. Thus one may assume that $M = P_k^n$.

A closed segment with the ends a, b (in any space under consideration) and its length will be denoted sometimes by ab.

2.2. In §§2.2–2.8 we will "connect" D_0 and D_1 by a family D_t, $t \in [0, 1]$, of compact domains which are "convex to the same extent κ".

In what follows, κ always means a positive number.

Let $v \in TP_k^n$, $|v| > 0$. Later on, we denote by B_κ^v the closed domain in P_k^n such that $\partial B_\kappa^v \ni \pi(v)$, where π is the natural projection, v is an interior normal to ∂B_κ^v, and ∂B_κ^v is a surface of constant normal curvature κ on the side of v. (Sphere, orisphere or equidistant of a plane.)

$[\ .\ ]_\varepsilon$ will denote the closed ε-neighborhood of a set. Let $D \in P_k^n$ be a connected closed convex domain with a nonempty boundary. A vector $v \neq 0$ at a point $p \in \partial D$ will be called a *generalized interior normal* to ∂D if v is perpendicular to a supporting plane (P_k^{n-1}) passing through p and if v is directed into the half-space where D is located. The domain D will be called a *κ-convex body* if for any generalized interior normal v at any point $p \in \partial D$ there is a number $\varepsilon > 0$ such that $[p]_\varepsilon \cap D \subset B_\kappa^v$.

2.3. The following statements seem obvious.

At regular points, normal curvatures of the boundary of a κ-convex body are not less than κ.

D_0, D_1, B_κ^v are κ-convex bodies. D_0, D_1 are also $\bar{\kappa}$-convex for any positive $\bar{\kappa} \leqslant \kappa_m$ where κ_m is the minimum of normal curvatures of ∂D_0 and ∂D_1. ($\kappa_m > \kappa$; see §2.1.)

The intersection of any collection of κ-convex bodies is a κ-convex body if it has an interior point.

The section of a κ-convex body by $P_k^m \subset P_k^n$, $m < n$, is also an m-dimensional κ-convex body if it has an interior point.

2.4. Lemma. *Let D be a κ-convex body, and v its generalized interior normal. Then $D \subset B_\kappa^v$.*

(So the condition $[p]_\varepsilon \cap D \subset B_\kappa^v$ in §2.2 can be replaced by $D \subset B_\kappa^v$.)

Proof. Suppose the contrary, that is, there exists $q \in D$, $q \notin B_\kappa^v$. Then $q \neq p \overset{\text{def}}{=} \pi(v)$. One may assume that $q \subset \operatorname{int} D$. Consider the sections of D and B_κ^v by P_k^2 passing through q and p and tangent to v. Put $d = D \cap P_k^2$,

$b = B_\kappa^v \cap P_k^2$. Denote by $\beta \leqslant \pi/2$ the angle betweeen pq and ∂b. Let $pq(\alpha)$, $\alpha \in [0, \beta]$ be the arc of constant curvature with the ends p, q emanating from p within the angle β and forming an angle α with pq. As d is convex (see §2.3), $\breve{pq}(0) = pq \subset d$. Points in $\breve{pq}(\beta)$ close to p lie outside b and therefore do not belong to κ-convex body d. Thus $\breve{pq}(\beta) \not\subset d$. Let $\alpha_* \subset (0, \beta)$ be the maximum number such that $\breve{pq}(\alpha) \subset d$ for $\alpha \leqslant \alpha_*$. Obviously, the arc $pq(\alpha_*)$ "touches ∂d from inside" at a point $r \neq q$; see Fig. 1. (Possibly $r = p$.) This is impossible as d is a κ-convex body (see §2.3), and the curvature of $\breve{pq}(\alpha_*)$ is less than that of $\breve{pq}(\beta)$ which in turn is less than the curvature κ of ∂b.

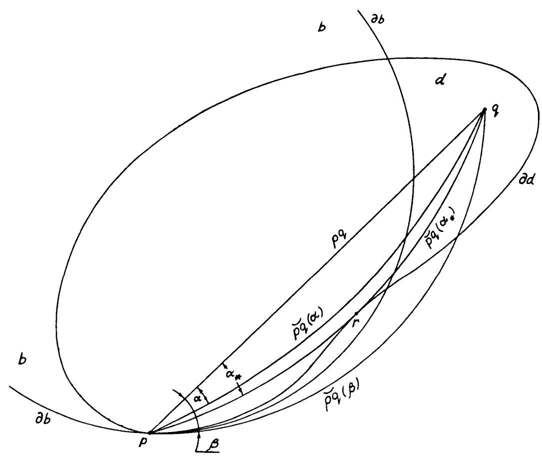

FIG. 1

2.5. The intersection of all κ-convex bodies containing a set $S \subset P_k^n$ will be called *κ-convex hull* and denoted by $H_\kappa(S)$. For example, for a unit open circle $S \subset P_0^2$, $H_\kappa(S) = \bar{S}$ (the closure of S) if $\kappa \leqslant 1$, and does not exist if $\kappa > 1$.

2.6. Let ϕ: $P_1^{n-1} \times [0, 1] \to P_k^n$ be a diffeomorphism such that $\phi(P_1^{n-1} \times 0) = \partial D_0$, $\phi(P_1^{n-1} \times 1) = \partial D_1$. For $t \in [0, 1]$, put $D_t = D_0 \cup \phi(P_1^{n-1} \times [0, t])$, $E_t = H_{\bar\kappa}(D_t)$ where $\bar\kappa = (\kappa + \kappa_m)/2$. ($E_t$ exists since the $\bar\kappa$-convex body $D_1 \supset D_t$; see §2.3.) Notice that for t close to 0 and 1, $E_t = D_t$. Since ϕ is a diffeomorphism, the family D_t and consequently E_t are increasing in the sense that $D_{t_1} \subset \operatorname{int} D_{t_2}$, $E_{t_1} \subset \operatorname{int} E_{t_2}$ for $t_1 < t_2$. We show in §§2.6, 2.7 that the family E_t is continuous.

Take $T \in (0, 1]$, and consider an increasing sequence $t_i \to T$, $i = 1, 2, \ldots$. Put $E_* = \overline{\bigcup_{i=1}^\infty E_{t_i}}$. Since the family E_t is increasing, $E_* \subset E_T$. The relation $D_{t_i} \subset E_{t_i} \subset E_*$ implies $D_T = \overline{\bigcup_{i=1}^\infty D_{t_i}} \subset E_*$. Using Lemma 2.4, it is easy to see that E_* is a $\bar{\kappa}$-convex body and therefore $E_* \supset E_T$. Thus $E_* = E_t$, i.e., the family E_t is continuous from below.

2.7. Take $T \in [0, 1)$, and consider a decreasing sequence $t_i \to T$. Put $E_* = \bigcap_{i=1}^{\infty} E_{t_i}$. Since the family E_t is increasing, $E_* \supset E_T$. Suppose now that the family E_t is not continuous from above, i.e., that $E_* \neq E_T$. Let a point $a \in \partial E_*$ be the most distant from ∂E_T, and let $b \in \partial E_T$ be the closest to a. Denote by g: $[0, s] \to P_k^n$, $s > 0$, the minimal geodesic parametrized by arc length such that $g(0) = a$, $g(s) = b$. Obviously, $\dot{g}(0)$ and $\dot{g}(s)$ are generalized interior normals to ∂E_* and ∂E_T at a and b respectively; see Fig. 2.

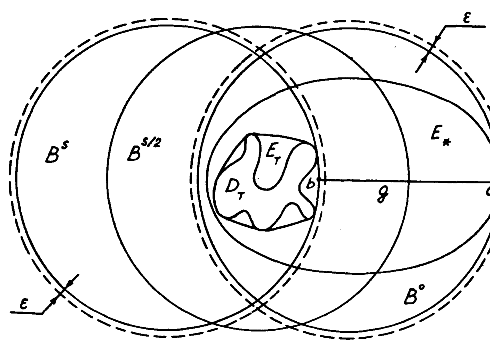

FIG. 2

Put for short $B_{\bar{\kappa}}^{\dot{g}(x)} = B^x$, $x \in [0, s]$. By Lemma 2.4, $E_t \subset B^s$ and $E_* \subset B^0$. Then for any $\varepsilon > 0$ and sufficiently large i, one has

$$D_{t_i} \subset [D_T]_\varepsilon \subset [E_T]_\varepsilon \subset [B^s]_\varepsilon,$$

$$D_{t_i} \subset E_{t_i} \subset [E_*]_\varepsilon \subset [B^0]_\varepsilon.$$

Thus $D_{t_i} \subset [B^0]_\varepsilon \cap [B^0]_\varepsilon$. Choose ε such that $[B^s]_\varepsilon \cap [B^0]_\varepsilon \subset B^{s/2}$; see Fig. 2. Now $D_{t_i} \subset B^{s/2}$. Since $a \in \partial E_* \subset \operatorname{int} E_{t_i}$ and $a \notin B^{s/2}$, the $\bar{\kappa}$-convex body $E_{t_i} \cap B^{s/2}$ which contains D_{t_i} is a part of the $\bar{\kappa}$-convex hull E_{t_i} of D_{t_i}; this is impossible.

2.8. The following lemma can be proved quite easily due to the increase and continuity of the family E_t.

Lemma. *There exists a* (C^∞) *diffeomorphism* d: $P_1^{n-1} \times [0, 1] \to P_k^n$ *such that*

(i) $d(P_1^{n-1} \times 0) = \partial D_0$, $d(P_1^{n-1} \times 1) = \partial D_1$, $d(P_1^{n-1} \times [0, 1]) = D_1 \setminus D_0$,

(ii) *the normal curvatures of the surfaces* $F_t \overset{\text{def}}{=} d(P_1^{n-1} \times t)$ *are not less than* κ.

The proof is reduced to a suitable smoothing of the boundaries ∂E_t of the $\bar{\kappa}$-convex bodies E_t by a standard technique. For t close to 0 and 1, where

$\partial E_t = \partial D_t$ (see §2.6) no smoothing is needed. For the proof, it is convenient to consider first the homeomorphism h: $P_1^{n-1} \times [0, 1] \to P_k^n$ such that $h(a, t)$ is the point in ∂E_t lying on the geodesic perpendicular to ∂D_0 and passing through $\phi(a, 0) \in \partial D_0$. Now d can be obtained by a proper smoothing of h.

2.9. Denote by G_t the integral Gauss curvature of F_t, and by N the interior normal to F_t. Put $Y = d_* D$ where D is the differentiation with respect to $t \in [0, 1]$. It follows from [2, (4)] that

$$\frac{d}{dt} G_t = \int_{F_t} kS_{n-2} \langle Y, N \rangle \, dS,$$

where S_{n-2} is $(n-2)$nd elementary symmetric function of the principal curvatures on the side of N. Due to §2.8(ii), $S_{n-2} \geqslant \kappa^{n-2}(n-1)$. Since $\langle Y, N \rangle < 0$, we have

$$\frac{d}{dt} G_t \geqslant -(n-1)\kappa^{n-2} k \int_{F_t} -\langle Y, N \rangle \, dS, \quad \text{if } k < 0,$$

$$\frac{d}{dt} G_t \leqslant -(n-1)\kappa^{n-2} k \int_{F_t} -\langle Y, N \rangle \, dS, \quad \text{if } k > 0.$$

Since $\int_0^1 \int_{F_t} -\langle Y, N \rangle \, ds \, dt = \int_{D_1 \setminus D_0} dV = V_1 - V_0$, integration over the segment $0 \leqslant t \leqslant 1$ yields (1.3.1) and (1.3.2).

3. The examples M_1 and M_2

3.1. We construct first a nonreducing space M_1 with the sectional curvatures varying within a segment $[1-\varepsilon, 1+\varepsilon]$ for any given $\varepsilon > 0$. The unit here can be easily replaced by any positive number.

Take $m > 0$ satisfying $m^2 \in [1 - \varepsilon/2, 1)$, and consider the manifold $P_1^{n-1} \times I$ where $I = (-\pi/2m, \pi/2m)$. Introduce a metric in the manifold by putting $ds^2 = d\sigma^2 \cos^2 mh + dh^2$, where $d\sigma^2$ is the metric in P_1^{n-1}, and $h \in I$. One can easily check that for any circle $P_1^1 \subset P_1^{n-1}$, the 2-dimensional submanifold $P_1^1 \times I$ ("vertical" plane) has constant interior curvature m^2. Due to symmetry, $P_1^d \times I$, $0 \leqslant d \leqslant n-2$, is a completely geodesic submanifold. (In particular, h-coordinate lines are geodesics.) Therefore the sectional curvatures in "vertical" directions are equal to m^2. Since the hypersurface $P_1^{n-1} \times 0$ is completely geodesic, the sectional curvature in any 2-dimensional direction tangent to $P_1^{n-1} \times 0$ ("horizontal") is equal to 1.

The symmetry of the space helps now to see that all sectional curvatures at any point in $P_1^{n-1} \times 0$ vary within the segment $[m^2, 1]$. Then in a small δ-neighborhood of $P_1^{n-1} \times 0$, the sectional curvatures vary within $(1-\varepsilon, 1+\varepsilon)$. Such a δ-neighborhood $P_1^{n-1} \times (-\delta, \delta)$ will serve as M_1.

3.2. Take $P_1^2 \subset P_1^{n-1}$ and set $M = P_1^2 \times (-\delta, \delta)$. Since M is a completely geodesic submanifold in M_1, it suffices to construct convex compact domains $D_0 \subset D_1 \subset M$, homeomorphic to a ball, with the integral Gauss curvatures G_0, G_1 of their boundaries satisfying $G_0 < G_1$. The reason is that D_0 and D_1 can be regarded as degenerate convex domains in M_1, and the inequality $G_0 < G_1$ in M results in the same inequality in M_1. The degeneracy of D_0 and D_1 in M_1 is not important as they can be "inflated" a little so that the strict inequality $G_0 < G_1$ in M_1 will still hold by continuity.

3.3. As D_0, we take a half of the sphere $P_1^2 \times 0 \subset M$ bounded by a circle P_1^1. So D_0 is again degenerate in M. Since P_1^1 is a geodesic, $G_0 = 0$. Take now an arbitrary point b in the geodesic perpendicular to $P_1^2 \times 0$ and passing through the center c of the circle D_0. Obviously, the minimal geodesics ba with $a \in P_1^1$ exist in M, when b ($\neq c$) is sufficiently close to c. We denote by D_1 the domain bounded by the circle D_0 and the cone composed of the segments ba; see Fig. 3. Obviously, $D_0 \subset D_1$, $D_0 \neq D_1$.

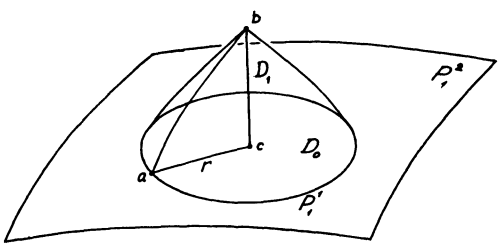

FIG. 3

Consider a triangle abc with $a \in P_1^1$ and sides lying in a vertical plane and bounding a figure isometric to a triangle in $P_{m^2}^2$. Since $ac = \pi/2 < \frac{1}{2}\pi m$ and $cb < \delta < \frac{1}{2}\pi m$, the angle $abc < \frac{1}{2}\pi$, so that D_1 is a convex domain, and Gauss curvature of ∂D_1 at the vertex b is positive. At the other points, the Gauss curvature is zero (in particular, along the rib P_1^1 as P_1^1 is a geodesic). Thus $G_1 > 0$, i.e., $G_1 > G_0$.

Notice that the example fails when $m = 1$, since then the angle $abc = \pi/2$ and $G_1 = G_0 = 0$.

3.4. The constructed domains D_0, D_1 can be easily approximated in M by nondegenerate domains with C^∞-boundaries; this also demonstrates that M is not reducing. It is convenient first to decrease a little the radius r ($= \pi/2$) of the circle D_0 so that the circumference of D_0 becomes strictly convex. Then the rulings of the cone and the diameters of its base can be replaced by the circumferences of small curvature lying in vertical planes to provide ∂D_0 and ∂D_1 with positive curvature at each point. (At this stage, D_0 will look like a lense.) A proper smoothing of ∂D_0 and ∂D_1 can be now easily constructed.

When the described variation is small enough, the inequality $G_1 > G_0$ will hold for the new D_0 and D_1 by continuity.

3.5. We construct now the space M_2 described in §1.6.

Take first a segment $[-p^2, -q^2] \subset (-u^2, -v^2)$ such that $p^2/q^2 > 2$. In the manifold $P^{n-1}_{-p^2} \times R$, let us introduce a metric $ds^2 = d\sigma^2 \operatorname{ch}^2 qh + dh^2$ where $d\sigma^2$ is the metric in $P^{n-1}_{-p^2}$ and $h \in R$. One can easily check that for any geodesic $P^1_{-p^2} \subset P^{n-1}_{-p^2}$, the 2-dimensional submanifold $P^1_{-p^2} \times R$ (a vertical plane) has constant interior curvature $-q^2$. Due to symmetry, any submanifold $P^d_{-p^2} \times R$, $0 \leqslant d \leqslant n-2$, is completely geodesic. Therefore the sectional curvatures in vertical directions are equal to $-q^2$. Since the hypersurface $P^{n-1}_{-p^2} \times 0$ is completely geodesic, the sectional curvature in any 2-dimensional direction tangent to that hypersurface is equal to $-p^2$. Due to symmetry, all sectional curvatures at any point in $P^{n-1}_{-p^2} \times 0$ vary within the segment $[-p^2, -q^2]$. Then, in a small δ-neighborhood of $P^{n-1}_{-p^2} \times 0$, the sectional curvatures vary within $[-u^2, -v^2]$. We denote such a δ-neighborhood $P^{n-1}_{-p^2} \times (-\delta, \delta) \subset P^{n-1}_{-p^2} \times R$ by M_2.

By a reason similar to §3.2, it suffices to consider only the case $n = 3$. We shall assume this further on.

3.6. As a (degenerate) domain D_0, we take a circle of a radius r in the surface $P^2_{-p^2} \times 0$. Then $G_0 = 4\pi \cosh pr$. (G_0 is proportional to the total curvature $t = 2\pi \cosh pr$ of the boundary of the circle D_0, and $G_0 = 4\pi$ for $t = 2\pi$, i.e., for $r = 0$ or $p = 0$.)

Put for short

$$\cosh pr = P, \quad \cosh qr = Q,$$

$$\frac{1}{2}\left(1 + \frac{q^2}{p^2 - q}\right) = y, \quad \frac{P^2 - 2P + Q^2}{P^2 - Q^2} = y_r.$$

One can check that $\lim_{r\to 0} y_r = q^2/(p^2 - q^2) \in (0, 1)$ since $p^2/q^2 > 2$. Then $0 < y_r < y < 1$ for sufficiently small $r > 0$.

3.7. Let c be the center of D_0. When r is sufficiently small, in M_2 there exists a point $b \neq c$ such that $bc \perp P^2_{-p^2} \times 0$, the triangle abc exists in M_2 for any point a in the circumference of D_0, and the angle $cab = \cos^{-1} y$. We denote by D_1 the domain bounded by the circle D_0 and the cone consisting of the segments ab. (See Fig. 3 disregarding P^1_1 and replacing P^2_1 by $P^2_{-p^2}$.)

The triangles abc lie in vertical planes and bound there figures isometric to a triangle in $P^2_{-q^2}$. Therefore the angle $abc = \sin^{-1}\sqrt{1 - Q^2 + y^2Q^2}$. Now $G_1 = 2\pi(1 - \sqrt{1 - Q^2 + y^2Q^2}\,) + 2\pi P(1 + y)$ where each of the two addends represents a portion of the integral Gauss curvature: first at the vertex b and second along the circular arc. (The second addend is calculated as G_0 in §3.6.)

3.8. Thus

$$(G_1 - G_0)/2\pi = 1 - P + Py - \sqrt{1 - Q^2 + y^2Q^2}\,. \tag{3.8.1}$$

Notice that y_r and 1 are roots of the polynomial $f(x) = (P^2 - Q^2)x^2 - 2P(P - 1)x + P^2 - 2P + Q^2$. Since $P^2 - Q^2 > 0$ and $y_r < y < 1$, one has $f(y) < 0$. This implies

$$1 + y^2P^2 - 2yP(P - 1) + P^2 - 2P < 1 - Q^2 + y^2Q^2,$$
$$(1 - P + yP)^2 < 1 - Q^2 + y^2Q^2,$$
$$1 - P + yP < \sqrt{1 - Q^2 + y^2Q^2}\,,$$

which together with (3.8.1) results in $G_1 < G_0$.

As in §3.4, D_0 and D_1 here can be easily replaced by nondegenerate domains with C^∞-boundaries showing that M_2 is nonenlarging.

References

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